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Concentration of Semi-Classical States for Nonlinear Dirac Equations of Space-Dimension n

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In the present paper, we study the semi-classical approximation of a massive Dirac equation in space-dimension $n \geq 2$ with some general nonlinear self-coupling. We prove that there exists a family of ground states of the semi-classical problem, for all $\hbar$ small, and show that the family concentrates around some certain sets determined by the competing potential functions as $\hbar \to 0$.

Keywords: Dirac equations, semi-classical states, concentration.

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1. Introduction and main result

It is commonly accepted that the free Dirac equation, apart from some peculiarities, describes free relativistic electrons. This free model gives approximate descriptions of many of the particles found in nature. In order to obtain a closer description of the real world, we must include some (new) nonlinear terms. A general form of nonlinear Dirac equations of space-dimension n can be written as

$$i\hbar\frac{\partial}{\partial t}\psi = -ic\hbar\sum_{k=1}^{n}\alpha_k\partial_k\psi + mc^2\beta\psi + U(x)\psi - f(x,\psi)\,, \tag{1}$$

where $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}^N$ stands for the wave, $\hbar > 0$ is the Planck's constant, c is the speed of light, m is the mass of the electron, $(\{\alpha_k\}_{k=1}^n, \beta)$ is an $(n+1)$-tuple of Dirac matrices:

(1) $\beta^* = \beta$ and $\alpha_k^* = \alpha_k$ for $k = 1, \cdots, n$, i.e. β and α_k are self-adjoint.
(2) $\beta^2 = 1$, $\alpha_k\beta + \beta\alpha_k = 0$ and $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$ for $i, j = 1, \cdots, n$.

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Proposition 1.1. [31] *There is an $(n+1)$-tuple of Dirac matrices in $M_N(\mathbb{C})$ when $N = 2^{[\frac{n+1}{2}]}$, where $[r]$ is the integral part of a nonnegative real number r. Moreover,we have $(\{\alpha_k\}_{k=1}^n, \beta)$ has the form*

$$\alpha_k = \begin{pmatrix} 0 & a_k \\ a_k^* & 0 \end{pmatrix} \quad \text{for } k = 1, \cdots, n, \text{ and } \beta = \begin{pmatrix} I_{N/2} & 0 \\ 0 & -I_{N/2} \end{pmatrix},$$

where the a_k are $\frac{N}{2} \times \frac{N}{2}$ matrices (which are Hermitian if n is odd).

There are many reasons why physicists concern theories in n-dimensional spacetime. By modeling spacetime in different dimensions, a physical theory becomes more mathematically tractable. For example, in string theories, extra dimensions of spacetime are required for their mathematical consistency. In bosonic string theory, physicists are interested in the equation with $n = 25$, while in superstring theory, it is with $n = 9$, and in M-theory, it is with $n = 10$. For the nonlinear Dirac equations of space-dimension n, $U(x)$ denotes the potential, and $f(x, \psi)$ models the nonlinear self-interaction involving a varying pointwise distribution. To preserve causality, we insist that the external force may involve only functions of fields at the same space-time point, for instance, $|\psi|\psi$ is fine, but $\psi(t, x) \cdot \psi(t, y)$ is not allowed (we used the notation $u \cdot v$ to express the inner product for the complex scalars u, v). These nonlinear terms, which describe the self-interaction in Quantum electrodynamics, are considered to be possible basis models for unified field theories (see [19], [20], [23] etc. and references therein). A first example is the "*psi-fourth*" theory, in which, the corresponding equation become

$$i\hbar\frac{\partial}{\partial t}\psi = -ic\hbar\sum_{k=1}^{2}\alpha_k\partial_k\psi + mc^2\beta\psi - \chi|\psi|^2\psi\,, \tag{2}$$

where χ is the dimensionless coupling constant. Although the "*psi-fourth*" theory may seems to be theoretical, we emphasize that models of the real world do contain a $|\psi|^2\psi$ interaction in some special situations (see [26], there also exists a $|\psi|\psi$ interaction). Meanwhile, more interesting (interaction) theories can be obtained by including several scalar fields.

In the two space-dimension case, dealing with the Cauchy problem of (1), a local well-posedness result for the Yukawa interaction model (in which the nonlinear term has the form $f(x, \psi)\psi = \phi\beta\psi$ with ϕ being a Klein-Gordon field determined by ψ) is proved by Bournaveas in [8] and is later improved by d'Ancona, Foschi and Selberg [9]. Their proof relied on the null structure of the nonlinear system. And a first global well-posedness result for large data in two space dimensions of Yukawa model is established by using Bourgain type function spaces, see Grünrock and Pecher [22]. However, the important global well-posedness problem remains open for other general interaction models. When a stationary solution (or soliton-like) of (1) is considered, we find that the problem is rarely studied so far. Here, by stationary solution, we mean that a solution of the type

$$\psi(t, x) = \varphi(x)e^{-i\omega t/\hbar}, \quad \xi \in \mathbb{R}, \tag{3}$$

where φ is a complex scalar function independent of the time coordinate.

We point out that the works mentioned above mainly concerned with the autonomous equations of Yukawa model. There are not so many results concern with the nonlinear fourth order interaction. And no results were related to the semi-classical approximation. To fill this gap, in the present paper, we are interested in the existence and concentration behavior of stationary semi-classical solutions to the equation (1) in space-dimension $n \geq 2$ with some general self-coupling nonlinearity involving the subcritical and critical exponent of the relevant Sobolev embedding.

To describe the transition from quantum to classical mechanics, the existence of solutions $\varphi_\hbar$, $\hbar$ small, possesses an important physical interest. More precisely, for ease of notations, denoting $\varepsilon = \hbar$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\alpha \cdot \nabla = \sum_{k=1}^n \alpha_k \partial_k$, we are concerned with (substitute (3) in (1)) the following stationary nonlinear Dirac equations:

$$-i\varepsilon\alpha \cdot \nabla\varphi + a\beta\varphi + V(x)\varphi = P(x)g(|\varphi|)\varphi\,, \tag{4}$$

$$-i\varepsilon\alpha \cdot \nabla\varphi + a\beta\varphi + V(x)\varphi = P(x)g(|\varphi|)\varphi + W(x)|\varphi|^{2^*-2}\varphi\,, \tag{5}$$

where $a = mc > 0$, $V(x) = U(x) - \omega$.

In order to state our results in a specific way, some notations are needed. We set

$$\kappa := \min_{x\in\mathbb{R}^n} V(x), \quad \mathscr{V} := \big\{x \in \mathbb{R}^n : V(x) = \kappa\big\}, \quad \kappa_\infty := \liminf_{|x|\to\infty} V(x),$$

$$m := \max_{x\in\mathbb{R}^n} P(x), \quad \mathscr{P} := \big\{x \in \mathbb{R}^n : P(x) = m\big\}, \quad m_\infty := \limsup_{|x|\to\infty} P(x),$$

$$l := \max_{x\in\mathbb{R}^n} W(x), \quad \mathscr{W} := \big\{x \in \mathbb{R}^n : W(x) = l\big\}, \quad l_\infty := \limsup_{|x|\to\infty} W(x),$$

$$\kappa_p := \min_{x\in\mathscr{P}} V(x), \quad m_v := \max_{x\in\mathscr{V}} P(x), \quad \kappa_w := \min_{x\in\mathscr{W}} V(x), \quad m_w := \max_{x\in\mathscr{W}} P(x)\,.$$

Then on the pointwise distribution, we will use the following assumptions:

(H1) *V and P are of C^1-smooth in $\mathbb{R}^n$, $|V|_\infty < a$ and* $\inf P > 0$.

(H2) *W is C^1 smooth in $\mathbb{R}^n$ with* $\inf W > 0$ *and $l_\infty \leq l$.*

(H3) *$\kappa_\infty > \kappa$ and there is $x_v \in \mathscr{V}$ such that $P(x_v) \geq P(x)$ for all $|x| \geq R$, some R large.*

(H4) *$m_\infty < m$ and there is $x_p \in \mathscr{P}$ such that $V(x_p) \leq V(x)$ for all $|x| \geq R$, some R large.*

To show the concentration phenomena, we introduce the following notations: in the case (H3) assume that $P(x_v) = m_v$ and set

$$\mathscr{A}_v := \big\{x \in \mathscr{V} : P(x) = m_v\big\} \cup \big\{x \notin \mathscr{V} : P(x) > m_v\big\}\,;$$

in the case (H4) assume that $V(x_p) = \kappa_p$ and set

$$\mathscr{A}_p := \big\{x \in \mathscr{P} : V(x) = \kappa_p\big\} \cup \big\{x \notin \mathscr{P} : V(x) < \kappa_p\big\}\,.$$

We find $\mathscr{A}_v$ and $\mathscr{A}_p$ are bounded sets since $\mathscr{V}$ and $\mathscr{W}$ are bounded. Moreover, $\mathscr{A}_v = \mathscr{A}_p = \mathscr{V} \cap \mathscr{P}$ provided $\mathscr{V} \cap \mathscr{P} \neq \emptyset$.

On the nonlinear self-coupling, writing $G(|w|) := \int_0^{|w|} g(s)sds$, we make the following hypotheses:

(G1) $g(0) = 0$, $g \in C^1(0,\infty)$, $g'(s) > 0$ *for* $s > 0$, *and there exist* $p \in (2, 2^*)$ *with* $2^* = \frac{2n}{n-1}$, $c_1 > 0$ *such that* $g(s) \le c_1(1 + s^{p-2})$ *for* $s \ge 0$;

(G2) *there exist* $\sigma > 2$, $\theta > 2$ *and* $c_0 > 0$ *such that* $c_0 s^\sigma \le G(s) \le \frac{1}{\theta} g(s)s^2$ *for all* $s > 0$.

A typical example is the power function $G(s) = s^p$.

Recall that a nonzero solution is called the least energy solution (or the ground state solution) means that it has the lowest energy among all nonzero solutions. The first existence and concentration result for the semiclassical least energy solution can be stated as follow.

Theorem 1.2. *Assume that* (H1) *and* (G1)–(G2) *are satisfied.*

(I) *Suppose that* (H3) *holds.*

Then, for $\varepsilon > 0$ *sufficiently small,* (4) *has a ground state solution*

$$\varphi_\varepsilon \in \cap_{q\ge 2} W^{1,q}(\mathbb{R}^n, \mathbb{C}^N).$$

If additionally ∇V *and* ∇P *are bounded, then* φ_ε *satisfies:*

(i) *There exists a maximum point* x_ε *of* $|\varphi_\varepsilon|$ *with* $\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathscr{A}_v) = 0$, *such that, for some* $C, c > 0$

$$|\varphi_\varepsilon(x)| \le C \exp\Big(-\frac{c}{\varepsilon}|x - x_\varepsilon|\Big) .$$

(ii) *Setting* $z_\varepsilon(x) = \varphi_\varepsilon(\varepsilon x + x_\varepsilon)$, *for any sequence* $x_\varepsilon \to x_0$ *as* $\varepsilon \to 0$, z_ε *converges in* H^1 *to a ground state solution of*

$$-i\alpha\cdot\nabla z + a\beta z + V(x_0)z = P(x_0)g(|z|)z .$$

If particularly $\mathscr{V}\cap\mathscr{P} \ne \emptyset$ *then* $\lim_{\varepsilon\to 0}\operatorname{dist}(x_\varepsilon, \mathscr{V}\cap\mathscr{P}) = 0$ *and* z_ε *converges in* H^1 *to a a ground state solution of*

$$-i\alpha\cdot\nabla z + a\beta z + \kappa z = mg(|z|)z .$$

(II) *Suppose* (H4) *holds. Then, replacing* $\mathscr{A}_v$ *with* $\mathscr{A}_p$, *all the conclusions under* (I) *remain true.*

The next result is concerned with (5), that is, the Dirac equation involving the critical exponent of the relevant Sobolev embedding (including the "*psi-fourth*"-type interaction in the space-dimension two). Firstly, by virtue of (G2), let $\gamma > 0$ denote the least energy (such energy is attained, see the proofs in Section 3) of

$$-i\alpha\cdot\nabla u + a\beta u = |u|^{\sigma-2}u .$$

Set

$$\mathcal{R}_\sigma := \left(\frac{c_0^{2/(\sigma-2)} S^n}{2n\cdot\gamma}\right)^{\sigma-2} ,$$

where σ and c_0 are the constants in (G2) and S denotes the Sobolev constant

$$S|u|_{2^*}^2 \le \|u\|_{\dot H^{1/2}}^2 \quad \text{for all } u \in \dot H^{1/2}(\mathbb{R}^n, \mathbb{C}^N) . \tag{6}$$

We used the notation $\dot{H}^{1/2}(\mathbb{R}^n, \mathbb{C}^N)$ to express the completion space of $C_c^\infty(\mathbb{R}^n, \mathbb{C}^N)$ with the norm

$$\|u\|^2_{\dot{H}^{1/2}} := \int_{\mathbb{R}^n} |\zeta| \cdot |\hat{u}(\zeta)|^2 \, d\zeta \,,$$

where $u \mapsto \hat{u}$ is the Fourier transform of $u \in L^2$.

We now make the following assumptions:

(H5) *$\kappa_\infty > \kappa_w$ and there is $x_v \in \mathscr{W}_v$ makes $P(x_v) \geq P(x)$ for all $|x| \geq R$, some $R > 0$ large, where $\mathscr{W}_v = \{x \in \mathscr{W} : V(x) = \kappa_w\}$.*

(H6) *$m_\infty < m_w$ and there is $x_p \in \mathscr{W}_p$ makes $V(x_p) \leq V(x)$ for all $|x| \geq R$, some $R > 0$ large, where $\mathscr{W}_p = \{x \in \mathscr{W} : P(x) = m_w\}$.*

(H7) $\big(\frac{a}{a+\kappa_\infty}\big)^{(2n-1)(\sigma-2)-2} m_\infty^{-2} \cdot l^{(n-1)(\sigma-2)} < \mathcal{R}_\sigma$.

(H8) *Either $\sigma < 3$ or $\big(\frac{a}{a+\kappa}\big)^{(2n-1)(\sigma-2)-2} m_\infty^{-2} \cdot l^{(n-1)(\sigma-2)} < \mathcal{R}_\sigma$.*

To describe the concentration, let us denote, in the case (H5),

$$\begin{aligned}\mathscr{C}_v := &\big\{x \in \mathscr{W}_v : P(x) \geq P(x_v)\big\} \cup \big\{x \in \mathscr{W} \setminus \mathscr{W}_v : P(x) > P(x_v)\big\} \\ &\cup \big\{x \notin \mathscr{W} : V(x) < \kappa_w \text{ or } P(x) > P(x_v)\big\};\end{aligned}$$

and similarly, in the case (H6),

$$\begin{aligned}\mathscr{C}_p := &\big\{x \in \mathscr{W}_p : V(x) \leq V(x_p)\big\} \cup \big\{x \in \mathscr{W} \setminus \mathscr{W}_p : V(x) < V(x_p)\big\} \\ &\cup \big\{x \notin \mathscr{W} : P(x) > m_w \text{ or } V(x) < V(x_p)\big\}.\end{aligned}$$

Then $\mathscr{C}_v$ and $\mathscr{C}_p$ are bounded nonempty sets. Moreover $\mathscr{C}_v = \mathscr{C}_p = \mathscr{V} \cap \mathscr{P} \cap \mathscr{W}$ provided $\mathscr{V} \cap \mathscr{P} \cap \mathscr{W} \neq \emptyset$. Generally speaking, $\mathscr{V} \cap \mathscr{P} \cap \mathscr{W} \neq \emptyset$ is not necessarily satisfied, by introducing $\mathscr{C}_v$ and $\mathscr{C}_p$ we describe the concentration behaviour in a competing sense.

Theorem 1.3. *Let $V(x) \leq 0$, and assume* (H1), (H2), (H7) *and* (G1)–(G2) *are satisfied.*

(I) *Suppose* (H5) *holds.*

Then for $\varepsilon > 0$ sufficiently small, (5) *has a ground state solution*

$$\varphi_\varepsilon \in \cap_{q \geq 2} W^{1,q}(\mathbb{R}^n, \mathbb{C}^N).$$

If additionally ∇V, ∇P and ∇W are bounded and (H8) *holds, then φ_ε satisfies:*

(i) *There exists a maximum point x_ε of $|\varphi_\varepsilon|$ with $\lim_{\varepsilon \to 0} \operatorname{dist}(x_\varepsilon, \mathscr{C}_v) = 0$, such that, for some $C, c > 0$*

$$|\varphi_\varepsilon(x)| \leq C \exp\Big(-\frac{c}{\varepsilon}|x - x_\varepsilon|\Big) \,.$$

(ii) *Setting $z_\varepsilon(x) = \varphi_\varepsilon(\varepsilon x + x_\varepsilon)$, for any sequence $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, z_ε converges in H^1 to a ground state solution of*

$$-i\alpha \cdot \nabla z + a\beta z + V(x_0) z = P(x_0) g(|z|) z + W(x_0)|z|^{2^*-2} z \,.$$

If particularly $\mathscr{V} \cap \mathscr{P} \cap \mathscr{W} \neq \emptyset$ then $\lim_{\varepsilon \to 0} \operatorname{dist}(x_\varepsilon, \mathscr{V} \cap \mathscr{P} \cap \mathscr{W}) = 0$ and z_ε converges in H^1 to a a ground state solution of

$$-i\alpha \cdot \nabla z + a\beta z + \kappa z = m g(|z|) z + l|z|^{2^*-2} z \,.$$

(II) *Suppose* (H6) *holds. Then, replacing* $\mathscr{C}_v$ *with* $\mathscr{C}_p$*, all the conclusions under* (I) *remain true.*

It is standard that (4) and (5) are equivalent to, by letting $u(x) = \varphi(\varepsilon x)$,

$$-i\alpha \cdot \nabla u + a\beta u + V_\varepsilon(x)u = P_\varepsilon(x)g(|u|)u\,, \tag{7}$$

$$-i\alpha \cdot \nabla u + a\beta u + V_\varepsilon(x)u = P_\varepsilon(x)g(|u|)u + W_\varepsilon(x)|u|^{2^*-2}u\,, \tag{8}$$

where $V_\varepsilon(x) = V(\varepsilon x)$, $P_\varepsilon(x) = P(\varepsilon x)$ and $W_\varepsilon(x) = W(\varepsilon x)$. We will in the sequel focus on these two equivalent problems. Our proofs are variational: the semiclassical solutions are obtained as critical points of an energy functional Φ_ε associated to the equivalent problems.

There have been a large number of works on existence and concentration phenomenon of semi-classical states of nonlinear Schrödinger or Schrödinger-Poisson systems arising in the *non-relativistic* quantum mechanics, see, for example, [2, 3, 4, 28] and their references. There are many results on existence and concentration phenomenon of semi-classical states of nonlinear Dirac equation with different space dimensions (Table 1). For the case of $n = 3$, the paper [12] studied the existence of a family of ground states of the problem

$$-i\varepsilon\alpha \cdot \nabla\varphi + a\beta\varphi = W(x)|\varphi|^{p-2}\varphi$$

and showed that the family concentrates around the maxima of $W(x)$ as $\varepsilon \to 0$. The paper [13] study the existence and concentration of a family of ground states of the following problem

$$-i\varepsilon\alpha \cdot \nabla\varphi + a\beta\varphi + V(x)\varphi = \sum_{j=1}^{J} W_j(x)|\varphi|^{p_j-2}\varphi.$$

The paper [14] studies the semiclassical ground states of the Dirac equation with critical nonlinearity

$$-i\varepsilon\alpha \cdot \nabla\varphi + a\beta\varphi + V(x)\varphi = W(x)(g(|\varphi|) + |\varphi|)\varphi.$$

And the papers [16, 17] studied the existence of a family of semi-classical ground states of nonlinear Dirac-Maxwell systems arising in the relativistic quantum mechanics in space-dimensional three and showed that the family concentrates around some certain sets as $\varepsilon \to 0$.

Table 1.1: Space Dimensions and Critical Indices

n	2	3	4	5	6	7	8	9	$\cdots$
N	2	4	4	8	8	16	16	32	$\cdots$
$2^* = \frac{2n}{n-1}$	4	3	$\frac{8}{3}$	$\frac{5}{2}$	$\frac{12}{5}$	$\frac{7}{3}$	$\frac{16}{7}$	$\frac{9}{4}$	$\cdots$

For the case of $n = 2$, Dirac matrices is given by Pauli matrices

$$\alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Many authors use σ_1, σ_2, σ_3 instead of α_1, α_2, β in two dimensional Dirac equation. Two dimensional Dirac equation has been widely used in Condensed Matter Physics to study the Dirac materials, i.e. graphene, topological insulators, superfluid phase of ^{3}He. The paper [6] shows the existence of an exponentially locallized stationary solution for the following Dirac equation with Kerr nonlinearity by using the shooting method.

$$-i\varepsilon\alpha_1\partial_1\varphi - i\varepsilon\alpha_2\partial_2\varphi + a\beta\varphi - \omega\varphi = |\varphi|^2\varphi.$$

The paper [7] studies the existence of infinitely many stationary solutions for a nonlinear Dirac equation with infinite mass boundary conditions. As for the concentration of semi-classical states for the two dimensional Dirac equation, it follows directly from Theorem 1.2,

Corollary 1.4. *Let $\omega \in (-a, a)$, $P, W \in C^1(\mathbb{R}^n, \mathbb{R}^+)$, $l_\infty \le l$, $m_\infty < m_w$, and*

$$(\frac{a}{a-\omega})^{3\sigma-8} m_\infty^{-2} l^{\sigma-2} < \mathcal{R}_\sigma.$$

Assume (G1)–(G2) *are satisfied. For $\varepsilon > 0$ sufficiently small, the following Dirac equation possesses a ground state solution $\varphi_\varepsilon \in \cap_{q\ge 2} W^{1,q}(\mathbb{R}^2, \mathbb{C}^2)$.*

$$-i\varepsilon\alpha_1\partial_1\varphi - i\varepsilon\alpha_2\partial_2\varphi + a\beta\varphi - \omega\varphi = P(x)g(|\varphi|)\varphi + W(x)|\varphi|^2\varphi \tag{9}$$

If additionally ∇P and ∇W are bounded, then φ_ε satisfies:

(i) *There exists a maximum point x_ε of $|\varphi_\varepsilon|$ with $\lim_{\varepsilon\to 0} \mathrm{dist}(x_\varepsilon, \mathscr{C}_p) = 0$, such that, for some $C, c > 0$*

$$|\varphi_\varepsilon(x)| \le C \exp\Big(-\frac{c}{\varepsilon}|x - x_\varepsilon|\Big),$$

where $\mathscr{C}_p = \mathcal{W}_p \cup \{x \notin \mathcal{W} : P(x) > m_w\}$.

(ii) *Setting $z_\varepsilon(x) = \varphi_\varepsilon(\varepsilon x + x_\varepsilon)$, for any sequence $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, z_ε converges in H^1 to a ground state solution of*

$$-i\alpha_1\partial_1 z - i\alpha_2\partial_2 z + a\beta z - \omega z = P(x_0)g(|z|)z + W(x_0)|z|^2 z\,.$$

If particularly $\mathscr{P} \cap \mathscr{W} \neq \emptyset$, then $\lim_{\varepsilon\to 0} \mathrm{dist}(x_\varepsilon, \mathscr{P} \cap \mathscr{W}) = 0$ and z_ε converges in H^1 to a a ground state solution of

$$-i\alpha_1\partial_1 z - i\alpha_2\partial_2 z + a\beta z - \omega z = mg(|z|)z + l|z|^2 z\,.$$

Mathematically, the problems in Dirac equations are much more difficult compared with Schrödinger equations because they are strongly indefinite in the sense that both the negative and positive parts of the spectrum of Dirac operator are unbounded and consist of essential spectra.

An outline of this paper is as follows: In Section 2 we treat the linking argument which gives us a min-max scheme. In Section 3, we study the limit equation and give some characterization of the least energy level. Lastly, in Section 4, the combination of the results in section 2, 3 lead to the proof of Theorem 1.2 and Theorem 1.3.

2. The variational framework

2.1. The functional setting and notations

In the sequel we denote by $|\cdot|_q$ the usual L^q-norm, and $(\cdot,\cdot)_2$ the usual L^2-inner product. Let $H_D = -i\alpha\cdot\nabla + a\beta$ denote the self-adjoint operator on $L^2 \equiv L^2(\mathbb{R}^n, \mathbb{C}^N)$. We find by Fourier analysis that the domain $\mathcal{D}(H_D) = H^1 \equiv H^1(\mathbb{R}^n, \mathbb{C}^N)$.

Let us first study H_D for its spectrum. As we have seen, H_D is a differential operator with constant coefficients. So, in the Fourier domain $\zeta = (\zeta_1, \cdots, \zeta_n)$, it becomes the operator of multiplication by the matrix

$$\hat{H}_D(\zeta) = \sum_{k=1}^{n} \zeta_k \alpha_k + a\beta = \begin{pmatrix} aI_{N/2} & \sum_{k=1}^{n} \zeta_k a_k \\ \sum_{k=1}^{n} \zeta_k a_k^* & -aI_{N/2} \end{pmatrix}.$$

Lemma 2.1. *$\sigma(H_D) = \sigma_e(H_D) = \mathbb{R} \setminus (-a, a)$, where $\sigma(\cdot)$ and $\sigma_e(\cdot)$ denote the spectrum and the essential spectrum.*

Proof. Since $\mathcal{F}H_D z = \hat{H}_D\mathcal{F}z$, therefore if $\lambda \in \sigma(H_D)$, then $\det(\lambda I_N - \hat{H}_D) = 0$.

Thus,
$$\det\begin{pmatrix} (\lambda - a)I_{\frac{N}{2}} & -\sum_{k=1}^{n} \zeta_k a_k \\ -\sum_{k=1}^{n} \zeta_k a_k^* & (\lambda + a)I_{\frac{N}{2}} \end{pmatrix} = 0.$$

For a_k, $k = 1, ..., N$, it is clear that $a_i a_j^* + a_j a_i^* = 0$, $a_i^* a_j + a_j^* a_i = 0$, and $a_k a_k^* = a_k^* a_k = I_{\frac{N}{2}}$, for $1 \le i, j, k \le n$. Then we have

$$(\sum_{k=1}^{n} \zeta_k a_k^*)(\sum_{k=1}^{n} \zeta_k a_k) = |\zeta|^2 I_{\frac{N}{2}},$$

where $\zeta = (\zeta_1, ..., \zeta_n)$. Therefore, if $\lambda \neq a$, we have

$$(\lambda - a)^{\frac{N}{2}} \det((\lambda + a)I_{\frac{N}{2}} - (\sum_{k=1}^{n} \zeta_k a_k^*)((\lambda - a)^{-1} I_{\frac{N}{2}})(\sum_{k=1}^{n} \zeta_k a_k)) = 0.$$

For the case of $\lambda = a$, we have $\lambda \neq -a$, then by the same way we have

$$((\lambda^2 - a^2) - |\zeta|^2)^{\frac{N}{2}} = 0.$$

Therefore, $\lambda^2 = a^2 + |\zeta|^2$, by the fact of

$$\sigma(H_D) = \{\lambda \in \mathbb{R} : \lambda^2 = a^2 + |\zeta|^2, \forall \zeta_k \in \mathbb{R}\},$$

we have $\sigma(H_D) = \sigma_e(H_D) = \mathbb{R} \setminus (-a, a)$. □

Now, the space L^2 possesses the orthogonal decomposition:

$$L^2 = L^+ \oplus L^-, \qquad u = u^+ + u^-, \tag{10}$$

so that H_D is positive definite (resp. negative definite) in L^+ (resp. L^-). Let $E := \mathcal{D}(|H_D|^{1/2}) = H^{1/2}(\mathbb{R}^n, \mathbb{C}^N)$ be equipped with the inner product

$$\langle u, v\rangle = \Re(|H_D|^{1/2}u, |H_D|^{1/2}v)_2$$

and the induced norm $\|u\| = \langle u, u\rangle^{1/2}$, where $|H_D|$ and $|H_D|^{1/2}$ denote respectively the absolute value of H_D and the square root of $|H_D|$. Since $\sigma(H_D) = \mathbb{R} \setminus (-a, a)$, one has

$$a|u|_2^2 \le \|u\|^2 \quad \text{for all } u \in E. \tag{11}$$

Note that this norm is equivalent to the usual $H^{1/2}$-norm, hence E embeds continuously into L^q for all $q \in [2, 2^*]$ and compactly into L^q_{loc} for all $q \in [1, 2^*)$. It is clear that E possesses the following decomposition

$$E = E^+ \oplus E^- \ \text{ with } \ E^\pm = E \cap L^\pm, \tag{12}$$

orthogonal with respect to both inner products $(\cdot,\cdot)_2$ and $\langle\cdot,\cdot\rangle$. This decomposition induces also a natural decomposition of L^p, hence there is $d_p > 0$ with

$$d_p|u^\pm|_p^p \le |u|_p^p \ \text{ for all } u \in E. \tag{13}$$

Associate with (7) and (8), we define the "energy" functional

$$\Phi_\varepsilon(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{1}{2}\int_{\mathbb{R}^n} V_\varepsilon(x)|u|^2 - \Psi_\varepsilon(u)$$

for $u = u^+ + u^-$, where

$$\Psi_\varepsilon(u) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} P_\varepsilon(x)G(|u|) & \text{for subcritical case (7)}\,, \\ \displaystyle\int_{\mathbb{R}^n} P_\varepsilon(x)G(|u|) + \frac{1}{2^*}\int_{\mathbb{R}^n} W_\varepsilon(x)|u|^{2^*} & \text{for critical case (8)}\,. \end{cases}$$

Plainly, under our assumptions, $\Phi_\varepsilon \in C^2(E, \mathbb{R})$ and any critical point of Φ_ε is a (weak) solution to the corresponding equation. For notation convenience, in the sequel, if there is no confusion we will use Φ_ε to represent the energy functional without explain the situation.

2.2. Technical results

In this subsection, we will give some general versions of results that valid in both subcritical and critical cases. To use a unified expression, we will use

$$\Psi_\varepsilon(u) = \int_{\mathbb{R}^n} P_\varepsilon(x)G(|u|) + \frac{1}{2^*}\int_{\mathbb{R}^n} W_\varepsilon(x)|u|^{2^*}\,.$$

When the subcritical case occurs, we simply set $W \equiv 0$.

For further consideration, for $r > 0$, set $B_r = \{u \in E : \|u\| \leq r\}$, and for $e \in E^+$,

$$E_e := E^- \oplus \mathbb{R}^+ e \ \text{ with } \mathbb{R}^+ = [0, +\infty).$$

By virtue of the assumptions (G1)–(G2), for any $\delta > 0$, there exist $r_\delta > 0, c_\delta > 0$ and $c'_\delta > 0$ such that

$$\begin{cases} g(s) < \delta \ \text{ for all } 0 \leq s \leq r_\delta; \\ G(s) \leq \delta\, s^2 + c'_\delta\, s^p \ \text{ for all } s \geq 0 \end{cases} \tag{14}$$

and $$\widehat{G}(s) := \frac{1}{2} g(s)s^2 - G(s) \geq \frac{\theta - 2}{2\theta} g(s)s^2 \geq \frac{\theta - 2}{2} G(s) \geq c_\theta s^\sigma \tag{15}$$

for all $s \geq 0$, where $c_\theta = c_0(\theta - 2)/2$.

Lemma 2.2. *For all $\varepsilon \in (0, 1]$, Φ_ε possess the linking structure:*

(1) *There are $r > 0$ and $\tau > 0$, both independent of ε, such that $\Phi_\varepsilon|_{B_r^+} \geq 0$ and $\Phi_\varepsilon|_{S_r^+} \geq \tau$, where*

$$B_r^+ = B_r \cap E^+ = \{u \in E^+ : \|u\| \leq r\},$$

and $$S_r^+ = \partial B_r^+ = \{u \in E^+ : \|u\| = r\}.$$

(2) *For any $e \in E^+ \setminus \{0\}$, there exist $R = R_e > 0$ and $C = C_e > 0$, both independent of ε, such that, for all $\varepsilon > 0$, there hold $\Phi_\varepsilon(u) < 0$ for all $u \in E_e \setminus B_R$ and $\max \Phi_\varepsilon(E_e) \leq C$.*

Proof. Recall that $|u|_p^p \leq C_p \|u\|^p$ for all $u \in E$ by Sobolev embedding theorem. (1) follows easily because, for $u \in E^+$ and $\delta > 0$ small enough

$$\begin{aligned} \Phi_\varepsilon(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int V_\varepsilon(x)|u|^2 - \Psi_\varepsilon(u) \\ &\geq \frac{a - |V|_\infty}{2a}\|u\|^2 - |P|_\infty\big(\delta|u|_2^2 + c'_\delta|u|_p^p\big) - \frac{|W|_\infty}{2^*}|u|_{2^*}^{2^*} \end{aligned}$$

with C_1 independent of u and $p > 2$ (see (14)).

For checking 2), take $e \in E^+ \setminus \{0\}$. In virtue of (13) and (14), one gets, for $u = se + v \in E_e$,

$$\begin{aligned} \Phi_\varepsilon(u) &= \frac{1}{2}\|se\|^2 - \frac{1}{2}\|v\|^2 + \frac{1}{2}\int V_\varepsilon(x)|se + v|^2 - \Psi_\varepsilon(u) \\ &\leq \frac{a + |V|_\infty}{2a} s^2\|e\|^2 - \frac{a - |V|_\infty}{2a}\|v\|^2 - c_0 d_\sigma \inf P \cdot s^\sigma |e|_\sigma^\sigma \end{aligned} \tag{16}$$

proving the conclusion. □

Now let us define (see [5, 30]) $$c_\varepsilon := \inf_{z \in E^+ \setminus \{0\}} \max_{u \in E_z} \Phi_\varepsilon(u)\,. \tag{17}$$

As a consequence of Lemma 2.2, we have

Lemma 2.3. *There is $C > 0$ independent of ε such that $\tau \le c_\varepsilon \le C$.*

Proof. It follows from (1) of Lemma 2.2 and the definition of c_ε that $c_\varepsilon \ge \tau$. Take $e \in E^+$ with $\|e\| = 1$. By (16) we deduce $c_\varepsilon \le C \equiv C_e$, completing the proof. □

Recall that a sequence $\{u_n\} \subset E$ is called to be a $(PS)_c$-sequence for the functional $\Phi \in C^1(E, \mathbb{R})$ if $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$, and is called to be $(C)_c$-sequence for Φ if $\Phi(u_n) \to c$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$. It is clear that if $\{u_n\}$ is a $(PS)_c$-sequence with $\{\|u_n\|\}$ bounded then it is also a $(C)_c$-sequence.

Recall that, by (G1), there exist $r_1 > 0$ and $a_1 > 0$ such that

$$g(s) \le \frac{a - |V|_\infty}{2\,|P|_\infty} \quad \text{for all } s \le r_1, \tag{18}$$

and, for $s \ge r_1$, $g(s) \le a_1 s^{p-2}$, so $g(s)^{\sigma_0 - 1} \le a_2 s^2$ with

$$\sigma_0 := \frac{p}{p-2} > 3$$

which, jointly with (G2), yields (see (15))

$$g(s)^{\sigma_0} \le a_2 g(s) s^2 \le a_3 \widehat{G}(s) \quad \text{for all } s \ge r_1. \tag{19}$$

Lemma 2.4. *For every pair of constants $c_1, c_2 > 0$, there exists a constant $\Lambda > 0$, depending only on c_1, c_2, such that for any $u \in E$ with*

$$|\Phi_\varepsilon(u)| \le c_1 \quad \text{and} \quad \|u\| \cdot \|\Phi_\varepsilon'(u)\| \le c_2, \tag{20}$$

we have

$$\|u\| \le \Lambda.$$

Lemma 2.4 has an immediate consequence which implies the boundness of a $(C)_c$-sequence:

Corollary 2.5. *Consider $\varepsilon \in (0, 1]$, and $\{u_n^\varepsilon\}$ is the corresponding $(C)_{c_\varepsilon}$-sequence for Φ_ε. If there exists $C > 0$ such that $|c_\varepsilon| \le C$ for all ε, then we have (up to a subsequence if necessary)*

$$\|u_n^\varepsilon\| \le \Lambda,$$

where Λ is found in Lemma 2.4 depends on the pair $c_1 = C$ and $c_2 = 1$.

Proof of Lemma 2.4. Take $u \in E$ such that (20) is satisfied. Without loss of generality we may assume that $\|u\| \ge 1$. The form of Φ_ε implies that

$$\int P_\varepsilon(x)\widehat{G}(|u|) + \frac{1}{2 \cdot 2^*}\int W_\varepsilon(x)|u|^{2^*} = \Phi_\varepsilon(u) - \frac{1}{2}\Phi_\varepsilon'(u)u \le c_1 + c_2, \tag{21}$$

and

$$\begin{aligned} &\|u\|^2 - \Re\int V_\varepsilon(x)u \cdot (u^+ - u^-) - \Re\int P_\varepsilon(x)g(|u|)u \cdot (u^+ - u^-) \\ &- \Re\int W_\varepsilon(x)|u|^{2^*-2}u \cdot (u^+ - u^-) = \Phi_\varepsilon'(u)(u^+ - u^-) \le c_2. \end{aligned} \tag{22}$$

By (15) and (21), one sees that $\max\{|u|_\sigma, |u|_{2^*}\} \le C_1$ where C_1 depends only on c_1, c_2. It follows from (22) that

$$\frac{a-|V|_\infty}{a}\|u\|^2 \le c_2 + \Re\int P_\varepsilon(x)g(|u|)u\cdot(u^+-u^-) + \Re\int W_\varepsilon(x)|u|^{2^*-2}u\cdot(u^+-u^-).$$

This, together with (18), (11) and the boundness of $|u|_{2^*}$, shows that

$$\frac{a-|V|_\infty}{2a}\|u\|^2 \le C_2 + \Re\int_{|u|\ge r_1} P_\varepsilon(x)g(|u|)u\cdot(u^+-u^-). \tag{23}$$

Recall that (G1) and (G2) imply $2<\sigma\le p$. Setting $t=\frac{p\sigma}{2\sigma-p}$, one sees

$$2<t<p, \quad \frac{1}{\sigma_0}+\frac{1}{\sigma}+\frac{1}{t}=1.$$

By Hölder's inequality, (19), (21) and the embedding of E to L^t, we have

$$\begin{aligned}&\int_{|u|\ge r_1} P_\varepsilon(x)\, g(|u|)|u|\cdot|u^+-u^-| \qquad (24)\\ &\le |P|_\infty\Big(\int_{|u|\ge r_1} g(|u|)^{\sigma_0}\Big)^{1/\sigma_0}\Big(\int|u|^\sigma\Big)^{1/\sigma}\Big(\int|u^+-u^-|^t\Big)^{1/t}\le C_3\|u\|\end{aligned}$$

where $C_3>0$ depends only on c_1, c_2.

Now the combination of (23) and (24) shows that

$$\|u\|^2 \le M_0 + M_1\|u\| \tag{25}$$

with M_0 and M_1 dependent only on the constants c_1, c_2. Therefore, either $\|u\|\le 1$ or there is $\Lambda\ge 1$ dependents only on c_1, c_2 such that $\|u\|\le\Lambda$, as desired. □

Let $\mathscr{K}_\varepsilon := \big\{u\in E\setminus\{0\} : \Phi_\varepsilon'(u)=0\big\}$ be the critical set of Φ_ε. By virtue of Lemma 2.4, using the same iterative argument of [18, Proposition 3.2] and [16, Lemma 3.19], we obtain the following

Lemma 2.6. *If $u\in\mathscr{K}_\varepsilon$ with $|\Phi_\varepsilon(u)|\le C$. Then*

(1) *if the assumptions of Theorem 1.2 are satisfied for any $q\ge 2$, we have $u\in W^{1,q}(\mathbb{R}^n,\mathbb{C}^N)$ with $\|u\|_{W^{1,q}}\le C_q$, where C_q depends only on C and q;*

(2) *if the assumptions of Theorem 1.3 are satisfied for any $q\ge 2$, we have $u\in W^{1,q}_{loc}(\mathbb{R}^n,\mathbb{C}^N)\cap L^\infty$ where $|u|_\infty$ depends only on C.*

Remark 2.7. Let $\mathscr{L}_\varepsilon$ be the set of all least energy solutions of Φ_ε. If $u\in\mathscr{L}_\varepsilon$, then $\Phi_\varepsilon(u)=c_\varepsilon$ (this will be proved in section 4). Therefore, as a consequence of Lemma 2.6, we see (together with the Sobolev embedding theorem) that there exist $C_\infty>0$ independent of ε such that

$$|u|_\infty \le C_\infty \quad \text{for all } u\in\mathscr{L}_\varepsilon\,.$$

3. Preliminary results

In this section, we will investigate some results that essentially relate to our main theorems.

Firstly, let us consider the following autonomous equations

$$-i\alpha\cdot\nabla u+a\beta u+\lambda u=\mu g(|u|)u \tag{26}$$

$$-i\alpha\cdot\nabla u+a\beta u+\lambda u=\mu g(|u|)u+\chi|u|^{2^*-2}u \tag{27}$$

for $u\in H^1(\mathbb{R}^n,\mathbb{C}^N)$, where $\lambda\in(-a,a)$, $\mu,\chi>0$ are constants (varying in different equations).

3.1. Equation (26)

The solutions of (26) are critical points of the functional

$$\begin{aligned}\mathscr{T}_{\lambda\mu}(u)&:=\frac{1}{2}\big(\|u^+\|^2-\|u^-\|^2\big)+\frac{\lambda}{2}|u|_2^2-\mu\int G(|u|)\\&=\frac{1}{2}\big(\|u^+\|^2-\|u^-\|^2\big)+\frac{\lambda}{2}|u|_2^2-\mathscr{G}_\mu(u)\end{aligned}$$

defined for $u=u^++u^-\in E=E^+\oplus E^-$. Denote the critical set and the least energy of $\mathscr{T}_{\lambda\mu}$ as follows

$$\mathscr{K}_{\lambda\mu}:=\big\{u\in E\setminus\{0\}:\mathscr{T}_{\lambda\mu}'(u)=0\big\},\quad\text{and}\quad\gamma_{\lambda\mu}:=\inf\big\{\mathscr{T}_{\lambda\mu}(u):u\in\mathscr{K}_{\lambda\mu}\big\}.$$

In order to find critical points of $\mathscr{T}_{\lambda\mu}$, we will use the the following abstract theorem which is taken from [5, 11].

Let E be a Banach space with direct sum decomposition $E=X\oplus Y$, $u=x+y$ and corresponding projections P_X,P_Y onto X,Y, respectively. For a functional $\Phi\in C^1(E,\mathbb{R})$ we write $\Phi_a=\{u\in E:\ \Phi(u)\ge a\}$.

Now we assume that X is separable and reflexive, and we fix a countable dense subset $\mathscr{S}\subset X^*$. For each $s\in\mathscr{S}$ there is a semi-norm on E defined by

$$p_s:E\to\mathbb{R},\quad p_s(u)=|s(x)|+\|y\|\quad\text{for } u=x+y\in X\oplus Y.$$

We denote by $\mathcal{T}_{\mathscr{S}}$ the induced topology. Let w^* denote the weak*-topology on E. Suppose:

(Φ_0) There exists $\xi>0$ such that $\|u\|<\xi\|P_Yu\|$ for all $u\in\Phi_0$.

(Φ_1) For any $c\in\mathbb{R}$, Φ_c is $\mathcal{T}_{\mathscr{S}}$-closed, and $\Phi':(\Phi_c,\mathcal{T}_{\mathscr{S}})\to(E^*,w^*)$ is continuous.

(Φ_2) There exists $\rho>0$ with $\kappa:=\inf\Phi(S_\rho Y)>0$, where $S_\rho Y:=\{u\in Y:\|u\|=\rho\}$.

The following theorem is a special case of [5], Theorem 3.4 (see also [11], Theorem 4.3).

Theorem 3.1. *Let* (Φ_0)–(Φ_2) *be satisfied and suppose there are* $R > \rho > 0$ *and* $e \in Y$ *with* $\|e\| = 1$ *such that* $\sup \Phi(\partial Q) \le \kappa$ *where*

$$Q = \{u = x + te : \, x \in X, t \ge 0, \|u\| < R\}.$$

Then Φ *has a* $(C)_c$*-sequence with* $\kappa \le c \le \sup \Phi(Q)$.

The following lemma is useful to verify (Φ_1) (see [5] or [11]).

Lemma 3.2. *Suppose* $\Phi \in C^1(E, \mathbb{R})$ *is of the form*

$$\Phi(u) = \frac{1}{2}\big(\|y\|^2 - \|x\|^2\big) - \Psi(u) \quad \text{for } u = x + y \in E = X \oplus Y$$

such that

(i) $\Psi \in C^1(E, \mathbb{R})$ *is bounded from below;*

(ii) $\Psi : (E, \mathcal{T}_w) \to \mathbb{R}$ *is sequentially lower semi-continuous, that is,* $u_n \rightharpoonup u$ *in* E *implies* $\Psi(u) \le \liminf \Psi(u_n)$*;*

(iii) $\Psi' : (E, \mathcal{T}_w) \to (E^*, w^*)$ *is sequentially continuous.*

(iv) $\nu : E \to \mathbb{R}$, $\nu(u) = \|u\|^2$, *is* C^1 *and* $\nu' : (E, \mathcal{T}_w) \to (E^*, w^*)$ *is sequentially continuous.*

Then Φ *satisfies* (Φ_1).

Next, we present the existence result for the limit equation (26).

Lemma 3.3. *Let* $\lambda \in (-a, a)$*, for each* $\mu > 0$*, we have*

(1) $\mathscr{K}_{\lambda\mu} \ne \emptyset$ *and* $\gamma_{\lambda\mu} > 0$,

(2) $\gamma_{\lambda\mu}$ *is attained and*
$$\gamma_{\lambda\mu} = \inf_{e \in E^+ \setminus \{0\}} \max_{u \in E_e} \mathscr{T}_{\lambda\mu}(u)\,. \tag{28}$$

Proof. Since the characterization of the least energy in (28) will be proved when we treat the non-autonomous equation (the situation here is much simpler), we only give the existence result.

With $X = E^-$ and $Y = E^+$ the condition (Φ_0) holds since $G(|u|) \ge 0$ for any $u \in E$. Together with the linking structure (see Lemma 2.2) we have all the assumptions of Theorem 3.1 verified. Therefore, there exists a sequence $\{u_m\}$ satisfying $\mathscr{T}_{\lambda\mu}(u_m) \to c > 0$ and $(1 + \|u_m\|)\mathscr{T}'_{\lambda\mu}(u_m) \to 0$ as $m \to \infty$. Using the same arguments in proving Lemma 2.4, we get $\{u_m\}$ is bounded. Now by the classical concentration compactness principle (cf. [24]) and the translation-invariance of $\mathscr{T}_{\lambda\mu}$, we infer there is $u \ne 0$ such that $\mathscr{T}'_{\lambda\mu}(u) = 0$.

If $u \in \mathscr{K}_{\lambda\mu}$, one has
$$\mathscr{T}_{\lambda\mu}(u) = \mathscr{T}_{\lambda\mu}(u) - \frac{1}{2}\mathscr{T}'_{\lambda\mu}(u)u = \mu \int \widehat{G}(u) \ge 0. \tag{29}$$

For proving $\gamma_{\lambda\mu} > 0$, assume by contradiction that $\gamma_{\lambda\mu} = 0$. Let $u_j \in \mathscr{K}_{\lambda\mu} \setminus \{0\}$ such that $\mathscr{T}_{\lambda\mu}(u_j) \to 0$. It is obvious that $\{u_j\}$ is bounded. Furthermore, by (15) and (29), we deduce $u_j \to 0$ in L^σ as $j \to \infty$.

On the other hand, $\mathscr{T}'_{\lambda\mu}(u_j)(u_j^+ - u_j^-) = 0$ and (11) imply that

$$\frac{a - |\lambda|}{a}\|u_j\|^2 \le \mu \Re \int g(|u_j|)u_j \cdot (u_j^+ - u_j^-) \le \mu \int g(|u_j|)|u_j| \cdot |u_j^+ - u_j^-|\,.$$

By (19) (with a suitable choice of r_1 smaller) and Hölder's inequality, one sees

$$\frac{a-|\lambda|}{2a}\|u_j\|^2 \le \mu C_2\Big(\int g(|u_j|)^{\sigma_0}\Big)^{1/\sigma_0}|u_j|_p^2 \le \mu C_3\big(\mathscr{T}_{\lambda\mu}(u_j)\big)^{1/\sigma_0}\|u_j\|^2 .$$

Hence $\frac{a-|\lambda|}{2a} \le 0$, which is a contradiction.

Lastly, again, by using the concentration compactness principle, we check easily that $\gamma_{\lambda\mu}$ is attained. Ending the proof. □

For later use, because $\sigma \in (2, 2^*)$ appeared in (G2), let us consider the equation

$$-i\alpha\cdot\nabla z + a\beta z = |z|^{\sigma-2}z$$

with the energy functional defined by

$$S_\sigma(z) := \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) - \frac{1}{\sigma}|z|_\sigma^\sigma$$

and the least energy denoted by γ. The following lemma is due to [14, Lemma 4.6] (see also [16, Lemma 3.5]) with some obvious modifications.

Lemma 3.4. *In the case $g(s) = c_0 s^{\sigma-2}$ and $\lambda \le 0$, the corresponding least energy of* (26)*, which denoted by $\gamma_{\lambda\mu}(\sigma)$, satisfies*

$$\gamma_{\lambda\mu}(\sigma) \le \Big(\frac{a}{a+\lambda}\Big)^{\frac{-2}{\sigma-2}+n-1}(c_0\,\mu)^{-2/(\sigma-2)}\gamma.$$

Proof. We only give a sketch of the proof. By observing that, setting $z(x) = u(ax/(a+\lambda))$, (26) is equivalent to

$$-i\alpha\cdot\nabla z + a\beta z + \frac{a\lambda}{a+\lambda}(I-\beta)z = \frac{a\mu}{a+\lambda}g(|z|)z$$

with the energy functional defined by

$$\mathscr{T}_{\mu/\lambda}(z) = \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) + \frac{a\lambda}{a+\lambda}|z_2|_2^2 - \frac{a\mu}{a+\lambda}\int G(|z|) .$$

Here, we used the notation $z(x) = (z_1(x), z_2(x)) \in \mathbb{C}^{\frac{N}{2}} \times \mathbb{C}^{\frac{N}{2}}$ for $z \in E$. Denoted by $\gamma_{\mu/\lambda}(\sigma)$ be the least energy of $\mathscr{T}_{\mu/\lambda}$, we have

$$\gamma_{\lambda\mu}(\sigma) = \big(\frac{a}{a+\lambda}\big)^{n-1}\gamma_{\mu/\lambda}(\sigma) .$$

Now, let $g(s) = c_0 s^{\sigma-2}$ and $\lambda \le 0$. Equation (28) makes us aware of $\gamma_{\mu/\lambda}(\sigma) \le \gamma_{\mu/\lambda}$ provided $\lambda \le 0$, where $\gamma_{\mu/\lambda}$ denotes the least energy of

$$-i\alpha\cdot\nabla u + a\beta u = \frac{ac_0\mu}{a+\lambda}|u|^{\sigma-2}u .$$

Following the lines in [14, 16], we find $\gamma_{\mu/\lambda} \le (\frac{a+\lambda}{ac_0\mu})^{2/(\sigma-2)}\gamma$, as desired. □

3.2. Equation (27)

The solutions of equation (27) are critical points of the functional

$$\begin{aligned}\mathscr{T}_{\lambda\mu\chi}(u) &:= \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{\lambda}{2}|u|_2^2 - \mu\int G(|u|) - \frac{\chi}{2^*}|u|_{2^*}^{2^*}\\ &= \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{\lambda}{2}|u|_2^2 - \mathscr{G}_{\mu\chi}(u)\end{aligned}$$

defined for $u = u^+ + u^- \in E = E^+ \oplus E^-$. Denote the critical set and the least energy of $\mathscr{T}_{\lambda\mu\chi}$ as

$$\mathscr{K}_{\lambda\mu\chi} := \big\{u \in E \setminus \{0\} : \mathscr{T}'_{\lambda\mu\chi}(u) = 0\big\}, \quad \text{and} \quad \gamma_{\lambda\mu\chi} := \inf\big\{\mathscr{T}_{\lambda\mu\chi}(u) : u \in \mathscr{K}_{\lambda\mu\chi}\big\}.$$

Firstly, we have

Lemma 3.5. *$\gamma_{\lambda\mu\chi}$ is attained if $\gamma_{\lambda\mu\chi} < \ell := \Big(\frac{1}{2} - \frac{1}{2^*}\Big) \cdot \chi^{\frac{-2}{2^*-2}}\Big(\frac{a-|\lambda|}{a\,S^{-1}}\Big)^{\frac{2^*}{2^*-2}}$.*

Proof. Let $\{u_n\}$ be a $(C)_c$-sequence with $c = \gamma_{\lambda\mu\chi}$. By the statements in Lemma 2.4, $\{u_n\}$ is bounded in E. From Lion's concentration principle [24], $\{u_n\}$ is either vanishing or non-vanishing.

Assume that $\{u_n\}$ is vanishing. Then $|u_n|_s \to 0$ for $s \in (2, 2^*)$. By (G1), (G2):

$$c + o(1) = \mathscr{T}_{\lambda\mu\chi}(u_n) - \frac{1}{2}\mathscr{T}'_{\lambda\mu\chi}(u_n)u_n = \frac{(2^*-2)\chi}{2\cdot 2^*}|u_n|_{2^*}^{2^*}.$$

Moreover,

$$\|u_n\|^2 + \lambda\Re\int u_n\cdot(u_n^+ - u_n^-) - \chi\cdot\Re\int |u_n|^{2^*-2}u_n\cdot(u_n^+ - u_n^-) = o(1).$$

Thus,

$$\frac{a-|\lambda|}{a}\|u_n\|^2 \le \chi|u_n|_{2^*}^{2^*-1}\cdot|u_n^+ - u_n^-|_{2^*} + o(1).$$

Observe that $S|u|_{2^*}^2 \le \|u\|^2$ (see (6)), we have

$$\gamma_{\lambda\mu\chi} = c \ge \Big(\frac{1}{2} - \frac{1}{2^*}\Big)\cdot\chi^{\frac{-2}{2^*-2}}\Big(\frac{a-|\lambda|}{a\,S^{-1}}\Big)^{\frac{2^*}{2^*-2}},$$

a contradiction.

Therefore, $\{u_n\}$ is non-vanishing, that is, there exist $r, \delta > 0$ and $x_n \in \mathbb{R}^n$ such that, setting $v_n(x) = u_n(x + x_n)$, along a subsequence,

$$\int_{B_r(0)} |v_n|^2 \ge \delta.$$

Without loss of generality we assume that $v_n \rightharpoonup v$. Then $v \neq 0$ and is a solution of (27). And so $\gamma_{\lambda\mu\chi}$ is attained. □

Lemma 3.6. *For $\lambda \le 0$, $\gamma_{\lambda\mu\chi}$ is attained if*

$$\Big(\frac{a}{a+\lambda}\Big)^{(2n-1)(\sigma-2)-2}\mu^{-2}\chi^{(n-1)(\sigma-2)} < \mathcal{R}_\sigma. \tag{30}$$

Proof. Observe that, for the nonlinearities, we have

$$\mathscr{G}_{\mu\chi}(u) \ge \mathscr{G}_\mu(u) \ge \frac{\mu c_0}{\sigma}\int |u|^\sigma.$$

So, from Lemma 3.3 ii), we deduce

$$\inf_{e\in E^+\setminus\{0\}}\max_{u\in E_e}\mathscr{T}_{\lambda\mu\chi}(u) \le \inf_{e\in E^+\setminus\{0\}}\max_{u\in E_e}\mathscr{T}_{\lambda\mu}(u) = \gamma_{\lambda\mu} \le \gamma_{\lambda\mu}(\sigma).$$

If
$$\Big(\frac{a}{a+\lambda}\Big)^{\frac{-2}{\sigma-2}+n-1}(c_0\,\mu)^{-2/(\sigma-2)}\gamma < \ell,$$

that is, (30) is satisfied, then $\gamma_{\lambda\mu\chi} < \ell$. So $\gamma_{\lambda\mu\chi}$ is attained by Lemma 3.5. □

3.3. Characterization for least energy level

Recall the minimax scheme (see (17)): $c_\varepsilon := \inf_{z\in E^+\setminus\{0\}}\max_{u\in E_z}\Phi_\varepsilon(u)$.

Although the value of c_ε can be different in subcritical case and critical case, we see it as an abstract value and give it a general characterization (let us remind that the next few results in this subsection are applicable to the limit equations with obvious modifications).

Following Ackermann [1] (also see [12, 13, 15]), for any fixed $u \in E^+$, let the map $\varphi_u : E^- \to \mathbb{R}$ be defined by $\varphi_u(v) = \Phi_\varepsilon(u+v)$. We have, for any $v, w \in E^-$,

$$\varphi_u''(v)[w,w] \le -\|w\|^2 .$$

Meanwhile, we additionally find that

$$\varphi_u(v) \le \frac{a+|V|_\infty}{2a}\|u\|^2 - \frac{a-|V|_\infty}{2a}\|v\|^2 \quad \text{for all } v \in E^- .$$

So there exists a unique bounded C^1 smooth map $h_\varepsilon : E^+ \to E^-$ such that

$$\Phi_\varepsilon(u+h_\varepsilon(u)) = \max_{v\in E^-}\Phi_\varepsilon(u+v)\,,$$

and
$$v \ne h_\varepsilon(u) \Leftrightarrow \Phi_\varepsilon(u+v) < \Phi_\varepsilon(u+h_\varepsilon(u))\,.$$

It is clear that, for all $v \in E^-$, $0 = \varphi_u'(h_\varepsilon(u))v$. In the sequel, we define

$$I_\varepsilon : E^+ \to \mathbb{R} \ \text{ by } \ I_\varepsilon(u) = \Phi_\varepsilon(u+h_\varepsilon(u))\,,$$

and
$$\mathscr{N}_\varepsilon = \big\{u \in E^+ \setminus \{0\} : I_\varepsilon'(u)u = 0\big\}\,.$$

Plainly, critical points of I_ε and Φ_ε are in one-to-one correspondence via the injective map $u \mapsto u + h_\varepsilon(u)$ from E^+ into E. For any $u \in E^+$ and $v \in E^-$, setting $z =$

$v - h_\varepsilon(u)$ and $l(t) = \Phi_\varepsilon(u + h_\varepsilon(u) + tz)$, one has $l(1) = \Phi_\varepsilon(u+v)$, $l(0) = \Phi_\varepsilon(u+h_\varepsilon(u))$ and $l'(0) = 0$. Thus $l(1) - l(0) = \int_0^1 (1-t)l''(t)dt$ implying

$$\Phi_\varepsilon(u+v) - \Phi_\varepsilon(u+h_\varepsilon(u)) = \int_0^1 (1-t)\Phi''_\varepsilon\big(u+h_\varepsilon(u)+tz\big)[z,z]\,dt$$

$$= -\int_0^1 (1-t)\left(\|z\|^2 + \int V_\varepsilon(x)|z|^2\,dx\right)dt - \int_0^1 (1-t)\Psi''_\varepsilon\big(u+h_\varepsilon(u)+tz\big)[z,z]\,dt,$$

and hence

$$\int_0^1 (1-t)\Psi''_\varepsilon(u+h_\varepsilon(u)+tz)[z,z]dt + \frac{1}{2}\|z\|^2 + \frac{1}{2}\int V_\varepsilon(x)|z|^2$$
$$= \Phi_\varepsilon(u+h_\varepsilon(u)) - \Phi_\varepsilon(u+v). \tag{31}$$

Remark 3.7. It is standard to see that, for the limit equations, the functionals $\mathscr{T}_{\lambda\mu}$ and $\mathscr{T}_{\lambda\mu\chi}$ have the similar properties mentioned above. Therefore, one can define correspondingly $h_{\lambda\mu}$, $h_{\lambda\mu\chi}$, $\mathscr{N}_{\lambda\mu}$, $\mathscr{N}_{\lambda\mu\chi}$, etc. These symbols will be used in the sequel without any more specification.

Lemma 3.8. *For any $u \in E^+ \setminus \{0\}$, there is an unique $t_\varepsilon = t_\varepsilon(u) > 0$ such that $t_\varepsilon u \in \mathscr{N}_\varepsilon$. Furthermore, there exists $T_u > 0$ independent of ε such that $t_\varepsilon \le T_u$.*

Proof. The proof is quite technical, for details we refer [1, 15]. We only give a sketch of the proof. Firstly, we observe that for any $u \in E \setminus \{0\}$ and $v \in E$,

$$\big(\Psi''_\varepsilon(u)[u,u] - \Psi'_\varepsilon(u)u\big) + 2\big(\Psi''_\varepsilon(u)[u,v] - \Psi'_\varepsilon(u)v\big) + \Psi''_\varepsilon(u)[v,v] > 0$$

Invoking the arguments in [1], if $z \in E^+ \setminus \{0\}$ with $I'_\varepsilon(z)z = 0$, we see by a delicate calculation that,

$$I''_\varepsilon(z)[z,z] < 0. \tag{32}$$

Now for a fixed $u \in E^+ \setminus \{0\}$, we set $f(t) = I_\varepsilon(tu)$. From Lemma 2.2, we see that $f(0) = 0$, $f(t) > 0$ for $t > 0$ sufficiently small, and $f(t) \to -\infty$ as $t \to \infty$. Thus there exists $t_\varepsilon = t_\varepsilon(u) > 0$ such that

$$I_\varepsilon(t_\varepsilon u) = \sup_{t\ge 0} I_\varepsilon(tu).$$

It is clear that $$\left.\frac{d\,I_\varepsilon(tu)}{dt}\right|_{t=t_\varepsilon} = I'_\varepsilon(t_\varepsilon u)u = \frac{1}{t_\varepsilon} I'_\varepsilon(t_\varepsilon u)t_\varepsilon u = 0,$$

and consequently by (32) $$I''_\varepsilon(t_\varepsilon u)[t_\varepsilon u, t_\varepsilon u] < 0.$$

Therefore, one sees that such $t_\varepsilon > 0$ is unique. Moreover, by noting that

$$\Phi_\varepsilon(t_\varepsilon u + h_\varepsilon(t_\varepsilon u)) = \max_{w\in E_u} \Phi_\varepsilon(w)\,,$$

together with (16) we have the existence of $T_u > 0$ proved. □

Proposition 3.9. *There holds:*

(1) $c_\varepsilon = \inf_{u\in\mathscr{N}} I_\varepsilon(u)$.

(2) *Let* $u \in \mathscr{N}$ *and set* $E_u = E^- \oplus \mathbb{R}^+ u$. *Then* $\max_{w\in E_u} \Phi_\varepsilon(w) = I_\varepsilon(u)$.

Proof. Denoting $d_\varepsilon = \inf_{u\in\mathscr{N}} I_\varepsilon(u)$, given $e \in E^+$, if $u = v + se \in E_e$ such that $\Phi_\varepsilon(u) = \max_{z\in E_e} \Phi_\varepsilon(z)$ then the restriction $\Phi_\varepsilon|_{E_e}$ of Φ_ε on E_e satisfies $(\Phi_\varepsilon|_{E_e})'(u) = 0$ which implies $v = h_\varepsilon(se)$ and $I'_\varepsilon(se)(se) = 0$, i.e. $se \in \mathscr{N}$. Thus $d_\varepsilon \le c_\varepsilon$. While, on the other hand, if $w \in \mathscr{N}$ then $(\Phi_\varepsilon|_{E_w})'(w + h_\varepsilon(w)) = 0$, hence, $c_\varepsilon \le \max_{u\in E_w} \Phi_\varepsilon(u) = I_\varepsilon(w)$. Thus $d_\varepsilon \ge c_\varepsilon$. It follow that $c_\varepsilon = d_\varepsilon$. Since (2) is a direct conclusion of Lemma 3.8, we complete the proof. □

As a consequence of Proposition 3.9, if $\mathscr{L}_\varepsilon$ (the set of least energy solutions) is not empty, then $\Phi_\varepsilon(u) = c_\varepsilon$ for all $u \in \mathscr{L}_\varepsilon$. And thus, from Lemma 2.6, we have $|u|_\infty$ is uniformly bounded for all $\varepsilon > 0$.

Going back to the limit equations discussed in subsection 3.1 and 3.2, here and in the sequel, when $\chi = 0$, by $\gamma_{\lambda\mu\chi}$ and $\mathscr{T}_{\lambda\mu\chi}$ we mean the least energy and the associate functional of the subcritical equation (26), i.e. $\gamma_{\lambda\mu}$ and $\mathscr{T}_{\lambda\mu}$. Following the previous proposition (with some modifications), we soon have

Lemma 3.10. *Let* $\lambda_1, \lambda_2 \in (-a, a)$, $\mu_1, \mu_2 > 0$ *and* $\chi_1, \chi_2 \ge 0$. *Suppose* $\lambda_1 \ge \lambda_2$, $\mu_1 \le \mu_2$ *and* $\chi_1 \le \chi_2$. *Assume additionally that the corresponding least energys are achieved. Then* $\gamma_{\lambda_1\mu_1\chi_1} \ge \gamma_{\lambda_2\mu_2\chi_2}$.
In addition, if $\max\{\lambda_1 - \lambda_2, \mu_2 - \mu_1, \chi_2 - \chi_1\} > 0$, *then* $\gamma_{\lambda_1\mu_1\chi_1} > \gamma_{\lambda_2\mu_2\chi_2}$.

Proof. Let u_1 be the ground state solution for $\mathscr{T}_{\lambda_1\mu_1\chi_1}$ and set $e = u_1^+$. Then

$$\gamma_{\lambda_1\mu_1\chi_1} = \mathscr{T}_{\lambda_1\mu_1\chi_1}(u_1) = \max_{w\in E_e} \mathscr{T}_{\lambda_1\mu_1\chi_1}(w).$$

Suppose $u_2 \in E_e$ is such that $\mathscr{T}_{\lambda_2\mu_2\chi_2}(u_2) = \max_{w\in E_e} \mathscr{T}_{\lambda_2\mu_2\chi_2}(w)$. We deduce that

$$\begin{aligned}
\gamma_{\lambda_1\mu_1\chi_1} &= \mathscr{T}_{\lambda_1\mu_1\chi_1}(u_1) \ge \mathscr{T}_{\lambda_1\mu_1\chi_1}(u_2) \\
&= \mathscr{T}_{\lambda_2\mu_2\chi_2}(u_2) + \frac{\lambda_1 - \lambda_2}{2}|u_2|_2^2 + (\mu_2 - \mu_1)\int G(|u_2|) + \frac{(\chi_2 - \chi_1)}{2^*}|u_2|_{2^*}^{2^*} \\
&\ge \gamma_{\lambda_2\mu_2\chi_2} + \frac{\lambda_1 - \lambda_2}{2}|u_2|_2^2 + (\mu_2 - \mu_1)\int G(|u_2|) + \frac{(\chi_2 - \chi_1)}{2^*}|u_2|_{2^*}^{2^*}.
\end{aligned}$$

This completes the proof. □

3.4. Auxiliary results

Assume that the sequence of functions $\hat{V}_\varepsilon$, $\hat{P}_\varepsilon$ and $\hat{W}_\varepsilon$ are in $C \cap L^\infty(\mathbb{R}^n, \mathbb{R})$, $0 < \varepsilon \le 1$, satisfy

($\star$) $\sup_{\varepsilon,x} |\hat{V}_\varepsilon(x)| < a$, $\inf_{\varepsilon,x} \hat{P}_\varepsilon(x) > 0$, $\inf_{\varepsilon,x} \hat{W}_\varepsilon(x) \ge 0$; $\hat{V}_\varepsilon(x) \to \lambda$, $\hat{P}_\varepsilon(x) \to \mu$ *and* $\hat{W}_\varepsilon(x) \to \chi$ *uniformly on bounded sets of* x *as* $\varepsilon \to 0$ *with* $\gamma_{\lambda\mu\chi}$ *achieved* (*e.g.* λ, μ, χ *satisfying* (30) *when* $\chi > 0$ *and* $\lambda \in (-a, 0]$).

Consider the equations

$$-i\alpha\cdot\nabla u + a\beta u + \hat V_\varepsilon(x)u = \hat P_\varepsilon(x)g(|u|)u + \hat W_\varepsilon(x)|u|^{2^*-2}u \tag{33}$$

and

$$-i\alpha\cdot\nabla u + a\beta u + \lambda u = \mu g(|u|)u + \chi|u|^{2^*-2}u \tag{34}$$

for $u\in H^1(\mathbb{R}^n,\mathbb{C}^N)$, $\lambda\in(-a,a)$, $\mu>0$, $\chi\geq 0$ and denote

$$\hat\Phi_\varepsilon(u) = \frac12\big(\|u^+\|^2-\|u^-\|^2\big)+\frac12\int \hat V_\varepsilon(x)|u|^2 - \hat\Phi_\varepsilon(u)$$

where

$$\hat\Phi_\varepsilon(u) = \int \hat P_\varepsilon(x)G(|u|) + \frac{1}{2^*}\int \hat W_\varepsilon(x)|u|^{2^*}.$$

As in the previous subsection, define the associate $\hat h_\varepsilon$, $\hat I_\varepsilon$, $\hat{\mathscr N}_\varepsilon$, $\hat c_\varepsilon$, etc.

Note that, by setting $V^0_\varepsilon(x)=\hat V_\varepsilon(x)-\lambda$, $P^0_\varepsilon(x)=\mu-\hat P_\varepsilon(x)$, $W^0_\varepsilon(x)=\chi-\hat W_\varepsilon(x)$, we have

$$\hat\Phi_\varepsilon(u) = \mathscr{T}_{\lambda\mu\chi}(u)+\frac12\int V^0_\varepsilon(x)|u|^2+\int P^0_\varepsilon(x)G(|u|)+\frac{1}{2^*}\int W^0_\varepsilon(x)|u|^{2^*}. \tag{35}$$

Lemma 3.11. $\limsup_{\varepsilon\to0}\hat c_\varepsilon\leq\gamma_{\lambda\mu\chi}$.

Proof. In virtue of the assumption $(\star)$, let $u=u^++u^-$ be a least energy solution of (34) and set $e=u^+$. It is clear that $e\in\mathscr N_{\lambda\mu\chi}$, $h_{\lambda\mu\chi}(e)=u^-$ and $I_{\lambda\mu\chi}(e)=\gamma_{\lambda\mu\chi}$. According to Lemma 3.8, there exists an unique $t_\varepsilon>0$ such that $t_\varepsilon e\in\hat{\mathscr N}_\varepsilon$.

Then we have

$$\hat c_\varepsilon\leq \hat I_\varepsilon(t_\varepsilon e)\,, \tag{36}$$

and $\{t_\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded. Without loss of generality, we assume that $t_\varepsilon\to t_0$ as $\varepsilon\to0$.

Claim. For any fixed $u\in E^+\setminus\{0\}$, $\hat h_\varepsilon(u)\to h_{\lambda\mu\chi}(u)$ as $\varepsilon\to0$.

Indeed, by (35), we deduce that

$$\begin{aligned}&\big(\hat\Phi_\varepsilon(z_\varepsilon)-\hat\Phi_\varepsilon(w)\big)+\big(\mathscr T_{\lambda\mu\chi}(w)-\mathscr T_{\lambda\mu\chi}(z_\varepsilon)\big) \qquad (37)\\ &=\frac12\int V^0_\varepsilon(x)\big(|z_\varepsilon|^2-|w|^2\big)+\int P^0_\varepsilon(x)\big(G(|z_\varepsilon|)-G(|w|)\big)+\frac{1}{2^*}\int W^0_\varepsilon(x)\big(|z_\varepsilon|^{2^*}-|w|^{2^*}\big)\end{aligned}$$

where $z_\varepsilon=u+\hat h_\varepsilon(u)$, $w=u+h_{\lambda\mu\chi}(u)$. Denoted by $v_\varepsilon=z_\varepsilon-w$, we find

$$\int V^0_\varepsilon(x)\big(|z|^2-|w|^2\big)=\int V^0_\varepsilon(x)|v_\varepsilon|^2+2\Re\int V^0_\varepsilon(x)w\cdot v_\varepsilon$$

and

$$\begin{aligned}&\int P^0_\varepsilon(x)\big(G(|z|)-G(|w|)\big)+\frac{1}{2^*}\int W^0_\varepsilon(x)\big(|z|^{2^*}-|w|^{2^*}\big)\\ &=\Re\int P^0_\varepsilon(x)g(|w|)w\cdot v_\varepsilon+\Re\int W^0_\varepsilon(x)|w|^{2^*-2}w\cdot v_\varepsilon\\ &+\int_0^1(1-s)\mathscr G''_{\mu\chi}(w+sv_\varepsilon)[v_\varepsilon,v_\varepsilon]\,ds-\int_0^1(1-s)\hat\Psi''_\varepsilon(w+sv_\varepsilon)[v_\varepsilon,v_\varepsilon]\,ds\,.\end{aligned}$$

Remark that, similar to (3.3), we infer

$$\int_0^1 (1-s)\hat{\Psi}''_\varepsilon(z_\varepsilon - sv_\varepsilon)[v_\varepsilon, v_\varepsilon]\,ds + \frac{1}{2}\|v_\varepsilon\|^2 + \frac{1}{2}\int \hat{V}_\varepsilon(x)|v_\varepsilon|^2 = \hat{\Phi}_\varepsilon(z_\varepsilon) - \hat{\Phi}_\varepsilon(w)$$

and

$$\int_0^1 (1-s)\mathscr{G}''_{\mu\chi}(w + sv_\varepsilon)[v_\varepsilon, v_\varepsilon]\,ds + \frac{1}{2}\|v_\varepsilon\|^2 + \frac{\lambda}{2}|v_\varepsilon|_2^2 = \mathscr{T}_{\lambda\mu\chi}(w) - \mathscr{T}_{\lambda\mu\chi}(z_\varepsilon)\,.$$

Then we get from (37) (jointly with (G1))

$$\begin{aligned}
&\|v_\varepsilon\|^2 + \lambda|v_\varepsilon|_2^2 \\
&\le \Re\int V_\varepsilon^0(x)w\cdot v_\varepsilon + \Re\int P_\varepsilon^0(x)g(|w|)w\cdot v_\varepsilon + \Re\int W_\varepsilon^0(x)|w|^{2^*-2}w\cdot v_\varepsilon \\
&\le \int |V_\varepsilon^0(x)|\cdot|w|\cdot|v_\varepsilon| + c_1\int |P_\varepsilon^0(x)|\cdot|w|\cdot|v_\varepsilon| \\
&\qquad + c_1\int |P_\varepsilon^0(x)|\cdot|w|^{p-1}\cdot|v_\varepsilon| + \int |W_\varepsilon^0(x)|\cdot|w|^{2^*-1}\cdot|v_\varepsilon| \\
&\le c_2\Big(\int \big(|V_\varepsilon^0(x)| + |P_\varepsilon^0(x)|\big)^2|w|^2\Big)^{1/2}|v_\varepsilon|_2 + c_1\Big(\int |P_\varepsilon^0(x)|^{p/(p-1)}|w|^p\Big)^{(p-1)/p}|v_\varepsilon|_p \\
&\qquad + \Big(\int |W_\varepsilon^0(x)|^{2^*/(2^*-1)}|w|^{2^*}\Big)^{(2^*-1)/2^*}|v_\varepsilon|_{2^*}\,.
\end{aligned} \tag{38}$$

Since w decays at infinity in the sense for $q = 2,\ p,\ 2^*$, we have

$$\limsup_{R\to\infty}\int_{|x|\ge R}|w|^q = 0\,.$$

We find (due to ($\star$))

$$\int \big(|V_\varepsilon^0(x)| + |P_\varepsilon^0(x)|\big)^2|w|^2 = o(1)\,,$$

$$\int |P_\varepsilon^0(x)|^{p/(p-1)}|w|^p = o(1)\,, \quad \text{and} \quad \int |W_\varepsilon^0(x)|^{2^*/(2^*-1)}|w|^{2^*} = o(1)\,,$$

as $\varepsilon \to 0$. Thus (38) leads to $\|v_\varepsilon\| = \|\hat{h}_\varepsilon(u) - h_{\lambda\mu\chi}(u)\| = o(1)$ as $\varepsilon \to 0$ provided $\lambda \in (-a, a)$. So the claim is proved.

From $t_\varepsilon \to t_0$ as $\varepsilon \to 0$ and the continuity of h_ε we obtain $\hat{h}_\varepsilon(t_\varepsilon e) \to h_{\lambda\mu\chi}(t_0 e)$.

Consequently,
$$\int V_\varepsilon^0(x)|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|^2 \to 0\,,$$

$$\int P_\varepsilon^0(x)G(|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|) \to 0 \quad \text{and} \quad \int W_\varepsilon^0(x)|t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)|^{2^*} \to 0\,,$$

as $\varepsilon \to 0$. This, jointly with (35), implies

$$\hat{\Phi}_\varepsilon(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) = \mathscr{T}_{\lambda\mu\chi}(t_\varepsilon e + \hat{h}_\varepsilon(t_\varepsilon e)) + o(1) = \mathscr{T}_{\lambda\mu\chi}(t_0 e + h_{\lambda\mu\chi}(t_0 e)) + o(1)\,,$$

that is, $$\hat{I}_\varepsilon(t_\varepsilon e) = \mathscr{T}_{\lambda\mu\chi}(t_0 e + h_{\lambda\mu\chi}(t_0 e)) + o(1)$$
as $\varepsilon \to 0$. Recalling that by Lemma 3.3

$$\mathscr{T}_{\lambda\mu\chi}(t_0 e + h_{\lambda\mu\chi}(t_0 e)) \le \max_{w\in E_e} \mathscr{T}_{\lambda\mu\chi}(w) = \mathscr{T}_{\lambda\mu\chi}(e + h_{\lambda\mu\chi}(e)) = \gamma_{\lambda\mu\chi}\,.$$

Therefore, by (36), we have, as desired,

$$\limsup_{\varepsilon\to 0} \hat{c}_\varepsilon \le \limsup_{\varepsilon\to 0} \hat{I}_\varepsilon(t_\varepsilon e) \le \gamma_{\lambda\mu\chi}. \qquad \square$$

In the sequel, for $\lambda \in [\kappa, \kappa_\infty]$ and $\mu \in [m_\infty, m]$, we set

$$V^\lambda(x) = \max\{\lambda, V(x)\} \quad \text{and} \quad P^\mu(x) = \min\{\mu, P(x)\}\,,$$

and let $V_\varepsilon^\lambda(x) = V^\lambda(\varepsilon x)$ and $P_\varepsilon^\mu(x) = P^\mu(\varepsilon x)$. Consider the functional

$$\Phi_\varepsilon^{\lambda\mu}(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) + \frac{1}{2}\int V_\varepsilon^\lambda(x)|u|^2 - \int P_\varepsilon^\mu(x)G(|u|) - \frac{1}{2^*}\int W_\varepsilon(x)|u|^{2^*}$$

with $h_\varepsilon^{\lambda\mu}$, $\mathscr{N}_\varepsilon^{\lambda\mu}$, $c_\varepsilon^{\lambda\mu}$, etc defined as before. By definition and Lemma 3.10, for $\chi \in [0, l]$, we know that $\gamma_{\kappa m l} \le \gamma_{V(0)P(0)\chi} \le \gamma_{V^\lambda(0)P^\mu(0)\chi}$.
This together with Lemma 3.11, if $V_\varepsilon^\lambda(x)$, $P_\varepsilon^\mu(x)$ and $W_\varepsilon(x)$ satisfy $(\star)$, then

$$\gamma_{\lambda\mu\chi} \le c_\varepsilon^{\lambda\mu} \quad \text{and} \quad \limsup_{\varepsilon\to 0} c_\varepsilon^{\lambda\mu} \le \gamma_{V^\lambda(0)P^\mu(0)\chi} \tag{39}$$

and particularly $$\lim_{\varepsilon\to 0} c_\varepsilon^{\lambda\mu} = \gamma_{V^\lambda(0)P^\mu(0)\chi} \quad \text{if } V(0) \le \lambda \text{ and } P(0) \ge \mu\,. \tag{40}$$

4. Proofs of the main results

We are now ready to present the proofs of the main results on the nonlinear (subcritical and critical) equation:

$$-i\alpha\cdot\nabla u + a\beta u + V_\varepsilon(x)u = P_\varepsilon(x)g(|u|)u + W_\varepsilon(x)|u|^{2^*-2}u\,. \tag{41}$$

Recall that, by the assumptions in Theorem 1.2 and Theorem 1.3, the condition $(\star)$ is always satisfied. Observe that, for any $x_0 \in \mathbb{R}^n$, setting $\tilde{V}_\varepsilon(x) = V(\varepsilon x + \varepsilon x_0)$, $\tilde{P}_\varepsilon(x) = P(\varepsilon x + \varepsilon x_0)$ and $\tilde{W}_\varepsilon(x) = W(\varepsilon x + \varepsilon x_0)$, if $\tilde{u}$ is a solution of

$$-i\alpha\cdot\nabla\tilde{u} + a\beta\tilde{u} + \tilde{V}_\varepsilon(x)\tilde{u} = \tilde{P}_\varepsilon(x)g(|\tilde{u}|)\tilde{u} + \tilde{W}_\varepsilon(x)|\tilde{u}|^{2^*-2}\tilde{u}\,,$$

then $u(x) = \tilde{u}(x - x_0)$ solves (41).

4.1. Proof of Theorem 1.2 in case (I)

Assume (H1), (H3) and (G1)–(G2) are satisfied. By virtue of above observation, without loss of generality, we can assume that $0 \in \mathscr{V}$ and $P(0) = m_v$. Then we find $(\star)$ is satisfied with $\lambda = \kappa$, $\mu = m_v$ and $\chi = 0$.

Lemma 4.1. *c_ε is attained for $\varepsilon > 0$ small enough.*

Proof. Given $\varepsilon > 0$, let $\{u_n\} \subset \mathscr{N}_\varepsilon$ be a minimizing sequence: $I_\varepsilon(u_n) \to c_\varepsilon$. By the Ekeland variational principle we can assume that $\{u_n\}$ is in fact a $(PS)_{c_\varepsilon}$-sequence for I_ε on E^+ (see [25, 32]). Then $w_n = u_n + h_\varepsilon(u_n)$ is a $(PS)_{c_\varepsilon}$-sequence for Φ_ε on E. It is clear that $\{w_n\}$ is bounded, hence is a $(C)_{c_\varepsilon}$-sequence. We can assume without loss of generality that $w_n \rightharpoonup w_\varepsilon = w_\varepsilon^+ + w_\varepsilon^- \in \mathscr{K}_\varepsilon$ in E. If $w_\varepsilon \neq 0$ then $\Phi_\varepsilon(w_\varepsilon) = c_\varepsilon$. So we are going to show that $w_\varepsilon \neq 0$ for all small $\varepsilon > 0$.

For this end, take $\nu \in (\kappa, \kappa_\infty)$, let us consider the functional $\Phi_\varepsilon^{\nu\mu}$. Following the proof of Lemma 3.11, one finds

$$\limsup_{\varepsilon \to 0} c_\varepsilon^{\nu\mu} \le \gamma_{\nu\mu}. \tag{42}$$

Assume by contradiction that there is a sequence $\varepsilon_j \to 0$ with $w_{\varepsilon_j} = 0$. Then $w_n = u_n + h_{\varepsilon_j}(u_n) \rightharpoonup 0$ in E, $u_n \to 0$ in L^q_{loc} for $q \in [1, 2^*)$, and $w_n(x) \to 0$ a.e. in $x \in \mathbb{R}^n$. Let $t_n > 0$ be such that $t_n u_n \in \mathscr{N}_{\varepsilon_j}^{\nu\mu}$. Since $u_n \in \mathscr{N}_\varepsilon$, it is not difficult to see that $\{t_n\}$ is bounded and one may assume $t_n \to t_0$ as $n \to \infty$. By (H3), the set $A_\varepsilon := \big\{x \in \mathbb{R}^n : V_\varepsilon(x) < \nu\big\}$ is bounded. Remark that $h_{\varepsilon_j}^{\nu\mu}(t_n u_n) \rightharpoonup 0$ in E and $h_{\varepsilon_j}^{\nu\mu}(t_n u_n) \to 0$ in L^q_{loc} for $q \in [1, 2^*)$ as $n \to \infty$ (see [1]). Moreover, by virtue of Proposition 3.9, $\Phi_{\varepsilon_j}(t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)) \le I_{\varepsilon_j}(u_n)$. We obtain

$$\begin{aligned}
c_{\varepsilon_j}^{\nu\mu} &\le I_{\varepsilon_j}^{\nu\mu}(t_n u_n) = \Phi_{\varepsilon_j}^{\nu\mu}(t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)) \\
&= \Phi_{\varepsilon_j}(t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)) + \frac{1}{2}\int \big(V_{\varepsilon_j}(x) - V_{\varepsilon_j}^{\nu}(x)\big)|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|^2 \\
&\qquad + \int \big(P_{\varepsilon_j}(x) - P_{\varepsilon_j}^{\mu}(x)\big) G\big(|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|\big) \\
&\le I_{\varepsilon_j}(u_n) + \frac{1}{2}\int_{A_{\varepsilon_j}} \big(V_{\varepsilon_j}(x) - V_{\varepsilon_j}^{\nu}(x)\big)|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|^2 \\
&\qquad + \int_{\left\{x:\, P_{\varepsilon_j}(x) \ge m_v\right\}} \big(P_{\varepsilon_j}(x) - m_v\big) G\big(|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|\big).
\end{aligned}$$

Since $\{t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)\}_{n\in\mathbb{N}}$ is bounded, and $m_v \ge m_\infty$, we have

$$\limsup_{R\to\infty} \int_{|x|\ge R} \big(P_{\varepsilon_j}(x) - m_v\big) G\big(|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|\big) \le 0\,.$$

Thus

$$\begin{aligned}
c_{\varepsilon_j}^{\nu\mu} &\le I_{\varepsilon_j}(u_n) + \frac{1}{2}\int_{A_{\varepsilon_j}} \big(V_{\varepsilon_j}(x) - V_{\varepsilon_j}^{\nu}(x)\big)|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|^2 \\
&\qquad + \int_{\left\{x:\, P_{\varepsilon_j}(x) \ge m_v\right\}} \big(P_{\varepsilon_j}(x) - m_v\big) G\big(|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|\big) = c_{\varepsilon_j} + o(1)
\end{aligned}$$

as $n \to \infty$. Hence $c_{\varepsilon_j}^{\nu\mu} \le c_{\varepsilon_j}$. By (39), $\gamma_{\nu\mu} \le c_{\varepsilon_j}^{\nu\mu}$, we see that $\gamma_{\nu\mu} \le c_{\varepsilon_j}$. Recall that $\mu = P(0)$ and in virtue of Lemma 3.11, letting $j \to \infty$ yields $\gamma_{\nu\mu} \le \gamma_{\kappa\mu}$, contradicting $\gamma_{\kappa\mu} < \gamma_{\nu\mu}$ (see Lemma 3.10). □

For the later use, letting $D = -i\alpha \cdot \nabla$, we rewrite (41) as

$$Du = -a\beta u - V_\varepsilon(x)u + f(\varepsilon x, u)u\,,$$

where $f(x,u)u$ is a abstract setting of the nonlinearities. Acting the operator D on the two sides of the above representation and noting that $D^2 = -\Delta$, we find

$$\Delta u = \big(a^2 - V_\varepsilon^2(x)\big)u - f^2(\varepsilon x, u)u + D\big(V_\varepsilon(x) - f(\varepsilon x, u)\big)u\,.$$

Letting
$$\operatorname{sgn} u = \begin{cases} \dfrac{u}{|u|} & \text{if } u \neq 0, \\ 0 & \text{if } u = 0, \end{cases}$$

by the Kato's inequality [10], there holds $\Delta|u| \geq \Re[\Delta u \cdot (\operatorname{sgn} u)]$. Observe that

$$\Re\Big[D\big(V_\varepsilon(x) - f(\varepsilon x, u)\big)u \cdot \frac{u}{|u|}\Big] = 0\,,$$

hence
$$\Delta|u| \geq \big(a^2 - V_\varepsilon^2(x)\big)|u| - f^2(\varepsilon x, u)|u|\,. \tag{43}$$

We remind that (43) together with the regularity results for u (see Lemma 2.6) implies that there is $M > 0$ (independent of ε) satisfying

$$\Delta|u| \geq -M|u|.$$

It then follows from the sub-solution estimate [21, 29] that

$$|u(x)| \leq C_0 \int_{B_1(x)} |u(y)|dy \tag{44}$$

with $C_0 > 0$ independent of x, ε and $u \in \mathscr{L}_\varepsilon$.

Lemma 4.2. *Suppose that ∇V and ∇P are bounded. There is a maximum point y_ε of $|u_\varepsilon|$ such that $\operatorname{dist}(\varepsilon y_\varepsilon, \mathscr{A}_v) \to 0$. Moreover, for any such y_ε, denoted by $\lim_{\varepsilon\to 0} \varepsilon y_\varepsilon = y_0$ and $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$, then v_ε converges in H^1 as $\varepsilon \to 0$ to a ground state solution of $-i\alpha \cdot \nabla u + a\beta u + V(y_0)u = P(y_0)g(|u|)u$.*

Proof. The proof will be carried out through several claims.

Claim 1. There exists $\{\bar{x}_\varepsilon\} \subset \mathbb{R}^n$ such that $\operatorname{dist}(\varepsilon\bar{x}_\varepsilon, \mathscr{A}_v) \to 0$. Moreover, denote by $\lim_{\varepsilon\to 0} \varepsilon\bar{x}_\varepsilon = y_0$ and $v_\varepsilon(x) := u_\varepsilon(x + \bar{x}_\varepsilon)$, then $v_\varepsilon \rightharpoonup v$ as $\varepsilon \to 0$ in E with v being a ground state solution of

$$-i\alpha \cdot \nabla u + a\beta u + V(y_0)u = P(y_0)g(|u|)u\,.$$

Indeed, let $\varepsilon_j \to 0$, $u_j \in \mathscr{L}_j$, where $\mathscr{L}_j = \mathscr{L}_{\varepsilon_j}$. Then $\{u_j\}$ is bounded. A standard concentration argument (see [24]) shows that there exist a sequence $\{\bar{x}_j\} \subset \mathbb{R}^n$ and constant $R > 0$, $\delta > 0$ such that

$$\liminf_{j\to\infty} \int_{B(\bar{x}_j, R)} |u_j|^2 \geq \delta\,.$$

Set $v_j(x) = u_j(x + \bar{x}_j)$ and define

$$\hat{V}_j(x) = V(\varepsilon_j(x + \bar{x}_j)) \text{ and } \hat{P}_j(x) = P(\varepsilon_j(x + \bar{x}_j)).$$

One easily checks that v_j solves

$$-i\alpha \cdot \nabla v_j + a\beta v_j + \hat{V}_j(x)v_j = \hat{P}_j(x) \cdot g(|v_j|)v_j\,, \tag{45}$$

with the energy

$$\begin{aligned} S(v_j) &:= \frac{1}{2}\big(\|v_j^+\|^2 - \|v_j^-\|^2\big) + \frac{1}{2}\int \hat{V}_j(x)|v_j|^2 - \int \hat{P}_j(x)G(|v_j|) \\ &= \Phi_j(u_j) = \int \hat{P}_j(x)\widehat{G}(|v_j|) = c_{\varepsilon_j}\,. \end{aligned}$$

Additionally, $v_j \rightharpoonup v$ in E and $v_j \to v$ in L^q_{loc} for $q \in [1, 2^*)$.

We now turn to prove that $\{\varepsilon_j \bar{x}_j\}$ is bounded. Arguing indirectly we assume that $\varepsilon_j|\bar{x}_j| \to \infty$ and get a contradiction.

Without loss of generality we may assume $V(\varepsilon_j\bar{x}_j) \to V_\infty$ and $P(\varepsilon_j\bar{x}_j) \to P_\infty$. By the boundness of ∇V and ∇P, one sees that $\hat{V}_j(x) \to V_\infty$ and $\hat{P}_j(x) \to P_\infty$ uniformly on bounded sets of x. Clearly, $\kappa < \kappa_\infty$ by (H3). Since for any $\psi \in C_c^\infty$

$$\begin{aligned} 0 &= \lim_{j\to\infty}\int \big(H_D\, v_j + \hat{V}_j(x)v_j - \hat{P}_j(x)g(|v_j|)v_j\big)\cdot\psi \\ &= \lim_{j\to\infty}\int \big(H_D\, v + V_\infty v - P_\infty g(|v|)v\big)\cdot\psi, \end{aligned}$$

we have that v solves $-i\alpha\cdot\nabla v + a\beta v + V_\infty v = P_\infty g(|v|)v$. Therefore,

$$S_\infty(v) := \frac{1}{2}\big(\|v^+\|^2 - \|v^-\|^2\big) + \frac{V_\infty}{2}|v|_2^2 - P_\infty\int G(|v|) \geq \gamma_{V_\infty P_\infty}.$$

It follows from $\kappa < \kappa_\infty \leq V_\infty$ and $\mu = m_v \geq m_\infty \geq P_\infty$, by Lemma 3.10, one has $\gamma_{\kappa\mu} < \gamma_{V_\infty P_\infty}$. Moreover, by the Fatou's lemma, one sees that

$$\lim_{j\to\infty}\int \hat{P}_j(x)\widehat{G}(|v_j|) \geq \int P_\infty \widehat{G}(|v|)\,.$$

Consequently we have the contradiction

$$\gamma_{\kappa\mu} < \gamma_{V_\infty P_\infty} \leq S_\infty(v) \leq \limsup_{j\to\infty} c_{\varepsilon_j} \leq \gamma_{\kappa\mu}\,.$$

Thus $\{\varepsilon_j\bar{x}_j\}$ is bounded. And hence, we can assume $\bar{y}_j = \varepsilon_j\bar{x}_j \to y_0$. At this moment, we see that v solves

$$-i\alpha\cdot\nabla v + a\beta v + V(y_0)v = P(y_0)g(|v|)v\,. \tag{46}$$

Meanwhile, we obtain

$$S_0(v) := \frac{1}{2}\big(\|v^+\|^2 - \|v^-\|^2\big) + \frac{V(y_0)}{2}|v|_2^2 - P(y_0)\int G(|v|) \geq \gamma_{V(y_0)P(y_0)}\,.$$

Further, use the fact that $V(\varepsilon_j x+\bar y_j)\to V(y_0)$, $P(\varepsilon_j x+\bar y_j)\to P(y_0)$, and

$$S_0(v)=S_0(v)-\frac{1}{2}S_0'(v)v=P(y_0)\int\widehat{G}(v),$$

by Fatou's Lemma and Lemma 3.11 (applied to S_j defined right below (45)), we get

$$\gamma_{V(y_0)P(y_0)}\le S_0(v)\le\liminf_{j\to\infty}c_{\varepsilon_j}\le\limsup_{j\to\infty}c_{\varepsilon_j}\le\gamma_{V(y_0)P(y_0)}\,. \tag{47}$$

Now, we are ready to show $\lim_{j\to\infty}\operatorname{dist}(\bar y_j,\mathscr{A}_v)=0$. In fact, it sufficient to check that $y_0\in\mathscr{A}_v$. Suppose that $y_0\notin\mathscr{A}_v$. It is easy to see that $\gamma_{P(y_0)W(y_0)}>\gamma_{\kappa\mu}$. Together with (47) and $\limsup_{j\to\infty}c_{\varepsilon_j}\le\gamma_{\kappa\mu}$ (see Lemma 3.11), we would have a contradiction.

Claim 2. $v_j\to v$ in E.

In order to prove Claim 2, recall that, by (47),

$$\lim_{j\to\infty}\int\hat P_j(x)\widehat{G}(|v_j|)=\int P(y_0)\widehat{G}(|v_j|)\,.$$

By the decay of v, using the Brezis-Lieb lemma, one obtains $|v_j-v|_\sigma\to0$, then $|v_j^\pm-v^\pm|_\sigma\to0$ by (13). Denote $z_j=v_j-v$. Remark that $\{z_j\}$ is bounded in E and $z_j\to0$ in L^σ, therefore $z_j\to0$ in L^q for all $q\in(2,2^*)$. The scale product of (45) with z_j^+ yields

$$\langle v_j^+,z_j^+\rangle=o(1).$$

Similarly, using the decay of v together with the fact that $z_j^\pm\to0$ in L^q_{loc} for $q\in[1,2^*)$, it follows from (46) that

$$\langle v^+,z_j^+\rangle=o(1).$$

Thus $\|z_j^+\|=o(1)$, and the same arguments show that

$$\|z_j^-\|=o(1),$$

we then get $v_j\to v$ in E.

Claim 3. $v_j(x)\to0$ as $|x|\to\infty$ uniformly in $j\in\mathbb{N}$.

Assume by contradiction that there exist $\delta>0$ and $x_\varepsilon\in\mathbb{R}^n$ with $|x_\varepsilon|\to\infty$ with

$$\delta\le|v_\varepsilon(x_\varepsilon)|\le C_0\int_{B_1(x_\varepsilon)}|v_\varepsilon(y)|dy\,.$$

Since $v_\varepsilon\to v$ in E, we obtain, as $\varepsilon\to0$,

$$\delta\le C_0\Big(\int_{B_1(x_\varepsilon)}|v_\varepsilon|^2\Big)^{1/2}\le C_0\Big(\int|v_\varepsilon-v|^2\Big)^{1/2}+C_0\Big(\int_{B_1(x_\varepsilon)}|v|^2\Big)^{1/2}\to0,$$

a contradiction.

By virtue of Claim 3 it is clear that one may assume the sequence $\{\bar{x}_j\}$ in Claim 1 to be the maximum points of $|u_j|$. Moreover, from the above argument we readily see that, any sequence of such points satisfies $\varepsilon_j \bar{x}_j$ converging to some point in $\mathscr{A}_v$ as $j \to \infty$.

In order to verify that $v_j \to v$ in H^1, observe that by (45) and (46), we find

$$H_D(v_j - v) = \hat{P}_j(x) g(|v_j|) v_j - P(y_0) g(|v|) v - \left(\hat{V}_j(x) v_j - V(y_0) v\right).$$

Using Claim 2 and the uniform estimate in Remark 2.7, it is easy to check that $|H_D(v_j - v)|_2 \to 0$ as $j \to \infty$. Therefore $v_j \to v$ in $H^1(\mathbb{R}^n, \mathbb{C}^N)$, completing the proof. □

Lemma 4.3. *There exist $C > 0$ such that for all $\varepsilon > 0$ small*

$$|u_\varepsilon(x)| \leq C \exp\Big(- \sqrt{\frac{a^2 - |V|_\infty}{2}} |x - x_\varepsilon| \Big)$$

Proof. By Claim 3 in of the proof Lemma 4.2, we may choose $\delta > 0$ and $R > 0$ such that $|v_\varepsilon(x)| \leq \delta$ and

$$f^2(\varepsilon x, v_\varepsilon(x)) \leq \frac{a^2 - |V|_\infty^2}{2}$$

for all $|x| \geq R$ and $\varepsilon > 0$ sufficiently small. This, together with (43), implies that

$$\Delta |v_\varepsilon| \geq \frac{a^2 - |V|_\infty^2}{2} |v_\varepsilon| \quad \text{for all } |x| \geq R \text{ and } \varepsilon > 0 \text{ small}.$$

And at this point, applying the maximum principle (see [27]), we easily have

$$|v_\varepsilon(x)| \leq C_1 \exp\Big(- \sqrt{\frac{a^2 - |V|_\infty}{2}} |x| \Big)$$

uniformly for ε small and $|x| \geq R$. By virtue of Lemma 2.6, we have

$$|v_\varepsilon(x)| \leq C \exp\Big(- \sqrt{\frac{a^2 - |V|_\infty}{2}} |x| \Big)$$

for $x \in \mathbb{R}^n$ and all ε small. The proof is hereby completed. □

Proof of Theorem 1.2 in case (I). Define

$$\varphi_\varepsilon(x) = u_\varepsilon(x/\varepsilon) \quad \text{and} \quad x_\varepsilon = \varepsilon y_\varepsilon.$$

Then φ_ε is a least energy solution of (4) for all ε small, x_ε is a maximum point of $|\varphi_\varepsilon|$, and the conclusions (i) and (ii) follow from Lemma 4.3 and Lemma 4.2. □

4.2. Proof of Theorem 1.2 in case (II)

Since the proof is similar to the case (I), we only give a sketch of the proof.

Assume (H1), (H4) and (G1)–(G2) are satisfied. Without loss of generality, we can assume that $0 \in \mathscr{P}$ and $V(0) = \kappa_p$. Then we find $(\star)$ is satisfied with $\lambda = \kappa_p$, $\mu = m$ and $\chi = 0$.

Lemma 4.4. *c_ε is attained for $\varepsilon > 0$ small enough.*

Proof. Let $\{u_n\} \subset \mathscr{N}_\varepsilon$ be a minimizing sequence: $I_\varepsilon(u_n) \to c_\varepsilon$ such that $w_n = u_n + h_\varepsilon(u_n)$ is a $(PS)_{c_\varepsilon}$-sequence for Φ_ε on E and $w_n \rightharpoonup w_\varepsilon$ in E as $n \to \infty$.

Take $\nu \in (m_\infty, m)$, let us consider the functional $\Phi_\varepsilon^{\lambda\nu}$. Following the proof of Lemma 3.11, one finds

$$\limsup_{\varepsilon\to 0} c_\varepsilon^{\lambda\nu} \le \gamma_{\lambda\nu} \tag{48}$$

Assume by contradiction that there is a sequence $\varepsilon_j \to 0$ with $w_{\varepsilon_j} = 0$. Then $w_n = u_n + h_{\varepsilon_j}(u_n) \rightharpoonup 0$ in E, $u_n \to 0$ in L^q_{loc} for $q \in [1, 2^*)$, and $w_n(x) \to 0$ a.e. in $x \in \mathbb{R}^n$. Let $t_n > 0$ be such that $t_n u_n \in \mathscr{N}_{\varepsilon_j}^{\lambda\nu}$. Since $u_n \in \mathscr{N}_\varepsilon$, it is not difficult to see that $\{t_n\}$ is bounded and one may assume $t_n \to t_0$ as $n \to \infty$. By (H4), the set $A_\varepsilon := \{x \in \mathbb{R}^n : P_\varepsilon(x) > \nu\}$ is bounded. Remark that $h_{\varepsilon_j}^{\lambda\nu}(t_n u_n) \rightharpoonup 0$ in E and $h_{\varepsilon_j}^{\lambda\nu}(t_n u_n) \to 0$ in L^q_{loc} for $q \in [1, 2^*)$ as $n \to \infty$. Moreover, by virtue of Proposition 3.9, $\Phi_{\varepsilon_j}(t_n u_n + h_{\varepsilon_j}^{\lambda\nu}(t_n u_n)) \le I_{\varepsilon_j}(u_n)$. We obtain

$$\begin{aligned}
c_{\varepsilon_j}^{\lambda\nu} &\le I_{\varepsilon_j}^{\lambda\nu}(t_n u_n) = \Phi_{\varepsilon_j}^{\lambda\nu}(t_n u_n + h_{\varepsilon_j}^{\lambda\nu}(t_n u_n)) \\
&= \Phi_{\varepsilon_j}(t_n u_n + h_{\varepsilon_j}^{\lambda\nu}(t_n u_n)) + \frac{1}{2}\int \big(V_{\varepsilon_j}(x) - V_{\varepsilon_j}^{\lambda}(x)\big)|t_n u_n + h_{\varepsilon_j}^{\lambda\nu}(t_n u_n)|^2 \\
&\qquad + \int \big(P_{\varepsilon_j}(x) - P_{\varepsilon_j}^{\nu}(x)\big)G\big(|t_n u_n + h_{\varepsilon_j}^{\lambda\nu}(t_n u_n)|\big) \\
&\le I_{\varepsilon_j}(u_n) + \frac{1}{2}\int_{\{x:\, V_{\varepsilon_j}(x)\le\kappa_p\}} \big(V_{\varepsilon_j}(x) - \kappa_p\big)|t_n u_n + h_{\varepsilon_j}^{\lambda\nu}(t_n u_n)|^2 \\
&\qquad + \int_{A_{\varepsilon_j}} \big(P_{\varepsilon_j}(x) - \nu\big)G\big(|t_n u_n + h_{\varepsilon_j}^{\lambda\nu}(t_n u_n)|\big).
\end{aligned}$$

We already have $\displaystyle\int_{\{x:\, V_{\varepsilon_j}(x)\le\kappa_p\}} \big(V_{\varepsilon_j}(x) - \kappa_p\big)|t_n u_n + h_{\varepsilon_j}^{\lambda\nu}(t_n u_n)|^2 \le 0$. Thus

$$\begin{aligned}
c_{\varepsilon_j}^{\lambda\nu} &\le I_{\varepsilon_j}(u_n) + \frac{1}{2}\int_{\{x:\, V_{\varepsilon_j}(x)\le\kappa_p\}} \big(V_{\varepsilon_j}(x) - \kappa_p\big)|t_n u_n + h_{\varepsilon_j}^{\lambda\nu}(t_n u_n)|^2 \\
&\qquad + \int_{A_{\varepsilon_j}} \big(P_{\varepsilon_j}(x) - \nu\big)G\big(|t_n u_n + h_{\varepsilon_j}^{\lambda\nu}(t_n u_n)|\big) = c_{\varepsilon_j} + o(1)
\end{aligned}$$

as $n \to \infty$. Hence $c_{\varepsilon_j}^{\lambda\nu} \le c_{\varepsilon_j}$. By (39), $\gamma_{\lambda\nu} \le c_{\varepsilon_j}^{\lambda\nu}$, we see that $\gamma_{\lambda\nu} \le c_{\varepsilon_j}$. Recall that $\lambda = V(0)$ and in virtue of Lemma 3.11, letting $j \to \infty$ yields $\gamma_{\lambda\nu} \le \gamma_{\lambda\mu}$, which stands in contradiction to $\gamma_{\lambda\mu} < \gamma_{\lambda\nu}$. □

Lemma 4.5. *Suppose that ∇V and ∇P are bounded. There is a maximum point y_ε of $|u_\varepsilon|$ such that $\mathrm{dist}(\varepsilon y_\varepsilon, \mathscr{A}_p) \to 0$. Moreover, for any such y_ε, denoted by $\lim_{\varepsilon\to 0} \varepsilon y_\varepsilon = y_0$ and $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$, then v_ε converges in H^1 as $\varepsilon \to 0$ to a ground state solution of $-i\alpha \cdot \nabla u + a\beta u + V(y_0)u = P(y_0)g(|u|)u$.*

Proof. The proof follows the same steps as in the proof of Lemma 4.2.

Claim 1. There exists $\{\bar{x}_\varepsilon\} \subset \mathbb{R}^n$ such that $\mathrm{dist}(\varepsilon\bar{x}_\varepsilon, \mathscr{A}_p) \to 0$ and denoted by $\lim_{\varepsilon\to 0} \varepsilon\bar{x}_\varepsilon = y_0$ and $v_\varepsilon(x) := u_\varepsilon(x + \bar{x}_\varepsilon)$, then $v_\varepsilon \rightharpoonup v$ as $\varepsilon \to 0$ in E with v being a ground state solution of $-i\alpha \cdot \nabla u + a\beta u + V(y_0)u = P(y_0)g(|u|)u$.

Indeed, let $\varepsilon_j \to 0$, $u_j \in \mathscr{L}_j$, where $\mathscr{L}_j = \mathscr{L}_{\varepsilon_j}$. Then $\{u_j\}$ is bounded. A standard concentration argument shows that there exist a sequence $\{\bar{x}_j\} \subset \mathbb{R}^n$ and constant $R > 0$, $\delta > 0$ such that

$$\liminf_{j\to\infty} \int_{B(\bar{x}_j,R)} |u_j|^2 \ge \delta\,.$$

Set $v_j(x) = u_j(x + \bar{x}_j)$. One easily checks that v_j solves

$$H_D\, v_j + \hat{V}_j(x)v_j = \hat{P}_j(x)\cdot g(|v_j|)v_j\,, \tag{49}$$

with the energy

$$\begin{aligned} S(v_j) &:= \frac{1}{2}\big(\|v_j^+\|^2 - \|v_j^-\|^2\big) + \frac{1}{2}\int \hat{V}_j(x)|v_j|^2 - \int \hat{P}_j(x)G(|v_j|) \\ &= \Phi_j(u_j) = \int \hat{P}_j(x)\widehat{G}(|v_j|) = c_{\varepsilon_j}\,. \end{aligned}$$

Additionally, $v_j \rightharpoonup v$ in E and $v_j \to v$ in L^q_{loc} for $q \in [1, 2^*)$.

We now turn to prove that $\{\varepsilon_j\bar{x}_j\}$ is bounded. Arguing indirectly we assume that $\varepsilon_j|\bar{x}_j| \to \infty$ and get a contradiction.

Without loss of generality assume $V(\varepsilon_j\bar{x}_j) \to V_\infty$ and $P(\varepsilon_j\bar{x}_j) \to P_\infty$. By the boundness of ∇V and ∇P, one sees that $\hat{V}_j(x) \to V_\infty$ and $\hat{P}_j(x) \to P_\infty$ uniformly on bounded sets of x. Clearly, $m > m_\infty$ by (H4). Since for any $\psi \in C_c^\infty$

$$\begin{aligned} 0 &= \lim_{j\to\infty} \int \big(H_D\, v_j + \hat{V}_j(x)v_j - \hat{P}_j(x)g(|v_j|)v_j\big)\cdot\psi \\ &= \lim_{j\to\infty} \int \big(H_D\, v + V_\infty v - P_\infty g(|v|)v\big)\cdot\psi, \end{aligned}$$

we have that v solves $-i\alpha\cdot\nabla v + a\beta v + V_\infty v = P_\infty g(|v|)v$. Therefore,

$$S_\infty(v) := \frac{1}{2}\big(\|v^+\|^2 - \|v^-\|^2\big) + \frac{V_\infty}{2}|v|_2^2 - P_\infty\int G(|v|) \ge \gamma_{V_\infty P_\infty}.$$

It follows from $\lambda = \kappa_p \le \kappa_\infty \le V_\infty$ and $\mu = m > m_\infty \ge P_\infty$, by Lemma 3.10, one has $\gamma_{\lambda\mu} < \gamma_{V_\infty P_\infty}$. Moreover, by the Fatou's lemma, one sees that

$$\lim_{j\to\infty} \int \hat{P}_j(x)\widehat{G}(|v_j|) \ge \int P_\infty\widehat{G}(|v|)\,.$$

Consequently we have the contradiction

$$\gamma_{\lambda\mu} < \gamma_{V_\infty P_\infty} \le S_\infty(v) \le \limsup_{j\to\infty} c_{\varepsilon_j} \le \gamma_{\lambda\mu}\,.$$

Thus $\{\varepsilon_j \bar{x}_j\}$ is bounded. And hence, we can assume $\bar{y}_j = \varepsilon_j \bar{x}_j \to y_0$. At this moment, we see that v solves

$$-i\alpha\cdot\nabla v + a\beta v + V(y_0)v = P(y_0)g(|v|)v\,. \tag{50}$$

Meanwhile, we obtain

$$S_0(v) := \frac{1}{2}\big(\|v^+\|^2 - \|v^-\|^2\big) + \frac{V(y_0)}{2}|v|_2^2 - P(y_0)\int G(|v|) \ge \gamma_{V(y_0)P(y_0)}\,.$$

Furthermore, we have

$$\gamma_{V(y_0)P(y_0)} \le S_0(v) \le \liminf_{j\to\infty} c_{\varepsilon_j} \le \limsup_{j\to\infty} c_{\varepsilon_j} \le \gamma_{V(y_0)P(y_0)}\,. \tag{51}$$

Now, we are ready to show $\lim_{j\to\infty}\operatorname{dist}(\bar{y}_j,\mathscr{A}_p)=0$. In fact, it sufficient to check that $y_0\in\mathscr{A}_p$. Suppose that $y_0\notin\mathscr{A}_v$. It is easy to see that $\gamma_{P(y_0)W(y_0)} > \gamma_{\lambda\mu}$. Together with (51) and $\limsup_{j\to\infty} c_{\varepsilon_j} \le \gamma_{\kappa\mu}$ (see Lemma 3.11), we would have a contradiction.

Claim 2. $v_j \to v$ in E.

Claim 3. $v_j(x)\to 0$ as $|x|\to\infty$ uniformly in $j\in\mathbb{N}$.

By virtue of Claim 3 it is clear that one may assume the sequence $\{\bar{x}_j\}$ in Claim 1 to be the maximum points of $|u_j|$. Moreover, from the above argument we readily see that, any sequence of such points satisfies $\varepsilon_j\bar{x}_j$ converging to some point in $\mathscr{A}_v$ as $j\to\infty$.

The proof is hereby completed. □

Repeat the arguments of Lemma 4.3, we will have

Lemma 4.6. *There exist $C>0$ such that for all $\varepsilon>0$ small*

$$|u_\varepsilon(x)| \le C\exp\Big(-\sqrt{\frac{a^2-|V|_\infty}{2}}|x-x_\varepsilon|\Big)\,.$$

Proof of Theorem 1.2 in case (II)**.** As proved in the case (I), define

$$\varphi_\varepsilon(x) = u_\varepsilon(x/\varepsilon) \quad\text{and}\quad x_\varepsilon = \varepsilon y_\varepsilon\,.$$

Then φ_ε is a least energy solution of (4) for all ε small, x_ε is a maximum point of $|\varphi_\varepsilon|$, and the conclusion (i) and (ii) follow from Lemma 4.6 and Lemma 4.5. □

4.3. Proof of Theorem 1.3

Since when (H6) occurs, the proof is much similar to the lines when dealing with the case (H5). We are going to prove the first part of Theorem 1.3. Assume $V(x)\le 0$, (H1), (H2), (H5), (H7) and (G1)–(G2) are satisfied.

Without loss of generality, we can assume that $0 \in \mathscr{W}$, $V(0) = \kappa_w$ and also $P(0) \geq m_\infty$. Then we find $(\star)$ is satisfied with $\lambda = \kappa_w$, $\mu = P(0)$ and $\chi = l$.

Lemma 4.7. *c_ε is attained for $\varepsilon > 0$ small enough.*

Proof. As before, let $\{u_n\} \subset \mathscr{N}_\varepsilon$ be a minimizing sequence: $I_\varepsilon(u_n) \to c_\varepsilon$ such that $w_n = u_n + h_\varepsilon(u_n)$ is a $(PS)_{c_\varepsilon}$-sequence for Φ_ε on E and $w_n \rightharpoonup w_\varepsilon$ in E as $n \to \infty$.

Take $\nu \in (\lambda, \kappa_\infty)$, let us consider the functional $\Phi_\varepsilon^{\nu\mu}$. Following the proof of Lemma 3.11, one finds

$$\limsup_{\varepsilon \to 0} c_\varepsilon^{\nu\mu} \leq \gamma_{\nu\mu\chi} \tag{52}$$

Assume by contradiction that there is a sequence $\varepsilon_j \to 0$ with $w_{\varepsilon_j} = 0$. Then $w_n = u_n + h_{\varepsilon_j}(u_n) \rightharpoonup 0$ in E, $u_n \to 0$ in L^q_{loc} for $q \in [1, 2^*)$, and $w_n(x) \to 0$ a.e. in $x \in \mathbb{R}^n$. Let $t_n > 0$ be such that $t_n u_n \in \mathscr{N}_{\varepsilon_j}^{\nu\mu}$. Since $u_n \in \mathscr{N}_\varepsilon$, it is not difficult to see that $\{t_n\}$ is bounded and one may assume $t_n \to t_0$ as $n \to \infty$. By (H5), the set $A_\varepsilon := \{x \in \mathbb{R}^n : V_\varepsilon(x) < \nu\}$ is bounded. Remark that $h_{\varepsilon_j}^{\nu\mu}(t_n u_n) \rightharpoonup 0$ in E and $h_{\varepsilon_j}^{\nu\mu}(t_n u_n) \to 0$ in L^q_{loc} for $q \in [1, 2^*)$ as $n \to \infty$. Moreover, by virtue of Proposition 3.9, $\Phi_{\varepsilon_j}(t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)) \leq I_{\varepsilon_j}(u_n)$. We obtain

$$\begin{aligned}
c_{\varepsilon_j}^{\nu\mu} &\leq I_{\varepsilon_j}^{\nu\mu}(t_n u_n) = \Phi_{\varepsilon_j}^{\nu\mu}(t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)) \\
&= \Phi_{\varepsilon_j}(t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)) + \frac{1}{2}\int \big(V_{\varepsilon_j}(x) - V_{\varepsilon_j}^{\nu}(x)\big)|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|^2 \\
&\quad + \int \big(P_{\varepsilon_j}(x) - P_{\varepsilon_j}^{\mu}(x)\big) G\big(|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|\big) \\
&\leq I_{\varepsilon_j}(u_n) + \frac{1}{2}\int_{A_{\varepsilon_j}} \big(V_{\varepsilon_j}(x) - \nu\big)|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|^2 \\
&\quad + \int_{\{x:\, P_{\varepsilon_j}(x) \geq \mu\}} \big(P_{\varepsilon_j}(x) - \mu\big) G\big(|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|\big)
\end{aligned}$$

Since $\{t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)\}_{n\in\mathbb{N}}$ is bounded, and $\mu = P(0) \geq m_\infty$, we have

$$\limsup_{R\to\infty} \int_{|x|\geq R} \big(P_{\varepsilon_j}(x) - \mu\big) G\big(|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|\big) \leq 0\,.$$

Thus

$$\begin{aligned}
c_{\varepsilon_j}^{\nu\mu} &\leq I_{\varepsilon_j}(u_n) + \frac{1}{2}\int_{A_{\varepsilon_j}} \big(V_{\varepsilon_j}(x) - \nu\big)|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|^2 \\
&\quad + \int_{\{x:\, P_{\varepsilon_j}(x) \geq \mu\}} \big(P_{\varepsilon_j}(x) - \mu\big) G\big(|t_n u_n + h_{\varepsilon_j}^{\nu\mu}(t_n u_n)|\big) = c_{\varepsilon_j} + o(1)
\end{aligned}$$

as $n \to \infty$. Hence $c_{\varepsilon_j}^{\nu\mu} \leq c_{\varepsilon_j}$. By (39), $\gamma_{\nu\mu\chi} \leq c_{\varepsilon_j}^{\nu\mu}$, we see that $\gamma_{\nu\mu\chi} \leq c_{\varepsilon_j}$. Recall that $\mu = P(0)$ and in virtue of Lemma 3.11, letting $j \to \infty$ yields

$$\gamma_{\nu\mu\chi} \leq \gamma_{\lambda\mu\chi},$$

which contradiction with $\gamma_{\lambda\mu\chi} < \gamma_{\nu\mu\chi}$ (see Lemma 3.10). $\square$

Lemma 4.8. *Suppose that ∇V, ∇P and ∇W are bounded and (H8) holds. There is a maximum point y_ε of $|u_\varepsilon|$ such that $\mathrm{dist}(\varepsilon y_\varepsilon, \mathscr{C}_v) \to 0$. Moreover, for any such y_ε, denoted by $\lim_{\varepsilon\to 0} \varepsilon y_\varepsilon = y_0$ and $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$, then v_ε converges in H^1 as $\varepsilon \to 0$ to a ground state solution of*

$$-i\alpha \cdot \nabla u + a\beta u + V(y_0)u = P(y_0)g(|u|)u + W(y_0)|u|^{2^*-2}u$$

Proof. The proof will be carried out through similar steps in Lemma 4.2.

Step 1. There exists $\{\bar{x}_\varepsilon\} \subset \mathbb{R}^n$ such that $\mathrm{dist}(\varepsilon\bar{x}_\varepsilon, \mathscr{C}_v) \to 0$ and denoted by $\lim_{\varepsilon\to 0} \varepsilon\bar{x}_\varepsilon = y_0$ and $v_\varepsilon(x) := u_\varepsilon(x + \bar{x}_\varepsilon)$, then $v_\varepsilon \rightharpoonup v$ as $\varepsilon \to 0$ in E with v being a ground state solution of

$$-i\alpha \cdot \nabla u + a\beta u + V(y_0)u = P(y_0)g(|u|)u + W(y_0)|u|^{2^*-2}u\,.$$

Indeed, let $\varepsilon_j \to 0$, $u_j \in \mathscr{L}_j$, where $\mathscr{L}_j = \mathscr{L}_{\varepsilon_j}$. Then $\{u_j\}$ is bounded. By (H8), a standard concentration argument (see [24]) shows that there exist a sequence $\{\bar{x}_j\} \subset \mathbb{R}^n$ and constant $R > 0$, $\delta > 0$ such that

$$\liminf_{j\to\infty} \int_{B(\bar{x}_j,R)} |u_j|^2 \ge \delta\,.$$

Set $v_j(x) = u_j(x + \bar{x}_j)$, and define

$$\hat{V}_j(x) = V(\varepsilon_j(x + \bar{x}_j)),\quad \hat{P}_j(x) = P(\varepsilon_j(x + \bar{x}_j)),\quad \hat{W}_j(x) = W(\varepsilon_j(x + \bar{x}_j)).$$

One easily checks that v_j solves

$$H_D\, v_j + \hat{V}_j(x)v_j = \hat{P}_j(x) \cdot g(|v_j|)v_j + \hat{W}_j(x)|v_j|^2 v_j\,, \tag{53}$$

with energy

$$\begin{aligned} S(v_j) &:= \frac{1}{2}\big(\|v_j^+\|^2 - \|v_j^-\|^2\big) + \frac{1}{2}\int \hat{V}_j(x)|v_j|^2 - \int \hat{P}_j(x)G(|v_j|) - \frac{1}{2^*}\int \hat{W}_j(x)|v_j|^{2^*} \\ &= \Phi_j(u_j) = \int \hat{P}_j(x)\widehat{G}(|v_j|) + \frac{1}{2^*}\int \hat{W}_j(x)|v_j|^{2^*} = c_{\varepsilon_j}\,. \end{aligned}$$

Additionally, $v_j \rightharpoonup v$ in E and $v_j \to v$ in L^q_{loc} for $q \in [1, 2^*)$.

We now turn to prove that $\{\varepsilon_j\bar{x}_j\}$ is bounded. Arguing indirectly we assume $\varepsilon_j|\bar{x}_j| \to \infty$ and get a contradiction.

Without loss of generality we assume $V(\varepsilon_j\bar{x}_j) \to V_\infty$, $P(\varepsilon_j\bar{x}_j) \to P_\infty$ and also $W(\varepsilon_j\bar{x}_j) \to W_\infty$. By the boundness of ∇V, ∇P and ∇W, one sees that $\hat{V}_j(x) \to V_\infty$, $\hat{P}_j(x) \to P_\infty$ and $\hat{W}_j(x) \to W_\infty$ uniformly on bounded sets of x. Clearly, $\kappa_w < \kappa_\infty$ by (H5). Since for any $\psi \in C_c^\infty$

$$\begin{aligned} 0 &= \lim_{j\to\infty} \int \big(H_D\, v_j + \hat{V}_j(x)v_j - \hat{P}_j(x)g(|v_j|)v_j - \hat{W}_j(x)|v_j|^{2^*-2}v_j\big) \cdot \psi \\ &= \lim_{j\to\infty} \int \big(H_D\, v + V_\infty v - P_\infty g(|v|)v - W_\infty|v|^{2^*-2}v\big) \cdot \psi, \end{aligned}$$

we have that v solves

$$-i\alpha\cdot\nabla v + a\beta v + V_\infty v = P_\infty g(|v|)v + W_\infty |v|^{2^*-2}v\,.$$

Therefore,

$$S_\infty(v) := \frac{1}{2}\big(\|v^+\|^2 - \|v^-\|^2\big) + \frac{V_\infty}{2}|v|_2^2 - P_\infty\int G(|v|) - \frac{W_\infty}{2^*}|v|_{2^*}^{2^*} \geq \gamma_{V_\infty P_\infty W_\infty}.$$

It follows from $\lambda = \kappa_w < \kappa_\infty \leq V_\infty$, $\mu = P(0) \geq m_\infty \geq P_\infty$ and $\chi = l \geq l_\infty \geq W_\infty$ by Lemma 3.10, one has $\gamma_{\lambda\mu\chi} < \gamma_{V_\infty P_\infty W_\infty}$. Moreover, by the Fatou's lemma, one sees that

$$\lim_{j\to\infty}\int \hat{P}_j(x)\widehat{G}(|v_j|) \geq \int P_\infty \widehat{G}(|v|) \quad\text{and}\quad \lim_{j\to\infty}\int \hat{W}_j(x)|v_j|^{2^*} \geq \int W_\infty |v|^{2^*}\,.$$

Consequently we have the contradiction

$$\gamma_{\lambda\mu\chi} < \gamma_{V_\infty P_\infty W_\infty} \leq S_\infty(v) \leq \limsup_{j\to\infty} c_{\varepsilon_j} \leq \gamma_{\lambda\mu\chi}\,.$$

Thus $\{\varepsilon_j\bar{x}_j\}$ is bounded. And hence, we can assume $\bar{y}_j = \varepsilon_j\bar{x}_j \to y_0$. At this moment, we see that v solves

$$-i\alpha\cdot\nabla v + a\beta v + V(y_0)v = P(y_0)g(|v|)v + W(y_0)|v|^{2^*-2}v\,. \tag{54}$$

Meanwhile, we obtain

$$\begin{aligned} S_0(v) &:= \frac{1}{2}\big(\|v^+\|^2 - \|v^-\|^2\big) + \frac{V(y_0)}{2}|v|_2^2 - P(y_0)\int G(|v|) - \frac{W(y_0)}{2^*}|v|_{2^*}^{2^*} \\ &\geq \gamma_{V(y_0)P(y_0)W(y_0)}\,. \end{aligned}$$

Further, use the fact that

$$V(\varepsilon_j x + \bar{y}_j) \to V(y_0),\;\; P(\varepsilon_j x + \bar{y}_j) \to P(y_0),\;\; W(\varepsilon_j x + \bar{y}_j) \to W(y_0),$$

and

$$S_0(v) = S_0(v) - \frac{1}{2}S_0'(v)v = P(y_0)\int \widehat{G}(v) + \frac{W(y_0)}{2^*}|v|_{2^*}^{2^*}\,,$$

by Fatou's lemma and Lemma 3.11 (apply to S_j defined right below (45)), we obtain

$$\gamma_{V(y_0)P(y_0)W(y_0)} \leq S_0(v) \leq \liminf_{j\to\infty} c_{\varepsilon_j} \leq \limsup_{j\to\infty} c_{\varepsilon_j} \leq \gamma_{V(y_0)P(y_0)W(y_0)}\,. \tag{55}$$

Now, we are ready to show $\lim_{j\to\infty}\mathrm{dist}(\bar{y}_j, \mathscr{C}_v) = 0$. In fact, it sufficient to check that $y_0 \in \mathscr{C}_v$. Suppose that $y_0 \notin \mathscr{C}_v$. It is easy to see that $\gamma_{V(y_0)P(y_0)W(y_0)} > \gamma_{\lambda\mu\chi}$. Together with (55) and $\limsup_{j\to\infty} c_{\varepsilon_j} \leq \gamma_{\lambda\mu\chi}$ (see Lemma 3.11), we would have a contradiction.

Step 2. $v_j \to v$ in E.

In order to prove Step 2, recall that, by (55),

$$\lim_{j\to\infty}\int \hat{W}_j(x)|v_j|^{2^*} = \int W(y_0)|v_j|^{2^*}\,.$$

By the decay of v, using the Brezis-Lieb lemma, one obtains $|v_j - v|_{2^*} \to 0$, then $|v_j^{\pm} - v^{\pm}|_{2^*} \to 0$ by (13). Denote $z_j = v_j - v$. Remark that $\{z_j\}$ is bounded in E and $z_j \to 0$ in L^{2^*}, therefore $z_j \to 0$ in L^q for all $q \in (2, 2^*)$. The scale product of (53) with z_j^+ yields

$$\langle v_j^+, z_j^+ \rangle = o(1).$$

Similarly, using the decay of v together with the fact that $z_j^{\pm} \to 0$ in L^q_{loc} for $q \in [1, 2^*)$, it follows from (54) that

$$\langle v^+, z_j^+ \rangle = o(1).$$

Thus $\|z_j^+\| = o(1)$, and the same arguments show $\|z_j^-\| = o(1)$, so that we get $v_j \to v$ in E.

Step 3. $v_j(x) \to 0$ as $|x| \to \infty$ uniformly in $j \in \mathbb{N}$.

By virtue of Step 3 it is clear that one may assume the sequence $\{\bar{x}_j\}$ in Claim 1 to be the maximum points of $|u_j|$. Moreover, from the above argument we readily see that, any sequence of such points satisfies $\varepsilon_j \bar{x}_j$ converging to some point in $\mathscr{C}_v$ as $j \to \infty$. □

Lemma 4.9. *There exists $C > 0$ such that for all $\varepsilon > 0$ small*

$$|u_\varepsilon(x)| \le C \exp\Big(-\sqrt{\frac{a^2 - |V|_\infty}{2}} |x - x_\varepsilon| \Big).$$

Proof of Theorem 1.3. Define

$$\varphi_\varepsilon(x) = u_\varepsilon(x/\varepsilon) \quad \text{and} \quad x_\varepsilon = \varepsilon y_\varepsilon .$$

Then φ_ε is a least energy solution of (5) for all ε small, x_ε is a maximum point of $|\varphi_\varepsilon|$, and the conclusions (i) and (ii) follow from Lemma 4.9 and Lemma 4.8. □

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