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Quasilinear Problems without the Ambrosetti-Rabinowitz Condition

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We show the existence of nontrivial solutions for a class of quasilinear problems in which the governing operators depend on the unknown function. By using a suitable variational setting and a weak version of the Cerami-Palais-Smale condition, we establish the desired result without assuming that the nonlinear source satisfies the Ambrosetti-Rabinowitz condition.

Keywords: Quasilinear equation, weak Cerami-Palais-Smale condition, failure of the Ambrosetti-Rabinowitz condition, p-superlinear problem, subcritical growth.

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1. Introduction

In this paper we investigate the existence of weak bounded solutions of the problem

$$\begin{cases} -\mathrm{div}(A(x,u)|\nabla u|^{p-2}\nabla u) + \frac{1}{p}\, A_t(x,u)|\nabla u|^p \;=\; g(x,u) & \text{in } \Omega, \\ u \;=\; 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

with $\Omega \subset \mathbb{R}^N$ bounded domain, $N \geq 1$, $p > 1$, where $A(x,t)$, $g(x,t)$ are given real functions on $\Omega \times \mathbb{R}$ and $A_t(x,t) = \frac{\partial}{\partial t} A(x,t)$.

Due to the fact that the divergence term depends also on the unknown function u, the quasilinear equation in (1) cannot be studied with standard variational techniques. For this reason, in the last years different approaches have been developed involving nonsmooth tools (see [10, 11, 12]) or a suitable definition of critical point, since weak solutions of (1) require as test functions only elements of $W_0^{1,p}(\Omega)$ which are also in $L^\infty(\Omega)$ (see, e.g., [3] and [17]).

More recently, the idea has come up to formulate problem (1) in a suitable Banach space X, namely $X = W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ equipped with the intersection norm

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$\|\cdot\|_X = \|\cdot\|_W + |\cdot|_\infty$, so that its weak solutions coincide with the *true* critical points of the associated functional

$$\mathcal{J}(u) \;=\; \frac{1}{p}\int_\Omega A(x,u)|\nabla u|^p dx - \int_\Omega G(x,u)dx, \qquad u \in X, \tag{2}$$

with $G(x,t) = \int_0^t g(x,\tau)d\tau$ (see [5, 6]).

Following such an approach we consider in this paper suitable assumptions, in particular those ones introduced in [15] for a superlinear (p,q)-equation, which allow us to prove the existence of at least one nontrivial critical point of $\mathcal{J}$ in X, i.e., a weak bounded solution of (1), when the nonlinear term $g(x,t)$ is $(p-1)$-superlinear but does not satisfies the Ambrosetti-Rabinowitz condition.

Problem (1) with a $(p-1)$-superlinear term $g(x,t)$ has been already studied if the Ambrosetti-Rabinowitz condition, or a similar slightly more general assumption, holds (see [3, 5, 6, 9, 10, 17]). Eventually, the term $A(x,t)|\xi|^p$ is replaced by some $\mathbb{A}(x,t,\xi)$, but both in [3] and [10] it is assumed $A_t(x,t)t \ge 0$ a.e. in Ω for all $t \in \mathbb{R}$. On the contrary, in [5, 6, 9, 17] such a product can also change sign while, here, with the failure of the Ambrosetti-Rabinowitz condition, we require $A_t(x,t)t \le 0$ (see Remark 4.5).

We note that, in order to find critical points of $\mathcal{J}$ in the intersection space X, we cannot apply the classical Mountain Pass Theorem in [2] as our functional $\mathcal{J}$ does not satisfy the Palais-Smale condition, or its Cerami's variant, in X (Palais-Smale sequences may converge in $W_0^{1,p}(\Omega)$ and be unbounded in $L^\infty(\Omega)$, see, e.g., [8, Example 4.3]). Hence, a weaker version of the Cerami-Palais-Smale condition is required and we can use a generalized version of the Mountain Pass Theorem (see Section 2).

Since our main theorem covers very general situations and a list of conditions is needed, we shall give the complete framework in Sections 3 and 4. However here, in order to highlight how our approach improves previous results, we consider the particular example

$$A(x,t) \;=\; a(x) - \operatorname{arctg}|t|^\theta, \tag{3}$$

so that problem (1) reduces to

$$\begin{cases} -\operatorname{div}\Big((a(x) - \operatorname{arctg}|u|^\theta)|\nabla u|^{p-2}\nabla u\Big) - \dfrac{\theta}{p}\dfrac{|u|^{\theta-2}u}{1+|u|^{2\theta}}|\nabla u|^p = g(x,u) & \text{in } \Omega, \\ u \;=\; 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Theorem 1.1. *Let $a \in L^\infty(\Omega)$ be such that*

$$a(x) \ge a_0 > \frac{\pi}{2} + \frac{\theta}{2p} \quad \textit{a.e. in } \Omega$$

and assume that $1 < \theta \le p$. If $g(x,t)$ satisfies the assumptions (G0)–(G4) *stated in Sections* 3 *and* 4*, for example,*

$$g(x,t) = g_1(t) = \begin{cases} |t|^{q-2}t & \textit{if } |t| \le 1 \\ |t|^{p-2}t\Big(\log|t| + 1\Big) & \textit{if } |t| > 1 \end{cases} \qquad \textit{with } 1 < p < q < p^*, \tag{5}$$

then problem (4) *admits at least one nontrivial bounded weak solution.*

We note that the assumptions on $a(x)$ and θ, given in Theorem 1.1, allow function $A(x,t)$ in (3) to verify all the conditions (H0)–(H4) required in Section 4. Thus, Theorem 1.1 is a corollary of Theorem 4.3 (see also Example 4.2).

2. Abstract tools

Throughout this section, we denote $\mathbb{N} = \{1, 2, \dots\}$ and assume that:

- $(X, \|\cdot\|_X)$ is a Banach space with dual space $(X', \|\cdot\|_{X'})$,
- $(W, \|\cdot\|_W)$ is a Banach space such that $X \hookrightarrow W$ continuously, i.e. $X \subset W$ and a constant $\rho_0 > 0$ exists such that

$$\|u\|_W \ \le\ \rho_0\ \|u\|_X \qquad \text{for all } u \in X, \tag{6}$$

- $J : \mathcal{D} \subset W \to \mathbb{R}$ and $J \in C^1(X, \mathbb{R})$ with $X \subset \mathcal{D}$.

Furthermore, fixing $c \in \mathbb{R}$, we define

- $K_J^c = \{u \in X :\ J(u) = c,\ dJ(u) = 0\}$ the set of the critical points of J in X at level c,
- $J^c = \{u \in X :\ J(u) \le c\}$ the sublevel of J with respect to c.

For simplicity, taking $c \in \mathbb{R}$, we say that a sequence $(u_n)_n \subset X$ is a *Cerami-Palais-Smale sequence at level* c, briefly $(CPS)_c$*-sequence*, if

$$\lim_{n\to+\infty} J(u_n) = c \quad \text{and} \quad \lim_{n\to+\infty} \|dJ(u_n)\|_{X'}(1 + \|u_n\|_X) = 0.$$

Moreover, c is a *Cerami-Palais-Smale level*, briefly (CPS)*-level*, if there exists a $(CPS)_c$-sequence.

Functional J satisfies the classical Cerami-Palais-Smale condition in X at level c if every $(CPS)_c$-sequence converges in X up to subsequences. Anyway, thinking about the setting of our problem, in general $(CPS)_c$-sequences may also exist which are unbounded in $\|\cdot\|_X$ but converge with respect to $\|\cdot\|_W$. Then, we can weaken the classical Cerami-Palais-Smale condition in the following way.

Definition 2.1. Given $c \in \mathbb{R}$, functional J satisfies the *weak Cerami-Palais-Smale condition at level* c, briefly $(wCPS)_c$ *condition*, if for every $(CPS)_c$-sequence $(u_n)_n$, a point $u \in X$ exists such that

(i) $\lim_{n\to+\infty} \|u_n - u\|_W = 0$ (up to subsequences),

(ii) $J(u) = c,\ dJ(u) = 0$.

We say that J satisfies the $(wCPS)$ condition in I, I real interval, if J satisfies the $(wCPS)_c$ condition at each level $c \in I$. □

Due to the convergence only in the norm $\|\cdot\|_W$, the $(wCPS)_c$ condition implies that the set of critical points of J at level c is compact with respect to $\|\cdot\|_W$. Anyway, this weaker "compactness" assumption allows one to prove the following Deformation Lemma (see [7, Lemma 2.3] which is stated in the weaker condition that each (CPS)-level is a critical level, too).

Lemma 2.2. (Deformation Lemma) *Let* $J \in C^1(X, \mathbb{R})$ *and consider* $c \in \mathbb{R}$ *such that*

- *J satisfies the $(wCPS)_c$ condition,*
- $K_J^c = \emptyset$.

Then, fixing any $\bar{\varepsilon} > 0$, there exist a constant $\varepsilon > 0$ and a homeomorphism $\psi : X \to X$ such that $2\varepsilon < \bar{\varepsilon}$ and

(i) $\psi(J^{c+\varepsilon}) \subset J^{c-\varepsilon}$,

(ii) *$\psi(u) = u$ for all $u \in X$ such that either $J(u) \le c - \bar{\varepsilon}$ or $J(u) \ge c + \bar{\varepsilon}$.*

Moreover, if J is even on X, then ψ can be chosen odd.

From Lemma 2.2 the following generalization of the Mountain Pass Theorem in [2, Theorem 2.1] can be stated (for the proof, see [7, Theorem 1.7]).

Theorem 2.3. (Mountain Pass Theorem) *Let $J \in C^1(X, \mathbb{R})$ be such that $J(0) = 0$ and the $(wCPS)$ condition holds in $\mathbb{R}$. Moreover, assume that two constants r_0, $\varrho_0 > 0$ and a point $e \in X$ exist such that*

$$u \in X, \ \|u\|_W = r_0 \quad \Longrightarrow \quad J(u) \ge \varrho_0, \tag{7}$$

$$\|e\|_W > r_0 \qquad \text{and} \qquad J(e) < \varrho_0. \tag{8}$$

Then, J has a Mountain Pass critical point $u^* \in X$ such that $J(u^*) \ge \varrho_0$.

3. Variational setting and first properties

Here and in the following, $|\cdot|$ is the standard norm on any Euclidean space as the dimension of the considered vector is clear and no ambiguity arises and meas(B) is the usual N-dimensional Lebesgue measure of a measurable set B in $\mathbb{R}^N$. Furthermore, let $\Omega \subset \mathbb{R}^N$ be an open bounded domain, $N \ge 1$, so we denote by:

- $L^\nu(\Omega)$ the Lebesgue space with norm $|u|_\nu = \left(\int_\Omega |u|^\nu dx\right)^{1/\nu}$ if $1 \le \nu < +\infty$;
- $L^\infty(\Omega)$ the space of Lebesgue-measurable and essentially bounded functions $u : \Omega \to \mathbb{R}$ with norm
$$|u|_\infty = \operatorname*{ess\,sup}_\Omega |u|;$$
- $W_0^{1,p}(\Omega)$ the classical Sobolev space with norm $\|u\|_W = |\nabla u|_p$ if $1 \le p < +\infty$.

From now on, let $A : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that the following conditions hold:

(H0) $A(x,t)$ is a C^1 Carathéodory function, i.e.,
$A(\cdot, t) : x \in \Omega \mapsto A(x,t) \in \mathbb{R}$ is measurable for all $t \in \mathbb{R}$,
$A(x, \cdot) : t \in \mathbb{R} \mapsto A(x,t) \in \mathbb{R}$ is C^1 for a.e. $x \in \Omega$ with $A_t(x,t) = \frac{\partial}{\partial t} A(x,t)$;

(H1) $A(x,t)$ and $A_t(x,t)$ are essentially bounded if t is bounded, i.e.,

$$\sup_{|t| \le r} |A(\cdot, t)| \in L^\infty(\Omega), \quad \sup_{|t| \le r} |A_t(\cdot, t)| \in L^\infty(\Omega) \qquad \text{for any } r > 0;$$

(G0) $g(x,t)$ is a Carathéodory function, i.e.,
$g(\cdot,t) : x \in \Omega \mapsto g(x,t) \in \mathbb{R}$ is measurable for all $t \in \mathbb{R}$,
$g(x,\cdot) : t \in \mathbb{R} \mapsto g(x,t) \in \mathbb{R}$ is continuous for a.e. $x \in \Omega$;

(G1) a_1, $a_2 > 0$ and $q \geq 1$ exist such that

$$|g(x,t)| \leq a_1 + a_2|t|^{q-1} \qquad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}.$$

Remark 3.1. By definition, it is $G(x,0) = 0$ a.e. in Ω; furthermore, from (G0)–(G1) it follows that $G(x,t)$ is a C^1 Carathéodory function in $\Omega \times \mathbb{R}$ and there exist a_3, $a_4 > 0$ such that

$$|G(x,t)| \leq a_3 + a_4|t|^q \qquad \text{a.e in } \Omega, \text{ for all } t \in \mathbb{R}. \tag{9}$$

We note that, unlike the classical assumption (G1) which requires $q < p^*$ for obtaining the regularity of the associated Nemytskii operator (see [1]), here no upper bound on q is actually assumed. □

In order to investigate the existence of weak solutions of the nonlinear problem (1), the notation introduced for the abstract setting at the beginning of Section 2 is referred to our problem with $W = W_0^{1,p}(\Omega)$ and the Banach space $(X, \|\cdot\|_X)$ defined as

$$X := W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \qquad \|u\|_X = \|u\|_W + |u|_\infty. \tag{10}$$

Moreover, from the Sobolev Embedding Theorem, for any $\nu \in [1, p^*[$, with $p^* = \frac{pN}{N-p}$ as $N > p$ otherwise $p^* = +\infty$, a constant $\rho_\nu > 0$ exists, such that

$$|u|_\nu \ \leq \ \rho_\nu \|u\|_W \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

and the embedding $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^\nu(\Omega)$ is compact.

From the definition of X, we have that $X \hookrightarrow W_0^{1,p}(\Omega)$ and $X \hookrightarrow L^\infty(\Omega)$ with continuous embeddings, and (6) holds with $\rho_0 = 1$.

We note that $X = W_0^{1,p}(\Omega)$ if $p > N > 1$ or $p \geq N = 1$, as in these cases $W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, so the abstract part is the standard one with the usual Mountain Pass Theorem.

Now, we consider the functional $\mathcal{J} : X \to \mathbb{R}$ defined as in (2).

Taking any u, $v \in X$, by direct computations it follows that its Gâteaux differential in u along the direction v is

$$\langle d\mathcal{J}(u), v\rangle = \int_\Omega A(x,u)|\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \frac{1}{p}\int_\Omega A_t(x,u)v|\nabla u|^p dx - \int_\Omega g(x,u)v \, dx. \tag{11}$$

The following regularity result holds (see [9, Proposition 3.2]).

Proposition 3.2. *Taking $p > 1$, assume that* (H0)–(H1), (G0)–(G1) *hold. If $(u_n)_n \subset X$, $u \in X$ are such that*

$$\|u_n - u\|_W \to 0, \quad u_n \to u \text{ a.e. in } \Omega \qquad \text{if } n \to +\infty$$

$$\text{and } M > 0 \text{ exists so that } |u_n|_\infty \leq M \text{ for all } n \in \mathbb{N},$$

$$\text{then} \qquad \mathcal{J}(u_n) \to \mathcal{J}(u) \quad \text{and} \quad \|d\mathcal{J}(u_n) - d\mathcal{J}(u)\|_{X'} \to 0 \quad \text{if } \ n \to +\infty.$$

Hence, $\mathcal{J}$ is a C^1 functional on X with Fréchet differential defined as in (11).

4. Statement of the main result

From now on, we assume that in addition to hypotheses (H0)–(H1) and (G0)–(G1), functions $A(x,t)$ and $g(x,t)$ satisfy the following further conditions:

(H2) a constant $\alpha_0 > 0$ exists such that

$$A(x,t) \geq \alpha_0 \qquad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R};$$

(H3) some constants $R_0 \geq 1$ and $\alpha_1 > 0$ exist such that

$$A(x,t) + \frac{1}{p} A_t(x,t)t \geq \alpha_1 A(x,t) \quad \text{a.e. in } \Omega \text{ if } |t| \geq R_0;$$

(H4) $A_t(x,st)s^{p+1}t \geq A_t(x,t)t$ for all $s \in [0,1]$, for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$;

(G2) $\displaystyle\lim_{|t|\to+\infty} \frac{G(x,t)}{|t|^p} = +\infty$ uniformly for a.e. $x \in \Omega$;

(G3) taking $\sigma(x,t) = g(x,t)t - pG(x,t)$, assume that $\beta \in L^1(\Omega)$ exists such that $\beta(x) \geq 0$ a.e. in Ω and

$$\sigma(x,t_1) \leq \sigma(x,t_2) + \beta(x) \quad \text{a.e. in } \Omega, \text{ for all } 0 \leq t_1 \leq t_2 \text{ or } t_2 \leq t_1 \leq 0;$$

(G4) $\displaystyle\lim_{t\to 0} \frac{G(x,t)}{|t|^p} = 0$ uniformly for a.e. $x \in \Omega$.

Remark 4.1. Condition (G3) was introduced in [15] in order to prevent the use of the Ambrosetti-Rabinowitz condition, and a slight improvement has been recently proposed in [16]. See also [4] for an application in a different framework. □

Example 4.2. We note that function $g_1(t)$ in (5) fails to satisfy the Ambrosetti-Rabinowitz condition but verifies conditions (G0)–(G4). On the contrary, function

$$g(x,t) = g_2(t) = |t|^{q-2}t \qquad \text{with } p < q < p^*,$$

satisfies both the Ambrosetti-Rabinowitz condition and hypotheses (G0)–(G4). □

Our main result reads as follows.

Theorem 4.3. *Assume* (H0)–(H4) *and* (G0)–(G4). *If* $1 < p < q < p^*$, *then problem* (1) *admits a nontrivial bounded weak solution.*

Remark 4.4. We note that condition (H3) allows $A_t(x,t)t \leq 0$. Note that, otherwise, condition (H3) is trivial. Our existence result extends the one proved in [17], where the Ambrosetti-Rabinowitz condition is required. □

Remark 4.5. Taking $s = 0$ in (H4) we have indeed that

$$A_t(x,t)t \leq 0 \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}. \tag{12}$$

Hence, from (H0) and (12) it follows that for a.e. $x \in \Omega$ the C^1 map $A(x,\cdot)$ is increasing in $]-\infty,0]$, decreasing in $[0,+\infty[$, then it attains its maximum in $t = 0$. On the other hand, (H1) implies that $A(\cdot,0) \in L^\infty(\Omega)$; hence, $\gamma_A > 0$ exists with

$$A(x,t) \leq \gamma_A \qquad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}. \tag{13}$$

Such a requirement was already assumed in [3] and [10]. □

Remark 4.6. From (G0)–(G2) and direct computations it follows that for all $\mu > 0$ a constant $L_\mu > 0$ exists, such that

$$G(x,t) \geq \mu|t|^p - L_\mu \qquad \text{a.e in } \Omega, \text{ for all } t \in \mathbb{R}. \tag{14}$$

We note that, for the arbitrariness of μ, (14) and (G1) imply $p < q$. □

Remark 4.7. Condition (G3) implies that $\sigma(x,0) = 0$ a.e in Ω, and then

$$\sigma(x,t) \geq -\beta(x) \qquad \text{a.e in } \Omega, \text{ for all } t \in \mathbb{R}.$$

Hence,

$$\int_\Omega \sigma(x,u)dx \ \geq \ -|\beta|_1 \qquad \text{for all } u \in X. \quad \square \tag{15}$$

5. Proof of the main result

The goal of this section is to prove the existence of a weak bounded nontrivial solution of problem (1), so, by using the variational principle which follows from Proposition 3.2, we want to apply Theorem 2.3 to the functional $\mathcal{J}$ in (2) on the Banach space X as in (10).

Proposition 5.1. *If* $1 < p < q < p^*$ *and* (H0)–(H4), (G0)–(G3) *hold, then functional* $\mathcal{J}$ *satisfies the weak Cerami-Palais-Smale condition in* X *at each level* $c \in \mathbb{R}$.

Proof. Let $c \in \mathbb{R}$ be fixed and consider a sequence $(u_n)_n \subset X$ such that

$$\mathcal{J}(u_n) = c + \varepsilon_n \quad \text{and} \quad \|d\mathcal{J}(u_n)\|_{X'}(1 + \|u_n\|_X) = \varepsilon_n, \tag{16}$$

where, for simplicity, throughout this proof, we use the notation $(\varepsilon_n)_n$ for any infinitesimal sequence depending only on $(u_n)_n$.

Firstly, we want to prove that

$$(u_n)_n \quad \text{is bounded in } W_0^{1,p}(\Omega). \tag{17}$$

The ideas of the proof of (17) are essentially contained in [13, Lemma 2.2] and [15, Proposition 3] (see also [14, Lemma 2.5]), but since some changes are required we include here all the details for the reader's convenience.

To this aim, arguing by contradiction, we assume that

$$\|u_n\|_W \to +\infty \qquad \text{if } \ n \to +\infty \tag{18}$$

and for any $n \in \mathbb{N}$ we define

$$v_n(x) = \frac{u_n(x)}{\|u_n\|_W} \quad \text{for a.e. } x \in \Omega, \tag{19}$$

so that $v_n \in X$. Since $(v_n)_n$ is bounded in $W_0^{1,p}(\Omega)$, a function $v \in W_0^{1,p}(\Omega)$ exists such that, up to subsequences,

$$v_n \rightharpoonup v \text{ weakly in } W_0^{1,p}(\Omega),$$

$$v_n \to v \text{ strongly in } L^\nu(\Omega) \text{ for each } \nu \in [1, p^*[, \tag{20}$$

$$v_n \to v \text{ a.e. in } \Omega. \tag{21}$$

Assume that $v \not\equiv 0$ in Ω, i.e.,

$$\mathrm{meas}(\Omega \setminus \Omega_0) > 0, \qquad \text{with } \Omega_0 = \{x \in \Omega : v(x) = 0\}. \tag{22}$$

From definition (19), the definition in (22) and (18), (21) it follows that

$$|u_n(x)| = |v_n(x)|\ \|u_n\|_W \ \to +\infty \quad \text{for a.e. } x \in \Omega \setminus \Omega_0;$$

hence, (G3) and (21) imply that

$$\frac{G(x, u_n(x))}{\|u_n\|_W^p} \;=\; \frac{G(x, u_n(x))}{|u_n(x)|^p}\ |v_n(x)|^p \;\to\; +\infty \quad \text{for a.e. } x \in \Omega \setminus \Omega_0.$$

Thus, from Fatou's Lemma and (22) it follows that

$$\int_{\Omega\setminus\Omega_0} \frac{G(x, u_n)}{\|u_n\|_W^p} dx \;\to\; +\infty,$$

which implies that

$$\int_{\Omega} \frac{G(x, u_n)}{\|u_n\|_W^p} dx \;\to\; +\infty, \tag{23}$$

as from (14) with, e.g., $\mu = 1$, and (18) we obtain that

$$\int_{\Omega_0} \frac{G(x, u_n)}{\|u_n\|_W^p} dx \;\geq\; -\frac{L_1 \mathrm{meas}(\Omega_0)}{\|u_n\|_W^p} = \varepsilon_n.$$

But (2), (16), (19) and (13) imply that

$$\varepsilon_n \;=\; -\frac{\mathcal{J}(u_n)}{\|u_n\|_W^p} \;=\; -\frac{\gamma_A}{p} + \int_{\Omega} \frac{G(x, u_n)}{\|u_n\|_W^p} dx$$

which contradicts (23). Hence, (22) cannot hold and it has to be $v(x) = 0$ a.e. in Ω.

Now, from Proposition 3.2 we have that the map

$$s \in [0, 1] \mapsto \mathcal{J}(su_n) \in \mathbb{R}$$

is C^1 in its domain for each $n \in \mathbb{N}$; then $s_n \in [0, 1]$ exists such that

$$\mathcal{J}(s_n u_n) \;=\; \max_{s\in[0,1]} \mathcal{J}(su_n). \tag{24}$$

If we fix any $\lambda > 0$ and define

$$w_n(x) \;=\; (2\lambda)^{\frac{1}{p}} v_n(x) \quad \text{for a.e. } x \in \Omega,$$

we have that $w_n \in X$; moreover, from (20) and (21) it follows that

$$w_n \to 0 \text{ strongly in } L^\nu(\Omega) \text{ for each } \nu \in [1, p^*[,$$
$$w_n \to 0 \text{ a.e. in } \Omega.$$

Hence, from Remark 3.1 with $q < p^*$, by using the continuity of the Nemytskii operator, we obtain that

$$\int_\Omega G(x, w_n)dx \ \to \ 0;$$

thus, $n_1 = n_1(\lambda) \in \mathbb{N}$ exists, such that

$$\left|\int_\Omega G(x, w_n)dx\right| \ < \ \frac{\lambda\alpha_0}{p} \qquad \text{for all } n \geq n_1, \tag{25}$$

with α_0 as in (H2). We note that (18) implies

$$\frac{(2\lambda)^{\frac{1}{p}}}{\|u_n\|_W} \ \to \ 0,$$

so $n_2 = n_2(\lambda) \geq n_1$ exists, such that

$$0 \ < \ \frac{(2\lambda)^{\frac{1}{p}}}{\|u_n\|_W} \ < 1 \qquad \text{for all } n \geq n_2;$$

then from (24), (H2), (25) and direct computations it follows that

$$\mathcal{J}(s_n u_n) \ \geq \ \mathcal{J}(w_n) \ \geq \frac{2\lambda\alpha_0}{p} \ - \ \int_\Omega G(x, w_n)dx \ \geq \ \frac{\lambda\alpha_0}{p} \quad \text{for all } n \geq n_2.$$

Whence, as $\lambda > 0$ is arbitrary, we obtain that

$$\mathcal{J}(s_n u_n) \ \to \ +\infty \qquad \text{if } n \to +\infty. \tag{26}$$

As $\mathcal{J}(0) = 0$, from (16), the limit (26) implies that $n_0 \in \mathbb{N}$ exists such that for all $n \geq n_0$ it has to be $s_n \in]0, 1[$ and then, from the Fermat Theorem, we have that

$$\frac{d}{ds}\mathcal{J}(su_n)|_{s=s_n} \ = \ 0 \qquad \text{for all } n \geq n_0,$$

which implies

$$\begin{aligned} 0 = s_n\frac{d}{ds}\mathcal{J}(su_n)|_{s=s_n} = \langle d\mathcal{J}(s_n u_n), s_n u_n\rangle = \int_\Omega A(x, s_n u_n)|\nabla(s_n u_n)|^p dx \\ +\frac{1}{p}\int_\Omega A_t(x, s_n u_n)s_n^{p+1}u_n|\nabla u_n|^p dx - \int_\Omega g(x, s_n u_n)s_n u_n dx, \end{aligned}$$

i.e.,

$$\begin{aligned} &\int_\Omega A(x, s_n u_n)|\nabla(s_n u_n)|^p dx \\ &= -\frac{1}{p}\int_\Omega A_t(x, s_n u_n)s_n^{p+1}u_n|\nabla u_n|^p dx + \int_\Omega g(x, s_n u_n)s_n u_n dx. \end{aligned} \tag{27}$$

Now, from one hand, we note that (2), (11), (16) and (12), (15), imply that

$$\begin{aligned} pc + \varepsilon_n \ &= \ p\mathcal{J}(u_n) - \langle d\mathcal{J}(u_n), u_n\rangle \\ &= \ -\frac{1}{p}\int_\Omega A_t(x, u_n)u_n|\nabla u_n|^p dx + \int_\Omega \sigma(x, u_n)dx \ \geq \ -|\beta|_1; \end{aligned}$$

hence,

$$\left(-\frac{1}{p}\int_\Omega A_t(x,u_n)u_n|\nabla u_n|^p dx + \int_\Omega \sigma(x,u_n)dx\right)_n \quad \text{is bounded;} \tag{28}$$

while, on the other hand, $s_n \in [0,1]$ and (G3) give

$$\sigma(x, s_n u_n(x)) \le \sigma(x, u_n(x)) + \beta(x) \quad \text{for a.e. } x \in \Omega, \text{ all } n \in \mathbb{N};$$

thus,

$$\int_\Omega \sigma(x, s_n u_n)dx \ \le \ \int_\Omega \sigma(x,u_n)dx + |\beta|_1 \quad \text{for all } n \in \mathbb{N}. \tag{29}$$

Summing up, from definition (2), estimates (27), (29), assumption (H4) and (28), for all $n \ge n_0$ we obtain that

$$\begin{aligned}\mathcal{J}(s_n u_n) &= -\frac{1}{p^2}\int_\Omega A_t(x, s_n u_n)s_n^{p+1}u_n|\nabla u_n|^p dx + \frac{1}{p}\int_\Omega \sigma(x, s_n u_n)dx \\ &\le -\frac{1}{p^2}\int_\Omega A_t(x,u_n)u_n|\nabla u_n|^p dx + \frac{1}{p}\int_\Omega \sigma(x,u_n)dx + |\beta|_1 \ \le \ b\end{aligned}$$

for some $b > 0$, in contradiction to (26). In consequence we conclude that (17) is true and $u \in W_0^{1,p}(\Omega)$ exists such that, up to subsequences, we have

$$\begin{aligned}&u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega),\\ &u_n \to u \text{ strongly in } L^\nu(\Omega) \text{ for each } \nu \in [1, p^*[,\\ &u_n \to u \text{ a.e. in } \Omega.\end{aligned}$$

Now, proceeding exactly as in Steps 2–5 of the proof of [6, Proposition 4.6], it has to be that $u \in L^\infty(\Omega)$, too, and not only $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$ but also u is a critical point of $\mathcal{J}$ in X such that $\mathcal{J}(u) = c$. □

Proof of Theorem 4.3. From (9), (G4) and direct computations we get that for every $\varepsilon > 0$ a constant $C_\varepsilon > 0$ exists such that

$$G(x,t) \le \varepsilon|t|^p + C_\varepsilon|t|^q \quad \text{for a.e. } x \in \Omega \text{ and for all } t \in \mathbb{R}. \tag{30}$$

Then, from (2), (30), (H2), the Poincaré and the Sobolev inequalities it follows that

$$\begin{aligned}\mathcal{J}(u) &\ge \frac{1}{p}\int_\Omega A(x,u)|\nabla u|^p dx - \varepsilon\int_\Omega |u|^p dx - C_\varepsilon \int_\Omega |u|^q dx \\ &\ge \left(\frac{\alpha_0}{p} - \frac{\varepsilon}{\lambda_1}\right)\|u\|_W^p - \tilde{C}_\varepsilon\|u\|_W^q\end{aligned}$$

for some $\tilde{C}_\varepsilon$ and for all $u \in X$, where

$$\lambda_1 = \inf_{u \ne 0,\, u \in W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p dx}{\displaystyle\int_\Omega |u|^p dx}.$$

Hence, if $\varepsilon < \frac{\lambda_1 \alpha_0}{p}$ and $\|u\|_W$ is small enough, we immediately deduce that 0 is a local minimum point for $\mathcal{J}$ and (7) in Proposition 2.3 holds for suitable r_0, $\varrho_0 > 0$.

On the other hand, denoting by φ_1 the first positive eigenfunction of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ with $|\varphi_1|_p = 1$, from (14) with any fixed $\mu > 0$, and from (2) and (13) we get

$$\mathcal{J}(s\varphi_1) \ \le \ s^p \left(\frac{\gamma_A}{p} \lambda_1 - \mu \right) + L_\mu |\Omega| \quad \text{for any } s > 0.$$

Hence, by choosing μ and s sufficiently large, we obtain that $\mathcal{J}$ satisfies also the geometrical assumption (8) of Theorem 2.3; thus, by Proposition 5.1 we can apply Theorem 2.3 and conclude with the existence of a nontrivial solution to problem (1). □

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