Minimax Theory and its Applications
Volume 1 (2016), No. 1, 51–63

Representation of Viscosity Solutions of Hamilton-Jacobi Equations*

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Received: May 01, 2014
Accepted: May 30, 2014

Hamilton Jacobi equations of the form $H(x, u, Du) = 0$ are considered with $H(x, r, p)$ nondecreasing in r and quasiconvex in p. A viscosity solution may be represented as the value function of a calculus of variations or control problem in L^∞, i.e., as a minimax problem. For time dependent problems of the form $u_t + H(t, x, u, Du) = 0$ we require that $H(t, x, r, p)$ is convex in p and nondecreasing in r. The viscosity solution is then given as the value of an L^∞ problem.

Keywords: Quasiconvex, Hamilton-Jacobi, representation.

2010 Mathematics Subject Classification: 35F21, 49L25

1. Introduction

One of the primary reasons to study control problems with L^∞ cost functions, apart from the intrinsic practical applications, is the representation of solutions of Hamilton-Jacobi equations. This is true also of classical, i.e., integral cost control problems which may be used to represent solutions of Hamilton-Jacobi equations with convex hamiltonians. The representation theorems in the classical case began from the very inception of viscosity solution theory. One may consult [3], [16], [19], and the references there.

Classical control problems lead to convex hamiltonians of the form

$$H(x, p) = \sup_z p \cdot z - L(x, z)$$

which naturally leads to the Legendre-Fenchel construction of

$$L(x, z) = \sup_p p \cdot z - H(x, p).$$

The connection is that $H(x, p)$ is recognized as the convex conjugate of the running cost $L(x, z)$. This construction allows one to represent a viscosity solution

*This project was supported by grant 1008602 from the National Science Foundation.

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

of $H(x, Du) = 0$ or $u_t + H(t, x, Du) = 0$. For example, a viscosity solution of $u + H(x, Du) = 0$ is

$$u(x) = \inf_{\xi(0)=x} \int_0^\infty e^{-r} L(\xi(r), \dot{\xi}(r))\, dr \ .$$

If one wanted to include u dependence in H, the only recourse was to add a discount factor term in the running cost. This greatly restricts the types of u dependence allowed. Quincampoix and Plaskascz [18] attacked this problem in a way that permits more involved u dependence, but it involves an implicit scheme using differential inclusions which includes u dependence in the running cost. The idea we use here is explicit.

It is the purpose of this paper to extend the classical results to general u dependence, except for the requirement that $u \mapsto H(x, u, p)$ be nondecreasing which is almost required for uniqueness. The context in which we will do this is to use L^∞ control and quasiconvex conjugates. Recall that a quasiconvex function is one which has all sublevel sets convex and so they are also known as level convex functions. A Fenchel type theory of quasiconvex conjugates may then be used. We identify $H(x, r, p)$ as the first conjugate of what turns out to be a quasiconvex running cost $L(x, z)$ as well as an essential sup cost. Quasiconvex conjugates have the property that they depend on two variables, r and p and this leads to the identification for u and Du dependence. That identification led to the Hopf-Lax formula with u dependence (see [1] and the references there). Also, quasiconvex conjugates may be defined in different ways and that gives us several representation theorems depending on whether $p \mapsto H(x, r, p)$ is quasiconvex or convex. Both conditions still involve L^∞ problems. For a background on L^∞ control and calculus of variations problems refer to [8],[9],[10].

1.1. L^∞ control

We begin by reviewing and extending the following optimal control problem in L^∞ first studied in [9],

$$u(t, x) = \inf_{\zeta \in \mathcal{Z}[0,t]} \operatorname*{ess\,sup}_{s \in [0,t]} \left(h(s, \xi(s), \zeta(s)) + \int_0^s k(\tau, \xi(\tau), \zeta(\tau))\, d\tau \right) \vee \left(g(\xi(t)) + \int_0^t k(\tau, \xi(\tau), \zeta(\tau))\, d\tau \right) \qquad (1)$$

$$\frac{d\xi}{ds} = f(s, \xi(s), \zeta(s)), \quad 0 < s < t, \quad \xi(0) = x \in \mathbb{R}^n \qquad (2)$$

$$\mathcal{Z}[0, t] = \{\zeta : [0, t] \to Z \subset \mathbb{R}^q \mid \zeta \text{ is Lebesgue measurable}\} \qquad (3)$$

where Z is given control set.

We will use the following assumption on the dynamics. These conditions are stronger than necessary.

$$\left.\begin{array}{l} (t,x,z) \mapsto \varphi(t,x,z) \text{ is continuous, } \varphi = f,h,k \\ |f(t,x,z) - f(t,y,z)| \le C_f|x-y|, \text{ and } |f(t,x,z)| \le C_f(1+|x|) \\ |k(t,x,z) - k(t,y,z)| \le C_k|x-y|, \text{ and } |k(t,x,z)| \le C_k(1+|x|) \\ |h(t,x,z) - h(t,y,z)| \le C_h|x-y|, \text{ and } \\ g \in BLSC(\mathbb{R}^n) = \{\text{bounded and lower semicontinuous on } \mathbb{R}^n\} \end{array}\right\} \quad \text{(H)}$$

for constants $C_f, C_h, C_k > 0$.

Theorem 1.1. *Assume condition (H). Then the lower semicontinuous (lsc) envelope u_{lsc} of u defined in* (1) *is the unique lsc BJ solution of*

$$H(t,x,u,u_t,D_x u) = 0, \;\; (t,x) \in [0,\infty) \times \mathbb{R}^n \tag{4}$$

and $u(0,x) = g(x) \vee \underline{h}(0,x), \; x \in \mathbb{R}^n$, where $\underline{h}(t,x) = \min_{z \in Z} h(t,x,z)$. Here we have the hamiltonian $H : [0,\infty) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$H(t,x,r,p_t,p_x) = \max_{z \in Z} \min\{p_t - p_x \cdot f(t,x,z) - k(t,x,z), \; r - h(t,x,z)\} \tag{5}$$

Remark 1.2. A BJ lsc viscosity solution is a lsc function u such that $(t,x) \in \arg\min(u - \varphi)$, $\varphi \in C^1$ implies that $H(t,x,u,\varphi_t,D_x\varphi) = 0$. These solutions were introduced by Barron and Jensen in [7] for use in the study of lsc solutions of problems with convex hamiltonians. The important feature of these solutions is the fact that a lsc function may not satisfy the subsolution condition $(t,x) \in \arg\max(u-\varphi), \varphi \in C^1$ implies that $H(t,x,u,\varphi_t,D_x\varphi) \le 0$. Consequently, the requirement that u be a BJ lsc solution requires the strengthened supersolution condition that not only must $(t,x) \in \arg\min(u-\varphi), \varphi \in C^1$ imply that $H(t,x,u,\varphi_t,D_x\varphi) \ge 0$, but in fact $H(t,x,u,\varphi_t,D_x\varphi) = 0$. It was shown in [7] that this condition is sufficient to guarantee existence and uniqueness of a lsc viscosity solution when the hamiltonian is convex in the gradient variable. This was recently extended in [6] to hamiltonians quasiconvex in the gradient.

The sense in which a lsc function assumes the initial condition is

$$u(0,x) = g(x) \vee \underline{h}(0,x) = \inf\{\liminf_{k\to\infty} u(t_k,x_k) \mid t_k \downarrow 0, x_k \to x\}$$

Remark 1.3. Note that assuming (H) means H is continuous in all variables. In fact, using the elementary inequality

$$|x \vee y - z \vee y| = |x - z| - |x \wedge y - z \wedge y| \le |x - z|$$

we may verify that H satisfies the conditions for some $C_H > 0$,

$$|H(t,x,r,p) - H(t,y,r,p)| \leq C_H(1+|p|)|x-y| \tag{6}$$
$$|H(t,x,r,p) - H(t,x,s,p)| \leq C_H|r-s| \tag{7}$$
$$|H(t,x,r,p) - H(t,x,r,q)| \leq C_H(1+|x|)|p-q| \tag{8}$$

In addition, it is clear that $r \mapsto H(t,x,r,p_t,p_x)$ is nonincreasing and $(p_t,p_x) \mapsto H(t,x,r,p_t,p_x)$ is quasiconvex. The quasiconvexity is most easily verified by showing that $\{(p_t,p_x) \mid H(\cdot,\cdot,\cdot p_t,p_x) \leq \alpha\}$ is convex for all $\alpha \in \mathbb{R}$.

Proof. It is shown as in Barron and Liu [8] (with $k = 0$ but the result easily extends) that u_{lsc} is the unique lsc BJ viscosity solution of

$$u_t + F(x,u,D_xu) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n \tag{9}$$
$$u(0,x) = g(x) \vee \underline{h}(0,x), \quad \text{where} \tag{10}$$
$$\mathcal{A}(t,x,r) = \{z \in Z \mid h(t,x,z) \leq r\}, \text{ shortly } \mathcal{A} \text{ in (12)} \tag{11}$$
$$F(t,x,r,p) = \begin{cases} \max\{-p \cdot f(t,x,z) - k(t,x,z) \mid z \in \mathcal{A}(t,x,r)\}, & \text{if } \mathcal{A} \neq \emptyset. \\ -\infty, & \text{if } \mathcal{A} = \emptyset. \end{cases} \tag{12}$$

Since this hamiltonian $F(t,x,r,p)$ is not continuous in r, computing the upper and lower semicontinuous envelopes of F, the definition of a lsc BJ viscosity solution means that if $(p_t,p_x) \in D^-u_{lsc}(t,x)$, $D^-u_{lsc}(t,x)$ the superdifferential of u_{lsc} at (t,x), we have

$$(a) \;\; p_t + F^{usc}(t,x,u,p_x) \geq 0 \quad \text{and} \quad (b) \;\; p_t + F_{lsc}(t,x,u,p_x) \leq 0. \tag{13}$$

In (13) we compute the upper and lower semicontinuous envelopes of F as in [9],

$$F^{usc}(t,x,u,p_x) = \limsup_{(s,y,v,q)\to(t,x,u,p_x)} F(s,y,v,q) = \lim_{v\downarrow u} F(t,x,v,p_x) := F(t,x,u+0,p_x)$$

and

$$F_{lsc}(t,x,u,p_x) = \liminf_{(s,y,v,q)\to(t,x,u,p_x)} F(s,y,v,q) = \lim_{v\uparrow u} F(t,x,v,p_x) := F(t,x,u-0,p_x).$$

Next we need the following lemma.

Lemma 1.4. *The function u is a lsc viscosity solution of* (13) *if and only if $(p_t,p_x) \in D^-u_{lsc}(t,x)$ implies*

$$H(t,x,u,p_t,p_x) = \max_{z\in Z} \min\{p_t - p_x \cdot f(t,x,z) - k(t,x,z),\ u(t,x) - h(t,x,z)\} \\ = 0 \tag{14}$$

That is, u_{lsc} *is a lsc viscosity solution of* (13) *if and only if* u_{lsc} *is an lsc BJ viscosity solution of* (14).

Proof. To this end suppose $(p_t, p_x) \in D^- u_{lsc}(t,x)$, (13) holds and, furthermore, $H(t,x,u,p_t,p_x) \le -\gamma < 0$. Then, for any $z \in Z$,

$$\min\{p_t - p_x \cdot f(t,x,z) - k(t,x,z),\ u(t,x) - h(t,x,z)\} \le -\gamma.$$

If $u \ge h$, then $p_t + \max_{z\in\mathcal{A}(t,x,u)}(-p_x . f(t,x,z) - k(t,x,z)) \le -\gamma$. On the other hand, if $u \le h - \gamma$ then $F = -\infty$. In either case we have $p_t + F(t,x,u+0,p_x) < 0$ which contradicts (13)(a).

Assume $H(t,x,u,p_t,p_x) \ge \gamma > 0$. Then there is a $z \in Z$ such that

$$\min\{p_t - p_x \cdot f(t,x,z) - k(t,x,z), u - h(t,x,z)\} \ge \gamma.$$

Thus $p_t + F(t,x,u-0,p_x) = p_t + \max_{z\in\mathcal{A}(t,x,u-0)}(-p_x \cdot f - k) \ge \gamma$ gives a contradiction of (13)(b).

Conversely, suppose (14) holds. If (13)(a) is false, there is $\gamma > 0$ so that $p_t + F(t,x,u+0,p_x) \le -\gamma < 0$. By definition of F, for every $z \in Z$ such that $h(t,x,z) \le u$, $p_t - p_x . f(t,x,z) - k(t,x,z) \le -\gamma$. If $\mathcal{A}(t,x,z) = \emptyset$, then $u < h(t,x,z)$. Therefore, in either case $\min\{p_t - p_x \cdot f - k, u - h\} < 0$ for every $z \in Z$. Consequently, $H(t,x,u,p_t,p_x) < 0$, a contradiction of (13). The remainder of the argument is similar. □

The proof that u is a lsc viscosity solution of $u_t + F(t,x,u,Du) = 0$ is based on the forward and backward dynamic programming principles. Refer to [8] and the next section.

Remark 1.5. 1. An important point in the proof is the equivalence (in the viscosity sense of semicontinuous solutions) of

$$H(t,x,u,u_t,D_x u) = \max_z \min\{u_t - f(t,x,z) \cdot D_x u - k(t,x,z),\ u - h(t,x,z)\} = 0$$

and

$$u_t + F(t,x,u,D_x u) = u_t + \max\{-f(t,x,z) \cdot D_x u - k(t,x,z) \mid z \in Z, h(t,x,z) \le u\} \\ = 0\ .$$

In the first case the hamiltonian $H(t,x,r,p_t,p_x)$ is continuous in all variables and quasiconvex in (p_t,p_x), while in the second it is not continuous (in u) but it is convex in (p_t, p_x) as long as $\{h \le u\} \neq \emptyset$. This equivalence for continuous viscosity solutions was observed in [14] for L^∞ problems without running cost and further used in [15] for max-plus stochastic control.

2. One may obtain a direct proof of the uniqueness of u_{lsc} using the formulation with $H = 0$ by using the recent result for semicontinuous viscosity solutions of quasiconvex hamiltonians in [6] and simple modifications of the proof of [3, Theorem 5.16] or [5, Theorem 5.14].

The relaxed version of the L^∞ control problem is given by

$$\widehat{u}(t,x) = \inf_{\mu\in\widehat{\mathcal{Z}}[0,t]} \operatorname*{ess\,sup}_{s\in[0,t]} \left(\widehat{h}(s,\widehat{\xi}(s),\mu(s)) + \int_0^s \widehat{k}(\tau,\widehat{\xi}(\tau),\mu(\tau))\,d\tau\right) \vee \left(g(\widehat{\xi}(t)) + \int_0^t \widehat{k}(\tau,\widehat{\xi}(\tau),\mu(\tau))\,d\tau\right)$$

where

$$\widehat{h}(t,x,\mu) = \mu - \operatorname*{ess\,sup}_{z\in Z} h(t,x,z), \quad \widehat{k}(t,x,\mu) = \int_Z k(t,x,z)\,d\mu(z)$$

$$\frac{d\widehat{\xi}}{ds} = \widehat{f}(s,\widehat{\xi}(s),\mu(s)),\ s>0,\ \ \widehat{\xi}(0) = x\in\mathbb{R}^n,\ \ \widehat{f}(t,x,\mu) = \int_Z f(t,x,z)\,d\mu(z)$$

$$\widehat{\mathcal{Z}}[0,t] = \{\mu : [0,t]\to Z\subset\mathbb{R}^q \mid \mu(s)\in\mathcal{M}(Z) \text{ is Lebesgue measurable for a.e. } s\in[0,t]\} \text{ and } \mathcal{M}(Z) = \{\mu \text{ regular probability measures on } Z\}$$

In other words, we extend the system so that the class of controls $\widehat{\mathcal{Z}}[0,t]$ are Young measures, or relaxed control functions. The relaxed L^∞ control problem was studied in [10] and [2, Theorem 3.2].

Theorem 1.6. *Assume (H). The lower semicontinuous envelope of the value function* $u : [0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ *is* $\widehat{u}$*; i.e.,* $\widehat{u} = u_{lsc}$.

Proof. This proof uses uniqueness of viscosity solutions to prove that the relaxed and ordinary values are the same. Writing shortly $\mathcal{A}$ for $\mathcal{A}(t,x,r)$ and $\mathcal{M}$ for $\mathcal{M}(t,x,r)$ in the following two formulas, we have

$$F(t,x,r,p) = \begin{cases} \max\{-p\cdot f(t,x,z) - k(t,x,z) \mid z\in\mathcal{A}\}, & \text{if } \mathcal{A}\neq\emptyset. \\ -\infty, & \text{if } \mathcal{A}=\emptyset. \end{cases}$$

$$\widehat{F}(t,x,r,p) = \begin{cases} \max\{-p\cdot \widehat{f}(t,x,\mu) - \widehat{k}(t,x,\mu) \mid \mu\in\mathcal{M}\}, & \text{if } \mathcal{M}\neq\emptyset. \\ -\infty, & \text{if } \mathcal{M}=\emptyset. \end{cases}$$

Here $\mathcal{M}(t,x,r) = \{\mu \in M(Z) \mid \widehat{h}(t,x,\mu) \le r\}$. Therefore, since it is easy to verify that $F = \widehat{F}$, u_{lsc} and $\widehat{u}$ are both BJ lsc viscosity solutions of $u_t + \widehat{F}(t,x,u,D_x u) = 0$ with $u_{lsc}(0,x) = \widehat{u}(0,x) = g(x) \vee \underline{h}(0,x)$. The fact that there is only one BJ lsc solution then implies $u_{lsc} = \widehat{u}$. □

Remark 1.7. While the original L^∞ problem may not have an optimal control, there exists an optimal relaxed control of the relaxed problem $\mu^* \in \widehat{\mathcal{Z}}[t,T]$. To see why, we set $\eta(s) = \int_t^s \widehat{k}(\tau, \widehat{\xi}(\tau), \mu(\tau))\, d\tau$. Since $\mu \mapsto \widehat{k}(t,x,\mu)$ is affine (and so convex), $\mu(\cdot) \mapsto \eta$ is weak* continuous. Then

$$\mu(\cdot) \mapsto \operatorname*{ess\,sup}_{s\in[t,T]} \left(\widehat{h}(s, \widehat{\xi}(s), \mu(s)) + \eta(s)\right)$$

is weak* lower semicontinuous in view of the fact that $\mu \mapsto \widehat{h}(t,x,\mu)$ is quasiconvex (see [10]). Existence of an optimal relaxed control then follows in the standard way.

2. Representation theorems for Hamilton-Jacobi equations

Solutions of Hamilton-Jacobi equations may be represented as value functions for control problems when the hamiltonian is convex or quasiconvex in the derivative variables. In this section we will summarize the representations of a wide variety of equations. The new feature in these representations is the appearance of dependence of the hamiltonian on the solution in a non trivial way.

Each representation which follows is motivated by L^∞ control or calculus of variations and quasiconvex conjugates of functions. In this paper we use three distinct conjugates:

Given a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ consider the condition:

$$\left.\begin{array}{c}\text{For every } \gamma > \inf f, \text{ there is a continuous affine function} \\ \text{which is a minorant of } f \text{ on } E_\gamma(f) = \{x \mid f(x) \le \gamma\}.\end{array}\right\} \quad \text{(C)}$$

Given such a function define

$$\begin{aligned} f^{\%}(r,p) &= \sup\{p\cdot x \wedge r - f(x) \mid x \in \mathbb{R}^n\} \\ f^{\%\%}(x) &= \sup\{(p\cdot x \wedge r - f^{\%}(r,p)) \mid r \in \mathbb{R}, p \in \mathbb{R}^n\} \end{aligned} \quad \text{(C1)}$$

$$\begin{aligned} f^{\sharp}(r,p) &= \sup\{p\cdot x - f(x) \mid x \in \{f(x) \le r\}\} \\ f^{\sharp\sharp}(x) &= \sup\{(p\cdot x - f^{\sharp}(r,p)) \wedge r \mid r \in \mathbb{R}, p \in \mathbb{R}^n\} \end{aligned} \quad \text{(C2)}$$

$$\begin{aligned} f^{\#}(r,p) &= \sup\{p\cdot x \mid x \in \{f(x) \le r\}\} \\ f^{\#\#}(x) &= \inf\{r \mid \sup_{p\in\mathbb{R}^n} p\cdot x - f^{\#}(r,p) \le 0\} \end{aligned} \quad \text{(C3)}$$

One can find a proof in [17] and [1] that a lower semicontinuous function f satisfying (C) is quasiconvex if and only if $f = f^{\%\%}$, $f = f^{\sharp\sharp}$, or $f = f^{\#\#}$. It is the form of the first conjugates $f^{\%}, f^{\sharp}, f^{\#}$ which lead to formulas for Hamilton-Jacobi equations with u dependence. The conjugates (C3) have been used in [11],[12] to develop Hopf-Lax explicit formulas for equations of the form $u_t + H(u, Du) = 0$, $u(0, x) = g(x)$. The main drawback of (C3) is that it requires the hamiltonian, which will be of the form $H(r, p)$, to be convex and homogenous degree one in p.

2.1. $H(x, u, Du) = 0$, H quasiconvex in Du and nondecreasing in u

We are given $H : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ assumed throughout to satisfy

(A1): $p \mapsto H(x, r, p)$ is quasiconvex and for each $K > 0$ there is a constant $C(K)$ such that $H(x, r, p) \geq K|p| - C(K),\ \forall(x, r, p)$,

(A2): $r \mapsto H(t, x, r, p)$ is nonincreasing.

Define $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$L(x, z) = \sup_{r\in\mathbb{R}, p\in\mathbb{R}^n} \{(p \cdot z \wedge r - H(x, r, p))\} \tag{15}$$

Using the quasiconvex conjugates (C1), we have

$$H(x, r, p) = \sup_{z\in\mathbb{R}^n} (p \cdot z \wedge r - L(x, z))$$

We have the following

Theorem 2.1. *Set* $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$,

$$u(x) = \inf_{\xi(0)=x} \operatorname*{ess\,sup}_{s\in[0,\infty)} \left(L(\xi(s), \dot{\xi}(s)) + \int_0^s L(\xi(r), \dot{\xi}(r))\, dr \right)$$

Then, if $u_{lsc} < +\infty$, u_{lsc} *is a lower semicontinuous BJ viscosity solution of* $H(x, u, -Du) = 0$, $x \in \mathbb{R}^n$.

Proof. We will sketch the proof. We may assume u is lower semicontinuous because otherwise we work with u_{lsc}. Suppose $\varphi \in C^1$, and $y \in \arg\min(u - \varphi)$. We must show that $H(y, u(y), D\varphi(y)) = 0$. The fact that $H(y, u(y), D\varphi(y)) \geq 0$ is proved in the same way as in [9]. We will show that $H(y, u(y), D\varphi(y)) \leq 0$.

The forward dynamic programming principle says that for each $\tau > 0$,

$$u(x) = \inf_{\xi(0)=x} \operatorname*{ess\,sup}_{s\in[0,\tau]} \left(L(\xi(s), \dot{\xi}(s)) + \int_0^s L(\xi(r), \dot{\xi}(r))\, dr \right) \vee \left(u(\xi(\tau)) + \int_0^\tau L(\xi(r), \dot{\xi}(r))\, dr \right). \tag{16}$$

The proof of (16) is straightforward. Indeed, for any $0 < \tau$,

$$\begin{aligned}
u(x) &= \inf_{\xi(0)=x} \operatorname*{ess\,sup}_{s\in[0,\infty)} \left(L(\xi(s),\dot\xi(s)) + \int_0^s L(\xi(r),\dot\xi(r))\,dr \right) \\
&= \inf_{\xi(0)=x} \operatorname*{ess\,sup}_{s\in[0,\tau]} \operatorname*{ess\,sup}_{s\in[\tau,\infty)} \left(L(\xi(s),\dot\xi(s)) + \int_0^s L\,dr \right) \\
&= \inf_{\xi(0)=x} \operatorname*{ess\,sup}_{s\in[0,\tau]} \left(L(\xi(s),\dot\xi(s)) + \int_0^s L\,dr \right) \\
&\qquad\qquad \vee \operatorname*{ess\,sup}_{s\in[\tau,\infty)} \left(L(\xi(s),\dot\xi(s)) + \int_0^\tau L\,dr + \int_\tau^s L\,dr \right) \\
&= \inf_{\xi(0)=x} \operatorname*{ess\,sup}_{s\in[0,\tau]} \left(L(\xi(s),\dot\xi(s)) + \int_0^s L\,dr \right) \vee \left(u(\xi(\tau)) + \int_0^\tau L\,dr \right).
\end{aligned}$$

One form of the backward dynamic programming principle says for each $\tau > 0$

$$\begin{aligned}
u(\xi_x(-\tau)) \le \operatorname*{ess\,sup}_{s\in[-\tau,0]} &\left(L(\xi_x(s),\dot\xi_x(s)) + \int_{-\tau}^s L(\xi_x(r),\dot\xi_x(r))\,dr \right) \\
&\vee \left(u(x) + \int_{-\tau}^0 L(\xi_x(r),\dot\xi_x(r))\,dr \right), \quad (17)
\end{aligned}$$

where $\xi_x(0) = x$.

Now suppose $H(y,u(y),-D\varphi(y)) \ge 5\gamma > 0$. Since $H(x,r,p) = \sup_z (p\cdot z - L(x,z)) \wedge (r - L(x,z))$, we have the existence of $z \in \mathbb{R}^n$ such that

$$-D\varphi(y)\cdot z - L(y,z) \ge 4\gamma \text{ and } u(y) - L(y,z) \ge 4\gamma.$$

Then, there is $\delta > 0$ so that $D\varphi(x)\cdot z + L(x,z) \le -3\gamma$ and $u(y) \ge L(x,z) + 3\gamma$ for all $x \in B_\delta(y)$. Setting $\xi_y(s) = y + s\,z$, there is $\sigma > 0$ so that $\xi_y(s) \in B_\delta(y), -\sigma \le s \le 0$. Thus

$$\frac{d}{ds}\varphi(\xi_y(s)) + L(\xi_y(s),\dot\xi_y(s)) \le -3\gamma \implies \varphi(y) - \varphi(\xi_y(-\sigma)) + \int_{-\sigma}^0 L \le -3\gamma\,\sigma.$$

Since $u - \varphi$ has a zero minimum at y, $u(y) - u(\xi_y(-\sigma)) \le \varphi(y) - \varphi(\xi_y(-\sigma))$. Consequently,

$$u(y) + \int_{-\sigma}^0 L + 3\gamma\,\sigma \le u(\xi_y(-\sigma)).$$

Using (17) we conclude that

$$u(\xi_y(-\sigma)) \le \operatorname*{ess\,sup}_{s\in[-\sigma,0]} \left(L(\xi_y(s),\dot\xi_y(s)) + \int_{-\sigma}^s L(\xi_y(r),\dot\xi_y(r))\,dr \right) \le L(y,z) + 2\gamma,$$

if σ is small enough. Finally, letting $\sigma \to 0$ and using the lower semicontinuity of u, we have $u(y) \le L(y,z) + \gamma$. This contradicts $u(y) \ge L(y,z) + 4\gamma$. The remainder of the proof is similar. □

The uniqueness of the solution of $H(x,u,-Du) = 0$ may not hold with our assumptions because of lack of coercivity. To guarantee uniqueness we consider the problem with a discount factor,

$$u(x) = \inf_{\{\xi(0)=x\}} \operatorname*{ess\,sup}_{s\in[0,\infty)} e^{-\delta\, s} L(\xi(s),\dot{\xi}(s)) + \int_0^s e^{-\delta\tau} L(\xi(\tau),\dot{\xi}(\tau))\, d\tau$$

Using a similar proof as in [9] and [8] one may readily prove that u is the unique BJ lower semicontinuous solution of

$$\sup_{z\in\mathbb{R}^n} \min\{\delta u - D_x u\cdot z - L(x,z), u - L(x,z)\} = 0, \quad x\in\mathbb{R}^n. \tag{18}$$

Note that

$$\begin{array}{l}
\sup_{z\in\mathbb{R}^n} \min\{\delta u - D_x u\cdot z - L(x,z),\ u(x) - L(x,z)\} = 0 \text{ iff} \\
\sup_{z\in\mathbb{R}^n} \min\{\delta u - D_x u\cdot z - L(x,z),\ \delta u(x) - \delta L(x,z)\} = 0 \text{ so that} \\
\delta u + F(x,-D_x u) = 0, \quad \text{where} \\
F(x,p) = \sup_{z\in\mathbb{R}^n} \min\{p\cdot z - L(x,z),\ -\delta L(x,z)\}
\end{array}$$

and $p \mapsto F(x,p)$ is quasiconvex.

Equivalent formulations are,

$$\left.\begin{array}{l}
\min\{\delta\, u + \sup\limits_{z\in\mathcal{A}(x,u)} -D_x u\cdot z - L(x,z),\ u - \underline{L}(x)\} = 0, \quad x\in\mathbb{R}^n \\
\mathcal{A}(x,r) = \{z\in\mathbb{R}^n \mid L(x,z)\le r\}, \underline{L}(x) = \min_{z\in\mathbb{R}^n} L(x,z).
\end{array}\right\} \tag{19}$$

and

$$\left.\begin{array}{l}
\delta u + G(x,u,-D_x u) = 0, \quad x\in \times\mathbb{R}^n \\
G(x,r,p) = \begin{cases} \sup_{z\in\mathcal{A}(x,r)} -p\cdot z - L(x,z), & \text{if } \mathcal{A}(x,r)\ne\emptyset \\ -\infty, & \text{if } \mathcal{A}(x,r)=\emptyset. \end{cases}
\end{array}\right\} \tag{20}$$

Observe that in the last formulation $p\mapsto G(x,r,p)$ is convex if $\mathcal{A}(x,r)\ne\emptyset$.

Remark 2.2. It is also possible to represent solutions of Dirichlet problems $H(x,u,-Du) = 0$, $x\in\Omega$, with $u = g$ on $\partial\Omega$. For instance, we have for $x\in\Omega$

$$u(x) = \inf_{\xi(0)=x} \operatorname*{ess\,sup}_{s\in[0,\tau_x]} L(\xi(s),\dot{\xi}(s)) + \int_0^s L(\xi(r),\dot{\xi}(r))\, dr$$
$$\vee \left(g(\xi(\tau_x)) + \int_0^{\tau_x} L(\xi(r),\dot{\xi}(r))\, dr \right),$$

is the solution of

$$H(x,u,-Du)=\sup_z \min\{-Du\cdot z-L(x,z),u-L(x,z)\}=0,\ \ x\in\Omega,$$

with $u(x)=g(x)\vee \inf_z L(x,z),\ x\in\partial\Omega$. Here, τ_x is the first exit time of $\xi(\cdot)$ from Ω. Note that $\partial\Omega$ requires a condition guaranteeing exit in finite time. Given $H(x,r,p)$ quasiconvex in p, and boundary data $g(x)\le \inf_z L(x,z),\ x\in\partial\Omega$, we may write down the solution u using quasiconvex conjugates of H.

2.2. $u+H(x,u,Du)=0$, H convex in Du

Given $H(x,r,p)$ convex and coercive in in p (see (A1)) and nondecreasing in r we will represent the solution of $u+H(x,u,Du)=0, x\in\mathbb{R}^n$ as the value function of an infinite horizon L^∞ control problem. Of course this is already known but without the general dependence on u we demonstrate below.

Here we use the conjugates (C2). Define

$$L(x,z)=\sup_{p\in\mathbb{R}^n,r\in\mathbb{R}}(p\cdot z-H(x,r,p))\wedge r$$

and then

$$H(x,r,p)=\sup\{p\cdot z-L(x,z)\mid z\in\mathcal{A}(x,r)\},\ \mathcal{A}(x,r)=\{z\in\mathbb{R}^n\mid L(x,z)\le r\}.$$

Then, under the condition $\mathcal{A}(x,r)\neq\emptyset$, for each r,

$$u+H(x,u,Du)=u+\sup\{-Du\cdot z-L(x,z)\mid z\in\mathcal{A}(x,u)\}=0. \tag{21}$$

The hamiltonian in (21) has the form of the hamiltonian of an L^∞ calculus of variation problem with running cost. The unique solution of (21) is given by

$$u(x)=\inf_{\xi(0)=x}\ \operatorname*{ess\,sup}_{s\in[0,\infty)} L(\xi(s),\dot\xi(s))e^{-s}+\int_0^s L(\xi(r),\dot\xi(r))\ e^{-r}\,dr. \tag{22}$$

As mentioned earlier, the discount factor e^{-t} guarantees coercivity and hence uniqueness, but even without this factor we have the representation of a viscosity solution.

2.3. Time dependent problems

The results of the preceding section for convex hamiltonians also apply to various time dependent problems. In this section we indicate the type of formulas we may obtain from L^∞ control.

We are interested in equations of the form

$$u_t+H(t,x,u,Du)=0,\ t>0,\ x\in\mathbb{R}^n,\qquad u(0,x)=g(x),x\in\mathbb{R}^n. \tag{23}$$

Given the hamiltonian $H(t,x,r,p)$, with $p \mapsto H(t,x,r,p)$ convex and $r \mapsto H(t,x,r,p)$ nondecreasing, define

$$L(t,x,z) = \sup_{p\in\mathbb{R}^n, r\in\mathbb{R}} (p\cdot z - H(t,x,r,p)) \wedge r.$$

This function is quasiconvex in $z \mapsto L(t,x,z)$. Then, using (C2), we have under the condition $\mathcal{A}(t,x,r) \neq \emptyset$ for each r,

$$H(t,x,r,p) = \sup_{z\in\mathcal{A}(t,x,r)} (p\cdot z - L(t,x,z)).$$

With this representation for H we have the Cauchy problem

$$u_t + H(t,x,u,-Du) = u_t + \sup\{-D_x u\cdot z - L(t,x,z) \mid z\in\mathcal{A}(t,x,u)\} = 0 \quad (24)$$

$$u(0,x) = g(x) \quad (25)$$

Theorem 2.3. *Assume that (i) L is continuous in all variables, (ii)* $\mathcal{A}(t,x,r)\neq\emptyset$, *for all* r, *and (iii)* $\inf_{z\in\mathbb{R}^n} L(0,x,z) \le g(x), x\in\mathbb{R}^n$. *The unique lsc viscosity solution of* (24) *with initial condition* $u(0,x) = g(x) \in BLSC(\mathbb{R}^n)$ *is given by*

$$u(t,x) = \inf_{\xi(0)=x} \operatorname*{ess\,sup}_{s\in[0,t]} \left(L(s,\xi(s),\dot\xi(s)) + \int_0^s L(\tau,\xi(\tau),\dot\xi(\tau))\,d\tau \right) \vee \left(g(\xi(t)) + \int_0^t L(\tau,\xi(\tau),\dot\xi(\tau))\,d\tau \right). \quad (26)$$

Remark 2.4. It is interesting to point out that if we want $H(t,x,r,p)$ to be merely quasiconvex in p then this representation does not hold and may not even be possible using L^∞. The reason is that the Hamilton Jacobi equation with time dependence really has hamiltonian $F(t,x,r,(p_t,p_x)) = p_t + H(t,x,r,p_x)$ and it is not hard to check that even if $p_x \mapsto H(t,x,r,p_x)$ is quasiconvex (and not convex), then F is not quasiconvex in (p_t,p_x), (see [12]).

Remark 2.5. Rampazzo [19] has shown that a convex hamiltonian can be represented as a classical optimal control problem rather than a calculus of variations problem. He constructs dynamics f, a running cost h, and compact control set (namely, the unit ball), so that the hamiltonian becomes $H(t,x,p) = \max_{z\in B(0,1)}(p\cdot f(t,x,z) + h(t,x,z))$. This representation allows one to express the solution of the equation as the value function of a standard control problem. Except for including a discount factor, more general u dependence is not allowed. It is an open problem to extend Rampazzo's result to express $H(t,x,r,p)$ as a standard L^∞ control problem.

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