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An Optimal Control Problem Governed by the Heat Equation with Nonconvex Constraints Applied to the Selective Laser Melting Process

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This paper deals with a PDE-constrained optimal control problem applied to an Additive Manufacturing process, namely a selective laser melting. Here, we want to control the temperature gradient inside the domain during a fixed time of heating, by acting on the trajectory of the dynamic Gaussian heating source. The nonconvex set of admissible controls reflects the fact that the control must fill the part of the boundary irradiated by the laser.

Keywords: Laser trajectory optimization, optimal control, non-convex constraints, first-order necessary optimality condition.

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1. Introduction

Selective laser melting (SLM) is an Additive Layer Manufacturing process used to produce three-dimensional objects from metal powders by melting the material in a layer-by-layer manner. First, a thin layer of powder is spread onto a build platform and simultaneously levelled or compacted to the required thickness. The laser beam scans the powder surface at an appropriate speed, heating the surface according to the desired scanning pattern and part profile. The mechanisms of SLM have been discussed in [25, 26, 27].

Thermal distortion of the fabricated part is one serious problem in SLM process [11], because of its fast laser scan rates and material transformations (solidification and liquifation) in a very short time frame. The temperature field was found to be inhomogeneous by many previous researchers [31, 33]. Meanwhile, the temperature evolution history in SLM process has significant effects on the quality of the final parts, such as density, dimensions, mechanical properties, microstructure, etc. For metals, rapid repeated heating and cooling cycles of the powder during SLM build process is responsible for large temperature gradients resulting in hight residual stresses and deformations, and may even lead to crack formation in the fabricated part.

The simulation results from [8, 29] show that thermal gradients are very different from one type of trajectory to another one, a similar observation can be made for thermal stresses [15, 22]. Also in [7, 20], fractal scanning strategies based upon ma-

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thematical fill curves, namely the Hilbert and Peano-Gosper curves were explored to stear the influence of laser trajectory on thermal stresses and temperature distribution. Finally in [1] the issue of thermal stresses in Additive Manufacturing is treated from a shape and topology optimization point of view.

However, a model incorporating trajectory optimization to minimize thermal gradients in SLM has not yet been studied in the literature. Our aim is to propose a mathematical model to find an optimal trajectory minimizing thermal gradients within the produced part using optimal control theory of PDE's. Thus, we introduce the appropriate cost functional and the set of admissible controls taking into account the constraints on laser trajectory. To the best of our knowledge, we know only one recent paper dealing with this topic using shape optimization tools [5]. The main difference between our paper and [5] is that our approach is based on geometry: the geometrical constraint is imposed on the trajectories to cover the built structure, the optimal arbitrary and parametrizable trajectories being chosen to minimize the gradient temperature, while their approach is based on physics: the minimization functional is chosen in order that the temperature must attain a melting value and the paths are broken lines. As mentioned in [5] such an optimization could seem too costly to be used straightly in the industry but it may give some intuitions about an optimal path satisfying the industrial constraints, validating some patterns or proposing new ones. Furthermore, using a parametrizable control trajectories allow us to use different types of initial curves (for instance the ones employed by industry), hence the chosen algorithm will furnish different "optimal" curves that can be compared in order to chose the best one.

We consider the optimal control of a linear heat equation that models the distribution of temperature within one layer Ω heated on its upper surface by a Gaussian laser beam [16, 12, 29] with a linear heat transfer with the bottom layer:

$$\begin{cases} \rho\, c\, \partial_t y - \kappa\, \Delta y = 0 & \text{in} \quad Q = \Omega \times]0,T[\,, \\ -\kappa \frac{\partial y}{\partial \nu} = h\, y - g_\gamma & \text{on} \quad \Sigma_1 = \Gamma_1 \times]0,T[\,, \\ -\kappa\, \frac{\partial y}{\partial \nu} = h\, y & \text{on} \quad \Sigma_2 = \Gamma_2 \times]0,T[\,, \\ -\kappa \frac{\partial y}{\partial \nu} = h\,(y - y_B) & \text{on} \quad \Sigma_3 = \Gamma_3 \times]0,T[\,, \\ y(x,0) = y_0(x) & \text{for} \quad x \in \Omega. \end{cases} \tag{1}$$

In these equations $y(x,t)$ denotes the temperature at point $x \in \Omega$ and time $t \in]0,T[$, $y_0 \in L^2(\Omega)$ is the initial temperature, while $y_B \in L^2(\Gamma_3)$ corresponds to the temperature at the top of the previous layer. We may always suppose that the ambient temperature y_a is zero by taking as new dependent variable $y - y_a$ so that we are led to the system (1). Here, $\Omega \subset \mathbb{R}^3$ is a bounded and simply connected domain with a connected Lipschitz boundary Γ, that is supposed to be split up

$$\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3},$$

where Γ_i, $i = 1,2,3$, are disjoint open disjoint subsets of Γ. In the SLM process, Γ_1 corresponds to the upper surface of the added layer and is supposed to be included in a plane, that without loss of generality we assume to be $\mathbb{R}^2$. By $\nu(x)$, we denote the outward normal direction at the point $x \in \Gamma$. The positive constants ρ, c, κ and h are respectively mass density, heat capacity, thermal conductivity and the convective heat transfer coefficient.

The heat source g_γ is of the form

$$g_\gamma(x,t) = \alpha \frac{2P}{\pi R^2} \exp\left(-2\frac{| x-\gamma(t) |^2}{R^2}\right), \ \forall (x,t) \in \Sigma_1, \tag{2}$$

where the absorbance of the material α, the laser power P, and the radius of the laser spot R are positive constants. The control $\gamma\colon t \in [0,T] \to \Gamma_1$ represents the displacement of the laser beam center on Γ_1 with respect to time. Note that g_γ depends nonlinearly on γ.

As suggested before, our main goal is to find a trajectory γ (the control) in such a way as to minimize temperature gradients inside the layer Ω with the constraints that the laser beam runs over the whole surface Γ_1 and does not leave it. From a mathematical point of view, these constraints lead to a non convex admissible set of controls, which is the main difficulty to overcome.

Note that in (1) the boundary condition on Σ_3 can be replaced by a non-homogeneous Dirichlet boundary condition $y = y_B$ on Σ_3.

We have simply chosen the Robin type boundary condition because it is more realistic in the practical point of view. A two-dimensional version of (1) with volumic Gaussian laser source also exists, see [5, 21, 24, 27, 30], and can be treated with similar arguments used here.

The outline of this paper is as follows. In section 2, we introduce the optimal control problem. We explain the link between our non convex set of admissible controls and laser trajectory in SLM process. Then we prove existence of a solution to the optimal control problem. In section 3, we prove the differentiability of the control-to-state mapping, result from which we infer the differentiability of the reduced cost functional. In section 4, the adjoint state is introduced which allow us to compute the Fréchet derivative of this reduced cost functional. Therefore, we derive a first order necessary condition for a control to be optimal in the form of a variational inequality. The main difficulty is in the non convex constraints required on the control γ.

2. The optimal control problem

For further purposes, we introduce the following (Hilbert) space:

$$W(0,T) := \{u \in L^2(0,T;H^1(\Omega)) \text{ such that } \frac{du}{dt} \in L^2(0,T;(H^1(\Omega))^*)\}.$$

Recall that by Theorem 3.12 in [28], if y_0 belongs to $L^2(\Omega)$, $y_B \in L^2(\Sigma_3)$ and $\gamma \in H^1(0,T;\mathbb{R}^2)$ then the initial-boundary value problem (1) has a unique solution y in $W(0,T) \cap L^\infty(0,T;L^2(\Omega))$ in the sense that, for all $\varphi \in L^2(0,T;H^1(\Omega))$,

$$\begin{aligned}
&\rho\, c \int_0^T \langle \frac{dy}{dt}(\cdot,t)\,,\, \varphi(\cdot,t)\rangle_{(H^1(\Omega))^*,H^1(\Omega)}\, dt + \kappa \int_0^T \int_\Omega \nabla y(x,t)\cdot\nabla\varphi(x,t)\, dx\, dt \\
&+ h\int_0^T \int_\Gamma y(x,t)\,\varphi(x,t)\, dS(x)\, dt - \int_0^T \int_{\Gamma_3} h y_B(x,t)\,\varphi(x,t)\, dS(x)\, dt \\
&- \int_0^T \int_{\Gamma_1} g_\gamma(x,t)\varphi(x,t)\, dS(x)\, dt = 0.
\end{aligned} \tag{3}$$

Our goal is to find a trajectory γ (a control) in such a way as to minimize the temperature gradient inside the layer Ω. Also we want to allow to choose the control in such a way that the corresponding temperature distribution y in Q (the state) is the best possible approximation to a given temperature distribution $y_Q \in L^2(Q)$. To meet all requirements, we define the following cost functional

$$J(y,\gamma) := \frac{1}{2}\int_0^T \int_\Omega \mid \nabla y(x,t) \mid^2 dx\, dt + \frac{\lambda_Q}{2}\int_0^T \int_\Omega \mid y(x,t) - y_Q(x,t) \mid^2 dx\, dt$$
$$+ \frac{\lambda_\gamma}{2} \parallel \gamma \parallel^2_{H^1(0,T;\mathbb{R}^2)}, \tag{4}$$

where $\lambda_Q \geq 0$ and $\lambda_\gamma > 0$ are constants, while $y_Q \in L^2(Q)$ is a given function. Note that λ_γ is a regularization parameter and that if $\lambda_Q = 0$, the only goal is to minimize the temperature gradient. The optimal control problem is

$$\text{(OCP)} \qquad \min_{\gamma \in U_{ad}} J(y(\gamma), \gamma),$$

where $y(\gamma)$ denotes the weak solution of problem (1) associated with the control γ, and the set of admissible controls $U_{ad} \subset H^1(0,T;\mathbb{R}^2)$ is defined as follows. For $\epsilon \geq R$, and a fixed positive constant c, U_{ad} is defined by

$$U_{ad} := \left\{ \gamma \in H^1(0,T;\mathbb{R}^2)\,; \begin{array}{l} R(\gamma) \subset \Gamma_{1,-\epsilon},\ R_\epsilon(\gamma) = \Gamma_1 \ \text{ and} \\ |\gamma'(t)| \leq c \ \text{ a.e. } t \in [0,T]\}, \end{array} \right\} \tag{5}$$

where $$R(\gamma) := \gamma([0,T]),\ \text{and}\ \Gamma_{1,-\epsilon} = \{x \in \Gamma_1; dist\,(x, \partial\Gamma_1) \geq \epsilon\}\,, \tag{6}$$

and $$R_\epsilon(\gamma) = \{x \in \Gamma_1;\ dist\,(x, R(\gamma)) \leq \epsilon\}. \tag{7}$$

Note that the condition $dist(x, \partial\Gamma_1) \geq \epsilon$ has a physical meaning because the control $t \mapsto \gamma(t)$ is nothing else than the path traced by the laser beam center which has R for radius. Practically, ϵ should be chosen in function of R. The constraint $R(\gamma) \subset \Gamma_{1,-\epsilon}$ is chosen to describe that γ must stay from an ϵ distance from the boundary of Γ_1. Moreover, $R_\epsilon(\gamma) = \Gamma_1$ is to constrain γ to cover Γ_1. Note that U_{ad} is not convex due to the two constraints $R(\gamma) \subset \Gamma_{1,-\epsilon}$ and $R_\epsilon(\gamma) = \Gamma_1$. By the theory of "tubes" [4], if $\partial\Gamma_1 \in C^2(\mathbb{R}^2)$, U_{ad} will be non void if $\epsilon > 0$ is chosen sufficently small and the constant c in definition (5) is chosen large enough.

Let us prove some preliminary results that will allow us to show that (OCP) has at least one optimal control.

Proposition 2.1. *U_{ad} is a weakly closed subset of $H^1(0,T;\mathbb{R}^2)$.*

Proof. Let $(\gamma_n)_{n\in\mathbb{N}} \subset U_{ad}$ be a weakly convergent sequence in $H^1(0,T;\mathbb{R}^2)$ and let us call γ its weak limit. As the embedding from $H^1(0,T;\mathbb{R}^2)$ into $C([0,T];\mathbb{R}^2)$ is compact, the weak convergence in $H^1(0,T;\mathbb{R}^2)$ of $(\gamma_n)_{n\in\mathbb{N}}$ to γ implies the strong convergence of $(\gamma_n)_{n\in\mathbb{N}}$ in $C([0,T];\mathbb{R}^2)$.

Given $x \in \Gamma_1$, there exists $(t_n)_{n\in\mathbb{N}} \subset [0,T]$ such that

$$\gamma_n(t_n) \in \Gamma_{1,-\epsilon} \quad \text{for every } n \in \mathbb{N}, \quad \text{and} \quad |x - \gamma_n(t_n)| \leq \epsilon \quad \text{for every } n \in \mathbb{N}.$$

As $[0,T]$ is compact there exists a subsequence $(t_{n_j})_{j\in\mathbb{N}} \subset [0,T]$ convergent to some $t \in [0,T]$. Thus we have

$$\begin{aligned} \mid \gamma_{n_j}(t_{n_j}) - \gamma(t) \mid &\leqslant \mid \gamma_{n_j}(t_{n_j}) - \gamma(t_{n_j}) \mid + \mid \gamma(t_{n_j}) - \gamma(t) \mid \\ &\leqslant \parallel \gamma_{n_j} - \gamma \parallel_\infty + \mid \gamma(t_{n_j}) - \gamma(t) \mid \longrightarrow 0 \ \text{ as } \ j \longrightarrow \infty. \end{aligned} \tag{8}$$

This implies that $|x - \gamma(t)| \leq \epsilon$. Thus, $R_\epsilon(\gamma) = \Gamma_1$ and $R(\gamma) \subset \Gamma_{1,-\epsilon}$ (recalling that $\Gamma_{1,-\epsilon}$ is closed). Using Mazur's theorem [32], for all $j \in \mathbb{N}^*$ there exists a convex combination

$$u_{n_j} = \sum_{k=1}^{n_j} \alpha_k \gamma_k, \left(\alpha_k \geq 0, \ \sum_{k=1}^{n_j} \alpha_k = 1 \right), \text{ such that } \parallel \gamma - u_{n_j} \parallel_{H^1(0,T;\mathbb{R}^2)} \leq \frac{1}{j}.$$

This implies that $\parallel \gamma' - u'_{n_j} \parallel_{L^2(0,T;\mathbb{R}^2)} \to 0$ as $j \to \infty$.

In particular, there exists a subsequence $(u'_{n_{j_k}})_{k\in\mathbb{N}}$ such that

$$u'_{n_{j_k}}(t) \to \gamma'(t) \text{ as } k \to \infty \text{ for a.e. } t \in [0,T].$$

As $\mid u'_{n_{j_k}}(t) \mid \leq c$ for a.e. $t \in [0,T]$, then also $\mid \gamma'(t) \mid \leq c$ for a.e. $t \in [0,T]$.

Thus $\gamma \in U_{ad}$. □

Proposition 2.2. *The control-to-state mapping $G : \gamma \in U_{ad} \longmapsto y(\gamma) \in W(0,T)$ is weakly sequentially continuous.*

Proof. Let $(\gamma_n)_{n\in\mathbb{N}} \subset U_{ad}$ be a weakly convergent sequence in $H^1(0,T;\mathbb{R}^2)$ and let $\gamma \in U_{ad}$ its weak limit. By the compact embedding from $H^1(0,T;\mathbb{R}^2)$ into $C([0,T];\mathbb{R}^2)$, $(\gamma_n)_{n\in\mathbb{N}}$ strongly converges to γ in $C([0,T];\mathbb{R}^2)$. From Theorem 3.13 in [28] it follows that the sequence $(y(\gamma_n))_{n\in\mathbb{N}}$ is a bounded sequence in the space $W(0,T)$. Consequently, it possesses a weakly convergent subsequence $(y(\gamma_{n_j}))_{j\in\mathbb{N}}$ in the space $W(0,T)$. Let y be the weak limit of $(y(\gamma_{n_j}))_{j\in\mathbb{N}}$.

For $0.5 < \varepsilon < 1$, the embedding from $W(0,T)$ into $L^2(0,T;H^\varepsilon(\Omega))$ is a linear continuous compact mapping by the Lions-Aubin compactness Lemma [18, p. 57]. The trace mapping

$$L^2(0,T;H^\varepsilon(\Omega)) \longrightarrow L^2(0,T;H^{\varepsilon-1/2}(\Gamma))\,; \quad y \longmapsto y_{|\Sigma}.$$

is linear and continuous [19], and thus the trace mapping from $L^2(0,T;H^\varepsilon(\Omega))$ into $L^2(\Sigma) = L^2(0,T;L^2(\Gamma))$ is also linear and continuous. This implies that the sequence of traces on Σ of $(y(\gamma_{n_j}))_{j\in\mathbb{N}}$ strongly converges to $y_{|\Sigma}$ in $L^2(\Sigma)$.

Now we recall that each $y(\gamma_{n_j})$ satisfies for all $v \in H^1(\Omega)$ and all $\varphi \in L^2(0,T)$ the equivalent weak formulation of problem (1), namely,

$$\begin{aligned} &\rho\, c \int_0^T \langle \frac{dy}{dt}(\gamma_{n_j})(.,t)\,,\, v\rangle_{(H^1(\Omega))^*,H^1(\Omega)}\, \varphi(t) dt + \kappa \int_0^T \int_\Omega \nabla y(\gamma_{n_j})(x,t) \cdot \nabla v(x) \varphi(t)\, dx\, dt \\ &+ h \int_0^T \int_\Gamma y(\gamma_{n_j})(x,t)\, v(x)\, \varphi(t)\, dS(x)\, dt - h \int_0^T \int_{\Gamma_3} y_B(x,t)\, v(x)\, \varphi(t)\, dS(x) dt \\ &- \alpha \frac{2P}{\pi R^2} \int_0^T \int_{\Gamma_1} \exp\left(-2 \frac{\mid x - \gamma_{n_j}(t) \mid^2}{R^2} \right) v(x)\, \varphi(t)\, dS(x)\, dt = 0. \end{aligned}$$

By the Lebesgue convergence theorem we have

$$\int_0^T \int_{\Gamma_1} exp\left(-2\frac{| x-\gamma_{n_j}(t) |^2}{R^2}\right) v(x)\varphi(t)dS(x)dt \longrightarrow$$
$$\int_0^T \int_{\Gamma_1} exp\left(-2\frac{| x-\gamma(t) |^2}{R^2}\right) v(x)\,\varphi(t)dS(x)dt$$

as $j \to \infty$, for all $v \in H^1(\Omega)$ and all $\varphi \in L^2(0,T)$.

Using all the previous convergence properties to pass to the limit in the above equation as $j \to \infty$, we obtain $y = y(\gamma)$. Thus $(y(\gamma_{n_j}))_{j\in\mathbb{N}}$ weakly converges to $y(\gamma)$ in $W(0,T)$. Therefore any subsequence of $(y(\gamma_n))_{n\in\mathbb{N}}$ contains a further subsequence which converges weakly to $y(\gamma)$ in $W(0,T)$. This implies that the whole sequence $(y(\gamma_n))_{n\in\mathbb{N}}$ converges weakly to $y(\gamma)$ in $W(0,T)$. This proves the proposition. □

Definition 2.3. The *reduced cost functional* is defined by

$$\hat{J} : U_{ad} \longrightarrow \mathbb{R}\,; \quad \gamma \longmapsto J(G(\gamma),\gamma).$$

We are now ready to prove our main result, namely the existence of at least one optimal control.

Theorem 2.4. (Existence of an optimal control) *Supposing $U_{ad} \neq \emptyset$, then the optimal control problem* (OCP) *admits at least one optimal control $\bar{\gamma} \in U_{ad}$.*

Proof. Since $\hat{J}(\gamma) \geq 0$, the infimum $L := \inf_{\gamma\in U_{ad}} \hat{J}(\gamma)$, exists and there is a sequence $(\gamma_n)_{n\in N} \subset U_{ad}$ such that $\hat{J}(\gamma_n) \to L$ as $n \to +\infty$. The sequence $(\gamma_n)_{n\in\mathbb{N}} \subset U_{ad}$ is bounded in $H^1(0,T;\mathbb{R}^2)$, because $\|\gamma_n\|^2_{H^1(0,T;\mathbb{R}^2)} \leq \frac{2}{\lambda_\gamma}\hat{J}(\gamma_n)$ for all $n \in \mathbb{N}$. Hence, it possesses a subsequence $(\gamma_{n_j})_{j\in\mathbb{N}}$ weakly convergent to some element $\bar{\gamma} \in U_{ad}$. This implies that

$$\| \bar{\gamma} \|_{H^1(0,T;\mathbb{R}^2)} \leqslant \lim_{j\to\infty} \inf \| \gamma_{n_j} \|_{H^1(0,T;\mathbb{R}^2)} \leq \sqrt{\frac{2L}{\lambda_\gamma}}. \tag{9}$$

By Proposition 2.2, $G(\gamma_{n_j}) \rightharpoonup G(\bar{\gamma})$ in $W(0,T)$ which implies that $G(\gamma_{n_j}) \rightharpoonup G(\bar{\gamma})$ in $L^2(0,T;H^1(\Omega))$, and thus

$$\| G(\bar{\gamma}) \|_{L^2(0,T;H^1(\Omega))} \leqslant \liminf_{j\to\infty} \| G(\gamma_{n_j}) \|_{L^2(0,T;H^1(\Omega))}\,. \tag{10}$$

The embedding from $W(0,T)$ into $L^2(0,T;L^2(\Omega))$ being compact [18, p.57], the sequence $G(\gamma_{n_j})$ also strongly converges to $G(\bar{\gamma})$ in $L^2(0,T;L^2(\Omega))$.

Using all the previous convergence properties and formula (4) we have

$$\begin{aligned} L \;\geqslant\; & \frac{1}{2}\liminf_{j\to\infty}\int_0^T \int_\Omega | \nabla G(\gamma_{n_j})(x,t) |^2\, dx\, dt + \frac{\lambda_Q}{2}\liminf_{j\to\infty}\| G(\gamma_{n_j}) - y_Q \|^2_{L^2(Q)} \\ & + \frac{\lambda_\gamma}{2}\liminf_{j\to\infty} \| \gamma_{n_j} \|^2_{H^1(0,T;\mathbb{R}^2)} \geqslant \hat{J}(\bar{\gamma}). \end{aligned}$$

By the definition of L we have also that $L \leqslant \hat{J}(\bar{\gamma})$. Thus $L = \hat{J}(\bar{\gamma})$. □

3. Differentiability of the control-to-state mapping

Our aim is to derive necessary optimality conditions for an admissible control to be optimal. We first have to discuss the differentiability of the control-to-state mapping.

Lemma 3.1. *The mapping*

$$G\colon U_{ad} \longrightarrow L^2(0,T;H^1(\Omega))\,; \quad \gamma \longmapsto y(\gamma)$$

is Fréchet differentiable.

Proof. We can write G as the restriction to U_{ad} of the composition of the Fréchet differentiable mappings w, g and q, where w, g and q are defined as follows:

$$w\colon H^1(0,T;\mathbb{R}^2) \longrightarrow C(\bar{\Gamma}_1 \times [0,T])\,; \quad \gamma \longmapsto -c_R \mid \tilde{\gamma}(\gamma) \mid^2, \tag{11}$$

where $c_R = \frac{2}{R^2}$ and $\tilde{\gamma}(\gamma)(x,t) := x - \gamma(t),\ \forall (x,t) \in \bar{\Gamma}_1 \times [0,T]$,

$$g\colon C(\bar{\Gamma}_1 \times [0,T]) \longrightarrow L^2(\Sigma_1)\,; \quad u \longmapsto a\exp(u), \tag{12}$$

where $a = \alpha\frac{2P}{\pi R^2}$, and

$$q\colon L^2(\Sigma_1) \longrightarrow L^2(0,T;H^1(\Omega))\,; \quad g \longmapsto y, \tag{13}$$

where y denotes the weak solution of the initial boundary value problem:

$$\begin{cases} \rho\, c\, \partial_t y - \kappa\, \Delta y = 0 & \text{in} \quad Q, \\ -\kappa\frac{\partial y}{\partial \nu} = h\, y - g & \text{on} \quad \Sigma_1, \\ -\kappa\, \frac{\partial y}{\partial \nu} = h\, y & \text{on} \quad \Sigma_2, \\ -\kappa\frac{\partial y}{\partial \nu} = h\,(y - y_B) & \text{on} \quad \Sigma_3, \\ y(x,0) = y_0(x) & \text{for} \quad x \in \Omega, \end{cases} \tag{14}$$

$y_0 \in L^2(\Omega)$ denoting a fixed initial condition.

• We start by proving that w is Fréchet differentiable, when $C(\bar{\Gamma}_1 \times [0,T])$ is endowed with its natural norm $\| \, u \, \|_\infty := \sup_{(x,t)\in\bar{\Gamma}_1\times[0,T]} |u(x,t)|$. For all $\delta\gamma \in H^1(0,T;\mathbb{R}^2)$ we have

$$\begin{aligned} [w(\gamma + \delta\gamma) - w(\gamma)]\,(x,t) &= -c_R\,[\,\mid x - (\gamma + \delta\gamma)(t) \mid^2 - \mid x - \gamma(t) \mid^2\,] \\ &= c_R\,\delta\gamma(t) \cdot (2x - 2\gamma(t) - \delta\gamma(t)) \\ &= 2\,c_R\,(x - \gamma(t)) \,\cdot \delta\gamma(t) - c_R \mid \delta\gamma(t) \mid^2,\ \forall (x,t) \in \bar{\Gamma}_1 \times [0,T], \end{aligned}$$

where $\cdot$ denotes here the inner product in $\mathbb{R}^2$. Since $H^1(0,T;\mathbb{R}^2) \hookrightarrow C([0,T];\mathbb{R}^2)$ there exists a constant $\eta \geq 0$ such that

$$\frac{\| \mid \delta\gamma \mid^2 \|_\infty}{\| \, \delta\gamma \, \|_{H^1(0,T;\mathbb{R}^2)}} \leq \eta \frac{\| \, \delta\gamma \, \|^2_{H^1(0,T;\mathbb{R}^2)}}{\| \, \delta\gamma \, \|_{H^1(0,T;\mathbb{R}^2)}} = \eta \, \| \, \delta\gamma \, \|_{H^1(0,T;\mathbb{R}^2)} \to 0 \text{ as } \| \, \delta\gamma \, \|_{H^1(0,T;\mathbb{R}^2)} \to 0.$$

Hence, w is Fréchet differentiable with Fréchet derivative

$$Dw(\gamma)\cdot\delta\gamma = 2\,c_R\,\tilde{\gamma}(\gamma)\cdot\delta\gamma, \text{ for all } \delta\gamma\in H^1(0,T;\mathbb{R}^2).$$

• The mapping q is Fréchet differentiable being an affine mapping and continuous by [28, (3.26) p. 140]. The mapping g being a superposition operator (also called Nemytskii operator) is known to be Fréchet differentiable by [28, p. 202] or [3]. We therefore skip the details and only give their Fréchet derivatives, which are respectively given by

$$Dq(\tilde{g}) = \tau,\ \forall\tilde{g}\in L^2(\Sigma_1),$$

with $\tau\colon L^2(\Sigma_1)\longrightarrow L^2(0,T;H^1(\Omega))\,;\quad g\longmapsto\tau(g):=y_2,$

where y_2 is the unique solution of

$$\begin{cases} \rho\,c\,\partial_t y_2-\kappa\,\Delta y_2=0 & \text{in} \quad Q,\\ -\kappa\frac{\partial y_2}{\partial\nu}=hy_2-g & \text{on} \quad \Sigma_1,\\ -\kappa\,\frac{\partial y_2}{\partial\nu}=hy_2 & \text{on} \quad \Sigma_2\cup\Sigma_3,\\ y_2(\cdot,0)=0 & \text{in} \quad \Omega. \end{cases} \tag{15}$$

and $Dg(u)\cdot\delta u = a\exp(u)\delta u, \text{ for all } \delta u\in C(\bar{\Gamma}_1\times[0,T]).$

In conclusion, G is Fréchet differentiable with Fréchet derivative

$$\begin{aligned} DG(\gamma)\cdot\delta\gamma &= D(q\circ g\circ w)(\gamma)\cdot\delta\gamma\\ &= Dq(g(w(\gamma)))\cdot(D(g\circ w)(\gamma)\cdot\delta\gamma)\\ &= (Dq(g(w(\gamma)))\circ Dg(w(\gamma)))\cdot(Dw(\gamma)\cdot\delta\gamma)\\ &= \tau((Dg(w(\gamma)))\circ Dw(\gamma))\cdot\delta\gamma)\\ &= 2ac_R\,\tau(\exp(w(\gamma))\,\tilde{\gamma}(\gamma)\cdot\delta\gamma),\ \text{for all } \delta\gamma\in H^1(0,T;\mathbb{R}^2). \end{aligned}$$ □

From the previous lemma and by composition we deduce that the reduced cost functional $\gamma\in H^1(0,T;\mathbb{R}^2)\mapsto J(G(\gamma),\gamma)\in\mathbb{R}$ is Fréchet differentiable. We define

$$v(\gamma,\delta\gamma):=2ac_R\,\tau(\exp(w(\gamma))\,\tilde{\gamma}(\gamma)\cdot\delta\gamma) \tag{16}$$

the Fréchet derivative of the control-to-state mapping

$$\gamma\in U_{ad}\mapsto G(\gamma)\in L^2(0,T;H^1(\Omega)).$$

Using the previous result, we obtain

$$\begin{aligned} D\hat{J}(\gamma)\cdot\delta\gamma &= \int_0^T\!\!\int_\Omega \nabla G(\gamma)(x,t)\cdot\nabla v(\gamma,\delta\gamma)(x,t)\,dxdt+\lambda_Q\,(G(\gamma)-y_Q,v(\gamma,\delta\gamma))_{L^2(Q)}\\ &+\lambda_\gamma\,(\gamma,\delta\gamma)_{H^1(0,T;\mathbb{R}^2)} = \int_0^T\int_\Omega \nabla G(\gamma)(x,t)\cdot\nabla v(\gamma,\delta\gamma)(x,t)\,dxdt\\ &+\lambda_Q\int_0^T\int_\Omega G(\gamma)(x,t)\,v(\gamma,\delta\gamma)(x,t)\,dxdt-\lambda_Q\int_0^T\int_\Omega y_Q(x,t)\,v(\gamma,\delta\gamma)(x,t)\,dxdt\\ &+\lambda_\gamma\int_0^T\gamma(t)\cdot\delta\gamma(t)\,dt+\lambda_\gamma\int_0^T\gamma'(t)\cdot\delta\gamma'(t)\,dt,\ \text{for all } \delta\gamma\in H^1(0,T;\mathbb{R}^2). \end{aligned} \tag{17}$$

4. Adjoint equation and necessary optimality conditions

It is well known that an optimal control $\bar{\gamma}$ minimizing $\hat{J}$ in U_{ad} has to obey the variational inequality

$$D\hat{J}(\bar{\gamma})(\gamma - \bar{\gamma}) \geq 0 \text{ for all } \gamma \in U_{ad}, \tag{18}$$

provided that $\hat{J}$ is Gâteaux differentiable at $\bar{\gamma}$ and U_{ad} convex. In our case $\hat{J}$ is Fréchet differentiable but U_{ad} is not convex, thus (18) is no more true. Therefore, we introduce at any point $\gamma \in U_{ad}$ the cone of admissible directions and we use the Kuhn-Tucker conditions. More precisely, we recall from [9, p.211] the following definition and result.

Definition 4.1. Let V a normed vector space and U_{ad} a non-empty subset of V. For every $\gamma \in U_{ad}$, the *cone of admissible directions at* γ is

$$C(\gamma) := \{0\} \cup \left\{ w \in V \setminus \{0\}\,; \begin{array}{c} \exists\, (\gamma_k)_{k\geqslant 0} \subset U_{ad},\ \gamma_k \neq \gamma\ \forall k \geq 0 \text{ s.t.} \\ \lim\limits_{k\to\infty} \gamma_k = \gamma \text{ and } \lim\limits_{k\to\infty} \dfrac{\gamma_k - \gamma}{||\gamma_k - \gamma||} = \dfrac{w}{||w||} \end{array} \right\}. \tag{19}$$

Theorem 4.2. (Kuhn-Tucker) *Let V be a normed vector space and U_{ad} a non-empty subset of V. Let $J\colon O \subset V \to \mathbb{R}$, a function defined on an open set O of V such that $U_{ad} \subset O$. If J has at $\bar{\gamma} \in U_{ad}$ a relative minimum compared to the subset U_{ad}, and if J is Fréchet differentiable at $\bar{\gamma}$ then*

$$DJ(\bar{\gamma}) \cdot (\delta\bar{\gamma}) \geq 0 \textit{ for every } \delta\bar{\gamma} \in C(\bar{\gamma}), \tag{20}$$

i.e. $DJ(\bar{\gamma})$ belongs to the dual cone of the cone of admissible directions $C(\bar{\gamma})$ at $\bar{\gamma}$.

Since $\hat{J}$ is Fréchet differentiable, Theorem 4.2 and (17) allow us to derive a necessary condition for an admissible control to be optimal. However, this necessary condition would not be practical due to the appearance of $v(\bar{\gamma}, \delta\bar{\gamma})$ in (17) which should be computed by solving the initial boundary value problem (15) for $g = \exp(w(\bar{\gamma}))\,\tilde{\gamma}(\bar{\gamma})\cdot \delta\bar{\gamma}$ each time we consider another $\delta\bar{\gamma} \in C(\bar{\gamma})$ to check if the necessary condition (20) is true at $\bar{\gamma}$. To resolve this difficulty, it is classical to introduce the "adjoint system" whose solution is called the adjoint state [28]. We claim that the adjoint system for our problem is the following linear backward boundary value problem

$$\begin{cases} \rho\, c\, \partial_t p + \kappa\, \Delta p = \Delta G(\bar{\gamma}) - \lambda_Q (G(\bar{\gamma}) - y_Q) & \text{in} \quad Q, \\ \kappa \frac{\partial p}{\partial \nu} + h\, p = \frac{\partial G(\bar{\gamma})}{\partial \nu} & \text{on} \quad \Gamma \times]0, T[, \\ p(., T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{21}$$

Definition 4.3. Let $\bar{\gamma}$ be an optimal control of (OCP) with associated state $G(\bar{\gamma})$. A function $p \in W(0,T)$ is said to be a *weak solution* to (21) if $p(\cdot, T) = 0$ in Ω and

$$\begin{aligned} -\rho c \int_0^T < \partial_t p(\cdot, t), \varphi(\cdot, t) >_{H^1(\Omega)^*, H^1(\Omega)} dt + \kappa \int_0^T \int_\Omega \nabla p(x,t) \cdot \nabla \varphi(x,t) dx dt \\ + h \int_0^T \int_\Gamma p(x,t) \varphi(x,t) dS(x) dt = \int_0^T \int_\Omega \nabla G(\bar{\gamma})(x,t) \nabla \varphi(x,t) dx dt \\ + \lambda_Q \int_0^T \int_\Omega (G(\bar{\gamma}) - y_Q)(x,t) \varphi(x,t) dx dt \end{aligned} \tag{22}$$

for every $\varphi \in L^2(0,T; H^1(\Omega))$.

Let us notice that (21) admits a unique weak solution in $W(0,T)$, see [13, pp. 512–513] for instance.

Theorem 4.4. *If $\bar{\gamma} \in U_{ad}$ is an optimal control of* (OCP) *with associated state $G(\bar{\gamma})$, and $p \in W(0,T)$ the corresponding adjoint state that solves* (21)*, then the variational inequality*

$$\lambda_\gamma \int_0^T \bar{\gamma}(t) \cdot (\delta\bar{\gamma})(t)dt + \lambda_\gamma \int_0^T \bar{\gamma}'(t) \cdot (\delta\bar{\gamma})'(t)dt$$
$$+2ac_R \int\int_{\Sigma_1} \exp(w(\bar{\gamma})(x,t))\tilde{\gamma}(\bar{\gamma})(x,t) \cdot (\delta\bar{\gamma})(t)p(x,t)dS(x)dt \geq 0 \qquad (23)$$

holds for all $\delta\bar{\gamma} \in C(\bar{\gamma})$.

Proof. If $\bar{\gamma}$ is an optimal control for the problem (OCP) then by Theorem 4.2

$$D\hat{J}(\bar{\gamma}) \cdot (\delta\bar{\gamma}) \geq 0 \text{ for all } \delta\bar{\gamma} \in C(\bar{\gamma}).$$

By (17) and (22) we have

$$\begin{aligned} D\hat{J}(\bar{\gamma}) \cdot (\delta\bar{\gamma}) = &- \rho c \int_0^T < \partial_t p(\cdot,t), v(\bar{\gamma},\delta\bar{\gamma})(\cdot,t) >_{H^1(\Omega)^*,H^1(\Omega)} dt \\ &+ \kappa \int_0^T \int_\Omega \nabla p(x,t) \cdot \nabla v(\bar{\gamma},\delta\bar{\gamma})(x,t)dxdt \\ &+ h \int_0^T \int_\Gamma p(x,t)v(\bar{\gamma},\delta\bar{\gamma})(x,t)dS(x)dt \\ &+ \lambda_{\bar{\gamma}} \int_0^T \bar{\gamma}(t) \cdot (\delta\bar{\gamma})(t)\, dt + \lambda_{\bar{\gamma}} \int_0^T \bar{\gamma}'(t) \cdot (\delta\bar{\gamma})'(t)\, dt \end{aligned} \qquad (24)$$

where we recall that $v(\bar{\gamma},\delta\bar{\gamma})$ given by (16) is the weak solution of

$$\begin{cases} \rho\, c\, \partial_t v(\bar{\gamma},\delta\bar{\gamma}) - \kappa\, \Delta v(\bar{\gamma},\delta\bar{\gamma}) = 0 & \text{in} \quad Q, \\ \kappa \frac{\partial v(\bar{\gamma},\delta\bar{\gamma})}{\partial\nu} + h\, v(\bar{\gamma},\delta\bar{\gamma}) = 2ac_R \exp(w(\bar{\gamma}))\tilde{\gamma}(\bar{\gamma}) \cdot \delta\bar{\gamma} & \text{on} \quad \Sigma_1, \\ \kappa \frac{\partial v(\bar{\gamma},\delta\bar{\gamma})}{\partial\nu} + h\, v(\bar{\gamma},\delta\bar{\gamma}) = 0 & \text{on} \quad \Sigma_2 \cup \Sigma_3, \\ v(\bar{\gamma},\delta\bar{\gamma})(x,0) = 0 & \text{for} \quad x \in \Omega, \end{cases} \qquad (25)$$

and p is the weak solution of (21). Using the fact that $v(\bar{\gamma},\delta\bar{\gamma})$ is the weak solution of (25), taking $p(\cdot,\cdot)$ as test function in the weak formulation of (25) and using the integration by parts formula in $W(0,T)$ [28, p. 148] taking into account that $v(\bar{\gamma},\delta\bar{\gamma})(\cdot,0) = 0$ and $p(\cdot,T) = 0$, we obtain:

$$-\rho c \int_0^T < \partial_t p(\cdot,t), v(\bar{\gamma},\delta\bar{\gamma})(\cdot,t) >_{H^1(\Omega)^*,H^1(\Omega)} dt$$
$$+\kappa \int_0^T \int_\Omega \nabla p(x,t) \cdot \nabla v(\bar{\gamma},\delta\bar{\gamma})(x,t)dxdt + h \int_0^T \int_\Gamma p(x,t)v(\bar{\gamma},\delta\bar{\gamma})(x,t)dS(x)dt$$
$$= 2ac_R \int_0^T \int_{\Gamma_1} \exp(w(\bar{\gamma})(x,t))\tilde{\gamma}\,(\bar{\gamma})\,(x,t) \cdot (\delta\bar{\gamma})(t)p(x,t)dS(x)dt.$$

By replacing this last identity in (24) we get

$$D\hat{J}(\bar{\gamma})\cdot(\delta\bar{\gamma}) = 2ac_R \int\int_{\Sigma_1} \exp(w(\bar{\gamma})(x,t))\tilde{\gamma}\,(\bar{\gamma})\,(x,t)\tilde{\gamma}(x,t)\cdot(\delta\bar{\gamma})(t)p(x,t)dS(x)dt$$
$$+\lambda_\gamma \int_0^T \bar{\gamma}(t)\cdot(\delta\bar{\gamma})(t)dt + \lambda_\gamma \int_0^T \bar{\gamma}'(t)\cdot(\delta\bar{\gamma})'(t)dt \geq 0.$$

This concludes the proof of the Theorem. □

5. Perspectives

It is clear that the set of admissible controls (5) is not convex due to the constraints on the control

$$R(\gamma)\subset \Gamma_{1,-\epsilon},\; R_\epsilon(\gamma)=\Gamma_1 \tag{26}$$

for which no efficient algorithm exists [2]. In a further work, we intend to replace the previous nonconvex constraints (26) by other conditions on the trajectory γ, by adding a penalization term to the cost functional (4) [6, 10]. We recall that the geometrical constraints on the path γ are set to describe the fact that the laser beam acting on Γ_1 must melt the powder to built the layer Ω. Therefore the penalization term is chosen in such a way that this latest condition is almost satisfied. Namely given a penalization parameter $\delta>0$, we add to $\hat{J}(\gamma)$ the term

$$\frac{1}{\delta^2}\left(2R\int_0^T \sqrt{\mid \gamma'(t)\mid^2+\delta^2}dt - \mid \Gamma_1 \mid\right)^2.$$

Formally as δ is close to zero, this will force the control to satisfy

$$2R\int_0^T \mid \gamma'(t)\mid\; dt \simeq \mid \Gamma_1\mid, \tag{27}$$

which means that the area covered by the laser is close to the area of Γ_1. This allows to hope that the laser path would have covered the whole Γ_1 avoiding self-intersection because, in the same time, we also minimize the temperature gradient. Hence we will consider the penalized optimal control problem: given $\delta>0$, find an optimal control $\bar{\gamma}^\delta$ in the new convex set of admissible controls (here we assume that Γ_1 is a convex subset of $\mathbb{R}^2$)

$$U_{ad}^p = \left\{\gamma\in H^2(0,T;\Gamma_1)\,;\; \begin{array}{l} |\gamma'(t)|\leq c \;\text{ a.e. } t\in[0,T] \text{ and} \\ 2R\int_0^T \mid\gamma'(t)\mid dt \leq \mid\Gamma_1\mid + 2\,diam(\Gamma_1)\,\epsilon \end{array}\right\} \tag{28}$$

such that

$$(\text{OCP}^\delta) \qquad \hat{J}^\delta(\bar{\gamma}^\delta) = \min_{\gamma\in U_{ad}} \hat{J}^\delta(\gamma).$$

The reduced cost functional $\hat{J}^\delta(\cdot)$ being now defined by

$$\hat{J}^\delta(\gamma) := \hat{J}(\gamma) + \frac{1}{\delta^2}\left(2R\int_0^T \sqrt{\mid\gamma'(t)\mid^2+\delta^2}dt - \mid\Gamma_1\mid\right)^2, \tag{29}$$

where

$$\hat{J}(\gamma) := \frac{1}{2}\int_0^T \int_\Omega |\nabla y(\gamma)(x,t)|^2 \, dx \, dt + \frac{\lambda_Q}{2}\int_0^T \int_\Omega | y(\gamma)(x,t) - y_Q(x,t)|^2 \, dx \, dt + \frac{\lambda_\gamma}{2} \| \gamma \|^2_{H^2(0,T;\mathbb{R}^2)} . \tag{30}$$

The choice of the control space $H^2(0,T;\mathbb{R}^2)$ is made for two main reasons: the first one is to guarantee the existence of an optimal control and the second one is to obtain smoother laser paths. From an industrial point of view, we believe that the use of C^1 curves will be more efficient than C^0 curves since smoother curves will diminish thermal gradients. Furthermore this kind of paths has been used in many additive manufacturing technologies [14, 34].

The parabolic state initial boundary value (1) and the adjoint parabolic backward boundary value problem will be discretized by using $\mathbb{P}^1$ finite element method in space and the Euler implicit sheme in time. We will consider the Hermite interpolating polynomial space to discretize the control space U^p_{ad}. Finally to solve this optimization problem with inequalities constraints we will use the augmented Lagrangian method [23, §17.4].

This approach will be properly investigated analytically and numerically in a forthcoming paper.

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