Minimax Theory and its Applications
Volume 8 (2023), No. 1, 121–138

Differential Games and Hamilton-Jacobi-Isaacs Equations in Metric Spaces

Qing Liu, Xiaodan Zhou
Okinawa Institute of Science and Technology Graduate University, Okinawa, Japan
qing.liu@oist.jp, xiaodan.zhou@oist.jp

Received: December 21, 2021
Accepted: May 10, 2022

This paper is concerned with a game-based interpretation of Hamilton-Jacobi-Isaacs equations in metric spaces. We construct a two-person continuous-time game in a geodesic space and show that the value function, defined by an explicit representation formula, is the unique solution of the Hamilton-Jacobi equation. Our result develops, in a general geometric setting, the classical connection between differential games and the viscosity solutions to possibly nonconvex Hamilton-Jacobi equations.

Keywords: Hamilton-Jacobi equations, metric space, differential games, viscosity solutions.

2010 Mathematics Subject Classification: 35Q91, 35R15, 49L25.

1. Introduction

We study a class of Hamilton-Jacobi equation

$$\partial_t u + H(x, t, |\nabla u|) = 0 \quad \text{in } \mathbf{X} \times (0, \infty) \tag{1}$$

in a complete metric space $(\mathbf{X}, d)$ with an initial value

$$u(x, 0) = u_0(x), \quad x \in \mathbf{X}, \tag{2}$$

where $H : \mathbf{X} \times (0, \infty) \times [0, \infty) \to \mathbb{R}$ is a given continuous function called the Hamiltonian of the equation, $u_0 : \mathbf{X} \to \mathbb{R}$ is assumed to be a uniformly continuous function in $\mathbf{X}$, and $|\nabla u|$ denotes a generalized notion, in general metric spaces, of the usual gradient norm of a differentiable function u in $\mathbb{R}^n$.

Throughout this paper, we assume that $(\mathbf{X}, d)$ is a geodesic space; namely for any $x, y \in \mathbf{X}$, there exists a Lipschitz curve $\gamma : [0, s] \to \mathbf{X}$ such that $\gamma(0) = x$, $\gamma(s) = y$ and $d(x, y) = s$. We denote the length s of γ by $\ell(\gamma)$. Also, in order to avoid trivial situation, we always consider non-singleton spaces, i.e., there are at least two distinct points in $\mathbf{X}$.

We aim to provide a differential game interpretation of (1) whose Hamiltonian H is of the form

$$H(x, t, p) = \max_{a \in A} \min_{b \in B} \{f(x, t, a, b)p - g(x, t, a, b)\} \tag{3}$$

for $x \in \mathbf{X}$, $t \in (0, \infty)$ and $p \in [0, \infty)$, where $A \subset \mathbb{R}^{n_1}, B \subset \mathbb{R}^{n_2}$ are compact subsets with $n_1, n_2 \geq 1$ and

$$f : \mathbf{X} \times (0, \infty) \times A \times B \to [0, \infty), \quad g : \mathbf{X} \times (0, \infty) \times A \times B \to \mathbb{R}$$

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / \$ 2.50

are given continuous functions satisfying more assumptions to be introduced later in Section 2. The connection between differential games and such type of Hamilton-Jacobi-Isaacs equations is well understood in the Euclidean spaces; see for example [8, 9, 14, 22]. Stochastic differential games associated to Hamilton-Jacobi equations in infinite-dimensional spaces are studied in [20, 23] etc.

In recent years, the study on Hamilton-Jacobi equations in general metric spaces has attracted great attention for applications in optimal transport [3, 24], mean field games [7], topological networks [1, 13, 21] and many other fields. Ambrosio and Feng [2] establish a viscosity approach to (1) for a class of convex Hamiltonians in complete geodesic spaces, while later in [10, 11] Gangbo and Święch propose a generalized notion of viscosity solutions and show uniqueness and existence of the solutions to more general equations in geodesic or length space without the convexity assumption on H. This definition is based on the local slope of locally Lipschitz functions. Stability and convexity of such solutions are studied respectively in [19] and in [15]. We also refer to [5] for further developments on viscous Hamilton-Jacobi equations in metric measure spaces.

Meanwhile, using curves in $\mathbf{X}$, Giga et al. in [12] provide a different notion of metric viscosity solutions to stationary eikonal equations and prove well-posedness for the Dirichlet problem. This curve-based definition of such viscosity solutions requires very weak structure of the space. The same approach is used in [18] to study the evolution equation (1) with the Hamiltonian $H(x, p)$ convex in p. It is also applied to construct unique solutions of the eikonal equation on fractals like the Sierpinski gasket [6].

In the case of eikonal equations, both notions of solutions described above turn out to be equivalent, as recently shown in [16]. For time-dependent equations with convex coercive Hamiltonian, the equivalence can be obtained via the Hopf-Lax formula adapted to metric spaces; we refer the reader to [11] for details.

The purpose of the present work is to develop the classical game-theoretic representation theorem for time-dependent Hamilton-Jacobi equations in the case of metric spaces. As shown in (3), the Hamiltonian we consider here is nor necessarily convex in p due to the max min structure. We construct a two-person continuous-time game in a geodesic space, whose value function agrees with the unique solution of (1)–(2). The notion of solutions we adopt is the slope-based one proposed in [11]. We will go over its precise definition and a comparison principle in Section 2.

Our result is in the spirit of [9, 22] etc. in the Euclidean space, but our settings here are different due to the general geometric structure of the space. To be more precise, let us briefly introduce our game below; more details can be found in Section 3. Set

$$\mathcal{A} = \{\alpha : (0, \infty) \to A \text{ measurable}\}, \quad \mathcal{B} = \{\beta : (0, \infty) \to B \text{ measurable}\} \tag{4}$$

and let Θ_B denote the set of non-anticipating strategies of Player B. Here, the definition of non-anticipating strategies follows the standard game setting (cf. [4]). We say that a mapping $\theta : \mathcal{A} \to \mathcal{B}$ is a non-anticipating strategy of Player B and write $\theta \in \Theta_B$ if the following property holds. For any $\alpha_1, \alpha_2 \in \mathcal{A}$ and $t > 0$, $\theta[\alpha_1](t) = \theta[\alpha_2](t)$ provided that $\alpha_1 = \alpha_2$ in $(0, t)$. It means that Player B can only make decisions based on the history without having any information about the future. One can symmetrically define the set of non-anticipating strategies of Player A, denoted by Θ_A.

In addition to controls from $\mathcal{A}$, Player A also has a right to choose a Lipschitz curve ξ that satisfies

$$|\xi'|(\tau) = f(\xi(\tau), t-\tau, \alpha(\tau), \theta[\alpha](\tau)) \quad a.e.\ \tau \in (0,t),$$

with $\xi(0) = x$. It plays the role of state equation in this game for the given controls $\alpha \in \mathcal{A}$ and $\theta[\alpha] \in \mathcal{B}$. In general one cannot expect such solutions (curves) are unique in metric spaces, in contrast to the Euclidean case, where we have an initial value problem for ordinary differential equations. In fact, there always exists a solution along every geodesic in $\mathbf{X}$ connected to x; see Proposition 3.1. We denote by $\mathbf{S}(\alpha, \theta[\alpha])$ the set of all solutions. In order for the game to proceed, we leave it to Player A to determine the curve ξ optimal for her, which makes a major difference from the standard game setting in the Euclidean space.

The value function of our game starting from $x \in \mathbf{X}$ with duration $t \geq 0$ is defined to be

$$u(x,t) = \sup_{\theta \in \Theta_B} \inf_{\substack{\alpha \in \mathcal{A} \\ \xi \in \mathbf{S}(\alpha, \theta[\alpha])}} J(x,t;\xi,\alpha,\theta[\alpha]), \tag{5}$$

where J, called the pay-off function of the game, is given by

$$J(x,t;\xi,\alpha,\beta) = u_0(\xi(t)) + \int_0^t g(\xi(\tau), t-\tau, \alpha(\tau), \beta(\tau))\, d\tau. \tag{6}$$

Our main result, Theorem 4.1, states that u given by (5) is a solution of (1)–(2). We prove this result by adapting the standard arguments in the Euclidean case. We first establish the so-called dynamic programming principle for our game and then show the uniform continuity of u. The dynamic programming principle, which connects the game to the PDE, enables us to verify the subsolution and supersolution property of u. We finally apply the comparison principle for (1) to conclude the entire proof. Although the proof streamlines that in the Euclidean case, one needs to overcome difficulty caused by the general structure of the space. In order to show the uniform continuity of the value function u, we further impose an assumption on the uniform positivity of f (see (H3) below) and use a different method of constructing strategies or controls due to the possible loss of linear structure of space. Besides, when verifying the property of viscosity solutions, one also needs to carefully handle the additional choice of curves ξ, which does not appear in the Euclidean case. We refer the reader to Section 4 for details.

Our representation formula for (1) with H given by (3) complements the results for convex Hamiltonians in [12, 18]. However, as mentioned above, the game is quite different from its Euclidean prototype due to the extra determination of curves ξ. For example, it is not clear to us how one can understand the Nash equilibrium in the current situation. If we consider (5) as an upper value function, then a straightforward definition of a lower game value should read

$$\tilde{u}(x,t) = \inf_{\theta \in \Theta_A} \sup_{\substack{\beta \in \mathcal{B} \\ \xi \in \mathbf{S}(\theta[\beta], \beta)}} J(x,t;\xi,\theta[\beta],\beta) \tag{7}$$

for $x \in \mathbf{X}$ and $t \geq 0$, where Θ_A is the set of non-anticipating strategies of Player A.

However, since Player B is entitled to pick ξ this time, $\tilde{u}$ satisfies a totally different Hamilton-Jacobi-Isaacs equation. One cannot expect that $u = \tilde{u}$ holds and a Nash equilibrium exists under our current assumptions. See Example 4.6 for a concrete example about this issue.

Although we assume $(\mathbf{X}, d)$ to be geodesic in this work, it is actually possible to generalize our results for any complete length space. One can further extend them to a complete rectifiably connected metric space by using the intrinsic metric to introduce proper notions solutions; see [16] for detailed arguments.

Acknowledgments. The work of the first author was supported by JSPS Grants-in-Aid for Scientific Research (No. 19K03574, No. 22K03396). The work of the second author was supported by JSPS Grant-in-Aid for Research Activity Start-up (No. 20K22315).

2. Assumptions and definition of solutions

Let us review the notion of viscosity solutions of (1) in a complete metric space $\mathbf{X}$. We impose the following assumptions related to the Hamiltonian (3).

(H1) $f, g : \mathbf{X} \times (0, \infty) \times A \times B \to \mathbb{R}$ are both continuous with $f \geq 0$ and there exists a modulus of continuity ω and $L > 0$ such that

$$\max\{|f(x,t,a,b) - f(x,s,a,b)|,\ |g(x,t,a,b) - g(x,s,a,b)|\} \leq \omega(|t-s|),$$

and

$$\max\{|f(x,t,a,b) - f(y,t,a,b)|,\ |g(x,t,a,b) - g(y,t,a,b)|\} \leq Ld(x,y)$$

for all $x, y \in \mathbf{X}$, $t, s \in (0, \infty)$, $a \in A$ and $b \in B$.

(H2) $f, g : \mathbf{X} \times (0, \infty) \times A \times B \to \mathbb{R}$ are both uniformly bounded; namely, there exists $M > 0$ such that $\max\{f(x,t,a,b), |g(x,t,a,b)|\} \leq M$ for all $x \in \mathbf{X}$, $t \in (0, \infty)$, $a \in A$ and $b \in B$.

(H3) There exists $m > 0$ such that $f(x,t,a,b) \geq m$ for all $x \in \mathbf{X}$, $t \in (0, \infty)$, $a \in A$ and $b \in B$.

The above assumptions immediately imply that

$$H(x,t,p) - H(y,s,p) \leq (p+1)(Ld(x,y) + \omega(|t-s|))$$

and

$$H(x,t,p) - H(x,t,q) \leq M|p-q|$$

for all $x, y \in \mathbf{X}$, $t, s \in (0, \infty)$ and $p, q \in [0, \infty)$.

Our games later can be set up for f and g under weaker assumptions than (H1). We assume this strong condition in order to apply the comparison principle established in [11] to get the connection between the game value and the unique solution.

We next go over the definition proposed in [10]. Hereafter, for simplicity of notation, we write $Q := \mathbf{X} \times (0, \infty)$. For $t \geq 0$ and $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbf{X} \times \{t\})$, define the *local spatial slope* of u, at $(x,t) \in Q$ as

$$|\nabla u|(x,t) := \limsup_{y \to x} \frac{|u(y,t) - u(x,t)|}{d(x,y)}.$$

Moreover, we define the *local spatial super-* and *subslopes* to be

$$|\nabla^{\pm} u|(x,t) := \limsup_{y\to x} \frac{[u(y,t)-u(x,t)]_{\pm}}{d(x,y)}$$

with $[a]_+ := \max\{a, 0\}$ and $[a]_- := -\min\{a, 0\}$ for any $a \in \mathbb{R}$.

For any open set $\mathcal{O} \subset Q$, let

$$\mathcal{C}(\mathcal{O}) := \{\psi \in \mathrm{Lip_{loc}}(\mathcal{O}) : \partial_t \psi \text{ is continuous in } \mathcal{O}\}$$

$$\overline{\mathcal{C}}(\mathcal{O}) := \{\psi \in \mathcal{C}(\mathcal{O}) : |\nabla^+\psi| = |\nabla\psi| \text{ and } |\nabla\psi| \text{ is continuous in } \mathcal{O}\},$$

$$\underline{\mathcal{C}}(\mathcal{O}) := \{\psi \in \mathcal{C}(\mathcal{O}) : |\nabla^-\psi| = |\nabla\psi| \text{ and } |\nabla\psi| \text{ is continuous in } \mathcal{O}\}.$$

In this work we call $|\nabla^+ u|$ and $|\nabla^- u|$ the (local) super- and sub-slopes of u respectively; they are also named super- and sub-gradient norms in the literature (cf. [17]).

Below we recall from [11, Definition 2.5] the definition of slope-based metric viscosity solutions of a general class of Hamilton-Jacobi equations. In this work we call them *solutions* for simplicity. The Hamiltonian $H(x,t,p)$ in (3) is originally defined only for $p \geq 0$, but we can naturally extend it to $p < 0$ without changing its form. As pointed out in [11], the definition of solutions actually works for any continuous extension of H. Under our extension of H, we take

$$H_a(x,t,p) = \inf_{|q-p|\leq a} H(x,t,q), \quad H^a(x,t,p) = \sup_{|q-p|\leq a} H(x,t,q)$$

for any $(x,t,p) \in Q \times [0,\infty)$ and $a \geq 0$.

Definition 2.1. (Slope-based solutions) An upper semicontinuous (resp., lower semicontinuous) function u in Q is called a *subsolution* (resp., supersolution) of (1) if

$$\partial_t\psi_1(x_0,t_0) + \partial_t\psi_2(x_0,t_0) + H_{|\nabla\psi_2|^*(x_0,t_0)}(x_0, t_0, |\nabla\psi_1|(x_0,t_0)) \leq 0 \tag{8}$$

$$\big(\text{resp., } \partial_t\psi_1(x_0,t_0) + \partial_t\psi_2(x_0,t_0) + H^{|\nabla\psi_2|^*(x_0,t_0)}(x_0, t_0, |\nabla\psi_1|(x_0,t_0)) \geq 0\big) \tag{9}$$

holds for any $\psi_1 \in \underline{\mathcal{C}}(Q)$ (resp., $\psi_1 \in \overline{\mathcal{C}}(Q)$) and $\psi_2 \in \mathcal{C}(Q)$ such that $u - \psi_1 - \psi_2$ attains a local maximum (resp., minimum) at a point $(x_0,t_0) \in Q$, where, for any $(x,t,p) \in Q \times [0,\infty)$,

$$|\nabla\psi_2|^*(x,t) := \lim_{\delta\to 0} \sup \{|\nabla\psi_2|(y,s) : \ d(x,y) + |t-s| \leq \delta\}.$$

We say that a continuous function u in Q is a solution of (1) if it is both a subsolution and a supersolution of (1).

Concerning the test functions in a general geodesic or length space $(\mathbf{X}, d)$, it is known that, for any $k \geq 0$, $x_0 \in \mathbf{X}$, the function $(x,t) \mapsto \psi(t) + k\varphi(d(x,x_0))$ (resp., $(x,t) \mapsto \psi(t) - k\varphi(d(x,x_0))$) belongs to the class $\underline{\mathcal{C}}(\mathcal{O})$ (resp., $\overline{\mathcal{C}}(\mathcal{O})$) in $\mathcal{O} \subset \mathbf{X} \times (0,\infty)$ provided that $\psi \in C^1((0,\infty))$ and $\varphi \in C^1([0,\infty))$ satisfies $\varphi'(0) = 0$ and $\varphi' \geq 0$; see details in [10, Lemma 7.2] and [11, Lemma 2.3].

Under (H1) and (H2), our Hamiltonian H (continuously extended to $\mathbf{X}\times(0,\infty)\times\mathbb{R}$) is uniformly continuous and satisfies

$$|H(x,t,p)-H(y,t,p)|\leq Ld(x,y)(|p|+1)$$

and

$$|H(x,t,p)-H(x,t,q)|\leq M|p-q|$$

for all $x,y\in\mathbf{X}$ and $t>0$ and $p,q\in\mathbb{R}$. Then the comparison principle given in [11, Proposition 3.3] holds for (1). The following comparison result is an adapted (stronger) version of [11, Proposition 3.3] that applies to our problem.

Theorem 2.2. (Comparison principle) *Let* $(\mathbf{X},d)$ *be a complete geodesic space. Assume that* (H1) *and* (H2) *hold. Let* u *and* v *be a subsolution and a supersolution of* (1) *respectively. Assume that for any* $T>0$, *there exist* $x_0\in\mathbf{X}$ *and* $C>0$ *such that*

$$\max\{u(x,t),-v(t,x)\}\leq C(1+d(x,x_0))$$

for all $(x,t)\in\mathbf{X}\times[0,T)$. *Let* u_0 *be uniformly continuous in* $\mathbf{X}$. *If*

$$\sup_{x\in K}\left(|u(x,t)-u_0(x)|+|v(x,t)-u_0(x)|\right)\to 0\quad \textit{as } t\to 0$$

holds for any bounded set $K\subset\mathbf{X}$, *then* $u\leq v$ *holds in* $\mathbf{X}\times[0,\infty)$.

3. The game setting

Suppose that there are two players, Player A and Player B, choosing functions respectively from the control sets $\mathcal{A}$ and $\mathcal{B}$ given by (4). For any given $x\in\mathbf{X}$ and fixed $\alpha\in\mathcal{A}$ and $\beta\in\mathcal{B}$, let $\xi:[0,\infty)\to\mathbf{X}$ be a Lipschitz curve satisfying

$$|\xi'|(\tau)=f(\xi(\tau),t-\tau,\alpha(\tau),\beta(\tau))\quad a.e.\ \tau\in(0,t), \tag{10}$$

and

$$\xi(0)=x. \tag{11}$$

If $\mathbf{X}$ is not a singleton, then such ξ does exist for any given $\alpha\in\mathcal{A}$ and $\beta\in\mathcal{B}$. We can reparametrize a geodesic connecting x to any different point.

Proposition 3.1. (Existence of game trajectories) *Let* $(\mathbf{X},d)$ *be a complete geodesic space. Assume that* (H1), (H2) *and* (H3) *hold. Let* $x\in\mathbf{X}$, $t\geq 0$, $\alpha\in\mathcal{A}$ *and* $\beta\in\mathcal{B}$. *Then there exists a Lipschitz solution* ξ *of* (10) *and* (11). *Moreover, for any* $y\in\mathbf{X}\setminus\{x\}$, *there exist* $\tau_1\in[d(x,y)/M,d(x,y)/m]$ *and a solution* ξ *of* (10) *and* (11) *satisfying*

$$\int_0^{\tau_1}|\xi'|(\tau)\,d\tau=\ell(\xi|_{[0,\tau_1]})=d(x,y). \tag{12}$$

Proof. Recall that we have $\mathbf{X}\neq\{x\}$. Then there exists at least one geodesic γ connecting x to a different point y. Without loss of generality, we may assume that $\ell(\gamma)=d(x,y)=1$ and $d(x,\gamma(\sigma))=\sigma$ for all $\sigma\in[0,1]$. Since γ is isomeric to the interval $[0,1]\subset\mathbb{R}$, we essentially need to solve the ODE

$$z'(\tau)=F(z(\tau),t-\tau,\alpha(\tau),\beta(\tau))$$

with $z(0)=0$ for a Lipchitz solution $z(\tau)\in[0,1]$, where

$$F:[0,1]\times(0,\infty)\times A\times B\to[0,\infty)$$

is given by $F(\sigma,s,a,b)=f(\gamma(\sigma),s,a,b)$.

The existence of solutions is guaranteed by the continuity of f in (H1). Note that (H2) and (H3) imply the existence of $d(x,y)/M \le \tau_1 \le d(x,y)/m$ such that

$$z(\tau_1) = 1. \tag{13}$$

We then turn to solve $z'(\tau) = -F(z(\tau), t-\tau, \alpha(\tau), \beta(\tau))$ with (13) to extend the domain of the function z to $(0,\tau_2)$ if there exists $\tau_2 \in (\tau_1, t)$ satisfying $z(\tau_2) = 0$. We can repeat this process to construct a global Lipschitz function $z : [0,t] \to [0,1]$.

Setting $$\xi(\tau) = \gamma(z(\tau)) \quad \text{for } \tau \in [0,t],$$

we easily see that ξ is a Lipschitz solution of (10)–(11). The equality (12) follows immediately from the fact that ξ is a reparametrization of the geodesic γ. □

In general, we cannot expect uniqueness of solutions because of the arbitrariness of y, which is one major difference from the usual state equation in the Euclidean space.

For any $(x,t) \in \mathbf{X} \times [0,\infty)$, $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$, we collect all such ξ by taking

$$\mathbf{S}(x,t;\alpha,\beta) = \{\xi \in \mathrm{Lip}\,([0,t] : \xi \text{ is a solution of (10)–(11) for given } \alpha \in \mathcal{A} \text{ and } \beta \in \mathcal{B}\}.$$

Below we write $\mathbf{S}(\alpha,\beta)$ to denote this set if there is no ambiguity.

Let Θ_B denote the set of all non-anticipating strategies for Player B. The pay-off function and value function of the game are given by (6) and (5), that is,

$$J(x,t;\xi,\alpha,\beta) = u_0(\xi(t)) + \int_0^t g(\xi(\tau), t-\tau, \alpha(\tau), \beta(\tau))\, d\tau$$

for any $(x,t) \in \overline{Q}$, $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $\xi \in \mathbf{S}(x,t;\alpha,\beta)$ and

$$u(x,t) = \sup_{\theta \in \Theta_B} \inf_{\substack{\alpha \in \mathcal{A} \\ \xi \in \mathbf{S}(\alpha,\theta[\alpha])}} J(x,t;\xi,\alpha,\theta[\alpha]).$$

In comparison with the standard Euclidean case, Player A has an additional right to choose a curve ξ in this game.

Let us consider the dynamic programming principle, which actually resembles that in the Euclidean case. We again need to pay extra attention to the presence of ξ chosen by Player A.

Theorem 3.2. (Dynamic programming principle) *Let* $(\mathbf{X}, d)$ *be a complete geodesic space. Assume that* (H1) *and* (H2) *hold. Let* u *be the value function as defined in* (5)*. Then for each fixed* $(x,t) \in Q$ *we have*

$$u(x,t) = \sup_{\theta \in \Theta_B} \inf_{\substack{\alpha \in \mathcal{A} \\ \xi \in \mathbf{S}(x,s;\alpha,\theta[\alpha])}} \left\{ u(\xi(s), t-s) + \int_0^s g(\xi(\tau), t-\tau, \alpha(\tau), \theta[\alpha](\tau))\, d\tau \right\} \tag{14}$$

holds for all $0 \le s < t$.

Proof. We denote by R the right hand side of (14). We first show that $u(x,t) \ge R$.

For any $\varepsilon > 0$, there exists $\theta_1 \in \Theta_B$ such that for any $\alpha_1 \in \mathcal{A}$ and $\xi_1 \in \mathbf{S}(x, t; \alpha_1, \theta_1[\alpha_1])$,

$$R \leq u(\xi_1(s), t-s) + \int_0^s g(\xi_1(\tau), t-\tau, \alpha_1(\tau), \theta_1[\alpha_1](\tau))\, d\tau + \varepsilon. \tag{15}$$

By (5), from $\xi_1(s) \in \mathbf{X}$ with duration $t-s$, there also exists a strategy $\theta_2 \in \Theta_B$ such that for any $\alpha_2 \in \mathcal{A}$ and $\xi_2 \in \mathbf{S}(\xi_1(s), t-s; \alpha_2, \theta_2[\alpha_2])$, there holds

$$u(\xi_1(s), t-s) \leq u_0(\xi_2(t-s)) + \int_0^{t-s} g(\xi_2(\tau), t-s-\tau, \alpha_2(\tau), \theta_2(\alpha_2)(\tau))\, d\tau + \varepsilon. \tag{16}$$

Fix an arbitrary $\alpha \in \mathcal{A}$. Let

$$\alpha_1(\tau) = \alpha(\tau) \text{ for } \tau \in (0, s), \quad \text{and} \quad \alpha_2(\tau) = \alpha(\tau + s) \text{ for } \tau \in (0, t-s).$$

We can thus take $\hat{\theta} \in \Theta_B$ to satisfy

$$\hat{\theta}[\alpha](\tau) = \begin{cases} \theta_1[\alpha_1](\tau) & \text{for } \tau \in (0, s], \\ \theta_2[\alpha_2](\tau - s) & \text{for } \tau \in (s, t). \end{cases}$$

For any $\xi \in \mathbf{S}(x, t; \alpha, \theta[\alpha])$, applying (15) and (16) respectively with

$$\xi_1(\tau) = \xi(\tau) \text{ for } \tau \in (0, s), \quad \text{and} \quad \xi_2(\tau) = \xi(\tau + s) \text{ for } \tau \in (0, t-s),$$

we obtain $$R \leq u(\xi(s), t-s) + \int_0^s g(\xi(\tau), t-\tau, \alpha(\tau), \hat{\theta}[\alpha](\tau))\, d\tau + \varepsilon.$$

and $$u(\xi(s), t-s) \leq u_0(\xi(t)) + \int_s^t g(\xi(\tau), t-\tau, \alpha(\tau), \hat{\theta}[\alpha](\tau))\, d\tau + \varepsilon.$$

We combine these two relations to get

$$R \leq u_0(\xi(t)) + \int_0^t g(\xi(\tau), t-\tau, \hat{\theta}(\beta)(\tau), \beta(\tau))\, d\tau + 2\varepsilon,$$

which yields $R \leq u(x, t) + 2\varepsilon$ by (5) again. We obtain the desired inequality by sending $\varepsilon \to 0$.

We next show $u(x, t) \leq R$. In view of (5), for any $\varepsilon > 0$, there exists $\hat{\theta} \in \Theta_B$ such that for any $\alpha \in \mathcal{A}$ and $\xi \in \mathbf{S}(x, t; \alpha, \hat{\theta}[\alpha])$,

$$u(x, t) \leq u_0(\xi(t)) + \int_0^t g(\xi(\tau), t-\tau, \alpha(\tau), \hat{\theta}[\alpha](\tau))\, d\tau + \varepsilon.$$

Then, for any $\alpha_1 \in \mathcal{A}$ and $\xi_1 \in \mathbf{S}(\xi(s), t-s; \alpha_1, \hat{\theta}[\alpha_1])$, taking

$$\alpha(\tau) = \alpha_1(\tau - s), \quad \xi(\tau) = \xi_1(\tau - s) \quad \text{for } \tau \in (s, t],$$

we have $$\begin{aligned} u(x, t) &- \int_0^s g(\xi(\tau), t-\tau, \alpha(\tau), \hat{\theta}[\alpha](\tau))\, d\tau - \varepsilon \\ &\leq u_0(\xi(t)) + \int_s^t g(\xi(\tau), t-\tau, \alpha(\tau), \hat{\theta}[\alpha](\tau))\, d\tau \\ &\leq u_0(\xi_1(t-s)) + \int_0^{t-s} g(\xi_1(\tau), t-s-\tau, \alpha_1(\tau), \hat{\theta}[\alpha_1](\tau))\, d\tau. \end{aligned}$$

It follows from (5) once again that

$$u(x,t) - \int_0^s g(\xi(\tau), t-\tau, \alpha(\tau), \hat{\theta}[\alpha](\tau))\, d\tau - \varepsilon \le u(\xi(s), t-s).$$

We therefore obtain

$$u(x,t) \le \inf_{\substack{\alpha\in\mathcal{A} \\ \xi\in\mathbf{S}(x,s;\alpha,\hat{\theta}[\alpha])}} \left\{ u(\xi(s), t-s) + \int_0^s g(\xi(\tau), t-\tau, \alpha(\tau), \hat{\theta}[\alpha](\tau))\, d\tau \right\} + \varepsilon,$$

which further implies $u(x,t) \le R + \varepsilon$. We thus obtain $u(x,t) \le R$ by letting $\varepsilon \to 0$. □

4. Representation theorem

This section is devoted to showing our main result as below.

Theorem 4.1. *Let* $(\mathbf{X}, d)$ *be a complete geodesic space. Assume that* (H1), (H2) *and* (H3) *hold. Assume that* u_0 *is uniformly continuous in* $\mathbf{X}$*. Let* u *be the value function defined as in* (5)*. Then* u *is a solution of* (1) *and* (2) *with* H *given by* (3)*. In addition, it is the unique solution that is uniformly continuous in* $\mathbf{X} \times [0,T]$ *for any* $T > 0$*.*

4.1. Uniform continuity

We first look into the uniform continuity of u.

Proposition 4.2. (Uniform continuity of game value) *Assume that the assumptions in Theorem* 4.1 *hold. Then* u *is uniformly continuous in* $\mathbf{X} \times [0,T]$ *for any* $T > 0$*.*

Proof. Let ω_0 denote the modulus of continuity for u_0. We first prove the uniform continuity of u in space. Fix $(x_1, t), (x_2, t) \in \mathbf{X} \times [0,T]$ arbitrarily. By definition of u, for any $\varepsilon > 0$ and $\theta \in \Theta_B$, there exist $\alpha_1 \in \mathcal{A}$ and $\xi_1 \in \mathbf{S}(x_1, t; \alpha_1, \theta[\alpha_1])$ such that

$$u(x_1, t) \ge u_0(\xi_1(t)) + \int_0^t g(\xi_1(\tau), t-\tau, \alpha_1(\tau), \theta[\alpha_1](\tau))\, d\tau - \varepsilon. \tag{17}$$

If $d(x_1, x_2) \le mt$, following Proposition 3.1, we can take $\alpha_0 \in \mathcal{A}$ and reparametrize a geodesic γ joining x_1 and x_2 to find a Lipschitz solution ξ_0 of

$$|\xi_0'|(\tau) = f(\xi_0(\tau), t-\tau, \alpha_0(\tau), \theta[\alpha_0](\tau)) \ge m \quad a.e.\ \tau \in (0, \tau_1)$$

with $\xi_0(0) = x_2$ and $\xi_0(\tau_1) = x_1$, where $\tau_1 = \inf\{\tau > 0 : \xi_0(\tau) = x_1\}$.

It is clear that
$$\tau_1 \le \frac{d(x_1, x_2)}{m} \quad \text{for any } \theta \in \Theta_B. \tag{18}$$

Let us take

$$\alpha_2(\tau) = \begin{cases} \alpha_0(\tau) & \text{for } \tau \in (0, \tau_1), \\ \alpha_1(\tau - \tau_1) & \text{for } \tau \in (\tau_1, t), \end{cases} \qquad \xi_2(\tau) = \begin{cases} \xi_0(\tau) & \text{for } \tau \in [0, \tau_1], \\ \xi_1(\tau - \tau_1) & \text{for } \tau \in (\tau_1, t]. \end{cases}$$

It is clear that $\xi_2 \in \mathbf{S}(x_2, t; \alpha_2, \theta[\alpha_2])$ with $\xi_2(t) = \xi_1(t-\tau_1)$. It thus follows from (H2) that

$$\begin{aligned} u(x_2,t) &\leq u_0(\xi_2(t)) + \int_0^t g(\xi_2(\tau), t-\tau, \alpha_2(\tau), \theta[\alpha_2](\tau))\, d\tau \\ &\leq u_0(\xi_1(t-\tau_1)) + M\tau_1 + \int_{\tau_1}^t g(\xi_2(\tau), t-\tau, \alpha_2(\tau), \theta[\alpha_2](\tau))\, d\tau \\ &= u_0(\xi_1(t-\tau_1)) + M\tau_1 + \int_0^{t-\tau_1} g(\xi_1(\tau), t-\tau_1-\tau, \alpha_1(\tau), \theta[\alpha_1](\tau))\, d\tau. \end{aligned}$$

Combining this with (17), we use (H1) and (H2) to deduce that

$$u(x_1,t) - u(x_2,t) \geq -\omega_0(d(\xi_1(t), \xi_1(t-\tau_1))) - 2M\tau_1 - |t-\tau_1|\omega(\tau_1) - \varepsilon.$$

Applying (H2) again together with (18) and sending $\varepsilon \to 0$, we obtain

$$u(x_1,t) - u(x_2,t) \geq -\omega_0(M\tau_1) - 2M\tau_1 - T\omega(\tau_1) \geq -\omega_x(d(x_1,x_2)),$$

where ω_x is a modulus of continuity defined by

$$\omega_x(\rho) = \omega_0\left(\frac{M\rho}{m}\right) + \frac{2M\rho}{m} + T\omega\left(\frac{\rho}{m}\right) \quad \text{for } \rho \geq 0.$$

Exchanging the roles of x_1 and x_2 in the estimate above, we have

$$u(x_1,t) - u(x_2,t) \leq \omega_x(d(x_1,x_2))$$

for any $x_1, x_2 \in \mathbf{X}$ with $d(x_1,x_2) \leq mt$. Since $(\mathbf{X}, d)$ is a geodesic space, this estimate can be generalized for any $x_1, x_2 \in \mathbf{X}$. We thus get the uniform continuity of $x \mapsto u(x,t)$ in $\mathbf{X} \times [0,T]$.

We next show the continuity of u in time. For any $(x,t) \in \mathbf{X} \times [0,\infty)$ and $\varepsilon > 0$, we still can find $\alpha_1 \in \mathcal{A}$ and $\xi_1 \in \mathbf{S}(x,t;\alpha_1,\theta[\alpha_1])$ for any $\theta \in \Theta_B$ such that (17) holds. By (H2) again, we get

$$u(x,t) \geq u_0(\xi(t)) - \int_0^t M - \varepsilon.$$

Since $d(\xi(t), x) \leq Mt$, it follows that $u(x,t) - u_0(x) \geq -\omega_0(Mt) - Mt - \varepsilon$.

Letting $\varepsilon \to 0$, we are led to $u(x,t) - u_0(x) \geq -\omega_0(Mt) - Mt$.

One can use a similar argument to show that

$$u(x,t) - u_0(x) \leq \omega_0(Mt) + Mt.$$

Hence, we have, for any $(x,t) \in \mathbf{X} \times (0,\infty)$,

$$|u(x,t) - u_0(x)| \leq \omega_0(Mt) + Mt. \tag{19}$$

In general, for any $(x,t_1), (x,t_2) \in \mathbf{X} \times [0,T]$ with $t_1 \leq t_2$, we can apply the same argument above to (14) to deduce that

$$|u(x,t_2) - u(x,t_1)| \leq \omega_x(M(t_2-t_1)) + M(t_2-t_1).$$

Our proof is now complete. □

Our above proof for uniform continuity of value functions is quite different from those in the Euclidean case ([4, Proposition VIII. 1.8] for instance). In the Euclidean space, for a nearly optimal trajectory starting from x_1, we usually choose a “parallel” trajectory from x_2 so that the game values can be compared directly. Such kind of parallel structures no longer exist in general geodesic spaces. In order to achieve our estimate, we instead take a different trajectory, following the one from x_1 as much as possible. This trajectory turns out to be available when the dynamics f satisfies the positivity assumption (H3).

4.2. Supersolution and subsolution properties

We next prove Theorem 4.1, showing that u given by (5) is both a supersolution and a subsolution of (1). At the end we use the comparison principle to conclude the proof.

Although our proof is analogous to the Euclidean version, the control set for Player A is actually different. Player A not only picks a control from $\mathcal{A}$ but also needs to choose a Lipschitz curve ξ from those solutions to (10). (In the Euclidean case, there are no such choices, since the game trajectory is uniquely determined by the state equation once the controls are fixed.) We need to handle this additional control carefully in our verification of subsolution and supersolution properties.

Proposition 4.3. (Supersolution property of game value) *Assume that the assumptions in Theorem* 4.1 *hold. Then* u *is a supersolution of* (1)*.*

Proof. Let us first show that u is a superolution of (1). Suppose that there exist $(x_0, t_0) \in Q = \mathbf{X} \times (0, \infty)$, $\psi_1 \in \overline{\mathcal{C}}(Q)$ and $\psi_2 \in \mathcal{C}(Q)$ such that $u - \psi_1 - \psi_2$ attains a local minimum at (x_0, t_0). Take

$$p_0 = |\nabla^- \psi_1|(x_0, t_0) + |\nabla \psi_2|^*(x_0, t_0). \tag{20}$$

For any $a \in A$, there exists $\hat{b}(a) \in B$ such that

$$\min_{b \in B} \{f(x_0, t_0, a, b)p_0 - g(x_0, t_0, a, b)\} = f(x_0, t_0, a, \hat{b}(a))p_0 - g(x_0, t_0, a, \hat{b}(a)).$$

Since A is compact it follows that for every $r > 0$ small, there exist finitely many points $a_1, a_2, \ldots, a_k \in A$ such that

$$A \subset \bigcup_{i=1}^{k} B_r(a_i).$$

Here $B_r(a_i)$ denotes the open ball centered at a_i with radius r. For any $\varepsilon > 0$, we let $r > 0$ sufficiently small so that $b_i = \hat{b}(a_i)$ satisfies

$$f(x_0, t_0, a, b_i)p_0 - g(x_0, t_0, a, b_i) \le \min_{b \in B} \{f(x_0, t_0, a, b)p_0 - g(x_0, t_0, a, b)\} + \varepsilon$$

for all $a \in B_r(a_i)$. Define $h : A \to B$ by

$$h(a) = b_i, \quad \text{if } a \in B_r(a_i) \setminus \bigcup_{j=1}^{i-1} B_r(a_j)$$

for $i = 1, 2, \ldots, k$.

For any $\alpha \in \mathcal{A}$, we see that $h(\alpha(\cdot)) \in \mathcal{B}$. Therefore by letting

$$\hat{\theta}[\alpha](s) := h(\alpha(s)), \quad s > 0,$$

we have $\hat{\theta} \in \Theta_B$ and

$$\begin{aligned} &f(x_0, t_0, \alpha(s), \hat{\theta}[\alpha](s))p_0 - g(x_0, t_0, \alpha(s), \hat{\theta}[\alpha](s)) \\ &\quad \le \min_{b \in B} \{f(x_0, t_0, \alpha(s), b)p_0 - g(x_0, t_0, \alpha(s), b)\} + \varepsilon \\ &\quad \le \max_{a \in A} \min_{b \in B} \{f(x_0, t_0, a, b)p_0 - g(x_0, t_0, a, b)\} + \varepsilon \end{aligned}$$

for almost all $s \in (0, t_0)$ and any $\alpha \in \mathcal{A}$. This amounts to saying that

$$\begin{aligned} &f(x_0, t_0, \alpha(s), \hat{\theta}[\alpha](s)) - g(x_0, t_0, \alpha(s), \hat{\theta}[\alpha](s)) \\ &\quad \le H(x_0, t_0, p_0) + \varepsilon \quad a.e.\ s \in (0, t_0) \end{aligned} \tag{21}$$

for any $\alpha \in \mathcal{A}$. Now applying (14) with $(x, t) = (x_0, t_0)$ and $\theta = \hat{\theta}$, we can find $\hat{\alpha} \in \mathcal{A}$ and a Lipschitz curve $\xi : [0, t_0] \to \mathbf{X}$ such that

$$|\xi'|(\tau) = f(\xi(\tau), t_0 - \tau, \hat{\alpha}(\tau), \hat{\theta}[\hat{\alpha}](\tau)) \quad a.e.\ \tau \in (0, t_0), \tag{22}$$

and, for $s > 0$ small,

$$u(x_0, t_0) \ge u(y, t_0 - s) + \int_0^s g(\xi(\tau), t_0 - \tau, \hat{\alpha}(\tau), \hat{\theta}[\hat{\alpha}](\tau))\, d\tau - s^2 \tag{23}$$

holds with $y = \xi(s)$. (Note that $y \ne x$ due to (H3).) Since the minimality of $u - \psi_1 - \psi_2$ at (x_0, t_0) implies

$$u(x_0, t_0) - \psi_1(x_0, t_0) - \psi_2(x_0, t_0) \le u(y, t_0 - s) - \psi_1(y, t_0 - s) - \psi_2(y, t_0 - s),$$

it thus follows from (23) that

$$\psi_1(x_0, t_0) + \psi_2(x_0, t_0) \ge \psi_1(y, t_0 - s) + \psi_2(y, t_0 - s) + \int_0^s \hat{g}[\tau]\, d\tau - s^2,$$

where we set
$$\hat{g}[\tau] = g\left(\xi(\tau), t_0 - \tau, \hat{\alpha}(\tau), \hat{\theta}[\hat{\alpha}](\tau)\right). \tag{24}$$

For simplicity of notation, we will also write

$$\hat{f}[\tau] = f\left(\xi(\tau), t_0 - \tau, \hat{\alpha}(\tau), \hat{\theta}[\hat{\alpha}](\tau)\right).$$

We thus get $s_1, s_2 \in (t_0 - s, t_0)$ such that

$$\begin{aligned} &\partial_t \psi_1(y, s_1)s + \partial_t \psi_2(y, s_2)s \\ &\quad \ge -(\psi_1(x_0, t_0) - \psi_1(y, t_0)) - (\psi_2(x_0, t_0) - \psi_2(y, t_0)) + \int_0^s \hat{g}[\tau]\, d\tau - s^2, \end{aligned}$$

which yields

$$\partial_t\psi_1(y,s_1)+\partial_t\psi_2(y,s_2)$$
$$\geq -\left(\frac{[\psi_1(x_0,t_0)-\psi_1(y,t_0)]_+}{d(x_0,y)}+\frac{|\psi_2(x_0,t_0)-\psi_2(y,t_0)|}{d(x_0,y)}\right)\frac{d(x_0,y)}{s}+\frac{1}{s}\int_0^s \hat{g}[\tau]\,d\tau - s.$$

In view of (22), we also have

$$d(x_0,y)\leq \int_0^s \hat{f}[\tau]\,d\tau. \tag{25}$$

We thus get

$$\partial_t\psi_1(y,s_1)+\partial_t\psi_2(y,s_2) \tag{26}$$
$$\geq -\frac{1}{s}\int_0^s \left\{\left(\frac{[\psi_1(x_0,t_0)-\psi_1(y,t_0)]_+}{d(x_0,y)}+\frac{|\psi_2(x_0,t_0)-\psi_2(y,t_0)|}{d(x_0,y)}\right)\hat{f}[\tau]-\hat{g}[\tau]\right\}d\tau - s.$$

Noticing that

$$\limsup_{s\to 0}\frac{[\psi_1(x_0,t_0)-\psi_1(y,t_0)]_+}{d(x_0,y)}\leq |\nabla^-\psi_1|(x_0,t_0),$$

$$\limsup_{s\to 0}\frac{|\psi_2(x_0,t_0)-\psi_2(y,t_0)|}{d(x_0,y)}\leq |\nabla\psi_2|^*(x_0,t_0),$$

and f, g satisfy (H1) and (H2), we have

$$\partial_t\psi_1(y,s_1)+\partial_t\psi_2(y,s_2)$$
$$\geq -\frac{1}{s}\int_0^s \Big(f(x_0,t_0,\hat{\alpha}(\tau),\hat{\theta}[\hat{\alpha}](\tau))p_0 - g(x_0,t_0,\hat{\alpha}(\tau),\hat{\theta}[\hat{\alpha}](\tau))\Big)\,d\tau - o(1),$$

which by (21) yields

$$\partial_t\psi_1(y,s_1)+\partial_t\psi_2(y,s_2)\geq -\frac{1}{s}\int_0^s H(x_0,t_0,p_0)\,d\tau-\varepsilon-o(1)$$

when $s>0$ is small. We recall that p_0 is given as in (20).

Since $\partial_t\psi_1$ and $\partial_t\psi_2$ are continuous, letting $s\to 0$ with $y=\xi(s)\to x_0$ and then $\varepsilon\to 0$, we obtain

$$\partial_t\psi_1(x_0,t_0)+\partial_t\psi_2(x_0,t_0)\geq -H(x_0,t_0,|\nabla^-\psi_1|(x_0,t_0)+|\nabla\psi_2|^*(x_0,t_0)).$$

Since $\psi_1\in\overline{\mathcal{C}}(Q)$ implies $|\nabla^+\psi_1|=|\nabla\psi_1|\geq|\nabla^-\psi_1|$ and $b\in B$ is arbitrarily chosen, it follows that

$$\partial_t\psi_1(x_0,t_0)+\partial_t\psi_2(x_0,t_0)+H(x_0,t_0,|\nabla\psi_1|(x_0,t_0)+|\nabla\psi_2|^*(x_0,t_0))\geq 0, \tag{27}$$

which yields (9). □

Remark 4.4. In the proof above, one actually obtains a stronger result than (27), which reads

$$\partial_t\psi_1(x_0,t_0)+\partial_t\psi_2(x_0,t_0)+H(x_0,t_0,|\nabla^-\psi_1|(x_0,t_0)+|\nabla\psi_2|^*(x_0,t_0))\geq 0.$$

In fact, if $u-\psi$ attains a local minimum at a point $(x_0,t_0)\in Q$ for $\psi\in\mathcal{C}(Q)$, we can follow the same argument in the proof to deduce

$$\partial_t\psi(x_0,t_0)+H(x_0,t_0,|\nabla^-\psi|(x_0,t_0))\geq 0. \tag{28}$$

This amounts to saying that in fact the value function u satisfies the following strengthened property of supersolutions: u is lower semicontinuous in Q and (28) holds whenever there exists $\psi\in\mathcal{C}(Q)$ such that $u-\psi$ attains a local minimum at $(x_0,t_0)\in Q$.

We next verify the subsolution property of u. The proof is simpler than the supersolution case.

Proposition 4.5. (Subsolution property of game value) *Assume that the assumptions in Theorem* 4.1 *hold. Then u is a subsolution of* (1).

Proof. Suppose that there exist $(x_0,t_0)\in Q$, $\psi_1\in\underline{\mathcal{C}}(Q)$ and $\psi_2\in\mathcal{C}(Q)$ such that $u-\psi_1-\psi_2$ attains a local maximum at (x_0,t_0). In this case, we consider $\hat\theta\in\Theta_B$ such that

$$u(x_0,t_0)\leq u(\xi(s),t_0-s)+\int_0^s g(\xi(\tau),t-\tau,\alpha(\tau),\hat\theta[\alpha](\tau))\,d\tau+s^2$$

holds for all $\alpha\in\mathcal{A}$, $\xi\in\mathbf{S}(x_0,t_0;\alpha,\hat\theta[\alpha])$ and $s>0$ small. Letting $\hat\alpha\equiv a$ in $(0,\infty)$ for an arbitrary $a\in A$, we can use the maximality of $u-\psi_1-\psi_2$ at (x_0,t_0) to get

$$\begin{aligned}&\psi_1(x_0,t_0)+\psi_2(x_0,t_0)\\&\leq\psi_1(\xi(s),t_0-s)+\psi_2(\xi(s),t_0-s)+\int_0^s\hat g[\tau]\,d\tau+s^2\end{aligned}$$

for all $\xi\in\mathbf{S}(x_0,t_0;\hat\alpha,\hat\theta[\hat\alpha])$ and $s>0$ small, where $\hat g$ is defined as in (24) but with different $\hat\theta$ and $\hat\alpha$ chosen here. We also similarly use the notation $\hat f$ below as well.

Note that by Proposition 3.1, for any $y\neq x_0$ near x_0, we can find $\xi\in\mathbf{S}(x_0,t_0,\hat\alpha,\hat\theta[\hat\alpha])$ and

$$d(x_0,y)/M\leq\tau_1\leq d(x_0,y)/m \tag{29}$$

such that $y=\xi(\tau_1)$. It thus follows that

$$\psi_1(x_0,t_0)+\psi_2(x_0,t_0)\leq\psi_1(y,t_0-\tau_1)+\psi_2(y,t_0-\tau_1)+\int_0^{\tau_1}\hat g[\tau]\,d\tau+\tau_1^2$$

for any $y\neq x_0$ near x_0 with some τ_1 satisfying (29).

We can then expand the inequality around (x_0,t_0) to get

$$\begin{aligned}&\partial_t\psi_1(y,s_1)\tau_1+\partial_t\psi_2(y,s_2)\tau_1+\psi_1(x_0,t_0)-\psi_1(y,t_0)\\&\leq\psi_2(y,t_0)-\psi_2(x_0,t_0)+\int_0^{\tau_1}\hat g[\tau]\,d\tau+\tau_1^2.\end{aligned} \tag{30}$$

We divide our discussion into the following two cases.

Case 1. $|\nabla^-\psi_1|(x_0, t_0) = 0$. Then $|\nabla\psi_1|(x_0, t_0) = 0$ holds as well due to the condition that $\psi_1 \in \underline{\mathcal{C}}(Q)$. It follows that

$$\lim_{y\to x} \frac{|\psi_1(y, t_0) - \psi_1(x_0, t_0)|}{d(x_0, y)} = 0.$$

By (30), we thus that

$$\partial_t\psi_1(y, s_1) + \partial_t\psi_2(y, s_2) - \left|\frac{\psi_1(x_0, t_0) - \psi_1(y, t_0)}{d(x_0, y)}\right| \frac{d(x_0, y)}{\tau_1}$$
$$\leq \left|\frac{\psi_2(x_0, t_0) - \psi_2(y, t_0)}{d(x_0, y)}\right| \frac{d(x_0, y)}{\tau_1} + \int_0^{\tau_1} \hat{g}[\tau]\, d\tau + \tau_1^2.$$

Letting $y \to x_0$ with $\tau_1 \to 0$, we deduce

$$\partial_t\psi_1(x_0, t_0) + \partial_t\psi_2(x_0, t_0) \leq \max_{b\in B} \{|\nabla\psi_2|^*(x_0, t_0) f(x_0, t_0, a, b) + g(x_0, t_0, a, b)\}.$$

Due to the arbitrariness of $a \in A$, it follows that

$$\partial_t\psi_1(x_0, t_0) + \partial_t\psi_2(x_0, t_0) + H(x_0, t_0, -|\nabla\psi_2|^*(x_0, t_0)) \leq 0,$$

where we adopt the extended Hamiltonian $H(x, t, p)$ for $p \in \mathbb{R}$. This gives (8) with $|\nabla\psi_1|(x_0, t_0) = 0$.

Case 2. $|\nabla^-\psi_1|(x_0, t_0) > 0$. Then we can find a sequence $y_j (\neq x_0)$ such that $y_j \to x_0$ and

$$\frac{\psi_1(x_0, t_0) - \psi_1(y_j, t_0)}{d(x_0, y_j)} \to |\nabla^-\psi_1|(x_0, t_0) \tag{31}$$

as $j \to \infty$. Let $\xi_j \in \mathbf{S}(x_0, t_0; \hat{\alpha}, \hat{\theta}[\hat{\alpha}])$ be the corresponding curve with $h_j > 0$ satisfying $\xi_j(h_j) = y_j$.

Applying (30) at $y = y_j$ and $\tau_1 = h_j$ for all $j \geq 1$, we get

$$\partial_t\psi_1(y_j, s_1) + \partial_t\psi_2(y_j, s_2)$$
$$\leq \frac{1}{h_j}\int_0^{h_j} \left\{\left(\frac{\psi_1(y_j, t_0) - \psi_1(x_0, t_0)}{d(x_0, y_j)} + \frac{|\psi_2(x_0, t_0) - \psi_2(y_j, t_0)|}{d(x_0, y_j)}\right) \hat{f}[\tau] + \hat{g}[\tau]\right\} d\tau + h_j.$$

Sending $j \to \infty$, by (31) we deduce that

$$\partial_t\psi_1(x_0, t_0) + \partial_t\psi_2(x_0, t_0)$$
$$\leq \max_{b\in B} \{(-|\nabla^-\psi_1|(x_0, t_0) + |\nabla\psi_2|^*(x_0, t_0)) f(x_0, t_0, a, b) + g(x_0, t_0, a, b)\}.$$

Since $|\nabla^-\psi_1| = |\nabla\psi_1|$ holds and $a \in A$ is arbitrarily taken, we are led to

$$\partial_t\psi_1(x_0, t_0) + \partial_t\psi_2(x_0, t_0) + H(x_0, t_0, |\nabla\psi_1|(x_0, t_0) - |\nabla\psi_2|^*(x_0, t_0)) \leq 0,$$

which gives the desired relation (8) again. □

We are now able to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. We have shown in Proposition 4.2 that u is uniformly continuous in $\mathbf{X} \times [0,T)$ for any $T > 0$; in particular, there exist $x_0 \in \mathbf{X}$ and $C > 0$ such that

$$|u(x,t)| \leq C(1 + d(x_0, x))$$

for all $x \in \mathbf{X}$ and $t \in [0,T)$. Moreover, u satisfies (19) and therefore

$$\sup_{x \in \mathbf{X}} |u(x,t) - u_0(x)| \to 0 \quad \text{as } t \to 0.$$

Since u is a solution of (1), as proved in Proposition 4.3 and Proposition 4.5, we conclude by Theorem 2.2 that u is the unique solution of (1) and (2). $\square$

4.3. Further discussions

Let us present one typical simple example of the Hamiltonian H that satisfies the assumptions (H1)–(H3) and give some further discussions.

Example 4.6. Let $A = [1,2]$, $B = [0,1]$, and

$$f(x,t,a,b) = a - b + 1, \quad g(x,t,a,b) = \frac{1}{2}a^2 - \frac{1}{2}b^2$$

for $(x,t) \in Q$, $a \in A$, $b \in B$. In this case, the Hamiltonian given in (3) does not depend on x and t. In fact, by direct calculations, we can express the Hamiltonian as

$$H(x,t,p) = \begin{cases} -\frac{1}{2}p^2 + 2p - \frac{1}{2} & \text{for } p \in [0,1), \\ \frac{1}{2}p^2 + \frac{1}{2} & \text{for } p \in [1,2), \\ 2p - \frac{3}{2} & \text{for } p \in [2,\infty). \end{cases}$$

It is clear that H is neither convex nor concave with respect to p.

However, as reflected by this example, due to our assumptions on f and g especially (H3), the Hamiltonian we consider in this work is strictly increasing in $p \in (0,\infty)$ and (weakly) coercive in the sense that $\inf_{(x,t) \in K} H(x,t,p) \to \infty$ uniformly as $p \to \infty$ for any bounded set $K \subset \mathbb{R}^n \times (0,\infty)$. We used (H3) to obtain the uniform continuity of the value function. It is not clear to us if the representation formula is still valid in general metric spaces without this assumption.

We finally discuss the situation when we have inf sup rather than sup inf in the definition of value function (5). In the Euclidean space, it is related to the notion of Nash equilibrium. For a pay-off function $J(x,t;\alpha,\beta)$, we call u and $\tilde{u}$ defined by

$$u(x,t) = \sup_{\theta \in \Theta_B} \inf_{\alpha \in \mathcal{A}} J(x,t;\alpha,\theta[\alpha])$$

$$\tilde{u}(x,t) = \inf_{\theta \in \Theta_A} \sup_{\beta \in \mathcal{B}} J(x,t;\theta[\beta],\beta)$$

upper value function and lower value function respectively. We say that the game has a *Nash equilibrium* if u and $\tilde{u}$ coincide in $\mathbb{R}^n \times [0,\infty)$ for a common initial value function. In fact, to prove the existence of the Nash equilibrium, we usually need to show that u and $\tilde{u}$ satisfy the same equation.

We are not able to generalize this notion to our general geometric setting. As mentioned several times above, in the game with value function (5), Player A needs to determine a Lipschitz curve ξ in addition to a control from $\mathcal{A}$. If we attempt to consider the lower value function by exchanging the roles of both players in our game, then $\tilde{u}$ will be given by (7). It means that this time Player B will choose the curve most favorable to him to continue the game. This will change significantly the corresponding Hamilton-Jacobi-Isaacs equation. Following the proofs of our previous results, one can show that $\tilde{u}$ defined by (7) is a viscosity solution of

$$\partial_t u + \tilde{H}(x, t, |\nabla u|) = 0 \quad \text{in } \mathbf{X} \times (0, \infty),$$

where $\tilde{H}$ is given by

$$\tilde{H}(x, t, p) = \min_{b \in B} \max_{a \in A} \{-f(x, t, a, b)p - g(x, t, a, b)\}$$

for $(x, t) \in \mathbf{X} \times (0, \infty)$ and $p \geq 0$. In general, one cannot expect that $\tilde{H} = H$ holds. For A, B, f and g given in Example 4.6, it is easily seen that

$$\tilde{H}(x, t, p) = -2p - \frac{1}{2}$$

for $x \in \mathbf{X}$, $t \in (0, \infty)$ and $p \geq 0$, which is different from H. In general, for any $(x, t) \in \mathbf{X} \times (0, \infty)$, as $p \to \infty$, we have

$$H(x, t, p) \to \infty \quad \text{but} \quad \tilde{H}(x, t, p) \to -\infty.$$

References

[1] Y. Achdou, F. Camilli, A. Cutrì, N. Tchou: *Hamilton-Jacobi equations constrained on networks*, Nonlinear Diff. Equations Appl. 20/3 (2013) 413–445.

[2] L. Ambrosio, J. Feng: *On a class of first order Hamilton-Jacobi equations in metric spaces*, J. Diff. Equations 256/7 (2014) 2194–2245.

[3] L. Ambrosio, N. Gigli, G. Savaré: *Heat flow and calculus on metric measure spaces with Ricci curvature bounded below – the compact case*, in: *Analysis and Numerics of Partial Differential Equations*, INdAM Ser. Vol. 4, Springer, Milan (2013) 63–115.

[4] M. Bardi, I. Capuzzo-Dolcetta: *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Systems & Control: Foundations & Applications, Birkhäuser, Boston (1997).

[5] U. Bessi: *Hamilton-Jacobi in metric spaces with a homological term*, Comm. Partial Diff. Equations 45/7 (2020) 776–819.

[6] F. Camilli, R. Capitanelli, C. Marchi: *Eikonal equations on the Sierpinski gasket*, Math. Ann. 364/3-4) (2016) 1167–1188.

[7] P. Cardaliaguet: *Notes on Mean Field Games (from P.-L. Lions' lectures at the Collège de France)*, https://www.ceremade.dauphine.fr/∼cardaliaguet/MFG2013-0420.pdf (2013).

[8] M. G. Crandall, H. Ishii, P.-L. Lions: *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) 27/1 (1992) 1–67.

[9] L. C. Evans, P. E. Souganidis: *Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations*, Indiana Univ. Math. J. 33/5 (1984) 773–797.

[10] W. Gangbo, A. Święch: *Optimal transport and large number of particles*, Discrete Contin. Dyn. Syst. 34/4 (2014) 1397–1441.

[11] W. Gangbo, A. Święch: *Metric viscosity solutions of Hamilton-Jacobi equations depending on local slopes*, Calc. Var. Partial Diff. 54/1 (2015) 1183–1218.

[12] Y. Giga, N. Hamamuki, A. Nakayasu: *Eikonal equations in metric spaces*, Trans. Amer. Math. Soc. 367/1 (2015) 49–66.

[13] C. Imbert, R. Monneau, H. Zidani: *A Hamilton-Jacobi approach to junction problems and application to traffic flows*, ESAIM Control Optim. Calc. Var. 19/1 (2013) 129–166.

[14] H. Ishii: *Representation of solutions of Hamilton-Jacobi equations*, Nonlinear Analysis 12/2 (1988) 121–146.

[15] Q. Liu, A. Nakayasu: *Convexity preserving properties for Hamilton-Jacobi equations in geodesic spaces*, Discrete Contin. Dyn. Syst. 39/1 (2019) 157–183.

[16] Q. Liu, N. Shanmugalingam, X. Zhou: *Equivalence of solutions of eikonal equation in metric spaces*, J. Diff. Equations 272 (2021) 979–1014.

[17] J. Lott, C. Villani: *Hamilton-Jacobi semigroup on length spaces and applications*, J. Math. Pures Appl. (9) 88/3 (2007) 219–229.

[18] A. Nakayasu: *Metric viscosity solutions for Hamilton-Jacobi equations of evolution type*, Adv. Math. Sci. Appl. 24/2 (2014) 333–351.

[19] A. Nakayasu, T. Namba: *Stability properties and large time behavior of viscosity solutions of Hamilton-Jacobi equations on metric spaces*, Nonlinearity 31/11 (2018) 5147–5161.

[20] M. Nisio: *On value function of stochastic differential games in infinite dimensions and its application to sensitive control*, Osaka J. Math. 36/2 (1999) 465–483.

[21] D. Schieborn, F. Camilli: *Viscosity solutions of Eikonal equations on topological networks*, Calc. Var. Partial Diff. Equations 46/3-4 (2013) 671–686.

[22] P. E. Souganidis: *Max-min representations and product formulas for the viscosity solutions of Hamilton-Jacobi equations with applications to differential games*, Nonlinear Analysis 9/3 (1985) 217–257.

[23] A. Święch: *Risk-sensitive control and differential games in infinite dimensions*, Nonlinear Analysis Ser. A: Theory Methods 50/4 (2002) 509–522.

[24] C. Villani: *Optimal Transport*, Grundlehren der Mathematischen Wissenschaften Vol. 338, Springer, Berlin (2009).