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Multiplicity Results for a Non-Homogeneous Neumann Problem via a Variational Principle of Ricceri

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In this paper, the existence and multiplicity of weak solutions are obtained for a class of non-homogeneous Neumann problems. The proof of the main results relies on a recent variational principle due to Ricceri.

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1. Introduction

This paper treats the following problem involving non-homogeneous differential operators

$$\begin{cases} -\operatorname{div}\big(\alpha(x,|\nabla u|)\nabla u\big) + \alpha(x,|u|)u = \lambda a(x)f(u) + b(x)g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous non-constant functions, $\lambda \in \mathbb{R}$, $a, b \in L^1(\Omega)$ are non-negative non-constant functions and the function $\alpha(x,t) : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ will be specified later.

The main features of this paper are the following:

(I) the presence of a non-homogeneous differential operator and the treatment in a suitable Orlicz-Sobolev function space;

(II) the use of the Ricceri four critical point theorem, which is a powerful analytic tool for multiplicity results in nonlinear problems with a variational structure.

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Orlicz-Sobolev spaces have been used in the last decades to model various phenomena. These function spaces play a significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, non-linear potential theory, the theory of quasi-conformal mappings, non-Newtonian fluids, image processing, differential geometry, geometric function theory, and probability theory. These spaces consist of functions that have weak derivatives and satisfy certain integrability conditions.

We refer to Chen, Levine and Rao [4], who proposed a framework for image restoration based on a Laplace operator with variable exponent. A second major application of non-homogeneous differential operators with variable exponent is the modeling of some materials with inhomogeneities, for instance electrorheological (non-Newtonian) fluids (sometimes referred to as 'smart fluids'), cf. [3, 8, 9, 20, 21, 22]. Materials requiring such more advanced theory have been studied experimentally since the middle of the last century. The first major discovery in electrorheological fluids is due to Willis Winslow in 1949. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. Winslow noticed that in such fluids (for instance lithium polymetachrylate) viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This phenomenon is known as the *Winslow effect.* For a general account of the underlying physics we refer to Halsey [11] and Pfeiffer et al. [16]. An overview of Orlicz–Sobolev spaces is given in the monographs by Rao and Ren [17] and Rădulescu and Repovš [21].

The starting point of our approach to problem (1) has been [18], in which the author considers the following ordinary Neumann problem

$$\begin{cases} -u'' + u = \lambda a(x) f(u) + b(x) g(u) & \text{in } [0,1], \\ u'(0) = u'(1) = 0, \end{cases} \tag{2}$$

where $\lambda \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$, $a, b : [0,1] \to [0, +\infty[$ are four continuous non-constant functions. Thanks to a new multiplicity result established in the same [18], Ricceri has proved that the problem above admits at least three non-zero solutions.

In [2], the authors extended problem (2) to the following $p(x)$-Laplacian case

$$\begin{cases} -\Delta_{p(x)} u + |u|^{p(x)-2} u = \lambda a(x) f(u) + b(x) g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative, $f, g : \mathbb{R} \to \mathbb{R}$ are continuous non-constant functions, $\lambda \in \mathbb{R}$, $a, b \in L^1(\Omega)$ are non-negative non-constant functions and $p \in L^\infty(\Omega)$ is such that

$$N < p^- := \inf_{x \in \Omega} p(x) \leq p^+ := \sup_{x \in \Omega} p(x) < +\infty.$$

It is useful to note that, in the particular case when in (1) we have $\alpha(x,t) = t^{p(x)-2}$ for all $t \in \mathbb{R}$, $p > 1$ with $p \in L^\infty(\Omega)$, then problem (1) becomes the $p(x)$-Laplacian equation, namely equation (3). It is clear that this is a natural extension from the earlier studies on nonlocal problems in classical Sobolev spaces and on nonlinear non-homogeneous problems in Orlicz-Sobolev spaces.

In this paper, motivated by the above facts and the recent paper [18], we establish some sufficient conditions under which the problem (1) possesses three non-zero weak solutions in the Orlicz-Sobolev space.

This paper is organized as follows. In Section 2, some preliminaries and the abstract Orlicz-Sobolev spaces setting are presented. In Section 3, we present our main result and its proof with some remarks and applications.

2. Functional setting

As already said in the introduction, our approach in facing problem (1) is based on the very recent multiplicity result established in [18], that we recall below for the reader's convenience (see also Theorem 1 of [19] for a similar result).

Theorem 2.1 ([18, Theorem 1]). *Let X be a reflexive real Banach space; $J : X \to \mathbb{R}$ be a coercive and sequentially weakly lower semicontinuous C^1 functional whose derivative admits a continuous inverse on X^*, $I_1, I_2 : X \to \mathbb{R}$ two C^1 functionals with compact derivative. Assume that there exist two points $u_0, v_0 \in X$ with the following properties:*

(i) *u_0 is a strict local minimum of J and $J(u_0) = I_1(u_0) = I_2(u_0) = 0$;*

(ii) *$J(v_0) \leq I_1(v_0)$ and $I_2(v_0) > 0$.*

Moreover, suppose that, for some $\rho \in \mathbb{R}$, one has either

$$\sup_{\lambda>0} \inf_{u\in X} (\lambda(J(u) - I_1(u) - \rho) - I_2(u)) \\ < \inf_{u\in X} \sup_{\lambda>0} (\lambda(J(u) - I_1(u) - \rho) - I_2(u)) \tag{4}$$

or

$$\sup_{\lambda>0} \inf_{u\in X} (J(u) - I_1(u) - \lambda(\rho + I_2(u))) \\ < \inf_{u\in X} \sup_{\lambda>0} (J(u) - I_1(u) - \lambda(\rho + I_2(u))). \tag{5}$$

Finally, assume that

$$\max\left\{ \limsup_{\|u\|\to+\infty} \frac{I_1(u)}{J(u)},\ \limsup_{u\to u_0} \frac{I_1(u)}{J(u)} \right\} < 1 \tag{6}$$

and

$$\max\left\{ \limsup_{\|u\|\to+\infty} \frac{I_2(u)}{J(u)},\ \limsup_{u\to u_0} \frac{I_2(u)}{J(u)} \right\} \leq 0. \tag{7}$$

Under such hypotheses, there exists $\lambda^* > 0$ *such that the equation* $J'(u) = I_1'(u) + \lambda^* I_2'(u)$ *has at least four solutions in* X. *More precisely, among them, one is* u_0 *as a strict local, not global minimum and two are global minima of the functional* $J - I_1 - \lambda^* I_2$.

Remark 2.2. It is important to remark that, in view of Theorem 1 of [6], condition (4) is equivalent to the existence of $u_1, v_1 \in X$ satisfying

$$J(u_1) - I_1(u_1) < \rho < J(v_1) - I_1(v_1), \quad \text{and}$$

$$\frac{\sup_{(J-I_1)^{-1}(]-\infty,\rho])} I_2 - I_2(u_1)}{\rho - J(u_1) + I_1(u_1)} < \frac{\sup_{(J-I_1)^{-1}(]-\infty,\rho])} I_2 - I_2(v_1)}{\rho - J(v_1) + I_1(v_1)} .$$

Likewise, condition (5) is equivalent to the existence of $u_1, v_1 \in X$ satisfying

$$I_2(v_1) < \rho < I_2(u_1), \quad \text{and}$$

$$\frac{J(u_1) - I_1(u_1) - \inf_{I_2([\rho,+\infty[)^{-1}}(J - I_1)}{I_2(u_1) - \rho} < \frac{J(v_1) - I_1(v_1) - \inf_{I_2([\rho,+\infty[)^{-1}}(J - I_1)}{I_2(v_1) - \rho}.$$

In order to study problem (1), let us introduce the functional spaces where it will be discussed. We will give just a brief review of some basic concepts and facts of the theory of Orlicz-Sobolev spaces, useful for what follows; for more details we refer the reader to Adams [1], Diening [7], Musielak [15] and Rao and Ren [17].

We now recall some facts on the theory of Orlicz-Sobolev spaces that will be used in the present paper. Suppose that the function $\alpha(x,t) : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is such that the mapping $\varphi(x,t) : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi(x,t) = \begin{cases} \alpha(x,|t|)t & \text{for} \quad t \neq 0, \\ 0 & \text{for} \quad t = 0 \end{cases}$$

satisfies the condition

(φ) for all $x \in \Omega$, $\varphi(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and

$$\Phi(x,t) = \int_0^t \varphi(x,s)ds, \quad \forall x \in \overline{\Omega}, \ t \geq 0$$

belongs to class Φ (see [15], p. 33), i.e. the function Φ satisfies the following conditions:

(Φ_1) for all $x \in \Omega$, $\Phi(x,\cdot) : [0,+\infty) \to \mathbb{R}$ is a non-decreasing continuous function, with $\Phi(x,0) = 0$ and $\Phi(x,t) > 0$ whenever $t > 0$, $\lim_{t\to\infty} \Phi(x,t) = \infty$,

(Φ_2) for every $t \geq 0$, $\Phi(\cdot,t) : \Omega \to \mathbb{R}$ is a measurable function.

Since $\varphi(x,\cdot)$ satisfies condition (φ), we deduce that $\Phi(x,\cdot)$ is convex and increasing from $\mathbb{R}^+$ to $\mathbb{R}^+$.

For the function Φ, we define the *generalized Orlicz class*,

$$K_\Phi(\Omega) = \Big\{u : \Omega \to \mathbb{R},\ \text{measurable};\ \int_\Omega \Phi(x, |u(x)|)\,dx < \infty\Big\}$$

and the *generalized Orlicz space*,

$$L^\Phi(\Omega) = \Big\{u : \Omega \to \mathbb{R},\ \text{measurable};\ \lim_{\lambda\to 0^+} \int_\Omega \Phi(x, \lambda|u(x)|)\,dx = 0\Big\}.$$

The space $L^\Phi(\Omega)$ is a Banach space endowed with the *Luxemburg norm*

$$|u|_\Phi = \inf\left\{\mu > 0;\ \int_\Omega \Phi\left(x, \frac{|u(x)|}{\mu}\right)\,dx \le 1\right\}$$

or the equivalent norm (the *Orlicz norm*)

$$|u|_{(\Phi)} = \sup\left\{\left|\int_\Omega uv\,dx\right|;\ v \in L^{\overline{\Phi}}(\Omega),\ \int_\Omega \overline{\Phi}(x, |v(x)|)\,dx \le 1\right\},$$

where $\overline{\Phi}$ denotes the *conjugate Young* function of Φ, that is,

$$\overline{\Phi}(x,t) = \sup_{s>0}\{ts - \Phi(x,s);\ s \in \mathbb{R}\}, \quad \forall\, x \in \overline{\Omega},\ t \ge 0\,.$$

Furthermore, for Φ and $\overline{\Phi}$ conjugate Young functions, the Hölder type inequality holds true:

$$\left|\int_\Omega uv\,dx\right| \le B\,|u|_\Phi\,|v|_{\overline{\Phi}}, \quad \forall\, u \in L^\Phi(\Omega),\ v \in L^{\overline{\Phi}}(\Omega)\,, \tag{8}$$

where B is a positive constant (see [15], Theorem 13.13). In this paper we assume that there exist two positive constants φ_0 and φ^0 such that

$$1 < \varphi_0 \le \frac{t\varphi(x,t)}{\Phi(x,t)} \le \varphi^0 < \infty, \quad \forall\, x \in \overline{\Omega},\ t \ge 0\,. \tag{9}$$

The above relation implies that Φ satisfies the Δ_2-*condition*, i.e.

$$\Phi(x, 2t) \le K\,\Phi(x,t), \quad \forall\, x \in \overline{\Omega},\ t \ge 0\,, \tag{10}$$

where K is a positive constant (see [14, Proposition 2.3]). Relation (10) and Theorem 8.13 in [15] imply that $L^\Phi(\Omega) = K_\Phi(\Omega)$. Furthermore, we assume that Φ satisfies the following condition:

$$\text{for each } x \in \overline{\Omega},\ \text{the function } [0,\infty) \ni t \to \Phi(x, \sqrt{t}) \text{ is convex}\,. \tag{11}$$

Relation (11) assures that $L^\Phi(\Omega)$ is an uniformly convex space and thus, a reflexive space (see [14, Proposition 2.2]).

On the other hand, we point out that assuming that Φ and Ψ belong to class Φ and

$$\Psi(x,t) \leq K_1\, \Phi(x, K_2\, t) + \eta(x), \quad \forall\, x \in \overline{\Omega},\ t \geq 0, \tag{12}$$

where $\eta \in L^1(\Omega)$, $\eta(x) \geq 0$ a.e. $x \in \Omega$ and K_1, K_2 are positive constants, then by Theorem 8.5 in [15] there exists the continuous embedding $L^{\Phi}(\Omega) \subset L^{\Psi}(\Omega)$. Next, we define the *generalized Orlicz-Sobolev space*

$$W^{1,\Phi}(\Omega) = \left\{ u \in L^{\Phi}(\Omega);\ \frac{\partial u}{\partial x_i} \in L^{\Phi}(\Omega),\ i = 1, ..., N \right\}.$$

On $W^{1,\Phi}(\Omega)$ we define the equivalent norms

$$\|u\|_{1,\Phi} = |\ |\nabla u|\ |_{\Phi} + |u|_{\Phi}, \qquad \|u\|_{2,\Phi} = \max\{|\ |\nabla u|\ |_{\Phi}, |u|_{\Phi}\},$$

$$\|u\| = \inf \left\{ \mu > 0;\ \int_{\Omega} \left[\Phi\left(x, \frac{|u(x)|}{\mu}\right) + \Phi\left(x, \frac{|\nabla u(x)|}{\mu}\right) \right] dx \leq 1 \right\}.$$

More precisely, for every $u \in W^{1,\Phi}(\Omega)$, we have

$$\|u\| \leq 2\|u\|_{2,\Phi} \leq 2\|u\|_{1,\Phi} \leq 4\|u\| \tag{13}$$

(see [14, Proposition 2.4]). The generalized Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ endowed with one of the above norms is a reflexive Banach space. In the following, we will use the norm $\|\cdot\|$ on $E := W^{1,\Phi}(\Omega)$.

The following lemma is useful in the proof of our results.

Lemma 2.3. *Let* $u \in E$. *Then*

$$\int_{\Omega} \Big(\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|) \Big)\, dx \geq \|u\|^{\varphi_0} \quad \text{if } \|u\| > 1; \tag{14}$$

$$\int_{\Omega} \Big(\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|) \Big)\, dx \leq \|u\|^{\varphi_0} \quad \text{if } \|u\| < 1; \tag{15}$$

$$\int_{\Omega} \Big(\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|) \Big)\, dx \leq \|u\|^{\varphi^0} \quad \text{if } \|u\| > 1; \tag{16}$$

$$\int_{\Omega} \Big(\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|) \Big)\, dx \geq \|u\|^{\varphi^0} \quad \text{if } \|u\| < 1. \tag{17}$$

For the proof of the previous result see, for instance, Lemma 2.3 of [13].

We point out that assuming that Φ and Ψ belong to class Φ, satisfying relation (12) and $\inf_{x\in\Omega} \Phi(x, 1) > 0$, $\inf_{x\in\Omega} \Psi(x, 1) > 0$, then there exists the continuous embedding $W^{1,\Phi}(\Omega) \hookrightarrow W^{1,\Psi}(\Omega)$.

In this paper we study the problem (1) in the particular case when Φ satisfies

$$M\,|t|^{p(x)} \leq \Phi(x,t), \quad \forall\, x \in \overline{\Omega},\ t \geq 0\,, \tag{18}$$

where $p(x) \in C(\overline{\Omega})$ with $p^- := \inf_{x\in\Omega} p(x) > N$ for all $x \in \overline{\Omega}$, and $M > 0$ is a constant.

By the relation (18) we deduce that E is continuously embedded in $W^{1,p(x)}(\Omega)$ (see relation (12) with $\Psi(x,t) = |t|^{p(x)}$).

Moreover, as pointed out in [10] and [12], $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ and since $p^- > N$, we deduce that $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, E is compactly embedded in $C^0(\overline{\Omega})$ and there exists a constant $c > 0$ such that

$$\|u\|_\infty \leq c\,\|u\|, \quad \forall\, u \in E, \tag{19}$$

where $\|u\|_\infty := \sup_{x\in\overline{\Omega}} |u(x)|$.

Example 2.4. Define

$$\varphi(x,t) = p(x)\frac{|t|^{p(x)-2}t}{\log(1+|t|)} \quad \text{for} \quad t \neq 0, \ \text{ and } \ \varphi(x,0) = 0,$$

where $p(x) \in C(\overline{\Omega})$ satisfying $N < p(x) < +\infty$ for all $x \in \overline{\Omega}$. Some simple computations imply

$$\Phi(x,t) = \frac{|t|^{p(x)}}{\log(1+|t|)} + \int_0^{|t|} \frac{s^{p(x)}}{(1+s)(\log(1+s))^2}\,ds\,,$$

and the relations (φ), (Φ_1) and (Φ_2) are verified. For each $x \in \overline{\Omega}$ fixed, by Example 3 on p. 243 in [5], we have

$$p(x) - 1 \leq \frac{t\varphi(x,t)}{\Phi(x,t)} \leq p(x), \quad \forall\, t \geq 0\,.$$

Thus, the relation (9) holds true with $\varphi_0 = p^- - 1$ and $\varphi^0 = p^+ := \sup_{x\in\Omega} p(x)$. Next, Φ satisfies the condition (18) since

$$\Phi(x,t) \geq t^{p(x)-1}, \quad \forall\, x \in \overline{\Omega},\ t \geq 0\,.$$

Finally, we point out that trivial computations imply that

$$\frac{d^2(\Phi(x,\sqrt{t}))}{dt^2} \geq 0$$

for all $x \in \overline{\Omega}$ and $t \geq 0$. Thus, the relation (11) is satisfied.

The following lemma is useful in the proof of our results.

Lemma 2.5. *If* $\gamma : \Omega \to [0, +\infty[$ *be a non-zero function in* $L^1(\Omega)$ *and* $h : \mathbb{R} \to \mathbb{R}$ *a continuous non-zero function, we denote by* $T_{\gamma,h}$ *the functional defined on* E *by putting*

$$T_{\gamma,h}(u) = \int_\Omega \gamma(x) h(u(x)) dx$$

for all $u \in E$. *Then, one has*

$$\limsup_{u \to 0} \frac{T_{\gamma,h}(u)}{\|u\|^{\varphi^0}} \leq c^{\varphi^0} \|\gamma\|_{L^1(\Omega)} \max\left\{0, \ \limsup_{\xi \to 0} \frac{h(\xi)}{|\xi|^{\varphi^0}}\right\} \tag{20}$$

and

$$\limsup_{\|u\| \to +\infty} \frac{T_{\gamma,h}(u)}{\|u\|^{\varphi_0}} \leq c^{\varphi_0} \|\gamma\|_{L^1(\Omega)} \max\left\{0, \ \limsup_{|\xi| \to +\infty} \frac{h(\xi)}{|\xi|^{\varphi_0}}\right\} . \tag{21}$$

Also, the functional $T_{\gamma,H}$, *where* H *is defined by* $H(\xi) = \int_0^\xi h(t)\, dt$ *for all* $\xi \in \mathbb{R}$, *turns out to be in* $C^1(E, \mathbb{R})$ *and its derivative is given by*

$$T'_{\gamma,H}(u) = \int_\Omega \gamma(x) h(u(x))\, v(x) dx$$

for all $u, v \in E$. *Moreover, the compact embedding of* E *in* $C^0(\bar{\Omega})$ *implies that* $T'_{\gamma,H} : E \to E^*$ *is compact.*

The proof of the previous lemma is similar to proof of Lemma 2.1 in [2].

We can deduce that $u \in E$ is a weak solution to problem (1) if and only if u is a critical point of the functional

$$\int_\Omega \Big(\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|) \Big)\, dx - \lambda T_{a,F} - T_{b,G}$$

which represents, therefore, the energy functional related to problem (1).

3. Main results

Now, we present our main result.

Theorem 3.1. *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be two continuous non-constant functions and let* $a, b : \Omega \to [0, +\infty[$ *be two non-constant functions in* $L^1(\Omega)$. *Assume that*

(f_1) $\quad \max\left\{ \limsup_{|\xi| \to +\infty} \frac{\int_0^\xi f(t)\, dt}{|\xi|^{\varphi_0}}, \limsup_{\xi \to 0} \frac{\int_0^\xi f(t)\, dt}{|\xi|^{\varphi^0}} \right\} \leq 0,$

(g_1) $\quad \sup_{\xi \in \mathbb{R}} \int_0^\xi g(t)\, dt < +\infty, \qquad \limsup_{\xi \to 0} \frac{\int_0^\xi g(t)\, dt}{|\xi|^{\varphi^0}} < \frac{1}{c^{\varphi^0} \|b\|_{L^1(\Omega)}}.$

Finally, suppose that there exist $\sigma > c \max\left\{1,\ (\|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G)^{\frac{1}{\varphi_0}}\right\}$ *and* $\xi_1 \in \mathbb{R}$ *such that*

(f_2) $0 < \int_0^{\xi_1} f(t)\,dt = \sup_{|\xi|\le\sigma} \int_0^{\xi} f(t)\,dt < \sup_{\xi\in\mathbb{R}} \int_0^{\xi} f(t)\,dt,$

(g_2) $\max\left\{|\xi_1|^{\varphi_0}, |\xi_1|^{\varphi^0}\right\} \le \dfrac{\|b\|_{L^1(\Omega)}}{|\Omega|} \int_0^{\xi_1} g(t)\,dt.$

Under such hypotheses, there exists $\lambda^* > 0$ *such that the problem*

$$\begin{cases} -\mathrm{div}\big(\alpha(x,|\nabla u|)\nabla u\big) + \alpha(x,|u|)u = \lambda^* a(x) f(u) + b(x) g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \end{cases}$$

has at least three non-zero solutions, two of which are global minima of the associated energy functional.

Proof. Our aim is to apply Theorem 2.1. Take I_1, I_2 equal, respectively, to $T_{b,G}, T_{a,F}$ defined in Section 2. For each $u \in E$, let the functional $J : E \to \mathbb{R}$ be defined by

$$J(u) := \int_\Omega \Big(\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|) \Big)\, dx.$$

Since Φ is convex, it follows that J is a convex functional, and hence it is sequentially weakly lower semi-continuous. By Lemma 2.3, we see that J is coercive, and arguing as in the proof of [13, Lemma 3.2], we have that $J' : E \to E^*$ is a uniformly monotone operator in E. By applying the Minty-Browder theorem (Theorem 26.A of [23]), J' admits a continuous inverse on E^*.

Moreover, similar arguments as those used in [14, Lemma 4.2] imply that $J \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle J'(u), v\rangle = \int_\Omega \alpha(x, |\nabla u(x)|)\nabla u(x)\, \nabla v(x)\, dx + \int_\Omega \alpha(x, |u(x)|) u(x)\, v(x)\, dx$$

for every $v \in E$. Also J is bounded from below.

Now, take $u_0 = 0$ and $v_0 = \xi_1$; of course, (i) is evident. Moreover, thanks to (f_2), (g_2) and Lemma 2.3, one has

$$\begin{aligned} J(v_0) = \int_\Omega \Phi(x, |\xi_1|)\, dx \ &\le\ |\Omega| \max\{|\xi_1|^{\varphi_0}, |\xi_1|^{\varphi^0}\} \\ &\le\ \|b\|_{L^1(\Omega)} \int_0^{\xi_1} g(t)\, dt = T_{b,G}(v_0) \end{aligned}$$

and

$$T_{a,F}(v_0) = \|a\|_{L^1(\Omega)} \int_0^{\xi_1} f(t)\, dt > 0,$$

which proves (ii). Now, if $\|u\| > 1$, due to Lemma 2.3 we have

$$\frac{T_{b,G}(u)}{J(u)} \leq \frac{T_{b,G}(u)}{\|u\|^{\varphi_0}}$$

and thanks to Lemma 2.5 we get

$$\limsup_{\|u\|\to+\infty} \frac{T_{b,G}(u)}{J(u)} \leq c^{\varphi_0}\|b\|_{L^1(\Omega)} \max\left\{0,\ \limsup_{|\xi|\to+\infty} \frac{\int_0^\xi g(t)\,dt}{|\xi|^{\varphi_0}}\right\} \leq 0.$$

If $\|u\| < 1$, due to Lemma 2.3 we have

$$\frac{T_{b,G}(u)}{J(u)} \leq \frac{T_{b,G}(u)}{\|u\|^{\varphi^0}}$$

and thanks to Lemma 2.5 and (g_1) we get

$$\limsup_{u\to0} \frac{T_{b,G}(u)}{J(u)} \leq c^{\varphi^0}\|b\|_{L^1(\Omega)} \max\left\{0,\ \limsup_{\xi\to0} \frac{\int_0^\xi g(t)\,dt}{|\xi|^{\varphi^0}}\right\} < 1.$$

So (6) is fulfilled. Likewise, from Lemma 2.5 and (f_1), we get

$$\limsup_{\|u\|\to+\infty} \frac{T_{a,F}(u)}{J(u)} \leq c^{\varphi_0}\|a\|_{L^1(\Omega)} \max\left\{0,\ \limsup_{|\xi|\to+\infty} \frac{\int_0^\xi f(t)\,dt}{|\xi|^{\varphi_0}}\right\} \leq 0$$

and

$$\limsup_{u\to0} \frac{T_{a,F}(u)}{J(u)} \leq c^{\varphi^0}\|a\|_{L^1(\Omega)} \max\left\{0,\ \limsup_{\xi\to0} \frac{\int_0^\xi f(t)\,dt}{|\xi|^{\varphi^0}}\right\} \leq 0,$$

which satisfy (7). Finally, let us check that (4) holds. being $\sigma > c$, we have that

$$1 - \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G < \left(\frac{\sigma}{c}\right)^{\varphi_0} - \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G$$

and so it is possible to choose $\rho \in \mathbb{R}$ such that

$$\max\left\{0,\ 1 - \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G\right\} < \rho < \left(\frac{\sigma}{c}\right)^{\varphi_0} - \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G.$$

Let $u \in E$ such that $J(u) - T_{b,G}(u) \leq \rho$. If $\|u\| \leq 1$, by the definition of σ and the choice of ρ we get

$$\|u\|^{\varphi^0} \leq J(u) \leq \rho + \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G$$

and then, being $\rho + \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G > 1$, we obtain

$$\|u\| \leq \left(\rho + \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G\right)^{\frac{1}{\varphi^0}} \leq \left(\rho + \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G\right)^{\frac{1}{\varphi_0}} \leq \frac{\sigma}{c}\ .$$

On the other hand, if $\|u\| > 1$, we get

$$\|u\|^{\varphi_0} \leq J(u) \leq \rho + \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G\,,$$

and then

$$\|u\| \leq \big(\rho + \|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G\big)^{\frac{1}{\varphi_0}} \leq \frac{\sigma}{c}\ .$$

Thus, owing to the embedding of E in $C^0(\bar{\Omega})$ we have the inclusion

$$\big\{u \in E \,:\, J(u) - T_{b,G}(u) \leq \rho\big\} \subseteq \left\{u \in E \,:\, \sup_{x \in \bar{\Omega}} |u(x)| \leq \sigma\right\}. \tag{22}$$

Now, in order to fulfill the equivalent formulation of (4) recalled in Remark 2.2, choose $u_1 = v_0$ and take as v_1 any constant d such that $F(d) > \sup_{[-\sigma,\sigma]} F$. Such a d does exist by (f_2). Thanks to (g_2) we have

$$J(u_1) - T_{b,G}(u_1) \leq |\Omega| \max\left\{|\xi_1|^{\varphi_0}, |\xi_1|^{\varphi^0}\right\} - \|b\|_{L^1(\Omega)} \int_0^{\xi_1} g(t)\, dt \leq 0 < \rho.$$

Moreover, $J(v_1) - T_{b,G}(v_1)$ has to be necessarily strictly greater than ρ, otherwise, by (22) we would have $|d| \leq \sigma$ and $F(d) \leq \sup_{[-\sigma,\sigma]} F$, a contradiction. Then, due to (f_2) and to the choice of d, we readily obtain that

$$\sup_{(J - T_{b,G})^{-1}(]-\infty,\rho])} T_{a,F} \leq T_{a,F}(u_1)$$

and

$$\sup_{(J - T_{b,G})^{-1}(]-\infty,\rho])} T_{a,F} \leq T_{a,F}(v_1).$$

Thus, the following inequalities hold

$$\frac{\sup_{(J - T_{b,G})^{-1}(]-\infty,\rho])} T_{a,F} - T_{a,F}(u_1)}{\rho - J(u_1) + T_{b,G}(u_1)} < 0 < \frac{\sup_{(J - T_{b,G})^{-1}(]-\infty,\rho])} T_{a,F} - T_{a,F}(v_1)}{\rho - J(v_1) + T_{b,G}(v_1)}$$

and, each assumption of Theorem 2.1 being satisfied, our problem admits at least three non-zero weak solutions, two of which are global minima of the energy functional. ☐

Remark 3.2. It is worth pointing out that, if assumption (f_2) of Theorem 3.1 is replaced by the existence of

$$\sigma > c \max\left\{1, (\|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G)^{\frac{1}{\varphi_0}}\right\}$$

and $\xi_1, \xi_2 \in \mathbb{R}$, with $\xi_1 \xi_2 > 0$, such that

$$0 < \int_0^{\xi_1} f(t)\, dt = \sup_{|\xi| \leq \sigma} \int_0^{\xi} f(t)\, dt < \int_0^{\xi_2} f(t)\, dt, \tag{23}$$

under the additional assumption $|\Omega| \geq \frac{1}{c^{\varphi_0}}$, we can ensure that the three non-zero solutions of the thesis of Theorem 3.1 are non-negative (respectively non-positive) provided $\xi_1 > 0$ (respectively $\xi_2 < 0$). To see this, it suffices to apply Theorem 3.1 to the functions $f_0, g_0 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_0(\xi) = \begin{cases} f(\xi) & \text{if } \xi \geq 0, \\ 0 & \text{if } \xi < 0, \end{cases} \qquad \text{and} \qquad g_0(\xi) = \begin{cases} g(\xi) & \text{if } \xi \geq 0, \\ 0 & \text{if } \xi < 0, \end{cases}$$

when $\xi_1 > 0$ or by

$$f_0(\xi) = \begin{cases} f(\xi) & \text{if } \xi \leq 0, \\ 0 & \text{if } \xi > 0, \end{cases} \qquad \text{and} \qquad g_0(\xi) = \begin{cases} g(\xi) & \text{if } \xi \leq 0, \\ 0 & \text{if } \xi > 0, \end{cases}$$

when $\xi_1 < 0$. In fact, conditions (f_1) and (g_1) ensure that $f(0) = 0$ and $g(0) = 0$, namely, that f_0 and g_0 are continuous functions. Therefore, condition $|\Omega| \geq \frac{1}{c^{\varphi_0}}$ guarantees that $|\xi_1| \leq \sigma$, in fact by (g_2) one has

$$|\xi_1|^{\varphi_0} \leq \frac{\|b\|_{L^1(\Omega)} \sup_{\mathbb{R}} G}{|\Omega|} \leq \sigma^{\varphi_0}.$$

Hence, the function f_0 satisfies assumption (f_2).

From Theorem 3.1, applied with $f = g$ and $a = b$, via Remark 3.2, we get:

Corollary 3.3. *Let $|\Omega| \geq \frac{1}{c^{\varphi_0}}$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

(f_3) $\sup_{\xi\in\mathbb{R}} \int_0^{\xi} f(t)\,dt < +\infty, \quad \limsup\limits_{\xi\to 0} \dfrac{\int_0^{\xi} f(t)\,dt}{|\xi|^{\varphi^0}} \leq 0.$

Moreover, suppose that there exist $\sigma > 0$ and $\xi_1, \xi_2 \in \mathbb{R}$, with $\xi_1\xi_2 > 0$, such that

(f_4) $0 < \int_0^{\xi_1} f(t)\,dt = \sup_{|\xi|\leq\sigma} \int_0^{\xi} f(t)\,dt < \int_0^{\xi_2} f(t)\,dt$ *and*

(f_5) $\dfrac{|\Omega| \max\left\{|\xi_1|^{\varphi_0}, |\xi_1|^{\varphi^0}\right\}}{\int_0^{\xi_1} f(t)\,dt} < \dfrac{\sigma^{\varphi_0}}{c^{\varphi_0} \sup_{\xi\in\mathbb{R}} \int_0^{\xi} f(t)\,dt}.$

Under such hypotheses, for every non-constant function $a : \Omega \to [0, +\infty[$ in $L^1(\Omega)$ satisfying

(f_6) $\max\left\{\dfrac{1}{\sup_{\mathbb{R}} F}, \dfrac{|\Omega|\left\{|\xi_1|^{\varphi_0}, |\xi_1|^{\varphi^0}\right\}}{\int_0^{\xi_1} f(t)\,dt}\right\} \leq \|a\|_{L^1(\Omega)} < \dfrac{\sigma^{\varphi_0}}{c^{\varphi_0} \sup_{\xi\in\mathbb{R}} \int_0^{\xi} f(t)\,dt},$

there exists $\widehat{\lambda} > 1$ such that the problem

$$\begin{cases} -\mathrm{div}\big(\alpha(x, |\nabla u|)\nabla u\big) + \alpha(x, |u|)u = \widehat{\lambda} a(x) f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

has at least three non-zero solutions which are non-negative or non-positive according to whether $\xi_1 > 0$ or $\xi_1 < 0$.

Another consequence of Theorem 3.1 is as follows:

Proposition 3.4. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a continuous function and* $\varsigma,\tau,\sigma,\upsilon$ *four positive constants, with* $\varsigma<\tau<\sigma<\upsilon$, *such that* $f(\xi)\geq 0$ *for all* $\xi\in]-\infty,-\upsilon]\cup[-\sigma,0]\cup[\varsigma,\tau]$, *while* $f(\xi)\leq 0$ *for all* $\xi\in]-\upsilon,-\sigma]\cup[0,\varsigma]\cup[\tau,+\infty[$, *and*

$$0<\int_0^{\tau} f(t)\,dt<\int_0^{-\upsilon} f(t)\,dt.$$

Moreover, let $g:\mathbb{R}\to\mathbb{R}$ *be a continuous function and let* $b:\Omega\to[0,+\infty[$ *be a function in* $L^1(\Omega)$ *such that*

$$\int_0^{\tau} g(t)\,dt>0,\quad \sup_{\xi\in\mathbb{R}}\int_0^{\xi} g(t)\,dt<+\infty,\quad \limsup_{\xi\to 0}\frac{\int_0^{\xi} g(t)\,dt}{|\xi|^{\varphi^0}}<\frac{1}{c^{\varphi^0}\|b\|_{L^1(\Omega)}}$$

and

$$\max\left\{\frac{1}{\sup_{\mathbb{R}} G},\ \frac{|\Omega|\max\left\{|\tau|^{\varphi_0},|\tau|^{\varphi^0}\right\}}{\int_0^{\tau} g(t)\,dt}\right\}\leq\|b\|_{L^1(\Omega)}<\frac{\sigma^{\varphi_0}}{c^{\varphi_0}\sup_{\xi\in\mathbb{R}}\int_0^{\xi} g(t)\,dt}.$$

Under such hypotheses, for each non-zero function $a:\Omega\to[0,+\infty[$ *in* $L^1(\Omega)$, *there exists* $\lambda^*>0$ *such that the problem*

$$\begin{cases} -\mathrm{div}\big(\alpha(x,|\nabla u|)\nabla u\big)+\alpha(x,|u|)u=\lambda^* a(x)f(u)+b(x)g(u) & \text{in }\Omega,\\ \dfrac{\partial u}{\partial\nu}=0 \qquad \text{on }\partial\Omega \end{cases}$$

has at least three non-zero solutions, two of which are global minima of the associated energy functional.

Proof. The same assumptions also imply that $\int_0^{\xi} f(t)\,dt\leq 0$ for all $\xi\in[-\sigma,\varsigma]$, and so (f_1) holds. Consequently, if we take $\xi_1=\tau$, the assumptions of Theorem 3.1 are satisfied and the conclusion follows. □

Example 3.5. Let $\Omega=\{(x,y)\in\mathbb{R}^2; x^2+y^2<3\}$. Put

$$\varphi(x,y,t)=p(x,y)\frac{|t|^{p(x,y)-2}t}{\log(1+|t|)}\quad\text{for}\quad t\neq 0,\ \text{ and }\ \varphi(x,y,0)=0,$$

where $p(x,y)=x^2+y^2+3$ for all $(x,y)\in\Omega$. Some simple computations imply

$$\Phi(x,y,t)=\frac{|t|^{p(x,y)}}{\log(1+|t|)}+\int_0^{|t|}\frac{s^{p(x,y)}}{(1+s)(\log(1+s))^2}\,ds,$$

such that the relations (φ), (Φ_1) and (Φ_2) are verified. The relation (9) holds also true with $\varphi_0=2$ and $\varphi^0=6$.

Define the function $f:\mathbb{R}\to\mathbb{R}$ by $f(t)=-t^5-7t^4+7t^3+43t^2-42t$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $b : \Omega \to [0, +\infty[$ be a function in $L^1(\Omega)$ such that

$$\int_0^2 g(t)\,dt > 0, \quad \sup_{\xi\in\mathbb{R}} \int_0^\xi g(t)\,dt < +\infty, \quad \limsup_{\xi\to 0} \frac{\int_0^\xi g(t)\,dt}{|\xi|^6} < \frac{1}{c^6\|b\|_{L^1(\Omega)}}$$

and

$$\max\left\{\frac{1}{\sup_{\mathbb{R}} G}, \frac{192\pi}{\int_0^2 g(t)\,dt}\right\} \le \|b\|_{L^1(\Omega)} < \frac{9}{c^2 \sup_{\xi\in\mathbb{R}} \int_0^\xi g(t)\,dt}.$$

Then, the conclusion of Proposition 3.4 does hold for the following problem

$$\begin{cases} \operatorname{div}\left(p(x,y)\frac{|\nabla u|^{p(x,y)-2}\nabla u}{\log(1+|\nabla u|)}\right) + p(x,y)\frac{|u|^{p(x,y)-2}u}{\log(1+|u|)} = \lambda a(x) f(u) + b(x) g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

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