Minimax Theory and its Applications
Volume 8 (2023), No. 2, 409–422

A Note on Fenchel Duality for Equilibrium Problems

Monica Bianchi
Università Cattolica del Sacro Cuore, Milano, Italy monica.bianchi@unicatt.it

Rita Pini
Università degli Studi di Milano-Bicocca, Milano, Italy
rita.pini@unimib.it

Received: January 23, 2023
Accepted: August 31, 2023

By means of the Fenchel duality, we introduce a dual problem associated to an equilibrium problem that turns out to be an equilibrium problem itself in the dual space. We present conditions which entail the solvability of the primal and dual problem. Moreover, we introduce the notion of robust and optimistic solution for parametric equilibrium problems, and we show that the solutions of the dual of the robust problem coincide with the optimistic solutions of the dual parametric equilibrium problem.

Keywords: Convex analysis, duality, equilibrium problems, Fenchel conjugation, robust solutions.

2020 Mathematics Subject Classification: 49J40, 47H05, 47J20.

1. Introduction

It is well known that an equilibrium problem, (EP) for short, is defined as follows:

$$\text{(EP)} \qquad \text{find } \overline{x} \in K \text{ such that } F(\overline{x}, y) \geq 0 \quad \text{for all } y \in K$$

where X is a Euclidean space, $F : X \times X \to \mathbb{R}$ is a bifunction, and K is a non empty convex subset of X. Despite these problems are often studied in a more general setting like, for instance, real locally convex Hausdorff topological vector spaces, in this paper we will focus on the finite-dimensional setting. Furthermore, the bifunction F can be taken with values within $(-\infty, +\infty]$, like in [1].

In the seminal paper [9] Blum and Oettli pointed out that equilibrium problems include as special cases optimization problems, variational inequalities, complementarity problems, fixed point problems and Nash equilibria. For this reason in recent years many works on equilibrium problems were interested in extending results of these particular instances to the more general and unifying setting of equilibrium problems.

One of the most interesting topic in optimization is certainly duality.

Duality for equilibrium problems was initially studied by introducing the so called *Minty equilibrium problem* (MEP), a natural counterpart of the Minty variational

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

inequality (see for instance [16]):

(MEP) find $\overline{x} \in K$ such that $F(y, \overline{x}) \leq 0$ for all $y \in K$

Anyway, this approach is not able to recover any of the well known dual problems in case of optimization. For this reason, inspired by duality for convex optimization, some authors proposed a dual problem associated to an equilibrium problem (see [8] and [17]). In [17] the authors build, via Fenchel conjugation, a dual problem to (EP) that is an optimization problem involving a gap function. On the other hand, the approach proposed by Bigi et al. in [8] is inspired by the known relationship between subdifferentials of a function and its conjugate, and is applied to the so-called diagonal subdifferential of a bifunction. This approach recovers, in particular, the duality approach for variational inequalities developed by Mosco [18]. Lalitha, in [14], pointed out that both dual problems are indeed equivalent under mild conditions.

The main drawback of the formulations in [8] and [17] is that the dual problem is not an equilibrium problem itself. To fill this gap, in this paper we propose a dual formulation of an equilibrium problem associated to a saddle function, which is again an equilibrium problem in the framework of the dual space. Observing that the solvability of (EP) is equivalent to the minimization of a gap function associated to (EP), the dual equilibrium problem arises naturally as the Fenchel dual of the minimization problem. We present conditions which entail the solvability of both the primal and the dual equilibrium problems.

Our approach recovers the special case of dual Fenchel optimization problems.

In the second part of the paper, we introduce the notion of robust and optimistic solution for parametric equilibrium problems starting from the well known notions in case of optimization problems (see for instance [5]). By means of our formulation of duality, we are able to show that the solutions of the dual of the robust problem coincide with the optimistic solutions of the dual parametric equilibrium problem, thereby extending a well known result in case of optimization.

The paper is organized as follows: in Section 2 we recall some preliminary notions and results. In Section 3 we present the dual equilibrium problem (DEP) for saddle functions. In Section 4 we study the solution set of (EP) and (DEP) and in Section 5 we address the special case of minimization problems. Finally, in Section 6 we deal with robust and optimistic equilibrium problems.

2. Preliminaries

In this section we recall some preliminaries on Fenchel conjugation, bifunctions and equilibrium problems.

Let X be a Euclidean space and denote by X^* its dual (even if they are isomorphic, we prefer to distinguish them for the sake of clarity in the forthcoming notations). Given a function $f : X \to (-\infty, +\infty]$ the *Fenchel conjugate* $f^* : X^* \to [-\infty, +\infty]$ is defined as

$$f^*(x^*) = \sup_{x \in X}(\langle x^*, x\rangle - f(x)),$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product. For any function f, the conjugate f^* is always a lower semicontinuous and convex function.

If f is proper, i.e. $f(x') < +\infty$ for some $x' \in X$, then f^* never takes the value $-\infty$. Moreover, if f is bounded from below, then $f^*(0) < +\infty$ and therefore f^* is proper.

Note that for every $x \in X$, $x^* \in X^*$,

$$f(x) + f^*(x^*) \geq \langle x^*, x \rangle$$

and the equality holds if and only if $x^* \in \partial f(x)$, where ∂f denote the subdifferential of f (see Proposition 5.3.1 in [15]).

Given the indicator function of the set $K \subset X$

$$\delta_K(x) = \begin{cases} +\infty & \text{if } x \notin K \\ 0 & \text{if } x \in K, \end{cases}$$

we have that
$$i_K(x^*) = \inf_{x \in K} \langle x^*, x \rangle = -\delta_K^*(-x^*).$$

Proposition 2.1. (see Proposition 13.41 in [4]) *Let $\{f_i\}_{i \in I}$ be a family of proper, lower semicontinuous and convex functions defined on X, such that $\sup_{i \in I} f_i$ is not identically $+\infty$. Then*

$$\left(\sup_{i \in I} f_i \right)^* = \left(\inf_{i \in I} f_i^* \right)^{\smile}$$

where $f^{\smile}$ denotes the lower semicontinuous, convex and proper envelope of f.

The *Fitzpatrick transform* associated to a bifunction $F : X \times X \to \mathbb{R}$ is the bifunction $\varphi_F : X \times X^* \to (-\infty, +\infty]$ given by

$$\varphi_F(y, x^*) = (-F(\cdot, y))^*(x^*) = \sup_{x \in X} (\langle x^*, x \rangle + F(x, y))$$

(see [6]). Note that, by construction,

(i) if $F(x, \cdot)$ is convex for every $x \in X$, then φ_F is lower semicontinuos and convex on $X \times X^*$;

(ii) if $F(\cdot, \overline{y})$ is upper bounded for some $\overline{y}$ in X, then φ_F is proper;

(iii) if, for every $y \in X$, the function $-F(\cdot, y)$ is lower semicontinuous and *super coercive*, i.e.,
$$\lim_{\|x\| \to +\infty} \frac{-F(x, y)}{\|x\|} = +\infty, \tag{1}$$
(see Definition 11.10 in [4]), then dom(φ_F)= $X \times X^*$, i.e. φ_F is real-valued; in particular, $F(\cdot, y)$ is upper bounded, for every $y \in X$;

(iv) $\varphi_F(y, x^*) \geq \langle x^*, x \rangle + F(x, y)$, for all $x, y \in X$.

Given a bifunction F and its Fitzpatrick transform φ_F, for any nonempty and compact subset K of X we can define the function

$$\ell_{F,K}(x^*) = \inf_{y \in K} \varphi_F(y, x^*).$$

The following result follows from Propositions 1.7 and 2.5 in [2]:

Proposition 2.2. *Let X, Y be Euclidean spaces, and $f : X \times Y \to (-\infty, +\infty]$ be convex and lower semicontinuous. Let K be a nonempty compact subset of X and assume that there exists $(\overline{x}, \overline{y}) \in K \times Y$ such that $f(\overline{x}, \overline{y}) < +\infty$. Then, the function*

$$\ell(y) = \inf_{x \in K} f(x, y)$$

is proper, convex and lower continuous on Y.

The next properties can be easily proved taking into account the previous result:

(a) if $F(x, x) = 0$ for every $x \in X$, then $\ell_{F,K}(x^*) > -\infty$ for every $x^* \in X^*$; indeed, by iv. we have that $\varphi_F(y, x^*) \geq \langle x^*, y \rangle$ for all $y \in X$ and

$$\ell_{F,K}(x^*) = \inf_{y \in K} \varphi_F(y, x^*) \geq \inf_{y \in K} \langle x^*, y \rangle = i_K(x^*);$$

(b) if $F(x, \cdot)$ is convex on X for every $x \in X$, and $F(\cdot, \overline{y})$ is upper bounded for some $\overline{y}$ in K, then by (i) and (ii) φ_F is proper, lower semicontinuous and convex on $X \times X^*$. Therefore, by Proposition 2.2 applied to the function φ_F on the set $X \times X^*$, the function $\ell_{F,K}$ is proper, lower semicontinuos and convex on X^*.

In the sequel, to entail the solvability of (EP) on a non empty, compact and convex set K, we will assume that the bifunction $F : X \times X \to \mathbb{R}$ satisfies the following assumptions (see Theorem 1 and Remark 3 in [7]):

(C) $F(x, x) = 0$ for every $x \in X$, $F(x, \cdot)$ convex for every $x \in X$ (and therefore continuous on X), and $F(\cdot, y)$ is upper semicontinuous for every $y \in X$.

Remark 2.3. The assumptions in (**C**) provide also the well known fact that that any solution of (MEP) is a solution of (EP) (see, for instance, Lemma 3 in [9]).

Finally, note that (MEP) is solvable on a non empty, compact and convex set K, if the bifunction $(x, y) \mapsto -F(y, x)$ satisfies (**C**).

3. Primal and dual equilibrium problems for saddle functions

The purpose of our study is to provide a "reasonable" definition of duality for the classical equilibrium problem. In particular, we would like to highlight the relationships between the bifunctions involved in the two problems. Our approach will be to look at the equilibrium problem as a particular optimization problem, and then apply the Fenchel duality theory to get the dual problem.

From now on we will consider the equilibrium problem (EP) on a nonempty, compact and convex set $K \subseteq X$. The solution set of (EP) will be denoted by S_{EP}.

Starting from the problem (EP), we define the *gap function* $g : X \to (-\infty, +\infty]$ as a natural extension of the one considered for variational inequalities (see [3]):

$$g(x) = \sup_{y \in K} (-F(x, y)).$$

To justify the name of gap function note that, by assuming $F(x,x)=0$ for all $x\in X$, it is easy to verify that $g(x)\geq 0$ for all $x\in K$, and that

$$S_{\mathrm{EP}}=\{x\in K:\ g(x)=0\}. \tag{2}$$

Associated to g, let us consider the following optimization problem, that will be called *primal* in the sequel:

$$\text{(P)} \qquad p=\inf_{x\in K} g(x)=\inf_{x\in X}(g(x)+\delta_K(x))$$

Note that, in general, p could be strictly positive, but if g attains its minimum, and $p=\min_{x\in K} g(x)=0$, then $S_{\mathrm{EP}}\neq\emptyset$. In particular, if (**C**) holds, (EP) is solvable and therefore $\min_{x\in K} g(x)=0$. In this case,

$$g(x)=\max_{y\in K}(-F(x,y)),$$

and g is a real-valued function defined on the whole of X.

Let us now recall the following

Definition 3.1. (see [19]) A bifunction $F:X\times X\to[-\infty,+\infty]$ is said to be a *saddle function* if $F(x,\cdot)$ is convex for every $x\in X$, and $F(\cdot,y)$ is concave for every $y\in X$. □

Any bifunction $F(x,y)=f(y)-f(x)$ where $f:X\to\mathbb{R}$ is convex is an example of a saddle function. Another example is given by

$$F(x,y)=\langle Ax,x\rangle+\langle By,y\rangle-\langle (A+B)x,y\rangle, \tag{3}$$

where A is negative semidefinite and B is positive semidefinite. Note that every saddle function is continuous with respect to each of its variables in the interior of its domain, in particular on the whole of X if it is real-valued.

If F is a real-valued saddle function, with $F(x,x)=0$ for every $x\in K$, the function $g:X\to\mathbb{R}$ is convex, and the primal problem (P) is a convex minimization problem with minimum equal to 0.

In the next result, through the Fenchel dual of the convex minimization problem (P) we build in a natural way a bifunction Φ_F associated to F and defined in the dual space $X^*\times X^*$ to which we can associate, under suitable assumptions on F, an equilibrium problem.

Theorem 3.2. *Let $F:X\times X\to\mathbb{R}$ be a saddle function, such that $F(x,x)=0$ for all $x\in X$. Suppose that there exists $\overline{y}\in K$ such that $F(\cdot,\overline{y})$ is upper bounded. Then*

$$0\leq\sup_{x^*\in X^*}\inf_{y^*\in Y^*}\Phi_F(x^*,y^*),$$

where $\Phi_F:X^*\times X^*\to[-\infty,+\infty]$ is defined as follows:

$$\Phi_F(x^*,y^*)=(\varphi_F(\cdot,x^*))^*(y^*)+i_K(x^*)-i_K(y^*). \tag{4}$$

Proof. According to Fenchel (see, for instance, [10], Chapter 4), the dual (D) of the convex optimization problem (P) is given by

$$\text{(D)} \qquad \sup_{x^*\in X^*}\left(-g^*(x^*)-\delta_K^*(-x^*)\right)$$

Note that

$$\sup_{x^*\in X^*}\left(-g^*(x^*)-\delta_K^*(-x^*)\right)=\sup_{x^*\in X^*}\left(i_K(x^*)-\inf_{y\in K}\varphi_F(y,x^*)\right). \tag{5}$$

Indeed, taking into account Proposition 2.1, we have that

$$\begin{aligned} g^*(x^*) &= \left(\sup_{y\in K}(-F(\cdot,y))\right)^*(x^*)=\left(\inf_{y\in K}(-F(\cdot,y))^*\right)^{\smile}(x^*)\\ &= \left(\inf_{y\in K}\varphi_F(y,\cdot)\right)^{\smile}(x^*)=(\ell_{F,K}(\cdot))^{\smile}(x^*). \end{aligned}$$

Moreover, from the assumptions on F, condition **(b)** in Section 2 implies that $(\ell_{F,K}(\cdot))^{\smile}(x^*)=\ell_{F,K}(x^*)$, and hence

$$g^*(x^*)=\ell_{F,K}(x^*). \tag{6}$$

Let us now consider, for every $x^*\in X^*$, the addend in (5)

$$\inf_{y\in K}\varphi_F(y,x^*)=\inf_{y\in X}(\varphi_F(y,x^*)+\delta_K(y)).$$

By the Fenchel Weak Duality Theorem (see Theorem 4.4.2 in [10]) we have that

$$\inf_{y\in X}(\varphi_F(y,x^*)+\delta_K(y))\geq\sup_{y^*\in X^*}\left(-(\varphi_F(\cdot,x^*))^*(y^*)+i_K(y^*)\right). \tag{7}$$

Therefore, we get that

$$\begin{aligned} i_K(x^*)-\inf_{y\in K}\varphi_F(y,x^*) &\leq i_K(x^*)-\sup_{y^*\in X^*}\left(-(\varphi_F(\cdot,x^*))^*(y^*)+i_K(y^*)\right)\\ &=\inf_{y^*\in X^*}\left((\varphi_F(\cdot,x^*))^*(y^*)+i_K(x^*)-i_K(y^*)\right)\\ &=\inf_{y^*\in X^*}\Phi_F(x^*,y^*) \end{aligned} \tag{8}$$

and consequently

$$\sup_{x^*\in X^*}\left(i_K(x^*)-\inf_{y\in K}\varphi_F(y,x^*)\right)\leq\sup_{x^*\in X^*}\inf_{y^*\in X^*}\Phi_F(x^*,y^*). \tag{9}$$

Now, under the assumptions on F, the domain of g is the whole of X, and the Fenchel Strong Duality Theorem (see Theorem 4.4.3 in [10]) gives

$$0=\inf_{x\in X}(g(x)+\delta_K(x))=\sup_{x^*\in X^*}\left(-g^*(x^*)-\delta_K^*(-x^*)\right).$$

Taking (9) into account, the assertion follows. □

Under the more restrictive condition of super coercivity of F given in (1), we can easily obtain the following:

Corollary 3.3. *Let $F : X \times X \to \mathbb{R}$ be a saddle function, such that $F(x,x) = 0$ for all $x \in X$, and suppose that the function $-F(\cdot, y)$ is super coercive for every $y \in X$. Then*

$$0 = \inf_{x\in X}(g(x) + \delta_K(x)) = \sup_{x^*\in X^*} \inf_{y^*\in X^*} \Phi_F(x^*, y^*). \tag{10}$$

Proof. Under the super coercivity assumption, the domain of φ_F is given by $X \times X^*$, and therefore in (7) the Fenchel Strong Duality Theorem can be applied and (8) becomes

$$i_K(x^*) - \inf_{y\in K} \varphi_F(y, x^*) = \inf_{y^*\in X^*} \Phi_F(x^*, y^*). \qquad \square \tag{11}$$

Let us highlight some properties of the bifunction Φ_F defined in (4).

Proposition 3.4. *Let us consider the bifunction $\Phi_F : X^* \times X^* \to [-\infty, +\infty]$;*

(i) *if $F(x, \cdot)$ is convex for every $x \in X$, then $\Phi_F(\cdot, \cdot)$ is a saddle function;*

(ii) *if $-F(\cdot, y)$ is lower semicontinuous and super coercive for every $y \in X$, then*

$$\Phi_F(x^*, y^*) > -\infty, \quad \forall (x^*, y^*) \in X^* \times X^*;$$

(iii) *under the assumptions of Corollary 3.3, $\inf_{y^*\in X^*} \Phi_F(x^*, y^*) \leq 0$ for every $x^* \in X^*$. In particular, $\Phi_F(x^*, \cdot)$ is proper for all $x^* \in X^*$.*

Proof. (i) From the assumptions, φ_F is convex and, therefore,

$$(\varphi_F(\cdot, x^*))^*(y^*) = \sup_{y\in X}(\langle y^*, y\rangle - \varphi_F(y, x^*))$$

is concave on X^* for every $y^* \in X^*$. Since i_K is concave, $\Phi_F(\cdot, y^*)$ is concave, too. The convexity of $\Phi_F(x^*, \cdot)$ is trivial.

(ii) The assumptions imply that φ_F is real-valued, and thus $\Phi_F(x^*, y^*) > -\infty$ for every $(x^*, y^*) \in X^* \times X^*$.

(iii) It is an easy consequence of (10). $\square$

If we consider a bifunction F such that $-F(\cdot, y)$ is super coercive and lower semicontinuous for every $y \in X$, then Φ_F takes values within $(-\infty, +\infty]$. We can then consider the following equilibrium problem in the dual space X^*, that will be denoted as *dual equilibrium problem* (DEP): find $\overline{x}^* \in X^*$ such that

$$\Phi_F(\overline{x}^*, y^*) \geq 0, \qquad \forall y^* \in X^*. \tag{DEP}$$

Remark 3.5. Under the assumptions of Corollary 3.3, the function

$$\gamma(x^*) = -\inf_{y^*\in X^*} \Phi_F(x^*, y^*) = \sup_{y^*\in X^*} (-\Phi_F(x^*, y^*))$$

is a gap function for the dual equilibrium problem. Indeed, by (iii) in Proposition 3.4, $\gamma(x^*) \geq 0$ for all $x^* \in X^*$. In addition, it is easy to verify that

$$\gamma(\overline{x}^*) = 0 \Longleftrightarrow \overline{x}^* \text{ is a solution of (DEP)}. \qquad \square$$

Example 3.6. In the particular case of the saddle function F given in (3), under the additional assumption that both A and B are non singular, the Fitzpatrick function φ_F has the following expression:

$$\varphi_F(y, x^*) = \frac{1}{2}\langle x^*, Dy\rangle + \frac{1}{4}\langle y, Cy\rangle - \frac{1}{4}\langle x^*, A^{-1}x^*\rangle,$$

where the matrix $C = -(A-B)A^{-1}(A-B)$ is symmetric and positive definite, and $D = A^{-1}(A+B)$. Therefore

$$\begin{aligned}\Phi_F(x^*, y^*) &= \sup_{y\in X}(\langle y^*, y\rangle - \varphi_F(y, x^*)) + i_K(x^*) - i_K(y^*)\\ &= \langle y^*, C^{-1}y^*\rangle + \frac{1}{4}\langle x^*, (A^{-1} + D^T C^{-1} D)x^*\rangle - \langle x^*, DC^{-1}y^*\rangle +\\ &\quad + i_K(x^*) - i_K(y^*).\end{aligned}$$

If we take, for instance, $K = \{x \in X : \|x\| \le 1\}$, then $i_K(x^*) = -\|x^*\|$, and thus

$$\Phi_F(x^*, y^*) = \langle y^*, C^{-1}y^*\rangle + \frac{1}{4}\langle x^*, (A^{-1} + DC^{-1}D)x^*\rangle - \langle x^*, DC^{-1}y^*\rangle - \|x^*\| + \|y^*\|.$$

4. Solvability of (EP) and (DEP)

Let us denote by S_{MEP} and S_{DEP} the solution sets of (MEP) and (DEP), respectively. If $F : X \times X \to \mathbb{R}$ is a saddle function such that $F(x, x) = 0$, then from Remark 2.3 applied to the bifunction $G(x, y) = -F(y, x)$, we easily get that $S_{\text{EP}} \subseteq S_{\text{MEP}}$, and therefore for this class of bifunctions F, the solution sets S_{EP} and S_{MEP} coincide and are nonempty and convex. In addition, by i. of Proposition 3.4, $\Phi_F(\cdot, y^*)$ is concave for all $y^* \in X^*$ and the set S_{DEP} is convex.

Consider the set S defined as follows:

$$S = \{(x, x^*) \in K \times X^* : i_K(x^*) = \varphi_F(x, x^*)\}.$$

Note that, if $F(x, x) = 0$, from **(a)** of Section 2 the function $i_K - \varphi_F$ is nonpositive on $K \times X^*$. Moreover, if $F(x, \cdot)$ is convex and $F(\cdot, \overline{y})$ is upper bounded for some $\overline{y} \in K$, then $i_K - \varphi_F$ is concave and upper semicontinuous on $K \times X^*$. Then, in this case, since $S = \text{lev}_{\ge 0}\,(i_K - \varphi_F)$, S is a closed and convex set.

In the following result we highlight a relationship between the set S and the solutions sets S_{EP} and S_{DEP}.

Theorem 4.1. *Let $F : X \times X \to \mathbb{R}$ be a saddle function such that $F(x, x) = 0$ for every $x \in X$. If $F(\cdot, \overline{y})$ is upper bounded for some $\overline{y} \in K$, then*

$$\emptyset \ne S \subseteq S_{\text{EP}} \times S_{\text{DEP}}.$$

Proof. Let us first show that $S \ne \emptyset$. By the assumptions, (EP) is solvable; if $\overline{x} \in K$ is a solution of (EP), we have that $g(\overline{x}) = 0$ and $p = 0$. In particular, since g is convex and real-valued,

$$0 \in \partial(g + \partial\delta_K)(\overline{x}) = \partial g(\overline{x}) + \partial\delta_K(\overline{x}),$$

i.e. there exists $\overline{x}^*$ such that $\overline{x}^* \in \partial g(\overline{x})$ and $\langle \overline{x}^*, y - \overline{x} \rangle \geq 0$ for all $y \in K$, that is $i_K(\overline{x}^*) = \langle \overline{x}^*, \overline{x} \rangle$. By the equivalence

$$\overline{x}^* \in \partial g(\overline{x}) \Longleftrightarrow g(\overline{x}) + g^*(\overline{x}^*) = \langle \overline{x}^*, \overline{x} \rangle,$$

we get that $g^*(\overline{x}^*) = i_K(\overline{x}^*)$. Hence, from (6)

$$i_K(\overline{x}^*) - \inf_{y \in K} \varphi_F(y, \overline{x}^*) = 0.$$

The lower semicontinuity of the function φ_F implies the existence of $\hat{x} \in K$ such that $\inf_{y \in K} \varphi_F(y, \overline{x}^*) = \varphi_F(\hat{x}, \overline{x}^*)$, i.e. $(\hat{x}, \overline{x}^*) \in S$.

Suppose now that $(\overline{x}, \overline{x}^*)$ is a point in S and let us show that $\overline{x} \in S_{\text{EP}}$ and $\overline{x}^* \in S_{\text{DEP}}$. From the inequality

$$\varphi_F(\overline{x}, \overline{x}^*) \geq \langle \overline{x}^*, x \rangle + F(x, \overline{x}) \qquad \forall x \in X,$$

we get that

$$-F(x, \overline{x}) \geq \langle \overline{x}^*, x \rangle - i_K(\overline{x}^*) \geq 0, \quad \forall x \in K,$$

and therefore $F(x, \overline{x}) \leq 0$, for all $x \in K$, i.e. $\overline{x} \in S_{\text{MEP}} = S_{\text{EP}}$.

Moreover, since

$$0 \geq i_K(\overline{x}^*) - \inf_{y \in K} \varphi_F(y, \overline{x}^*) \geq i_K(\overline{x}^*) - \varphi_F(\overline{x}, \overline{x}^*) = 0,$$

we get $i_K(\overline{x}^*) - \inf_{y \in K} \varphi_F(y, \overline{x}^*) = 0$; from (8) we have that

$$\inf_{y^* \in X^*} \Phi_F(\overline{x}^*, y^*) \geq 0 \quad \forall y^* \in X^*,$$

and thus $\overline{x}^* \in S_{\text{DEP}}$. □

In the light of the previous result the natural question arises: are all the pairs in $(\overline{x}, \overline{x}^*) \in S_{\text{EP}} \times S_{\text{DEP}}$ solutions of the equation

$$i_K(\overline{x}^*) = \varphi_F(\overline{x}, \overline{x}^*)\,?$$

At the moment we have a positive answer only in some particular cases:

- when either the equilibrium problem (EP) or the dual problem (DEP) has a unique solution
- in the case of the optimization problem.

We discuss first the case when (EP) has a unique solution.

Proposition 4.2. *Under the assumption of Corollary* 3.3*, if we have* $S_{\text{EP}} = \{\overline{x}\}$*, then* $S = S_{\text{EP}} \times S_{\text{DEP}}$.

Proof. Indeed, take any $\overline{x}^* \in S_{\text{DEP}}$. From Remark 3.5 we have that $\gamma(\overline{x}^*) = 0$, i.e.,

$$\inf_{y^* \in X^*} \Phi_F(\overline{x}^*, y^*) = 0.$$

From (11) we have that

$$i_K(\overline{x}^*) - \inf_{y \in K} \varphi_F(y, \overline{x}^*) = 0.$$

In particular, there exists $\hat{x} \in K$ such that $\varphi_F(\hat{x}, \overline{x}^*) = i_K(\overline{x}^*)$, i.e., $(\hat{x}, \overline{x}^*) \in S$. This implies that $\hat{x} \in S_{\mathrm{EP}}$, and thus $\hat{x} = \overline{x}$. □

We address now the case when (DEP) has a unique solution. To this purpose, let us first prove the following result:

Proposition 4.3. *Let $F : X \times X \to \mathbb{R}$ be a bifunction such that $F(x, x) = 0$, and let $K \subseteq X$ be a nonempty, compact and convex set. Assume that $F(\cdot, x)$ is concave for all $x \in X$. Then $\overline{x} \in S_{\mathrm{MEP}}$ if and only if there exists $\overline{x}^* \in X^*$ such that $(\overline{x}, \overline{x}^*) \in S$.*

Proof. Denote by $G : K \times X \to (-\infty, +\infty]$ the bifunction given by given

$$G(x, y) = -F(y, x) + \delta_K(y),$$

and define the diagonal subdifferential operator ${}^{G}A : K \rightrightarrows X^*$ as follows (see [6]):

$${}^{G}A(x) = \{x^* \in X^* : G(x, y) \geq \langle x^*, y - x\rangle \text{ for all } y \in X\} = \partial(G(x, \cdot))(x).$$

Note that $\overline{x} \in S_{\mathrm{MEP}} \Longleftrightarrow 0 \in {}^{G}A(\overline{x}) \Longleftrightarrow 0 \in \partial(-F(\cdot, \overline{x}) + \delta_K(\cdot))(\overline{x})$.

By the assumptions, this is equivalent to the existence of a point $\overline{x}^* \in X^*$ such that $\overline{x}^* \in \partial(-F(\cdot, \overline{x}))(\overline{x})$ and $i_K(\overline{x}^*) = \langle \overline{x}^*, \overline{x}\rangle$. By the equality

$$-F(\overline{x}, \overline{x}) + (-F(\cdot, \overline{x}))^*(\overline{x}^*) = \langle \overline{x}^*, \overline{x}\rangle$$

which holds if and only if $\overline{x}^* \in \partial(-F(\cdot, \overline{x}))(\overline{x})$, it follows that $\varphi_F(\overline{x}, \overline{x}^*) = i_K(\overline{x}^*)$, i.e. $(\overline{x}, \overline{x}^*) \in S$. □

Proposition 4.4. *Under the assumptions of Corollary 3.3, if $S_{\mathrm{DEP}} = \{\overline{x}^*\}$, then $S = S_{\mathrm{EP}} \times S_{\mathrm{DEP}}$.*

Proof. Take a point $(\overline{x}, \overline{x}^*) \in S_{\mathrm{EP}} \times S_{\mathrm{DEP}}$. Since $\overline{x} \in S_{\mathrm{EP}} = S_{\mathrm{MEP}}$, the previous proposition entails that there exists $x^* \in X^*$ such that $(\overline{x}, x^*) \in S$. Thus, from Theorem 4.1 it follows that $x^* \in S_{\mathrm{DEP}}$, i.e. $x^* = \overline{x}^*$. □

We now consider the case of the minimization problem $\min_{x \in K} f(x)$, with $f : X \to \mathbb{R}$, and $F(x, y) = f(y) - f(x)$. Note that $F(x, x) = 0$, and F is a saddle function in case f is convex; moreover

$$g(x) = \max_{y \in K}(-F(x, y)) = \max_{y \in K}(f(x) - f(y)) = f(x) - \min_{y \in K} f(y).$$

In case f is convex, the primal problem (P), i.e. the minimization of g over K, has optimal value 0. Standard computations show that

$$\varphi_F(y, x^*) = f(y) + f^*(x^*), \quad (\varphi_F(\cdot, x^*))^*(y^*) = f^*(y^*) - f^*(x^*). \tag{12}$$

Then $\Phi_F(x^*, y^*) = f^*(y^*) - i_K(y^*) - (f^*(x^*) - i_K(x^*))$

and a solution $\overline{x}^*$ of the dual equilibrium problem (DEP) is, in this case, a solution of the optimization problem

$$\min_{x^* \in X^*} (f^*(x^*) - i_K(x^*)) = -\max_{x^* \in X^*} (i_K(x^*) - f^*(x^*)).$$

It is worthwhile noting that finding $\overline{x}^* \in X^*$ such that $\Phi_F(\overline{x}^*, y^*) \geq 0$ for all $y^* \in X^*$ amounts to maximizing the function $y^* \mapsto -f^*(y^*) + i_K(y^*)$, which is exactly the Fenchel dual problem of the original optimization problem.

Assuming, in addition, the super coercivity of the function f, from Corollary 3.3, both the primal and the dual problems have solutions with the same optimal value equal to 0. In particular, we have that $S = S_{\mathrm{EP}} \times S_{\mathrm{DEP}}$. Indeed, the assumptions of the Fenchel Strong Duality Theorem are satisfied; thus

$$\begin{aligned}\min_{x \in K} f(x) = \min_{x \in X}(f(x) + \delta_K(x)) &= \sup_{x^* \in X^*} (-f^*(x^*) - \delta_K^*(-x^*)) \\ &= -\inf_{x^* \in X^*} (f^*(x^*) - i_K(x^*)).\end{aligned}$$

In particular the optimal values are equal, i.e.

$$f(\overline{x}) = -f^*(\overline{x}^*) + i_K(\overline{x}^*)$$

and from (12) we have that $\varphi_F(\overline{x}, \overline{x}^*) = i_K(\overline{x}^*)$, that is $(\overline{x}, \overline{x}^*) \in S$.

5. Robust and optimistic equilibrium problem

The aim of this section is to introduce suitable notions of *robust* and *optimistic* solutions for parametric equilibrium problems, inspired by the well known notions for optimization problems. This will allow a comparison between the solutions of the dual of the robust problem and the optimistic solutions of the dual of the parametric problem, thereby getting a *primal worst equals dual best* result in line to what is known about optimization problems (see for instance [5], [13] and the references therein).

In the sequel X, Y will denote Euclidean spaces, $K \subseteq X$ and $A \subseteq Y$ nonempty, compact and convex sets, and $F : X \times X \times A \to \mathbb{R}$ will be a function fulfilling the following set of properties:

(C′) $F(x, x, a) = 0$ for every $(x, a) \in X \times A$, $F(\cdot, y, a)$ is concave for every $(y, a) \in X \times A$, $F(x, \cdot, \cdot)$ is convex and continuous for all $x \in X$.

For each $a \in A$, let us consider the *parametric equilibrium problem* (EP_a): find a point $\overline{x}_a \in K$ such that

$$(\mathrm{EP}_a) \qquad F(\overline{x}_a, y, a) \geq 0 \quad \forall y \in K$$

and the function $g : X \times A \to \mathbb{R}$, defined as

$$g(x, a) = \sup_{y \in K}(-F(x, y, a)).$$

Due to the properties in $(\mathbf{C}')$, we get $g(x,a) = \max_{y \in K}(-F(x,y,a))$ and $g(x,a) \geq 0$ for every $(x,a) \in K \times A$. In addition, $g(\cdot,a)$ is convex and lower semicontinuous, for every $a \in A$ and, thanks to Proposition 2.2, $g(x,\cdot)$ is concave and upper semicontinuous, for every $x \in X$.

The equilibrium problem (EP_a) is now related to the parametric optimization problem

$$p_a = \inf_{x \in K} g(x,a) = \min_{x \in K} g(x,a) = \inf_{x \in X} (g(x,a) + \delta_K(x)) \qquad (\mathrm{P}_a)$$

and (EP_a) is solvable if and only if there exists $\overline{x}_a \in K : \, g(\overline{x}_a) = 0 = \min_{x \in K} g(x)$.

In order to introduce a suitable notion of robust equilibriun problem, let us remind that in literature a *robust solution* of the parametric optimization problem $(\mathrm{P}_a)_{a \in A}$ is a point $\overline{x} \in K$ such that $\overline{x}$ is a minimizer on K of the function $s : X \to \mathbb{R}$ given by

$$s(x) = \max_{a \in A} g(x,a).$$

Note that the function s is well defined on the whole X since $g(x,\cdot)$ is upper semicontinuous on A for each $x \in X$, and it is non negative everywhere. Moreover, we have that

$$\begin{aligned} s(x) &= \max_{a \in A} g(x,a) = \max_{a \in A} \sup_{y \in K} (-F(x,y,a)) \\ &= \sup_{y \in K} \max_{a \in A} (-F(x,y,a)) = \sup_{y \in K} (-\min_{a \in A} F(x,y,a)). \end{aligned}$$

Setting

$$f(x,y) = \min_{a \in A} F(x,y,a),$$

under the assumptions $(\mathbf{C}')$ on F, the function f fulfills the following properties:

(i) $f(x,x) = 0$ for every $x \in X$;

(ii) $f(\cdot,y)$ is concave on X for every $y \in X$ (and therefore continuous on X);

(iii) $f(x,\cdot)$ is convex on X, for every $x \in X$, by Proposition 2.2; in particular, it is continuous on X.

The properties above entail the solvability of the equilibrium problem associated to f on K and, as in (2), the solution set is the set of points $\{x \in K : \, s(x) = 0\}$, i.e. the solutions of this equilibrium problem are the solutions of the robust minimization problem.

Starting from these considerations, given the family of equilibrium problems $(\mathrm{EP}_a)_{a \in A}$, it seems to be reasonable to define the *robust equilibrium problem* as the equilibrium problem associated to the bifunction f. We will denote it by R-$(\mathrm{EP}_a)_{a \in A}$.

The optimistic counterpart O-$(\mathrm{EP}_a)_{a \in A}$ of the $(\mathrm{EP}_a)_{a \in A}$ can be therefore given by the equilibrium problem associated to the bifunction $\sup_{a \in A} F(x,y,a)$.

By using the dual equilibrium problem discussed in Section 3, we are now interested in highlighting the relationship between the solutions of the dual of R-$(\mathrm{EP}_a)_{a \in A}$ and the solutions of O-$(\mathrm{DEP}_a)_{a \in A}$, which is the optimistic problem associated to the family of the dual equilibrium problems $(\mathrm{DEP}_a)_{a \in A}$. As we will see in the next result, we are able to extend to equilibrium problems a well known result that holds in other settings; in [5], for instance, the result was discussed either for linear programming problems or optimization problems with respect to the Lagrangian duality.

Theorem 5.1. *Let $K \subseteq X$ and $A \subseteq Y$ be nonempty, compact and convex sets, and let $F : X \times X \times A \to \mathbb{R}$ satisfy the assumptions* $(\mathbf{C}')$. *In addition suppose that the function $-F(\cdot, y, a)$ is super coercive, for every $a \in A$. Then the dual of* R-$(\mathrm{EP}_a)_{a\in A}$ *coincides with* O-$(\mathrm{DEP}_a)_{a\in A}$.

Proof. For every $a \in A$, set $F_a(\cdot,\cdot) = F(\cdot,\cdot,a)$. Let us first note that the optimistic solutions of $(\mathrm{DEP}_a)_{a\in A}$ are the solutions of the equilibrium problem associated to the bifunction $\sup_{a\in A} \Phi_{F_a}$, i.e. the points $\overline{x}^* \in X^*$ such that

$$\sup_{a\in A} \Phi_{F_a}(\overline{x}^*, y^*) \geq 0, \qquad \forall y^* \in X^*.$$

On the other hand, the dual of R-$(\mathrm{EP}_a)_{a\in A}$ is associated to the bifunction Φ_f where, again, $f(x,y) = \min_{a\in A} F_a(x,y)$. By the Sion Minimax Theorem (see [20], Corollary 3.3) we have that

$$\inf_{a\in A} \varphi_{F_a}(y, x^*) = \inf_{a\in A} \sup_{x\in X}(\langle x^*, x\rangle + F(x,y,a)) = \sup_{x\in X}(\langle x^*, x\rangle + \inf_{a\in A} F(x,y,a)) = \varphi_f.$$

Hence, starting from

$$\Phi_{F_a}(x^*, y^*) = (\varphi_{F_a}(\cdot, x^*))^*(y^*) + i_K(x^*) - i_K(y^*),$$

we have that

$$\begin{aligned}
\sup_{a\in A} \Phi_{F_a}(x^*, y^*) &= i_K(x^*) - i_K(y^*) + \sup_{a\in A}\left(\sup_{y\in X}(\langle y^*, y\rangle - \varphi_{F_a}(y, x^*))\right) \\
&= i_K(x^*) - i_K(y^*) + \sup_{y\in X}\sup_{a\in A}(\langle y^*, y\rangle - \varphi_{F_a}(y, x^*)) \\
&= i_K(x^*) - i_K(y^*) + \sup_{y\in X}(\langle y^*, y\rangle - \inf_{a\in A}\varphi_{F_a}(y, x^*)) \\
&= i_K(x^*) - i_K(y^*) + \sup_{y\in X}(\langle y^*, y\rangle - \varphi_f(y, x^*)) \\
&= i_K(x^*) - i_K(y^*) + (\varphi_f(\cdot, x^*))^*(y^*) \\
&= \Phi_f(x^*, y^*).
\end{aligned}$$

□

Acknowledgements. We would like to thank Juan Enrique Martínez-Legaz for valuable conversations on the topic of the paper, and to express our gratitude to the referee for his/her careful reading which improved the presentation of the paper.

References

[1] B. Alleche: *On hemicontinuity of bifunctions for solving equilibrium problems*, Adv. Nonlinear Anal. 3 (2014) 69–80.

[2] J.-P. Aubin: *Optima and Equilibria. An Introduction to Nonlinear Analysis*, 2nd ed., Graduate Texts in Mathematics 140, Springer, Berlin (1998).

[3] A. Auslender: *Optimization. Méthodes Numériques*, Masson, Paris (1976).

[4] H. H. Bauschke, P. L. Combettes: *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, Berlin (2011).

[5] A. Beck, A. Ben-Tal: *Duality in robust optimization: primal worst equals dual best*, Oper. Res. Lett. 37 (2009) 1–6.

[6] M. Bianchi, N. Hadjisavvas, R. Pini: *Representative functions of maximally monotone operators and bifunctions*, Math. Program. 168 (2018) 433–448.

[7] M. Bianchi, G. Kassay, R. Pini: *Brezis pseudomonotone bifunctions and quasi equilibrium problems via penalization*, J. Global Optim. 82 (2022) 483–498.

[8] G. Bigi, M. Castellani, G. Kassay: *A dual view of equilibrium problems*, J. Math. Anal. Appl. 342 (2008) 17–26.

[9] E. Blum, W. Oettli: *From optimization and variational inequalities to equilibrium problems*, Math. Student 63 (1994) 123–145.

[10] J. M. Borwein, Q. J. Zhu: *Techniques of Variational Analysis*, Springer, Berlin (2005).

[11] S. Boyd, L. Vandenberghe: *Convex Optimization*, Cambridge University Press, Cambridge (2004).

[12] G. Y. Chen, C. J. Goh, X. Q. Yang: *On gap functions and duality of variational inequality problems*, J. Math. Anal. Appl. 214 (1997) 658–673.

[13] V. Jeyakumar, G. Li, J. H. Wang: *Some robust convex programs without a duality gap*, J. Convex Anal. 20 (2013) 377–394.

[14] C. S. Lalitha: *A note on duality of generalized equilibrium problem*, Optim. Lett. 4 (2010) 57–66.

[15] R. Lucchetti: *Convexity and Well-Posed Problems*, Springer, Berlin (2006).

[16] I. Konnov, S. Schaible: *Duality for equilibrium problems under generalized monotonicity*, J. Optim. Theory Appl. 104 (2002) 395–408.

[17] J. E. Martinez-Legaz, W. Sosa: *Duality for equilibrium problems*, J. Global Optim. 35 (2006) 311–319.

[18] U. Mosco: *Dual variational inequalities*, J. Math. Anal. Appl. 40 (1972) 202–206.

[19] R. T. Rockafellar: *Convex Analysis*, Princeton Mathematical Series No. 28, Princeton University Press, Princeton (1970).

[20] M. Sion: *On general minimax theorems*, Pac. J. Math. 8 (1958) 171–176.