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Recovering Simultaneously a Potential and a Point Source from Cauchy Data

Gang Bao, Yuantong Liu
Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang 310027, China
baog@zju.edu.cn, ytliu@zju.edu.cn

Faouzi Triki
Laboratoire Jean Kuntzmann, UMR CNRS 5224, Université Grenoble-Alpes, 38401 Saint-Martin-d'Hères, France
faouzi.triki@univ-grenoble-alpes.fr

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This paper is devoted to the inverse problem of recovering simultaneously a potential and a point source in a Schrödinger equation from the associated nonlinear Dirichlet to Neumann map. The uniqueness of the inversion is proved and logarithmic stability estimates are derived. It is well known that the inverse problem of determining only the potential while knowing the source, is ill-posed. In contrast the problem of identifying a point source when the potential is given is well posed. The obtained results show that the nonlinear Dirichlet to Neumann map contains enough information to determine simultaneously the potential and the point source. However recovering a point source imbedded in an unknown background medium becomes an ill-posed inversion.

Keywords: Inverse potential, Dirichlet to Neumann map, stability estimate, point sources, Schrödinger equation.

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1. Introduction and main results

In this paper we study the issue of uniqueness and stability for determining simultaneously a smooth potential and a point source in a Schrödinger equation by boundary measurements. Motivation for investigating this inverse problem is provided by medical imaging as well as antenna synthesis [4, 7, 10, 11]. From the point view of mathematical modeling, many works have considered the simplification that the background medium in which the source is imbedded is known. Here we are interested in recovering both the source and the background medium from Cauchy data.

Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, be a bounded domain, with C^∞ boundary $\partial\Omega$. We consider the Schrödinger equation

$$\big(\Delta + q(x)\big)u(x) = a\delta_z(x) \quad \text{in} \quad \Omega, \tag{1}$$

where the real-valued function $q(x)$ is the potential, $z \in \Omega$ is the position of the point source, and $a \in \mathbb{R}^d \setminus \{0\}$ its amplitude.

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Assume that the kernel of the operator $\Delta + q(x)$ acting on $H_0^1(\Omega)$ is the trivial space. Associated with (1), we define the nonlinear Dirichlet-to-Neumann map (DtN) $\Phi[q,a,z]\colon H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Phi(f) = \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega},$$

where u is the solution to (1) with the Dirichlet condition $u = f$ on $\partial\Omega$, and ν is the unit outer normal vector of $\partial\Omega$.

The nonlinear map $f \to \Phi[q,a,z](f)$ is an affine function. It can be decomposed as $\Phi[q,a,z](f) = \Phi[q,a,z](0) + \Phi[q,0,0](f)$, where le latter map is the classical linear Dirichlet-to-Neumann map associated to the Schrödinger equation.

We further denote by $$\|\Phi\|_\star = \sup_{\|f\|_{H^{1/2}(\partial\Omega)}\leq 1} \|\Phi(f)\|_{H^{-1/2}(\partial\Omega)},$$

the norm that we will use to evaluate the strength of the nonlinear (DtN) map. Due to the affine property of the map, the trivial map $\Phi = 0$ is the unique solution to the equation $\|\Phi\|_\star = 0$.

The inverse problem we consider in this paper is *to recover the triplet* (q,a,z) *from the knowledge of the nonlinear (DtN) map* $\Phi[q,a,z]$. It is well known that the inverse problem of determining only the potential while knowing the source, is ill-posed. The uniqueness of this inverse problem is derived in [15]. Alessandrini proved that the stability estimate for this problem is of log type [1], and Mandache showed that the log type stability is optimal for smooth potentials [12]. For a given potential it is also well known that the identification of a general source function from full Cauchy data is not possible. Indeed the authors in [2] showed that many boundary measurements are not sufficient to fully identify a general source. Moreover it turns out that increasing the number of boundary measurements does not increase the information concerning the source. The identification may be achieved by considering many boundary measurements generated by different frequencies [5, 6]. Unlike general source functions, point sources are singular and has a lower dimensionality.

These specificities enable one to obtain uniqueness in the inverse source problem with a single Cauchy observed data [3]. Many Hölder type stability estimates have been derived for this inverse problem when the background medium is known and homogeneous [8, 9]. The inverse problem considered in this paper is quite new, and only few partial results are available. Recently the authors in [14] established a Hölder stability estimate on the reconstruction of point sources with respect to smooth changes of a known potential. Their results say that if the potential is known up to a small smooth perturbation, the recovered source is close to the true one with respect to a given nonconventional distance (not comparable to classical distances in Sobolev spaces for example). Now we state the main result of the paper.

Theorem 1.1. *Assume that* (q_1,a_1,z_1) *and* (q_2,a_2,z_2) *are two triplets with associated (DtN) nonlinear maps* $\Phi[q_1,a_1,z_1]$ *and* $\Phi_2[q_2,a_2,z_2]$*, respectively. Assume* $s > (d/2)+1$ *and* $M \geq 1$. *Suppose* $\|q_j\|_{H^s(\Omega)} \leq M$ $(j = 1,2)$ *and* $\mathrm{supp}(q_1 - q_2) \subset \Omega$. *Then if* $\|\Phi[q_1,a_1,z_1] - \Phi_2[q_2,a_2,z_2]\|_\star < 1$*, the following stability estimate*

$$\|a_1\delta_{z_1} - a_2\delta_{z_2}\|_{H^{-s}(\mathbb{R}^d)} + \|q_1 - q_2\|_{H^{-s}(\mathbb{R}^d)} \\ \leq C\left(-\log\|\Phi[q_1, a_1, z_1] - \Phi[q_2, a_2, z_2]\|_\star\right)^{-(s-d/2)} \tag{2}$$

holds, where $C > 0$ depends only on s, d, Ω, and M.

The stability of reconstructing the potential and the point source is of logarithmic type. This means that the inversion is ill-posed and small variations in the measured data can lead to large errors in the reconstructions. If the potential is given, that is $q_1 = q_2$, following the proof of the main Theorem leads to a Lipschitz stability estimate.

The proof of Theorem 1.1 is based on Alessandrini's arguments in [1] and the Complex geometrical optics (CGO) solutions constructed in [15]. The main idea is to first recover the potential by exploiting the nonlinearity of the DtN map (Proposition 3.1). Then the remaining inverse problem becomes a linear one, and we again use (CGO) solutions to construct a new type of special solutions of the equation (1) in order to determine the position and amplitude of the point source (Lemma 3.3). Using the same approach the obtained results can be extended to the inverse problem of recovering a potential and a finite number of point sources. In the rest of the paper, we introduce the (CGO) solutions in Section 2, and prove the main result in Section 3.

2. Complex geometrical optics solutions

In this section, we construct (CGO) solutions to the equation (1) by using the idea in [15]. We recall the following fundamental results due to Sylvester and Uhlmann in [15] concerning solutions of the equation

$$\Delta w + \xi \cdot \nabla w = F \tag{3}$$

where $\xi \in \mathbb{C}^d$ and $\xi \cdot \xi = 0$, and $F \in L^2_{loc}(\mathbb{R}^d)$. Given $\xi \in \mathbb{C}^d$ with $|\xi| \geq 2$ and $\xi \cdot \xi = 0$, define

$$\widehat{K_\xi}(k) = \frac{1}{-|k|^2 + i\xi \cdot k} \qquad \text{for } k \in \mathbb{R}^d,$$

where $\widehat{K}_\xi$ stands for the Fourier transform of the kernel K_ξ. Then for $F \in H^s(\mathbb{R}^d)$, $s \geq 0$ with a compact support

$$\mathcal{K}_\xi(F) := K_\xi * F(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{ik\cdot x}\widehat{F}(k)}{-|k|^2 + i\xi \cdot k} dk,$$

is a solution to the equation $\Delta w + \xi \cdot \nabla w = F$ in $\mathbb{R}^d$, and

$$\widehat{K_\xi * F} = \widehat{K}_\xi \cdot \widehat{F} \in H^s(\mathbb{R}^d).$$

Lemma 2.1. ([15]) *Let $-1 < \delta < 0$, $\xi \in \mathbb{C}^d$ with $|\xi| > 2$ and $\xi \cdot \xi = 0$, and let $F \in H^s_{\delta+1}(\mathbb{R}^d)$. Then*

$$\|K_\xi * F\|_{H^s_\delta} \leq \frac{C}{|\xi|}\|F\|_{H^s_{\delta+1}} \qquad \text{for } s \geq 0, \tag{4}$$

$$\|K_\xi * F\|_{H^{s+1}_\delta} \leq C\|F\|_{H^s_{\delta+1}} \qquad \text{for } s \geq 0. \tag{5}$$

for some positive constant $C > 0$ that only depends on δ, s, and d. Here

$$\|v\|_{L^2_\delta} := \|(1 + |\cdot|^2)^\delta v(\cdot)\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|v\|_{H^s_\delta} := \sum_{|\alpha|=0}^{s} \|(1 + |\cdot|^2)^\delta \partial^\alpha v(\cdot)\|_{L^2(\mathbb{R}^d)}.$$

These estimates are the corner stone of the proof of the uniqueness of smooth potentials [15] and of the proof of the stability estimates in [1]. By using this lemma, we can obtain a solution to the general equation

$$\Delta\psi + \xi \cdot \nabla\psi + q\psi = F \tag{6}$$

satisfying some decaying property as in the following lemma.

Lemma 2.2. *Let $s > (d/2) + 1$ be an integer. Let $\xi \in \mathbb{C}^d$ satisfy $\xi \cdot \xi = 0$ and $|\xi| \geq 2$. Let $F \in H^s(\Omega)$. Then there exist constants $C_1 > 0$ and $C_2 > 0$ depending only on d, s, and Ω such that if*

$$|\xi| \geq C_1 \|q\|_{H^s(\Omega)},$$

then there exists a solution $\psi \in H^s(\Omega)$ to equation (6) satisfying the estimates

$$\|\psi\|_{H^s(\Omega)} \leq \frac{C_2}{|\xi|}\|F\|_{H^s(\Omega)}, \quad \textit{and} \quad \|\psi\|_{H^{s+1}(\Omega)} \leq C_2 \|F\|_{H^s(\Omega)}. \tag{7}$$

Proof. Let χ be a C^∞ compactly supported function satisfying $\chi = 1$ on a neighborhood of $\overline{\Omega}$. Denote q_0 and F_0 respectively the extensions in $H^s(\mathbb{R}^d)$ of the functions q and F, satisfying $\|q_0\|_{H^s(\mathbb{R}^d)} \leq \|q\|_{H^s(\Omega)}$, $\|F_0\|_{H^s(\mathbb{R}^d)} \leq \|F\|_{H^s(\Omega)}$ [13], and denote $\tilde{q} = \chi q_0$ and $\widetilde{F} = \chi F_0$. Simple calculation shows that

$$\|\tilde{q}\|_{H^s(\mathbb{R}^d)} \leq C\|q\|_{H^s(\Omega)}, \quad \text{and} \quad \|\widetilde{F}\|_{H^s(\mathbb{R}^d)} \leq C\|F\|_{H^s(\Omega)}, \tag{8}$$

where $C > 0$ is a constant that only depends on s and d.

Let $\tilde{\psi}$ be a solution to the equation (6) in the whole space with $\tilde{F}$ and $\tilde{q}$ substituting f and q respectively. Lemma 2.1 shows that the linear operator $\mathcal{K}_\xi(\tilde{q}\cdot)$ is bounded from $H^s(\mathbb{R}^d)$ to itself. Let I_d be the identity operator from $H^s(\mathbb{R}^d)$ to itself. The decaying behavior (4) implies that the operator $\mathcal{K}_\xi(\tilde{q}\cdot)$ is a contraction for $|\xi|$ large enough, and hence $I_d - \mathcal{K}_\xi(\tilde{q}\cdot)$ becomes invertible. Then

$$\tilde{\psi} = \left(I_d - \mathcal{K}_\xi(\tilde{q}\cdot)\right)^{-1} \mathcal{K}_\xi(\tilde{q})\widetilde{F}.$$

The estimates (7) follow immediately by taking ψ the restriction of $\tilde{\psi}$ to the domain Ω, from the convergence of the Neumann series

$$\tilde{\psi} = -\left(I_d - \mathcal{K}_\xi(\tilde{q}\cdot)\right)^{-1} \mathcal{K}_\xi(\tilde{q})\widetilde{F} = -\sum_{p=0}^{\infty} \left(\mathcal{K}_\xi(\tilde{q}\cdot)\right)^{p+1} \widetilde{F},$$

for large $|\xi|$, and inequalities (8). □

Next, we introduce CGO solutions to the equation

$$\big(\Delta + q(x)\big)u(x) = 0 \quad \text{in} \quad \Omega. \tag{9}$$

These solutions will be used later to construct test functions for the equation (1).

Proposition 2.3. *Let $s > d/2$ be an integer. Let $\xi \in \mathbb{C}^d$ satisfy $\xi \cdot \xi = 0$ and $|\xi| \geq 2$. Define the constants C_1 and C_2 as in Lemma 2.2. Then if*

$$|\xi| \geq C_1 \|q\|_{H^s(\Omega)}$$

then there exists a solution u to equation (9) *with the form of*

$$u(x) = \exp\Big(\frac{\xi}{2}\cdot x\Big)\big(1+\psi(x)\big), \tag{10}$$

where ψ has the estimates

$$\|\psi\|_{H^s(\Omega)} \le \frac{C_2}{|\xi|}\|q\|_{H^s(\Omega)}, \quad \textit{and} \quad \|\psi\|_{H^{s+1}(\Omega)} \le C_2\|q\|_{H^s(\Omega)}.$$

Proof. Substituting (10) into (1), we have

$$\Delta\psi + \xi\cdot\nabla\psi + q\psi = -q.$$

Then by Lemma 2.2, we obtain this proposition. □

3. Proof of the stability estimate

This section is devoted to the proof of Theorem 1.1. For $f \in H^{1/2}(\partial\Omega)$, let v be a solution to

$$\big(\Delta + q(x)\big)v(x) = 0 \quad \text{in} \quad \Omega, \tag{11}$$

and satisfying the Dirichlet condition $u = f$ on $\partial\Omega$, and define linear map (DtN) $\Phi_0[q] : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Phi_0(f) = \frac{\partial v}{\partial\nu}\Big|_{\partial\Omega}.$$

We first observe that the following equality

$$\Phi[q,a,z](f) - \Phi[q,a,z](0) = \Phi_0[q](f), \tag{12}$$

holds for all $f \in H^{1/2}(\partial\Omega)$.

Proposition 3.1. *Assume that q_1 and q_2 are two potentials with associated* (DtN) *maps $\Phi_0[q_1]$ and $\Phi_0[q_2]$, respectively. Let $s > d/2$, $M \ge 1$. Suppose $\|q_j\|_{H^s(\Omega)} \le M$ $(j = 1,2)$, and $\operatorname{supp}(q_1 - q_2) \subset \Omega$. Then if $\|\Phi_0[q_1] - \Phi_0[q_2]\| < 1$, the following inequality*

$$\|q_1 - q_2\|_{H^{-s}(\mathbb{R}^d)} \le C\left(-\log\left(\|\Phi_0[q_1] - \Phi_0[q_2]\|_\star\right)\right)^{-(s-d/2)} \tag{13}$$

holds, where $C > 0$ only depends on $d, s, \Omega,$ and M.

Let v_j be a solution to (11) with $q = q_j$, $(j = 1,2)$, then we have

$$\int_\Omega (q_2 - q_1)v_1v_2\,dx = \big\langle(\Phi_0[q_1] - \Phi_0[q_2])v_1|_{\partial\Omega},\, v_2|_{\partial\Omega}\big\rangle_{H^{-1/2},H^{1/2}}. \tag{14}$$

Now we would like to estimate $\widehat{q_2 - q_1}(\eta)$, $\eta \in \mathbb{R}^d$ in terms of the boundary measurements. The principal idea is to estimate the low frequencies using products of CGO's solutions, and to approximate the high frequencies through the regularity of the difference.

Lemma 3.2. *Let $s > d/2$ be an integer and $M \geq 1$. Assume $\|q_l\|_{H^s(\Omega)} \leq M$,* $\text{supp}(q_1 - q_2) \subset \Omega$. *Then there exist constants $C_M \geq 1$ such that the following inequality*

$$\|q_2 - q_1\|^2_{H^{-s}(\mathbb{R}^d)} \leq C_M \left(\frac{1}{R^{2s-d}} + \exp(CR)\|\Phi_0[q_1] - \Phi_0[q_2]\|^2_\star \right), \tag{15}$$

holds for all $R > 0$, where $C > 0$ only depends on s, Ω and d.

Proof. In the following proof, C stands for a general constant strictly larger than one depending only on d, s and Ω.

By Proposition 2.3, we can construct CGO solutions $v_j(x)$ to the equation (11) with $q = q_j$, having the form of

$$v_j(x) = \exp\left(\frac{\xi_j}{2} \cdot x\right)\big(1 + \psi_j(x)\big)$$

for $j = 1, 2$, and we have

$$\int_\Omega (q_2 - q_1) \exp\left(\frac{1}{2}(\xi_1 + \xi_2) \cdot x\right)(1 + \psi_1 + \psi_2 + \psi_1\psi_2)\, dx$$
$$= \big\langle (\Phi_0[q_1] - \Phi_0[q_2])v_1|_{\partial\Omega},\, v_2|_{\partial\Omega}\big\rangle_{H^{-1/2},H^{1/2}}, \tag{16}$$

from identity (14), where ψ_j satisfies

$$\|\psi_j\|_{H^s(\Omega)} \leq \frac{C}{|\xi_j|}\|q_j\|_{H^s(\Omega)},$$

if $\xi_j \in \mathbb{C}^d$ satisfies $\xi_j \cdot \xi_j = 0$, $|\xi_j| \geq 2$ and

$$|\xi_j| \geq C_1\|q_j\|_{H^s(\Omega)}. \tag{17}$$

Now, let $\eta \in \mathbb{R}^d$ and $\rho > 0$. We assume that $\alpha, \zeta \in \mathbb{R}^d$ satisfy

$$\alpha \cdot \eta = \alpha \cdot \zeta = \eta \cdot \zeta = 0, |\alpha| = \rho, \text{ and } |\zeta|^2 = |\eta|^2 + \rho^2. \tag{18}$$

Define ξ_1 and ξ_2 as

$$\xi_1 = \zeta + i\alpha - i\eta \quad \text{and} \quad \xi_2 = -\zeta - i\alpha - i\eta.$$

Then we have

$$\xi_j \cdot \xi_j = 0,\ |\xi_j|^2 = |\zeta|^2 + |\eta|^2 + \rho^2 = 2|\zeta|^2\ (l = 1, 2) \text{ and } \frac{1}{2}(\xi_1 + \xi_2) = -i\eta.$$

Hence by (16), we immediately obtain that

$$\widehat{q_2 - q_1}(\eta) = -\int_\Omega (q_2 - q_1)\exp(-i\eta \cdot x)(\psi_1 + \psi_2 + \psi_1\psi_2)\, dx$$
$$+ \big\langle (\Phi_0[q_1] - \Phi_0[q_2])v_1|_{\partial\Omega},\, v_2|_{\partial\Omega}\big\rangle_{H^{-1/2},H^{1/2}}, \tag{19}$$

provided $|\xi_j| \geq 2$ and (17) are satisfied. Suppose now that $\chi \in C_0^\infty(\mathbb{R}^d)$ satisfies $\chi \equiv 1$ near Ω. We first estimate the first term on the right hand side of (19) by

$$\left|\int_\Omega (q_2 - q_1)\exp(-i\eta\cdot x)(\psi_1 + \psi_2 + \psi_1\psi_2)\,dx\right|$$
$$\leq \|q_2 - q_1\|_{H^{-s}(\Omega)}\big\|\chi(\psi_1 + \psi_2 + \psi_1\psi_2)\big\|_{H^s(\Omega)}.$$

The Sobolev embedding Theorem, and Theorem 3.20 in [13], lead to

$$\left|\int_\Omega (q_2 - q_1)\exp(-i\eta\cdot x)(\psi_1 + \psi_2 + \psi_1\psi_2)\,dx\right|$$
$$\leq C\|q_2 - q_1\|_{H^{-s}(\mathbb{R}^d)}\big(\|\psi_1\|_{H^s(\Omega)} + \|\psi_2\|_{H^s(\Omega)} + \|\psi_1\|_{H^s(\Omega)}\|\psi_2\|_{H^s(\Omega)}\big)$$
$$\leq \frac{CM^2}{|\zeta|}\|q_2 - q_1\|_{H^{-s}(\mathbb{R}^d)}.$$

On the other hand, by straightforward calculations, we have

$$\big\|v_j|_{\partial\Omega}\big\|_{L^2(\partial\Omega)}, \big\|\nabla v_j|_{\partial\Omega}\big\|_{L^2(\partial\Omega)} \leq CM\exp(C|\zeta|),$$

for $j = 1, 2$, which by interpolation [13], provide

$$\big\|v_l|_{\partial\Omega}\big\|_{H^{1/2}(\partial\Omega)} \leq CM\exp(C|\zeta|).$$

Therefore, we can estimate the second term of the right-hand side of (19) by

$$\big\langle(\Phi_0[q_1] - \Phi_0[q_2])v_1|_{\partial\Omega},\, v_2|_{\partial\Omega}\big\rangle_{H^{-1/2},H^{1/2}}$$
$$\leq \big\|v_1|_{\partial\Omega}\big\|_{H^{1/2}(\partial\Omega)}\big\|v_2|_{\partial\Omega}\big\|_{H^{1/2}(\partial\Omega)}\|\Phi_0[q_1] - \Phi_0[q_2]\|_\star,$$
$$\leq CM^2\exp(C|\zeta|)\|\Phi_0[q_1] - \Phi_0[q_2]\|_\star.$$

Summing up, we have shown that for $\eta \in \mathbb{R}^d$ if we take α and ζ satisfying the conditions (18), and

$$\rho \geq C_1M + 1, \tag{20}$$

then

$$|\widehat{q_2 - q_1}(\eta)| \leq \frac{CM^2}{\rho}\|q_2 - q_1\|_{H^{-s}(\mathbb{R}^d)} + CM^2\exp(C(\rho + |\eta|))\|\Phi_0[q_1] - \Phi_0[q_2]\|_\star, \tag{21}$$

holds. Integrating the last inequality with respect to η over

$$B_R(0) = \{\eta \in \mathbb{R}^d :\, |\eta| < R\},$$

and taking into account the fact that $s > d/2$, we obtain

$$\int_{B_R(0)}\left|\widehat{q_2 - q_1}(\eta)\right|^2(1 + |\eta|^2)^{-s}d\eta$$
$$\leq \frac{CM^4}{\rho^2}\|q_2 - q_1\|^2_{H^{-s}(\mathbb{R}^d)} + CM^4\exp(C(\rho + R))\|\Phi_0[q_1] - \Phi_0[q_2]\|^2_\star. \tag{22}$$

Now since $q_2 - q_1$ belong to $H^s(\mathbb{R}^d)$, we have

$$\|q_2 - q_1\|^2_{H^{-s}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left|\widehat{q_2 - q_1}(\eta)\right|^2 (1 + |\eta|^2)^{-s} d\eta, \tag{23}$$

$$\leq \int_{B_R(0)} \left|\widehat{q_2 - q_1}(\eta)\right|^2 (1 + |\eta|^2)^{-s} d\eta + \frac{CM^2}{R^{2s-d}}. \tag{24}$$

Taking $\rho = \rho_M$, with $\rho_M^2 = (C_1M + 1)^2 + 2C^2M^4$, and combining (22) with (23), we get

$$\|q_2 - q_1\|^2_{H^{-s}(\mathbb{R}^d)} \leq \frac{CM^2}{R^{2s-d}} + CM^4 \exp(C(\rho_M + R))\|\Phi_0[q_1] - \Phi_0[q_2]\|^2_\star,$$
$$\leq C_M \left(\frac{1}{R^{2s-d}} + \exp(CR)\|\Phi_0[q_1] - \Phi_0[q_2]\|^2_\star\right),$$

which finishes the proof of the Lemma. □

Now, we shall prove Proposition 3.1.

Proof. Since $2s - d > 0$, there exists a unique $R_0 > 0$ satisfying

$$\frac{1}{R_0^{2s-d}} = \exp(CR_0)\|\Phi_0[q_1] - \Phi_0[q_2]\|^2_\star,$$

where $C > 0$ is the constant appearing in Lemma (3.2). Since $\log(R)/R$ is bounded by e^{-1} for all $R > 0$, we have

$$R_0 \geq \frac{-2\log(\|\Phi_0[q_1] - \Phi_0[q_2]\|_\star)}{C + \frac{2s-d}{e}}.$$

We deduce from the estimate (15) with $R = R_0$, and the previous inequality

$$\|q_2 - q_1\|^2_{H^{-s}(\mathbb{R}^d)} \leq \frac{2C_M}{R_0^{2s-d}} \leq 2C_M \left(\frac{-2\log(\|\Phi_0[q_1] - \Phi_0[q_2]\|_\star)}{C + \frac{2s-d}{e}}\right)^{-(2s-d)},$$

completing the proof of the Proposition. □

Next, assuming that the potential q is known we identify the source $a\delta_z$ from the knowledge of $\Phi[q, a, z](0)$.

Lemma 3.3. *Let $s > (d/2) + 1$ be an integer. Let $\xi \in \mathbb{C}^d$ satisfy $\xi \cdot \xi = 0$, $|\xi| \geq 2$, and $\xi \cdot e_1 > 0$. There exist constants $C_3 > 0$ and $C_4 > 0$ that only depend on Ω, s, and d such that if*

$$|\xi| \geq C_3\|q\|_{H^s(\Omega)},$$

then there exist solutions v and w to the equations (9)*, respectively, satisfying*

$$\Delta w + \nabla \log(v^2) \cdot \nabla w = 0 \text{ in } \Omega,$$

having the form

$$v(x) = \exp\left(\frac{\xi}{2} \cdot x\right)\big(1 + \psi_v(x)\big), \qquad w(x) = \xi \cdot x + \psi_w(x), \tag{25}$$

where $v \neq 0$, $\partial_{x_1} w \neq 0$ in Ω, and where ψ_v and ψ_w have the estimates

$$\|\psi_v\|_{H^s(\Omega)} \leq \frac{C_4}{|\xi|}\|q\|_{H^s(\Omega)}, \qquad \|\psi_w\|_{H^s(\Omega)} \leq C_4\|q\|_{H^s(\Omega)}. \tag{26}$$

In addition the function $\phi := vw$ lies in $H^s(\Omega)$, and satisfies the equation (9).

Proof. Assuming that $|\xi| \geq C_1\|q\|_{H^s(\Omega)}$, we deduce from Proposition 2.3, the existence of a solution $v \in H^s(\Omega)$, to the equation (1) with the form (30), and ψ_v, verifying the following estimate

$$\|\psi_v\|_{H^s(\Omega)} \leq \frac{C_2}{|\xi|}\|q\|_{H^s(\Omega)}, \tag{27}$$

where C_1 and C_2 are as in Lemma 2.2.
Since $s > (d/2) + 1$, $H^s(\Omega)$ is compactly embedded in $C^1(\overline{\Omega})$, and the inequality

$$\|\varphi\|_{L^\infty(\Omega)} \leq C_0\|\varphi\|_{H^s(\Omega)}, \tag{28}$$

is valid for all $\varphi \in H^s(\Omega)$, where $C_0 > 0$ is a constant that only depends on Ω, s, and d. Combining the last two inequalities, we get

$$\|\psi_v\|_{L^\infty(\Omega)} \leq \frac{2C_0C_2}{|\xi|}\|q\|_{H^s(\Omega)}.$$

Since $\xi \cdot \xi = 0$, taking $|\xi| \geq \max\{C_1, 4C_0C_2\}\|q\|_{H^s(\Omega)}$, leads to

$$|v| \geq 2\left|\exp\left(\frac{\xi}{2}\cdot x\right)\right| > 0.$$

Since $v \neq 0$ in Ω, we have $\nabla \log(v^2) \in H^{s-1}(\Omega)$ and has the following decomposition

$$\nabla \log(v^2) = \xi + \frac{\nabla\psi_v}{1+\psi_v}.$$

Let $\tilde{\psi}_v \in H^s(\mathbb{R}^d)$ be a compact supported extension of v to the whole space as in the proof of Lemma 2.2, satisfying

$$\|\tilde{\psi}_v\|_{H^s(\mathbb{R}^d)} \leq C\|\psi_v\|_{H^s(\Omega)}, \tag{29}$$

where C is a constant than only depends on s and d.
Denote now $\tilde{\psi}_w$ the solution to

$$\Delta\tilde{\psi}_w + \xi\cdot\nabla\tilde{\psi}_w + \frac{1}{1+\tilde{\psi}_v}\nabla\tilde{\psi}_v\cdot\nabla\tilde{\psi}_w = -\frac{1}{1+\tilde{\psi}_v}\nabla\tilde{\psi}_v\cdot\xi \text{ in } \Omega,$$

Since $\frac{1}{1+\tilde{\psi}_v}\nabla\tilde{\psi}_v$ lies in $H^s(\mathbb{R}^d)$, we deduce from Lemma 2.1 that the linear operator $\mathcal{K}_\xi(\frac{1}{1+\tilde{\psi}_v}\nabla\tilde{\psi}_v\cdot)$ is bounded from $H^s(\mathbb{R}^d)$ to itself. Let I_d be the identity operator from $H^s(\mathbb{R}^d)$ to itself. The decaying behavior (27), and the inequality (29) imply that the operator $K_\xi(\frac{1}{1+\tilde{\psi}_v}\nabla\tilde{\psi}_v\cdot\nabla\cdot)$ is a contraction for $|\xi|$ large enough, and hence $I_d - \mathcal{K}_\xi(\frac{1}{1+\tilde{\psi}_v}\nabla\tilde{\psi}_v\cdot\nabla\cdot)$ becomes invertible. In fact we have

$$\begin{aligned}\|\mathcal{K}_\xi(\frac{1}{1+\tilde{\psi}_v}\nabla\tilde{\psi}_v\cdot\nabla\psi)\|_{H^s_\delta} &\leq C\|\frac{1}{1+\tilde{\psi}_v}\nabla\tilde{\psi}_v\cdot\nabla\psi\|_{H^{s-1}_{\delta+1}}\\ &\leq C'\|\psi_v\|_{H^s(\Omega)}\|\psi\|_{H^s(\mathbb{R}^d)} \leq \frac{C''}{|\xi|}\|q\|_{H^s(\Omega)}\|\psi\|_{H^s(\mathbb{R}^d)},\end{aligned}$$

for all $\psi \in H^s(\mathbb{R}^d)$, and where $C, C', C'' > 0$ only depends on Ω, s, and d. Then

$$\tilde{\psi}_w = -\left(I_d - \mathcal{K}_\xi(\frac{1}{1+\tilde{\psi}_v}\nabla\tilde{\psi}_v\cdot\nabla\cdot)\right)^{-1}\mathcal{K}_\xi(\frac{1}{1+\tilde{\psi}_v}\nabla\tilde{\psi}_v\cdot\xi),$$

The second estimate (26) follows immediately by taking ψ_w the restriction of $\tilde{\psi}_w$ to the domain Ω, from the results of Lemma 2.1, and the convergence of the Neumann series. □

Now we are ready to prove the main stability estimate. In the following proof, C stands for a general constant strictly larger than one depending only on d, s and Ω.

Let
$$\theta_1(x) = \frac{v(x)}{v(z_1)}\frac{(w(x)-w(z_2))}{w(z_1)-w(z_2)}, \quad \theta_2(x) = \frac{v(x)}{v(z_2)}\frac{(w(x)-w(z_1))}{w(z_2)-w(z_1)}. \tag{30}$$

Because of the construction of the functions θ_1 and θ_2 are solutions to the equation (9), and satisfy $\theta_i(z_j) = \delta_{ij}$, $i, j = 1, 2$, where δ_{ij} is the Kronecker delta, that is

$$\theta_i(z_i) = 1, \quad \text{and} \quad \theta_i(z_j) = 0 \text{ if } i \neq j.$$

Moreover, we have

$$\begin{aligned}a_1\varphi(z_1) - a_2\varphi(z_2) &= \langle a_1\delta_{z_1} - a_2\delta_{z_2}, \varphi\rangle_{H^{-s},H^s}\\ &= \langle a_1\delta_{z_1} - a_2\delta_{z_2}, \varphi(z_1)\theta_1 + \varphi(z_2)\theta_2\rangle_{H^{-s},H^s},\end{aligned} \tag{31}$$

for all $\varphi \in H^s(\Omega)$. Here $\langle,\rangle_{H^{-s},H^s}$ stands for the dual product between $H^{-s}(\Omega)$ and $H^s(\Omega)$. Then

$$\|a_1\delta_{z_1} - a_2\delta_{z_2}\|_{H^{-s}(\Omega)} = \sup_{\varphi\in H^s(\Omega)}\left|\langle a_1\delta_{z_1} - a_2\delta_{z_2}, \varphi(z_1)\theta_1 + \varphi(z_2)\theta_2\rangle_{H^{-s},H^s}\right|.$$

Proposition 3.4. *Let θ_i be defined as in* (30). *Then there exists a constant $C > 0$ that only depends on Ω, s, and d such that the following inequality*

$$\|\varphi(z_1)\theta_1 + \varphi(z_2)\theta_2\|_{H^{1/2}(\partial\Omega)} \leq C\|\varphi\|_{H^s(\Omega)},$$

is true for all $\varphi \in H^s(\Omega)$.

Proof. We first prove that there exists a constant $C > 0$ that only depends on Ω, s, and d such that

$$|w(z_2) - w(z_1)| \geq C|z_2 - z_1|. \tag{32}$$

Indeed without loss of generality we can choose z_1 and z_2 on the line $\{te_1;\, t \in \mathbb{R}\}$, that is $z_1 = (z_1 \cdot e_1)e_1$, $z_2 = (z_2 \cdot e_1)e_1$, and $(z_2 - z_1) \cdot e_1 = |z_2 - z_1|e_1$.

Hence
$$w(z_2) - w(z_1) = (\xi \cdot e_1)|z_2 - z_1| + \psi_w(z_2) - \psi_w(z_1). \tag{33}$$
Since $s > (d/2) + 1$, we have
$$|\psi_w(z_2) - \psi_w(z_1)| \le C\|q\|_{H^s(\Omega)}|z_2 - z_1|. \tag{34}$$
Combining equations (33) and (34), we get
$$|w(z_2) - w(z_1)| \ge \big|(\xi \cdot e_1) - C\|q\|_{H^s(\Omega)}\big|\, |z_2 - z_1|.$$
Then by choosing $(\xi \cdot e_1) > 0$ large enough we obtain (32).
Returning to the proof of the proposition, we have
$$\begin{aligned}\varphi(z_1)\theta_1(x) + \varphi(z_2)\theta_2(x) &= (\varphi(z_2) - \varphi(z_1))\theta_2(x) + \varphi(z_1)(\theta_1(x) + \theta_2(x))\\ &= \frac{v(x)}{v(z_2)}\frac{\varphi(z_2) - \varphi(z_1)}{w(z_2) - w(z_1)}(w(x) - w(z_1)) + \varphi(z_1)\frac{v(x)}{v(z_1)}\\ &\quad + \varphi(z_1)v(x)\Big(\frac{1}{v(x_2)} - \frac{1}{v(x_1)}\Big)\frac{1}{w(z_2) - w(z_1)}.\end{aligned}$$

We finally deduce from the inequality (32), and the regularity of the functions φ, v and w the desired inequality. □

For $j = 1, 2$, let u_j be the solution to the equation (1) with zero Dirichlet boundary condition, q_j, $a_j\delta_{z_j}$, as a potential, and a source respectively.
Then $U = u_2 - u_1$ is a solution to
$$\big(\Delta + q_2(x)\big)U(x) = a_2\delta_{z_2}(x) - a_1\delta_{z_1}(x) + u_1(x)(q_1(x) - q_2(x)) \quad \text{in} \quad \Omega. \tag{35}$$
We further assume that the functions θ_1 and θ_2 are constructed as previously with $q = q_2$.
For a given test function $\varphi \in H^s(\Omega)$, multiplying equation (35) by $\varphi(z_1)\theta_1 + \varphi(z_2)\theta_2$, and integrating by parts, we obtain
$$\begin{aligned}|\langle a_1\delta_{z_1} - a_2\delta_{z_2}, \varphi\rangle_{H^{-s},H^s}| &\le \|\frac{\partial U}{\partial \nu}\|_{H^{-1/2}(\partial\Omega)}\|\varphi(z_1)\theta_1\\ &+\varphi(z_2)\theta_2\|_{H^{1/2}(\partial\Omega)} + \|q_2 - q_1\|_{H^{-s}(\mathbb{R}^d)}\|u_1\|_{H^s(\Omega)}\|\varphi\|_{H^s(\Omega)}.\end{aligned} \tag{36}$$

Combining the results of Propositions 3.1 and 3.4, we finish the proof of the main stability estimate.

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