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On the Uniqueness of Solutions to One-Dimensional Constrained Hamilton-Jacobi Equations

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The goal of this paper is to study the uniqueness of solutions to a constrained Hamilton-Jacobi equation

$$\begin{cases} u_t = u_x^2 + R(x, I(t)) & \text{in } \mathbb{R} \times (0, \infty), \\ \max_{\mathbb{R}} u(\cdot, t) = 0 & \text{on } [0, \infty), \end{cases}$$

with an initial condition $u(x,0) = u_0(x)$ on $\mathbb{R}$. A reaction term $R(x, I(t))$ is given while $I(t)$ is an unknown constraint (Lagrange multiplier) that forces maximum of u to be always zero. In the paper, we prove uniqueness of a pair of unknowns (u, I) using dynamic programming principle for a particular class of non-separable reaction $R(x, I(t))$ when the space is one-dimensional.

Keywords: Hamilton-Jacobi equation with constraint, selection-mutation model.

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1. Introduction

It is known by Darwin's theory of evolution that biological individuals evolve under the competition between natural selection and mutation. The mathematical model describing this phenomenon has been studied in literatures such as [7, 9, 10, 11]. One can find an interesting feature so called Dirac concentration of density as the mutation rate in the model vanishes. To illustrate this, we consider the following evolution equation

$$\begin{cases} n_t^\varepsilon - \varepsilon \Delta n^\varepsilon = \frac{n^\varepsilon}{\varepsilon} R(x, I^\varepsilon(t)) & \text{in } \mathbb{R}^d \times (0, \infty), \\ n^\varepsilon(x, 0) = n_0^\varepsilon \in L^1(\mathbb{R}^d) & \text{on } \mathbb{R}^d, \\ I^\varepsilon(t) = \int_{\mathbb{R}^d} \psi(x) n^\varepsilon(t, x) dx, \end{cases}$$

where the spatial variable x denotes 'traits' in the environment. Furthermore, n^ε, $R(x, I^\varepsilon(t))$, ε and $\psi(x)$ describe density of the population, reproduction rate, mutation rate and consumption rate by a trait x. Here ψ is assumed to be a nonnegative compactly supported function. We then take the Hopf-Cole transformation

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$n^\varepsilon(x,t) = e^{u^\varepsilon(x,t)/\varepsilon}$. It was shown in many literatures (see [3, 4, 15]) that as mutation rate ε vanishes, u^ε converges locally uniformly to u which is a viscosity solution to

$$\begin{cases} u_t = |Du|^2 + R(x, I(t)) & \text{in } \mathbb{R}^d \times (0,\infty), \\ \max_{\mathbb{R}^d} u(\cdot,t) = 0 & \text{on } [0,\infty), \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^d. \end{cases} \tag{1}$$

The constraint of $\max_{\mathbb{R}^d} u(\cdot,t) = 0$ is obtained from the property that I^ε is non-negative and uniformly bounded. It was also shown that

$$n^\varepsilon(x,t)n(x,t) \rightharpoonup \overline{\rho}(x)(x(t) - \overline{x}(t)) \text{ weakly in the sense of measure as } \varepsilon \to 0, \tag{2}$$

where
$$u(\overline{x}(t),t) = \max_{\mathbb{R}^d} u(\cdot,t) = 0 \text{ and } \rho(t) = \frac{I(t)}{\psi(x)}$$

for the solution $n^\varepsilon(x,t)$ to (1) (see [8, 15]). Despite the existence of a solution to (1) has been understood quite well, the uniqueness of a solution is known less. The uniqueness was first proved in [15] for a separable reaction term by B. Perthame and G. Barles. Some non-separable cases were treated by S. Mirrahimi, J.-M. Roquejoffre in [14]. In their work, the uniqueness of a solution was shown when the reaction and initial condition $u_0(x)$ are strictly concave in x so that the regularity of a solution u at the maximizing point can be handled. However, the uniqueness for non-concave initial data and a non-separable reaction is still open. In this paper, the uniqueness property for a constrained Hamilton-Jacobi equation with a particular class of non-separable reaction term is obtained using dynamic programming principle when the space is one-dimensional.

1.1. Setting and main result

We need the following assumptions on

$$R(x,I) : \mathbb{R} \times [0,\infty) \to \mathbb{R} \text{ and } u_0(x) : \mathbb{R} \to \mathbb{R},$$

where the reaction term is defined as

$$R(x,I) = \begin{cases} b(x) - Q(I) & \text{for } x \geq 0, \\ r(x,I) & \text{for } x < 0. \end{cases}$$

Main Assumptions. We assume that

(A1) $R(x,I)$ is smooth and $r(\cdot,I) < 0$ on $(-\infty,0)$ for any positive I;

(A2) $\sup_{0 \leq I \leq I_M} \|R(\cdot,I)\|_{W^{2,\infty}(\mathbb{R})} < \infty$ and R is strictly decreasing in I;

(A3) $Q(0) = 0$;

(A4) $\sup_{\mathbb{R}} R(\cdot, I_M) = 0$ for some $I_M > 0$;

(A5) $\min_{\mathbb{R}} R(\cdot,0) = 0$;

(A6) $b(x)$ is strictly increasing on $[0,\infty)$ with $b(0) = 0$;

(A7) $b'(x)$ is Lipschitz continuous, hence, non-negative;

(A8) $u_0(x) \in C^2(\mathbb{R})$ with $\|u_0\|_{C^2(\mathbb{R})} < \infty$, $\max_{\mathbb{R}} u_0(\cdot) = u_0(0) = 0$ and $u_0(x) < 0$ elsewhere.

One simple reaction we can think of is

$$R(x,I) = \begin{cases} e^{-1/x^2} - I & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$$

It is straightforward to check that the reaction satisfies all the assumptions.

Now we are ready to state our main theorem. Under the assumptions above, we consider the following equation;

$$\begin{cases} u_t = u_x^2 + R(x, I(t)) & \text{in } \mathbb{R} \times (0,\infty), \\ \max_{\mathbb{R}} u(\cdot,t) = 0 & \text{on } [0,\infty), \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}. \end{cases} \tag{3}$$

Theorem 1.1. *If the assumptions* (A1)–(A8) *are satisfied, then there exists at most one pair* (u,I) *such that* $u(x,t) \in C(\mathbb{R} \times [0,\infty))$ *solves* (3) *in the viscosity sense and* $I(t) \in C([0,\infty))$ *is strictly increasing.*

2. Preliminary

Throughout the section, let us assume $(u,I) \in C(\mathbb{R} \times [0,\infty)) \times C([0,\infty))$ is a pair of solution to (3) in viscosity sense and $I(t)$ is strictly increasing. By Lipschitz estimate provided by the author in [12], one can assume further that u is Lipschitz continuous in $\mathbb{R} \times [0,T]$ for any positive T. Now we follow the dynamic programming principle argument presented in [14], which yields

$$u(x,t) = \sup_{\gamma(t)=x} \{F(\gamma) : \gamma \in C^1([0,t];\mathbb{R})\}, \tag{4}$$

where
$$F(\gamma) := u_0(\gamma(0)) + \int_0^t \left(-\frac{\dot{\gamma}^2}{4} + R(\gamma(s), I(s)) \right) ds.$$

Furthermore, one can actually show that there exists a path $\gamma(s) \in C^1([0,t];\mathbb{R})$ such that

$$u(x,t) = u_0(\gamma(0)) + \int_0^t \left(-\frac{\dot{\gamma}^2}{4} + R(\gamma(s), I(s)) \right) ds$$

with $\gamma(t) = x$ and it satisfies the Euler-Lagrange equation

$$\begin{cases} \ddot{\gamma}(s) + 2R_x(\gamma(s), I(s)) = 0, \\ \dot{\gamma}(0) + 2\dot{u}_0(\gamma(0)) = 0, \\ \gamma(t) = x. \end{cases} \tag{5}$$

For the details, see [14] and references therein.

There could be more than one path satisfying the Euler-Lagrange equation (5) derived above. However, it will turn out that the Euler-Lagrange equation reduces to a simpler equation and this property will result in the uniqueness of a pair solution (u,I) in our setting. We start with providing some generic properties of u, I and an optimizing path γ.

Proposition 2.1. *Assume that* $\max_{\mathbb{R}} u(\cdot,t) = u(x',t) = 0$. *Then* $R(x', I(t)) = 0$.

Proof. By viscosity subsolution test one can easily obtain $R(x', I(t)) \geq 0$. Now we assume that $R(x', I(t)) > 0$. Then there exists $t_0 > 0$ such that $R(x', I(s)) > 0$ for $s \in [t, t+t_0]$ by the continuity of I and R. Integrating (3) on both sides over $\{x'\} \times [t, t+t_0]$ yields

$$u(x', t+t_0) - u(x', t) \geq \int_t^{t+t_0} R(x', I(s))ds > 0.$$

Hence, we get $u(x', t+t_0) > 0$, which violates the maximum constraint. □

Definition 2.2. We define $x(t) \in \mathbb{R}$ to satisfy

$$R(x(t), I(t)) = 0 \quad \text{for } t > 0$$

where $I(t)$ is strictly increasing. Then, together with Proposition 2.1, we have

$$\max_{\mathbb{R}} u(\cdot, t) = u(x(t), t) = 0 \tag{6}$$

for a pair of solution (u, I).

Proposition 2.3. $I(0) = 0$ *and* $I(s) \leq I_M$ *on* $[0, \infty)$.

Proof. Let us first prove that $I(0) = 0$ when (u, I) is a pair of solution. By the structure of the reaction $R(x, I)$, it is clear that $x(t) > 0$ and it satisfies the equation $b(x(t)) - I(t) = 0$. Since b is strictly increasing and continuous, the existence of $x(0+)$ which is a right limit of x is clear. Now we assume first $I(0) > 0$. From the property (6), we deduce that

$$0 = \lim_{t \to 0+} u(x(t), t) = u(x(0+), 0).$$

However, $x(0+) = b^{-1}(I(0)) > 0$ which contradicts (A8). Therefore, $I(0) = 0$. The second part of the proposition, $I(s) \leq I_M$, is a straightforward consequence of Proposition 2.1 due to the assumption $\sup_{\mathbb{R}} R(\cdot, I_M) = 0$. □

We also need some regularity properties of $u(x, t)$, which play crucial roles in analyzing the trajectory $\gamma(s)$.

Definition 2.4. For a given real-valued function $w(x) : \mathbb{R}^d \to \mathbb{R}$, we define the *super-differential* and the *sub-differential* at x as

$$D^+ w(x) = \{p \in \mathbb{R}^d : \limsup_{y \to x} \frac{w(y) - w(x) - p \cdot (y - x)}{|y - x|} \leq 0\},$$

$$D^- w(x) = \{p \in \mathbb{R}^d : \liminf_{y \to x} \frac{w(y) - w(x) - p \cdot (y - x)}{|y - x|} \geq 0\}.$$

Lemma 2.5. *The solution* $u(x, t)$ *is semiconvex in* $x \in \mathbb{R}$ *for any fixed time* $t > 0$.

Proof. Let us define $v(x,t) = -u(x,t)$ and prove $v(x,t)$ is semiconcave in $\mathbb{R} \times [0,T]$. Clearly, v satisfies

$$\begin{cases} v_t + v_x^2 + R(x, I(t)) = 0 & \text{in } \mathbb{R} \times (0,T], \\ v(x,0) = -u(x,0) & \text{on } \mathbb{R} \end{cases}$$

in the viscosity sense. To prove semiconcavity of v, we first provide an a priori estimate for v^ε where v^ε is the unique solution to

$$\begin{cases} v_t^\varepsilon + (v_x^\varepsilon)^2 + R(x, I(t)) = \varepsilon v_{xx}^\varepsilon & \text{in } \mathbb{R} \times (0,T], \\ v^\varepsilon(x,0) = -u(x,0) := v_0(x) & \text{on } \mathbb{R}. \end{cases} \tag{7}$$

Using the argument in [13], we have the following Lipschitz estimate independent of ε:

$$\|v_t^\varepsilon\|_{L^\infty(\mathbb{R}\times[0,T])} + \|v_x^\varepsilon\|_{L^\infty(\mathbb{R}\times[0,T])} < C. \tag{8}$$

Now differentiating (7) twice with respect to x and substituting w for v_{xx}^ε yields

$$w_t + 2w^2 + 2v_x^\varepsilon w_x + R_{xx} = \varepsilon w_{xx}. \tag{9}$$

It is known that w is bounded but the bound might depend on ε. However, one can actually show that the bound is uniform in ε. To justify this, we first notice that w is a subsolution to the following parabolic equation

$$\begin{cases} \eta_t + 2v_x^\varepsilon \eta_x + R_{xx} = \varepsilon \eta_{xx} & \text{in } \mathbb{R} \times (0,T], \\ \eta(x,0) = v_0''(x) & \text{on } \mathbb{R}. \end{cases} \tag{10}$$

Moreover, we get $w \le \eta$ by the comparison principle where η is the unique bounded solution to (10). The uniform boundedness of v_x^ε and R_{xx} together with maximum principle for w in (9) yields that there exists $C > 0$ such that $w^2 < C$, hence, we get $w < C$ regardless of ε. On the other hand, due to the estimate (8), v^ε converges locally uniformly to v up to subsequence as ε goes to 0 by Arzela-Ascoli theorem. In addition to that owing to the uniqueness and stability of a viscosity solution([2, 13]), v^ε converges to v.

Now we are ready to provide semiconcavity of v. We have $w = v_{xx}^\varepsilon < C$ uniformly in x and ε , which implies that

$$v^\varepsilon(x,t) - K|x|^2$$

is concave in x for some positive K. Combining it with locally uniform convergence of v^ε, we get semiconcavity of v in x. Therefore, $u = -v$ is semiconvex in x for any fixed time $t > 0$. □

Lemma 2.6. *For each $t \in (0, \infty)$, $u(x,t)$ is differentiable at $(x(t), t)$ with respect to the space variable x and it satisfies*

$$0 = u_x(x(t), t) = -\frac{\dot{\gamma}_x(t)}{2}, \tag{11}$$

where $\gamma_x \in C^1([0,t]; \mathbb{R})$ is a maximizer to (4) with $x = x(t)$.

Proof. By Lemma 2.5, $v(x,t) := -u(x,t)$ is semiconcave in x. Hence, super-differential of v with respect to x at (x,t) is non-empty. On the other hand, $p = 0$ is a sub-differential of v at $(x(t),t)$ with respect to x since $\min_{\mathbb{R}} v(\cdot,t) = 0$. Therefore, u is differentiable with respect to x and $u_x(x(t),t) = 0$.

On the other hand, a classical result in [5] implies that

$$-\eta(t) \in D^+v(x,t),$$

where $\dot{\gamma}_x(s) = -2\eta(s)$ for $s \in (0,t)$ and $D^+v(x,t)$ denotes a set of super-differential of v with respect to the space at (x,t). For the sake of the reader's convenience, we provide the proof following the same lines in [5].

Let $-2\eta = \gamma_x$. Then, from the Euler-Lagrange equation, one obtains $\dot{\eta} = R_x$.

Next, we let $x_0 = \gamma(0)$ and observe that

$$u(x+h,t) \geq u_0(x_0+h) + \int_0^t \left(-\frac{\dot{\gamma}_x^2}{4} + R(\gamma_x(s)+h, I(s))\right) ds.$$

Then,

$$\begin{aligned}&\liminf_{h\to 0} \frac{u(x+h,t)-u(x,t)}{h} \\ &\geq \lim_{h\to 0} \frac{u_0(x_0+h)-u_0(x_0)}{h} + \lim_{h\to 0} \int_0^t \frac{1}{h} \cdot \left(R(\gamma_x(s)+h, I(s)) - R(\gamma_x(s), I(s))\right) ds,\end{aligned}$$

which implies that $$\liminf_{h\to 0} \frac{u(x+h,t)-u(x,t)}{h} \geq \eta$$

since $R_x = \dot{\eta}$. Therefore we have for each $t > 0$, $\eta(t) \in D^-u(x,t)$, or equivalently, $-\eta(t) \in D^+v(x,t)$, which finishes the proof. □

Now let us fix the time $t > 0$. We provide comparison between $\gamma(s)$ and $x(s)$ where $\gamma(s)$ is an optimal path with $\gamma(t) = x$ and $x(s)$ satisfies $R(x(s), I(s)) = 0$ for a strictly increasing function I given.

Proposition 2.7. *$\gamma(s) > x(s)$ for $s \in (0,t)$.*

Proof. Let us first take $t_0 > 0$ such that $\gamma(s) > 0$ for all $s \in (t_0,t)$. For this t_0 chosen, if $\gamma(s) < x(s)$ on (t_0,t), then $R(\gamma(s),I(s)) < R(x(s),I(s)) = 0$ on (t_0,t). Using the dynamic programming principle, we get

$$0 = u(x(t),t) = \int_{t_0}^t \left(-\frac{\dot{\gamma}^2}{4} + R(\gamma(s),I(s))\right) ds + u(\gamma(t_0),t_0) < 0,$$

which is a contradiction. Hence, there exists $t' \in (t_0,t)$ such that $\gamma(t') = x(t')$.

On the other hand, $\gamma(s)$ satisfies the Euler-Lagrange equation on (t',t), which is,

$$\ddot{\gamma}(s) + R_x(\gamma(s),I(s)) = \ddot{\gamma}(s) + b'(\gamma(s)) = 0. \tag{12}$$

Integrating the equation above from t' to t gives

$$0 = \dot{\gamma}(t') - \dot{\gamma}(t) = \int_{t'}^{t} b'(\gamma(s)) > 0$$

by Lemma 2.6. Therefore, $\gamma(s) > x(s)$ on (t_0, t). Hence, we may assume $\gamma(s) > 0$ on $(0, t)$ and we have $\gamma(s) > x(s)$ for all $s \in (0, t)$. $\square$

3. Proof of Theorem 1.1

We assume that we have two pairs of solutions (u_1, I_1) and (u_2, I_2) to (3). Let us fix the time T and consider the following two cases.

Case 1. $I_1(s)$ and $I_2(s)$ intersect only at the origin for $s \in [0, T]$.
Without loss of generality, let us assume $I_1(s) < I_2(s)$ for $s \in (0, T]$. Then u_1 is a viscosity supersolution to

$$\begin{cases} u_t = u_x^2 + R(x, I_2(t)) & \text{in } \mathbb{R} \times (0, T], \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}. \end{cases} \tag{13}$$

By the comparison principle, we have $u_1 \geq u_2$. In addition to that by the maximum constraint, we have $x_1(s) = x_2(s)$ for all s where $x_1(s)$ and $x_2(s)$ are defined as in Definition 2.2. However, we get a contradiction from

$$0 = R(x_1(s), I_1(s)) > R(x_2(s), I_2(s)) = 0.$$

Case 2. $I_1(s)$ and $I_2(s)$ intersect at more than one point including the terminal time T. Let $t_1, t_2 \in (0, T]$ be points such that $t_1 < t_2$ and

$$I_1(t_i) = I_2(t_i) \quad \text{for } i = 1, 2.$$

Hence, we have $x_1(t_1) = x_2(t_1) := \alpha$ and $x_1(t_2) = x_2(t_2) := \beta$. In addition, we may assume that

$$I_1(s) > I_2(s) \quad \text{for } s \in (t_1, t_2).$$

For t_i's above, we define $\gamma_1(s)$ and $\eta_1(s)$ as optimizing trajectories corresponding to I_1 whose terminal points are α and β respectively. Similarly, one can define $\gamma_2(s)$ and $\eta_2(s)$ to be optimizing trajectories corresponding to I_2 whose terminal points are α and β respectively. By Proposition 2.7 and Lemma 2.6, γ_i satisfies

$$\ddot{\gamma}_i + 2b'(\gamma_i) = 0, \quad \dot{\gamma}_i(t) = 0, \quad \text{and} \quad \gamma_i(t) = \alpha$$

for each $i = 1, 2$. Similarly, η_i satisfies

$$\ddot{\eta}_i + 2b'(\eta_i) = 0, \quad \dot{\eta}_i(t) = 0, \quad \text{and} \quad \eta_i(t) = \beta$$

for each $i = 1, 2$ as well. Therefore, we get $\gamma_1 = \gamma_2 := \gamma$ and $\eta_1 = \eta_2 = \eta$ from the Lipschitz regularity of b'.

Now, using this for the optimal control formula, one obtains

$$
\begin{aligned}
0 = u_1(\alpha, t_2) &= \int_0^{t_2} \left(-\frac{\dot\gamma(s)^2}{4} + b(\gamma(s)) - Q(I_1(s)) \right) ds + u_0(\eta(0), 0),\\
0 = u_2(\alpha, t_2) &= \int_0^{t_2} \left(-\frac{\dot\gamma(s)^2}{4} + b(\gamma(s)) - Q(I_2(s)) \right) ds + u_0(\eta(0), 0),\\
0 = u_1(\beta, t_1) &= \int_0^{t_1} \left(-\frac{\dot\eta(s)^2}{4} + b(\eta(s)) - Q(I_1(s)) \right) ds + u_0(\gamma(0), 0),\\
0 = u_2(\beta, t_1) &= \int_0^{t_1} \left(-\frac{\dot\eta(s)^2}{4} + b(\eta(s)) - Q(I_2(s)) \right) ds + u_0(\gamma(0), 0).
\end{aligned}
$$

From the first and the second equation, one obtains

$$0 = \int_0^{t_2} \{Q(I_1(s)) - Q(I_2(s))\} ds.$$

Similarly, from the third and the fourth equation, we have

$$0 = \int_0^{t_1} \{Q(I_1(s)) - Q(I_2(s))\} ds.$$

Combining these two results above we get

$$0 = \int_{t_1}^{t_2} \{Q(I_1(s)) - Q(I_2(s))\} ds,$$

which contradicts $I_1 > I_2$ on (t_1, t_2) since Q is strictly increasing in I. Hence, we reached the conclusion. □

4. Failure of the Euler-Lagrange equation when $I \in BV([0,T])$

A general constrained Hamilton-Jacobi equation for a superlinear Hamiltonian H can be written as

$$
\begin{cases}
u_t + H(I, u, Du) = 0 & \text{in } \mathbb{R} \times (0, T],\\
\min_{\mathbb{R}} u(\cdot, t) = 0 & \text{on } [0, T],\\
u(x, 0) = u_0(x) & \text{on } \mathbb{R},
\end{cases}
$$

where $I \in BV([0,T])$ is an unknown. The variational formulation associated with the equation above can be formulated as

$$u(x,t) = \inf_{\gamma(t)=x} \{F(\gamma) : \gamma \in AC([0,t]; \mathbb{R})\},$$

where
$$F(\gamma) := u_0(\gamma(0)) + \int_0^t L(I(s), \gamma, \dot\gamma(s)) ds.$$

Here, the Lagrangian L is defined by the Legendre transform, namely,

$$L(I, x, v) = \sup_{p \in \mathbb{R}} \{p \cdot v - H(I, x, v)\}.$$

In this section, we argue that the continuity assumption on I is needed to have the Euler-Lagrange equation for an optimizing path, which explains why I is assumed to be continuous for the analysis above.

Proposition 4.1. *There exist $T > 0$, $I \in BV([0,T])$ and a Lagrangian $L(I,x,v)$ such that an optimizing path does not satisfy the Euler-Lagrange equation in distributional sense.*

Proof. Let $T_k = \sum_{n=k}^{\infty} \frac{1}{1.5^k}$ for $k \geq 0$ and set $T = T_0$. We define $I(s)$ to be

$$I(s) = \begin{cases} 1 & \text{when } s = 0, \\ \frac{1}{2^k} & \text{when } s \in (T_{k+1}, T_k], \end{cases}$$

for $k \geq 0$. One can observe that $I(s)$ is increasing on $(0,T]$ so that $I \in BV([0,T])$. We consider the following variational problem

$$\inf_{\gamma(T)=x_0, \gamma \in AC[0,T]} \{ \int_0^T L(I(s), \gamma(s), \dot{\gamma}(s)) ds + u_0(\gamma(0)) \},$$

for some small $x_0 > 0$ where

$$L(I(s), x, v) = I(s)(v^2 + 1) \quad \text{and} \quad u_0(x) = 1000 \cdot e^{-|x|^2}.$$

We claim that the optimization problem above does not have a minimizer γ satisfying the Euler-Lagrange equation in the distributional sense. Clearly, $L_{vv} > 0$ and $L_I > 0$. Now we assume the contrary so that there exists a minimizer $\gamma \in AC([0,T])$ satisfying the Euler-Lagrange equation,

$$\frac{d}{ds} D_v L = D_x L = 0$$

in the distributional sense, which implies $\frac{d}{ds}(I(s)\dot{\gamma}(s)) = 0$. Therefore, we have $\dot{\gamma}(s) = C_k$ on $(T_{k+1}, T_k]$ where C_k is a constant depending on k, hence, γ is piecewise linear.

On the other hand, at each T_k for $k > 0$ where we have jumps for γ, we have

$$I(T_k^-)\dot{\gamma}(T_k^-) = I(T_k^+)\dot{\gamma}(T_k^+)$$

implying $\gamma(s) = C_1 2^k$ for $k \in (T_{k+1}, T_k]$. If $C_1 = 0$, then $\gamma \equiv x_0$. One can see from the structure of the initial condition that $\gamma(s) \equiv x_0$ cannot be a minimizer. Hence, we may assume $C_1 = 1$ and by observing that

$$\int_0^{t_0} |\dot{\gamma}| ds > \sum_{k=k_0}^{\infty} \frac{2^k}{1.5^k}$$

for $t_0, k_0 > 0$, we have $\gamma \notin AC([0,T])$. □

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References

[1] S. Armstrong, H. V. Tran: *Viscosity solutions of general viscous Hamilton-Jacobi equations*, Math. Annalen 361 (2014) 647–687.

[2] G. Barles: *Discontinuous viscosity solutions of first-order Hamilton-Jacobi equations: a guided visit*, Nonlinear Analysis: Theory, Methods Appl. 20/9 (1999) 1123–1134.

[3] G. Barles, S. Mirrahimi, B. Perthame: *Concentration in Lotka-Volterra parabolic or integral equations: a general convergence result*, Methods Appl. Analysis 16/3 (2009) 321–340.

[4] G. Barles, B. Perthame: *Concentrations and constrained Hamilton-Jacobi equations arising in adaptive dynamics*, Contemporary Math. 439 (2007) 57–68.

[5] P. Cannarsa, C. Sinestrari: *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control*, Progress in Nonlinear Differential Equations and their Applications 58, Birkhäuser, Boston (2004).

[6] M. G. Crandall, L. C. Evans, P.-L. Lions: *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. 282/2 (1984) 487–502.

[7] O. Diekmann: *Beginner's guide to adaptive dynamics*, Banach Center Publications 63 (2004) 47–86.

[8] O. Diekmann, P.-E. Jabin, S. Mischler, B. Perthame: *The dynamics of adaptation: an illuminating example and a Hamilton-Jacobi approach*, Theoretical Population Biology 67/4 (2005) 257–271.

[9] S. A. H. Geritz, E. Kisdi, M. Gyllenberg, F. J. Jacobs, J. A. J. Metz: *Link between population dynamics and dynamics of Darwinian evolution*, Phys. Review Letters 95/7 (2005), art. no. 078105.

[10] S. A. H. Geritz, E. Kisdi, G. Mészena, J. A. J. Metz: *Dynamics of adaptation and evolutionary branching*, Phys. Review Letters 78 (1997) 2024–2027.

[11] S. A. H. Geritz, E. Kisdi, G. Mészena, J. A. J. Metz: *Evolutionary singular strategies and the adaptive growth and branching of the evolutionary tree*, Evolutionary Ecology 12 (1998) 35–57.

[12] Y. Kim: *Well-posedness for constrained Hamilton-Jacobi equations*, Acta Appl. Math. 167 (2020) 39–57.

[13] N. Q. Le, H. Mitake, H. V. Tran: *Dynamical and Geometric Aspects of Hamilton-Jacobi and Linearized Monge-Ampère Equations*, Lecture Notes in Mathematics 2183, Springer, Cham (2017).

[14] S. Mirrahimi, J.-M. Roquejoffre: *A class of Hamilton-Jacobi equations with constraint: Uniqueness and constructive approach*, J. Diff. Equations 260/5 (2016) 4717–4738.

[15] B. Perthame, G. Barles: *Dirac concentrations in Lotka-Volterra parabolic PDEs*, Indiana Univ. Math. J. 57/7 (2008) 3275–3301.