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# Sharp Estimate of the Cost of Controllability for a Degenerate Parabolic Equation with Interior Degeneracy

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This work is motivated by the study of null controllability for the typical degenerate parabolic equation with interior degeneracy and one-sided control:

$$u_t - (|x|^\alpha u_x)_x = h(x,t)\chi_{(a,b)}, \quad x \in (-1,1),$$

with $0 < a < b < 1$. It was proved in [7] that this equation is null controllable (in any positive time $T$) if and only if $\alpha < 1$, and that the cost of null controllability blows up as $\alpha \to 1^-$. This is related to the following property of the eigenvalues: the gap between an eigenvalue of odd order and the consecutive one goes to 0 as $\alpha \to 1^-$ (see [7]).

The goal of the present work is to provide optimal upper and lower estimates of the null controllability cost, with respect to the degeneracy parameter (when $\alpha \to 1^-$) and in short time (when $T \to 0^+$). We prove that the null controllability cost behaves as $\frac{1}{1-\alpha}$ as $\alpha \to 1^-$ and as $e^{1/T}$ as $T \to 0^+$. Our analysis is based on the construction of a suitable family biorthogonal to the sequence $(e^{\lambda_n t})_n$ in $L^2(0,T)$, under some general gap conditions on the sequence $(\lambda_n)_n$, conditions that are suggested by a motivating example.

*Keywords:* Controllability, degenerate parabolic equation, biorthogonal family

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## 1. Introduction

### 1.1. General considerations

Degenerate parabolic equations have received increasing attention in recent years because of their connections with several applied domains such as climate science, populations genetics, vision, and mathematical finance (see, e.g., [7, 9] and the references therein). Indeed, in all these fields, one is naturally led to consider parabolic

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problems where the diffusion coefficients lose uniform ellipticity. Different situations may occur: degeneracy (of uniform ellipticity) may take place at the boundary or in the interior of the space domain. Moreover, the equation may be degenerate on a small set or even on the whole domain.

From the point of view of control theory, interesting phenomena have been pointed out for degenerate parabolic equations, in particular the existence of threshold values, where some property completely changes its nature (for examples, in several examples, null controllability holds below some critical value but not above). We refer the reader to [8, 9] for boundary-degenerate parabolic operators and to [2, 3] for interior-degenerate equations, associated with certain classes of hypoelliptic diffusion operators (and see also [4]) for Grushin type structures, and to [1] for the Heisenberg operator.

### 1.2. The problem considered here, and the main results

In [7], null controllability for the following parabolic equation

$$\begin{cases} u_t - (|x|^\alpha u_x)_x = h(x,t)\chi_{(a,b)}, & x \in (-1,1) \\ u(-1,t) = 0, & t \in (0,T), \\ u(1,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in (-1,1), \end{cases} \tag{1}$$

- with interior degeneracy (at point $x = 0$),
- and using a one-sided control (localized in $(a,b)$ with $0 < a < b < 1$),

has been studied from both the theoretical and numerical point of view. It turns out that null controllability,

- fails for $\alpha \in [1,2)$,
- holds true when $\alpha \in [0,1)$.

Consequently, the control acting on $(a,b)$ is sufficiently strong to cross the degeneracy point $x = 0$ if and only if $\alpha < 1$. A tool to measure the change of behavior for $\alpha = 1$ is to estimate the "null controllability cost", that is:

- given $u_0$, consider the set of admissible controls $h$ driving the solution of (1) to rest in time $T$:
$$\mathcal{U}^{ad}(\alpha,T;u_0) := \Big\{ h \in L^2((a,b)\times(0,T)) \mid u^{(h)}(T) = 0 \Big\}.$$
- then, given $u_0$, consider the norm of the best admissible control:
$$\inf_{h\in\mathcal{U}^{ad}(\alpha,T;u_0)} \|h\|_{L^2((a,b)\times(0,T))},$$
and maximize this quantity along $u_0$ in the unit ball or sphere of $L^2(-1,1)$, hence, roughly speaking, the smaller quantity that one needs to control all the initial conditions of the unit ball or sphere of $L^2(-1,1)$:
$$C_{NC}(\alpha,T) := \sup_{\|u_0\|_{L^2(-1,1)}=1} \Big( \inf_{h\in\mathcal{U}^{ad}(\alpha,T;u_0)} \|h\|_{L^2((a,b)\times(0,T))} \Big), \tag{2}$$
- finally, estimate the behavior of the null controllability cost $C_{NC}(\alpha,T)$ as $\alpha \to 1^-$.

It was proved in [7] that there exist constants $C, C' > 0$, independent of $\alpha \in [0,1)$ and of $T > 0$, such that

$$\frac{C}{(1-\alpha)\sqrt{T}} e^{-T/C} \leq C_{NC}(\alpha, T) \leq \frac{C'}{(1-\alpha)^2}\, e^{C/T'}\, e^{-T/C'}. \tag{3}$$

Therefore:

- $C_{NC}(\alpha, T)$ blows up as $\alpha \to 1^-$ (as expected since null controllability does not hold for $\alpha = 1$), with a blow-up rate between $\frac{1}{1-\alpha}$ and $\frac{1}{(1-\alpha)^2}$,
- and $C_{NC}(\alpha, T)$ blows up as $T \to 0^+$, with a blow-up rate between $\frac{1}{\sqrt{T}}$ and $e^{1/T}$.

The goal of the present paper is to improve the estimates (3), proving that

- $C_{NC}(\alpha, T)$ blows up exactly as $\frac{1}{1-\alpha}$ when $\alpha \to 1^-$,
- and $C_{NC}(\alpha, T)$ blows up exponentially as $e^{1/T}$ when $T \to 0^+$.

(See Theorems 2.1 and 2.2 for a precise statement.)

### 1.3. Main tools and comparison with the literature

Degenerate parabolic equations with one (or more) degeneracy point inside the domain have also been studied

- by the flatness method developed by Martin-Rosier-Rouchon in [31, 32, 33] (see also Moyano [35] for some strongly degenerate equations),
- by Carleman estimates, see Fragnelli-Mugnai in [18, 19] when the control region is on both sides of the space domain with respect to the degeneracy point.

However, our analysis of the cost in the weakly degenerate case does not seem to be attainable by these approaches. It is based on the spectral problem associated with (1). The eigenvalues of problem (1) are related to the zeros of Bessel functions, and exhibit the following behavior: the gap between an eigenvalue of odd order and the consecutive one goes to 0 as $\alpha \to 1^-$ (see [7])

$$\forall n \geq 1, \quad \lambda_{2n}(\alpha) - \lambda_{2n-1}(\alpha) \to 0 \quad \text{as } \alpha \to 1^-,$$

while there is a uniform gap for the other ones: there is some $C_u > 0$ independent of $\alpha \in [0,1)$ and of $n \geq 1$ such that

$$\forall \alpha \in [0,1), \forall n \geq 1, \quad \lambda_{2n+1}(\alpha) - \lambda_{2n}(\alpha) \geq C_u.$$

Let us observe that recently Benabdallah-Boyer-Morancey [5] provided general results concerning such problems where groups of eigenvalues are separated by a uniform gap but, inside each group, eigenvalues can be close. This motivated us to push further the techniques we developed earlier, designed to study precisely the effects of some parameters ($\alpha$ and $T$ here) on the null controllability cost, whereas [5] is focused on obtaining formulas giving the minimal time needed to control a given system.

Our approach is the following:

- to obtain a precise upper estimate of the null controllability cost:
  - first we prove a general result (Theorem 2.4) concerning biorthogonal families to exponentials in the case where pairs of eigenvalues are close; the

proof is based on complex analysis, and in the spirit of a similar result proved in [10] (but for which we needed to modify the starting point of the proof),
  - and then Theorem 2.4 allows us to deduce that $C_{NC}(\alpha, T)$ blows up at most as $\frac{1}{1-\alpha}$;
- to obtain an exponential lower estimate of the null controllability cost:
  - we add an artificial control region, the goal being to deal with a new eigenvalue problem that is easy to study, and that will of course make the related null controllability cost cheaper than the original one,
  - and then we estimate the new controllability cost with some Hilbertian techniques developed also in [10], in the spirit of a result of Guichal [22], and we conclude.

### 1.4. Plan of the paper

- In Section 2, we state our results: Theorem 2.1 (upper estimate), Theorem 2.2 (lower estimate), and Theorem 2.4 (general construction of a biorthogonal family under this assumption that pairs of eigenvalues condensate).
- In Section 3, we prove Theorem 2.4.
- In Section 4, we prove Theorem 2.1.
- In Section 5, we prove Theorem 2.2.

## 2. Main results

### 2.1. Upper bound of the null controllability cost

Our first result is the following upper estimate of the null controllability cost, more precise than the one in [7]:

**Theorem 2.1.** *There exists $C_u > 0$, independent of $\alpha \in [0,1)$ and $T > 0$, such that*

$$\forall \alpha \in [0,1), \forall T > 0, \quad C_{NC}(\alpha, T) \leq \frac{C_u}{1-\alpha} \, e^{\frac{C_u}{T}} \, e^{-\frac{T}{C_u}}. \tag{4}$$

The proof of Theorem 2.1 is based on a general result, stated in section 2.3.

### 2.2. Lower bound of the null controllability cost

The following result improves also the lower estimate of the null controllability cost obtained in [7], yielding the expected exponential behavior in short time:

**Theorem 2.2.** *There exists $c_u > 0$, independent of $\alpha \in [0,1)$ and of $T \in (0,1)$, such that*

$$\forall \alpha \in [0,1), \forall T \in (0,1), \quad C_{NC}(\alpha, T) \geq \frac{c_u}{1-\alpha}, \tag{5}$$

*and*

$$\forall \alpha \in [0,1), \forall T \in (0,1), \quad C_{NC}(\alpha, T) \geq c_u \, e^{\frac{c_u}{T}}. \tag{6}$$

**Remark 2.3.** (i) Note that, comparing to (4), it would have been natural to expect a lower bound of the form

$$\forall \alpha \in [0,1), \forall T \in (0,1), \quad C_{NC}(\alpha, T) \geq \frac{c_u}{1-\alpha} \, e^{\frac{c_u}{T}}, \tag{7}$$

which does not follow from our results. However, combining (5) and (6), we have

$$\forall \alpha \in [0,1), \forall T \in (0,1), \forall \theta \in [0,1], \quad C_{NC}(\alpha,T) \geq \frac{c_u}{(1-\alpha)^\theta}\, e^{\frac{c_u(1-\theta)}{T}},$$

which combines the blow-up in $\alpha$ whith the one in $T$.

(ii) Combining with (3), one has that there exists $c_u$ independent of $\alpha \in [0,1)$ and of $T > 0$ such that

$$\forall \alpha \in [0,1), \forall T \in (0,1), \quad C_{NC}(\alpha,T) \geq c_u\, e^{\frac{c_u}{T}}\, e^{-\frac{T}{c_u}},$$

and, once again,

$$\forall \alpha \in [0,1), \forall T > 0, \forall \theta \in [0,1], \quad C_{NC}(\alpha,T) \geq \frac{c_u}{(1-\alpha)^\theta}\, e^{\frac{c_u(1-\theta)}{T}}\, e^{-\frac{T}{c_u}}.$$

### 2.3. An adapted biorthogonal family

The proof of Theorem 2.1 is based on the following general result, which can be viewed as a particular version of Theorem 2.3 of [10], but that gives a more precise estimate, that will be useful in our present context:

**Theorem 2.4.** *Suppose $\lambda_1 \geq 0$ and assume that there exists $0 < \gamma_{min} \leq \gamma^*_{min}$ such that*

$$\begin{cases} \forall m \geq 1, \quad \sqrt{\lambda_{2m}} - \sqrt{\lambda_{2m-1}} \geq \gamma_{min}, \\ \forall m \geq 1, \quad \sqrt{\lambda_{2m+1}} - \sqrt{\lambda_{2m}} \geq \gamma^*_{min}, \end{cases} \tag{8}$$

*Then there exists a family $(\sigma^+_m)_{m\geq 1}$ which is biorthogonal to the family $(e^{\lambda_n t})_{n\geq 1}$ in $L^2(0,T)$:*

$$\forall m,n \geq 1, \quad \int_0^T \sigma^+_m(t) e^{\lambda_n t}\, dt = \delta_{m,n}. \tag{9}$$

*Moreover, there is a universal constant $C_u > 0$, independent of $T$, $\gamma_{min}$, $\gamma^*_{min}$ and $m$, such that, for all $m \geq 1$, we have*

$$\|\sigma^+_m\|^2_{L^2(0,T)} \tag{10}$$
$$\leq C_u \Big(1 + \frac{(\gamma_{min}+\gamma^*_{min})^2}{\gamma_{min}(\sqrt{\lambda_1}+\gamma_{min})}\Big)^2 e^{\frac{C_u}{(\gamma_{min}+\gamma^*_{min})^2 T}}\, e^{C_u \frac{\sqrt{\lambda_m}}{\gamma_{min}+\gamma^*_{min}}}\, e^{-2\lambda_m T} B(T,\gamma_{min},\gamma^*_{min}),$$

$$\textit{with} \qquad B(T,\gamma_{min},\gamma^*_{min}) = \begin{cases} \Big(\frac{1}{T} + \frac{1}{T^2(\gamma_{min}+\gamma^*_{min})^2}\Big) & \textit{if } T \leq \frac{1}{(\gamma_{min}+\gamma^*_{min})^2}, \\ C_u(\gamma_{min}+\gamma^*_{min})^2 & \textit{if } T \geq \frac{1}{(\gamma_{min}+\gamma^*_{min})^2}. \end{cases} \tag{11}$$

(We note that Theorem 2.4 is similar to the results we proved in [10, 12]. Each of these results has been adapted to different examples and applications. Assumptions change from a version to another, leading to precise results in the desired case. It seems difficult, however, to provide a general framework for this theory.

## 3. Proof of Theorem 2.4

### 3.1. The general strategy

We adapt the strategy developed in [10, 11] to our assumptions. This approach is adapted from the construction of Seidman-Avdonin-Ivanov [38], which has the advantage to be completely explicit, combined with some ideas coming from the construction of Tenenbaum-Tucsnak [39] and Lissy [28], adding some parameter, in order to obtain quite optimal results.

Our approach is a perturbation of the one used in [10, 11], based on the Paley-Wiener theorem ([41]): if $f : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type, such that there exist nonnegative constants $C, A$ such that

$$|f(z)| \leq Ce^{A|z|},$$

and if $f \in L^2(\mathbb{R})$, then there exists $\phi \in L^2(-A, A)$ such that

$$f(z) = \int_{-A}^{A} \phi(\tau)e^{iz\tau}\, d\tau.$$

In [10, 11], we adapted the general construction of [38] (see Theorem 2 and Lemma 3 in [38]) to construct a suitable sequence $(f_m)_m$ satisfying

$$\begin{cases} \forall m, n \geq 1, \quad f_m(-i\lambda_n) = \delta_{m,n}, \\ \forall z \in \mathbb{C}, \quad |f_m(-z)e^{-iz\frac{T}{2}}| \leq C_m e^{\frac{T}{2}|z|}, \\ \forall m \geq 1, \quad f_m \in L^2(\mathbb{R}). \end{cases} \tag{12}$$

Then the two last properties together with the Paley-Wiener theorem imply that there exists some $\phi_m \in L^2(-\frac{T}{2}, \frac{T}{2})$ such that

$$f_m(-z)e^{-iz\frac{T}{2}} = \int_{-T/2}^{T/2} \phi_m(\tau)e^{iz\tau}\, d\tau,$$

hence

$$f_m(z) = \int_0^T \phi_m(t - \frac{T}{2})e^{-izt}\, dt,$$

and then

$$\int_0^T \phi_m(t - \frac{T}{2})e^{-\lambda_n t}\, dt = f_m(-i\lambda_n) = \delta_{m,n},$$

hence $(\phi_m(t - \frac{T}{2}))_m$ is biorthogonal to the family $(e^{-\lambda_n t})_n$, and $(\sigma_m^+(t))_m$ defined by

$$\sigma_m^+(t) = \phi_m(\frac{T}{2} - t)e^{-\lambda_m T}$$

is biorthogonal to the family $(e^{\lambda_n t})_n$ in $L^2(0, T)$, as desired. Moreover

$$\|\sigma_m^+\|^2_{L^2(0,T)} = e^{-2\lambda_m T} \int_{-T/2}^{T/2} \phi_m(\tau)^2\, d\tau \leq Ce^{-2\lambda_m T}\|f_m\|^2_{L^2(\mathbb{R})}$$

using the Parseval theorem.

Now, it remains to construct such entire functions $f_m$. The idea is to consider the natural infinite product that satisfies the first condition of (12), $f_m(-i\lambda_n) = \delta_{m,n}$, and to multiply it by a so-called 'mollifier', in such a way that the other two conditions of (12) will be also satisfied. Hence one has to estimate the growth of the natural infinite product, and then to choose a suitable mollifier. With respect to [10, 11], we choose a more adapted infinite product, and then we perform all the necessary estimates.

### 3.2. The counting function

Consider $\qquad \forall \rho > 0, \quad N_m(\rho) := \text{ card } \{k, 0 < |\lambda_m - \lambda_k| \le \rho\}.$

We prove the following:

**Lemma 3.1.** *Assume that the gap assumptions* (8) *are satisfied. Then, for all* $m \ge 1$*, we have*

$$\rho \in (0, \gamma_{min}(\gamma_{min} + 2\sqrt{\lambda_1})) \quad \Longrightarrow \quad N_m(\rho) = 0, \tag{13}$$

*and*

$$\forall m \ge 1, \forall \rho > 0, \quad N_m(\rho) \le 4\frac{\sqrt{\rho}}{\gamma_{min} + \gamma^*_{min}} + 1. \tag{14}$$

**Proof.** Of course the "worst" situation is when we have equalities in (8), hence when

$$\begin{cases} \forall m \ge 1, \quad \sqrt{\lambda_{2m}} - \sqrt{\lambda_{2m-1}} = \gamma_{\min}, \\ \forall m \ge 1, \quad \sqrt{\lambda_{2m+1}} - \sqrt{\lambda_{2m}} = \gamma^*_{\min}, \end{cases} \tag{15}$$

"worst" in the sense that if (13) and (14) are true under (15), they will be true under (8).

Hence we assume (15). And then, we have easily the following formulas:

$$\begin{cases} \forall m \ge 1, \quad \sqrt{\lambda_{2m}} = \sqrt{\lambda_1} + m\gamma_{\min} + (m-1)\gamma^*_{\min}, \\ \forall m \ge 1, \quad \sqrt{\lambda_{2m+1}} = \sqrt{\lambda_1} + m\gamma_{\min} + m\gamma^*_{\min}, \end{cases} \tag{16}$$

and then, we obtain clearly that for all $m \ge 1$

$$\begin{aligned} \lambda_{2m} - \lambda_{2m-1} &= (\sqrt{\lambda_{2m}} - \sqrt{\lambda_{2m-1}})(\sqrt{\lambda_{2m}} + \sqrt{\lambda_{2m-1}}) \\ &= \gamma_{\min}(2\sqrt{\lambda_1} + (2m-1)\gamma_{\min} + 2(m-1)\gamma^*_{\min}) \ge \gamma_{\min}(2\sqrt{\lambda_1} + \gamma_{\min}), \end{aligned}$$

and

$$\begin{aligned} \lambda_{2m+1} - \lambda_{2m} &= (\sqrt{\lambda_{2m+1}} - \sqrt{\lambda_{2m}})(\sqrt{\lambda_{2m+1}} + \sqrt{\lambda_{2m}}) \\ &= \gamma^*_{\min}(2\sqrt{\lambda_1} + 2m\gamma_{\min} + (2m-1)\gamma^*_{\min}) \\ &\ge \gamma^*_{\min}(2\sqrt{\lambda_1} + \gamma_{\min}) \ge \gamma_{\min}(2\sqrt{\lambda_1} + \gamma_{\min}). \end{aligned}$$

Now, if $N_m(\rho) \neq 0$, there exists some $k$ such that $|\lambda_k - \lambda_m| \le \rho$, and of course the same holds true with $|k - m| = 1$. Then, $m$ is even and $k$ is odd, or the contrary, and there exists some $n$ such that

$$\lambda_{2n} - \lambda_{2n-1} \le \rho \quad \text{or} \quad \lambda_{2n+1} - \lambda_{2n} \le \rho.$$

In any case, $\gamma_{\min}(2\sqrt{\lambda_1} + \gamma_{\min}) \le \rho$, which implies (13).

Now we prove (14). We distinguish several cases. First we consider that $m$ is even: $m = 2m'$, and we estimate the number of $k$ such that

$$|\lambda_k - \lambda_{2m'}| \leq \rho.$$

- if $k > 2m'$ and $k$ is even, hence if $k = 2k'$ with $k' > m'$, then $\lambda_{2k'} \leq \rho + \lambda_{2m'}$, hence $\sqrt{\lambda_{2k'}} \leq \sqrt{\rho + \lambda_{2m'}} \leq \sqrt{\rho} + \sqrt{\lambda_{2m'}}$; and using (16), we obtain
$$\sqrt{\lambda_1} + k'\gamma_{\min} + (k'-1)\gamma^*_{\min} \leq \sqrt{\rho} + \sqrt{\lambda_1} + m'\gamma_{\min} + (m'-1)\gamma^*_{\min},$$
which gives $(k'-m')(\gamma_{\min} + \gamma^*_{\min}) \leq \sqrt{\rho}$, hence
$$1 \leq k' - m' \leq \frac{\sqrt{\rho}}{\gamma_{\min} + \gamma^*_{\min}},$$
and there are at most $\frac{\sqrt{\rho}}{\gamma_{\min}+\gamma^*_{\min}}$ such integers $k'$;
- the reasoning is symmetric if $k < 2m'$ and $k$ is even, hence there are at most $\frac{\sqrt{\rho}}{\gamma_{\min}+\gamma^*_{\min}}$ such integers $k'$;
- when we consider the case where $k > 2m'$ and $k$ is odd, hence $k = 2k' + 1$ with $k' \geq m'$; in the same way:
$$\sqrt{\lambda_{2k'+1}} \leq \sqrt{\rho + \lambda_{2m'}} \leq \sqrt{\rho} + \sqrt{\lambda_{2m'}},$$
and using (16), we obtain
$$\sqrt{\lambda_1} + k'\gamma_{\min} + k'\gamma^*_{\min} \leq \sqrt{\rho} + \sqrt{\lambda_1} + m'\gamma_{\min} + (m'-1)\gamma^*_{\min},$$
hence $(k'-m')(\gamma_{\min} + \gamma^*_{\min}) \leq \sqrt{\rho} - \gamma^*_{\min}$, hence
$$0 \leq k' - m' \leq \frac{\sqrt{\rho} - \gamma^*_{\min}}{\gamma_{\min} + \gamma^*_{\min}},$$
and there are at most $\frac{\sqrt{\rho}-\gamma^*_{\min}}{\gamma_{\min}+\gamma^*_{\min}} + 1$ such integers $k'$;
- finally, if $k < 2m'$ and $k$ is odd, hence $k = 2k' + 1$ with $k' < m'$, we have $\sqrt{\lambda_{2m'}} \leq \sqrt{\rho} + \sqrt{\lambda_{2k'+1}}$, and using (16), we obtain
$$\sqrt{\lambda_1} + m'\gamma_{\min} + (m'-1)\gamma^*_{\min} \leq \sqrt{\rho} + \sqrt{\lambda_1} + k'\gamma_{\min} + k'\gamma^*_{\min},$$
hence $(m'-k')(\gamma_{\min} + \gamma^*_{\min}) \leq \sqrt{\rho} + \gamma^*_{\min}$, hence
$$0 < m' - k' \leq \frac{\sqrt{\rho} + \gamma^*_{\min}}{\gamma_{\min} + \gamma^*_{\min}},$$
and there are at most $\frac{\sqrt{\rho}+\gamma^*_{\min}}{\gamma_{\min}+\gamma^*_{\min}}$ such integers $k'$.

Finally we obtain that

$$N_m(\rho) \leq \frac{\sqrt{\rho}}{\gamma_{\min} + \gamma^*_{\min}} + \frac{\sqrt{\rho}}{\gamma_{\min} + \gamma^*_{\min}} + \frac{\sqrt{\rho} - \gamma^*_{\min}}{\gamma_{\min} + \gamma^*_{\min}} + 1 + \frac{\sqrt{\rho} + \gamma^*_{\min}}{\gamma_{\min} + \gamma^*_{\min}}$$
$$= 4\frac{\sqrt{\rho}}{\gamma_{\min} + \gamma^*_{\min}} + 1,$$

which proves (14). □

### 3.3. A Weierstrass product

Motivated by [38], we used in [10, 11] the following product

$$\prod_{k=0,k\neq m}^{\infty}\Big(1-\Big(\frac{iz-\lambda_m}{\lambda_k-\lambda_m}\Big)^2\Big),$$

but under our assumptions it seems more clever to consider the following one:

$$\prod_{k=1,k\neq m}^{\infty}\Big(1-\frac{iz-\lambda_m}{\lambda_k-\lambda_m}\Big)=: F_m(z). \qquad (17)$$

Since
$$\ln\Big|1-\frac{iz-\lambda_m}{\lambda_k-\lambda_m}\Big|\leq \ln\Big|1+\frac{|z|+\lambda_m}{|\lambda_k-\lambda_m|}\Big|\sim_{k\to\infty}\frac{|z|+\lambda_m}{\lambda_k},$$

and since $\sum_k \frac{1}{\lambda_k}$ is convergent, we deduce that the infinite product defining $F_m$ converges uniformly over all the compacts sets. Hence $F_m$ is well-defined and entire over $\mathbb{C}$. Moreover

$$F_m(-i\lambda_n)=\prod_{k=1,k\neq m}^{\infty}\Big(1-\frac{\lambda_n-\lambda_m}{\lambda_k-\lambda_m}\Big)=\begin{cases}0 & \text{if } m\neq n,\\ 1 & \text{if } m=n,\end{cases} \quad ,$$

hence
$$\forall m,n\geq 1,\quad F_m(-i\lambda_n)=\delta_{m,n}. \qquad (18)$$

Now we are going to estimate the growth of $F_m$. We prove the following

**Lemma 3.2.** *Assume that the gap assumption* (8) *is satisfied. Then the function $F_m$ satisfies the following growth estimate:*

$$\forall m\geq 1,\forall z\in\mathbb{C},\quad |F_m(z)|\leq\Big(1+\frac{|z|+\lambda_m}{\gamma_{min}(2\sqrt{\lambda_1}+\gamma_{min})}\Big)\,e^{\frac{4\pi}{\gamma_{min}+\gamma^*_{min}}(\sqrt{|z|+\lambda_m})}. \qquad (19)$$

The proof of Lemma 3.2 is based on the following preliminary estimate, relating the growth of $F_m$ to the counting function $N_m$:

**Lemma 3.3.** *Assume that the gap assumption* (8) *is satisfied. Then the function $F_m$ satisfies the following growth estimate:*

$$\forall m\geq 1,\forall z\in\mathbb{C},\quad \ln|F_m(z-i\lambda_m)|\leq\int_0^{+\infty}N_m(\rho)\frac{|z|}{\rho^2+\rho|z|}\,d\rho. \qquad (20)$$

**Proof of Lemma 3.2, assuming Lemma 3.3.** Assume that (20) is true. Then using (13) and (14), we obtain that

$$\begin{aligned}\ln|F_m(z-i\lambda_m)| &\leq \int_{\gamma_{\min}(\gamma_{\min}+2\sqrt{\lambda_1})}^{+\infty}\Big(4\frac{\sqrt{\rho}}{\gamma_{\min}+\gamma^*_{\min}}+1\Big)\frac{|z|}{\rho^2+\rho|z|}\,d\rho\\ &\leq \frac{4}{\gamma_{\min}+\gamma^*_{\min}}\int_0^{+\infty}\frac{|z|\sqrt{\rho}}{\rho^2+\rho|z|}\,d\rho+\int_{\gamma_{\min}(\gamma_{\min}+2\sqrt{\lambda_1})}^{+\infty}\frac{|z|}{\rho^2+\rho|z|}\,d\rho,\end{aligned}$$

and these two integrals can be easily computed: for the last one, we have

$$\frac{|z|}{\rho^2+\rho|z|} = -\frac{d}{d\rho}\ln\Big(1+\frac{|z|}{\rho}\Big)$$

(this will appear in the proof of Lemma 3.3), and then

$$\begin{aligned}\int_{\gamma_{\min}(\gamma_{\min}+2\sqrt{\lambda_1})}^{+\infty} \frac{|z|}{\rho^2+\rho|z|}\,d\rho &= \Big[-\ln\Big(1+\frac{|z|}{\rho}\Big)\Big]_{\gamma_{\min}(\gamma_{\min}+2\sqrt{\lambda_1})}^{+\infty} \\ &= \ln\Big(1+\frac{|z|}{\gamma_{\min}(\gamma_{\min}+2\sqrt{\lambda_1})}\Big);\end{aligned}$$

for the other one, we first note that

$$\frac{|z|\sqrt{\rho}}{\rho^2+\rho|z|} = \Big(\frac{1}{\rho}-\frac{1}{\rho+|z|}\Big)\sqrt{\rho} = \frac{1}{\sqrt{\rho}}-\frac{\sqrt{\rho}}{\rho+|z|};$$

fix $X>0$, then we have (using the change of variables $\sigma=\sqrt{\rho}$)

$$\begin{aligned}&\int_0^X \frac{|z|\sqrt{\rho}}{\rho^2+\rho|z|}\,d\rho = \int_0^X \frac{1}{\sqrt{\rho}}\,d\rho - \int_0^X \frac{\sqrt{\rho}}{\rho+|z|}\,d\rho \\ &= 2\sqrt{X} - \int_0^{\sqrt{X}} \frac{\sigma}{\sigma^2+|z|}2\sigma\,d\sigma = 2\sqrt{X} - \int_0^{\sqrt{X}} \frac{2\sigma^2+2|z|-2|z|}{\sigma^2+|z|}\,d\sigma \\ &= 2\sqrt{X} - \int_0^{\sqrt{X}} 2-\frac{2|z|}{\sigma^2+|z|}\,d\sigma = \int_0^{\sqrt{X}} \frac{2|z|}{\sigma^2+|z|}\,d\sigma = \int_0^{\sqrt{X}} \frac{2|z|}{|z|}\frac{1}{1+\Big(\frac{\sigma}{\sqrt{|z|}}\Big)^2}\,d\sigma \\ &= 2\Big[\sqrt{|z|}\,\arctan\Big(\frac{\sigma}{\sqrt{|z|}}\Big)\Big]_0^{\sqrt{X}} = 2\sqrt{|z|}\,\arctan\Big(\frac{\sqrt{X}}{\sqrt{|z|}}\Big),\end{aligned}$$

and letting $X\to+\infty$, we obtain that

$$\int_0^{+\infty} \frac{|z|\sqrt{\rho}}{\rho^2+\rho|z|}\,d\rho = \pi\sqrt{|z|}.$$

Therefore

$$\ln|F_m(z-i\lambda_m)| \le \ln\Big(1+\frac{|z|}{\gamma_{\min}(\gamma_{\min}+2\sqrt{\lambda_1})}\Big) + \frac{4\pi}{\gamma_{\min}+\gamma_{\min}^*}\sqrt{|z|},$$

hence $$|F_m(z-i\lambda_m)| \le \Big(1+\frac{|z|}{\gamma_{\min}(\gamma_{\min}+2\sqrt{\lambda_1})}\Big)\,e^{\frac{4\pi}{\gamma_{\min}+\gamma_{\min}^*}\sqrt{|z|}},$$

and using $Z=z-i\lambda_m$ we have

$$\begin{aligned}\ln|F_m(Z)| &\le \Big(1+\frac{|Z+i\lambda_m|}{\gamma_{\min}(\gamma_{\min}+2\sqrt{\lambda_1})}\Big)\,e^{\frac{4\pi}{\gamma_{\min}+\gamma_{\min}^*}\sqrt{|Z+i\lambda_m|}} \\ &\le \Big(1+\frac{|Z|+\lambda_m}{\gamma_{\min}(\gamma_{\min}+2\sqrt{\lambda_1})}\Big)\,e^{\frac{4\pi}{\gamma_{\min}+\gamma_{\min}^*}(\sqrt{|Z|+\lambda_m})}\end{aligned}$$

which gives (19) and concludes the proof of Lemma 3.2, assuming Lemma 3.3. □

**Proof of Lemma 3.3.** First we note that

$$\ln|F_m(z-i\lambda_m)| = \sum_{k=1,k\neq m}^{\infty} \ln\Big|1-\frac{iz}{\lambda_k-\lambda_m}\Big| \leq \sum_{k=1,k\neq m}^{\infty} \ln\Big(1+\frac{|z|}{|\lambda_k-\lambda_m|}\Big),$$

and the proof of Lemma 3.3 will follow from the following identity:

$$\sum_{k=1,k\neq m}^{\infty} \ln\Big(1+\frac{|z|}{|\lambda_k-\lambda_m|}\Big) = \int_0^{+\infty} N_m(\rho)\frac{|z|}{\rho^2+\rho|z|}\,d\rho. \tag{21}$$

Hence it remains to prove (21). Let us prove it first when $m=1$: it comes from the definition of $N_1(\rho)$ that

$$N_1(\rho) = \text{card}\,\{k>1, \lambda_k-\lambda_1 \leq \rho\},$$

$$\text{hence}\qquad \begin{cases} 0\leq \rho < \lambda_2-\lambda_1 & \Longrightarrow \quad N_1(\rho)=0,\\ \lambda_2-\lambda_1 \leq \rho < \lambda_3-\lambda_1 & \Longrightarrow \quad N_1(\rho)=1,\\ \lambda_3-\lambda_1 \leq \rho < \lambda_4-\lambda_1 & \Longrightarrow \quad N_1(\rho)=2,\end{cases}$$

and more generally

$$\forall k\geq 1: \quad \lambda_k-\lambda_1 \leq \rho < \lambda_{k+1}-\lambda_1 \quad \Longrightarrow \quad N_1(\rho)=k-1.$$

Then, given $N\geq 1$, we have

$$\begin{aligned}
&\int_0^{\lambda_{N+1}-\lambda_1} N_1(\rho)\frac{|z|}{\rho^2+\rho|z|}\,d\rho\\
&= \sum_{k=1}^{N}\int_{\lambda_k-\lambda_1}^{\lambda_{k+1}-\lambda_1} N_1(\rho)\frac{|z|}{\rho^2+\rho|z|}\,d\rho = \sum_{k=1}^{N}\int_{\lambda_k-\lambda_1}^{\lambda_{k+1}-\lambda_1} (k-1)\frac{|z|}{\rho^2+\rho|z|}\,d\rho\\
&= \sum_{k=2}^{N}(k-1)\int_{\lambda_k-\lambda_1}^{\lambda_{k+1}-\lambda_1} -\frac{d}{d\rho}\ln\Big(1+\frac{|z|}{\rho}\Big)\,d\rho = \sum_{k=2}^{N}(k-1)\Big[-\ln\Big(1+\frac{|z|}{\rho}\Big)\Big]_{\lambda_k-\lambda_1}^{\lambda_{k+1}-\lambda_1}\\
&= \sum_{k=2}^{N}(k-1)\Big(\ln\Big(1+\frac{|z|}{\lambda_k-\lambda_1}\Big)-\ln\Big(1+\frac{|z|}{\lambda_{k+1}-\lambda_1}\Big)\Big)\\
&= \Big[\sum_{k=2}^{N}(k-1)\ln\Big(1+\frac{|z|}{\lambda_k-\lambda_1}\Big)\Big] - \Big[\sum_{k=2}^{N}(k-1)\ln\Big(1+\frac{|z|}{\lambda_{k+1}-\lambda_1}\Big)\Big]\\
&= \Big[\sum_{k=2}^{N}(k-1)\Big(\ln\Big(1+\frac{|z|}{\lambda_k-\lambda_1}\Big)\Big] - \Big[\sum_{k=3}^{N+1}(k-2)\Big(\ln\Big(1+\frac{|z|}{\lambda_k-\lambda_1}\Big)\Big]\\
&= \Big[\sum_{k=2}^{N}\ln\Big(1+\frac{|z|}{\lambda_k-\lambda_1}\Big)\Big] - (N-1)\ln\Big(1+\frac{|z|}{\lambda_{N+1}-\lambda_1}\Big).
\end{aligned}$$

To conclude, let $N\to\infty$:

$$\ln\Big(1+\frac{|z|}{\lambda_{N+1}-\lambda_1}\Big) \sim_{N\to\infty} \frac{|z|}{\lambda_{N+1}-\lambda_1} \sim_{N\to\infty} \frac{|z|}{\lambda_{N+1}},$$

hence $\quad (N-1)\ln\Big(1+\dfrac{|z|}{\lambda_{N+1}-\lambda_1}\Big) \sim_{N\to\infty} |z|\dfrac{N}{\lambda_{N+1}} \to 0$ as $N\to\infty$

since there is $c>0$ such that $\lambda_N \geq mN^2$. Therefore we obtain

$$\int_0^{+\infty} N_1(\rho)\frac{|z|}{\rho^2+\rho|z|}\,d\rho = \sum_{k=2}^{\infty} \ln\Big(1+\frac{|z|}{\lambda_k-\lambda_1}\Big),$$

hence (21) in the case $m=1$.

The case $m=2$ can be studied in a similar way, or performing the following change: denote $\tilde\lambda_1$ the symmetric of $\lambda_1$ with respect to $\lambda_2$:

$$\tilde\lambda_1-\lambda_2=\lambda_2-\lambda_1,$$

and reorder the sequence $\{\lambda_n, n\geq 2\}\cup\{\tilde\lambda_1\}$ in the increasing order: then we are in the situation of the previous case, and we obtain that

$$\begin{aligned}\int_0^{+\infty} N_2(\rho)\frac{|z|}{\rho^2+\rho|z|}\,d\rho &= \ln\Big(1+\frac{|z|}{\tilde\lambda_1-\lambda_2}\Big)+\sum_{k=3}^{\infty}\ln\Big(1+\frac{|z|}{\lambda_k-\lambda_2}\Big),\\ &= \sum_{k=1,k\neq 2}^{\infty}\ln\Big(1+\frac{|z|}{|\lambda_k-\lambda_2|}\Big),\end{aligned}$$

which is (21) when $m=2$. And the same reasoning allows to prove (21) in full generality. This concludes the proof of Lemma 3.3. □

### 3.4. The direct consequences

We derive from Lemma 3.2 that, for all $m\geq 1$, for all $z\in\mathbb{C}$, we have

$$|F_m(z)| \leq \Big(1+\frac{|z|+\lambda_m}{(\gamma_{\min}+\gamma^*_{\min})^2}\frac{(\gamma_{\min}+\gamma^*_{\min})^2}{\gamma_{\min}(2\sqrt{\lambda_1}+\gamma_{\min})}\Big)\, e^{\frac{4\pi}{\gamma_{\min}+\gamma^*_{\min}}(\sqrt{|z|+\lambda_m})},$$

hence there exists some $C_u>0$ independent of $m$, $\gamma_{\min}$, $\gamma^*_{\min}$ and $z$ such that

$$|F_m(z)| \leq C_u\Big(1+\frac{(\gamma_{\min}+\gamma^*_{\min})^2}{\gamma_{\min}(2\sqrt{\lambda_1}+\gamma_{\min})}\Big)\, e^{\frac{8\pi}{\gamma_{\min}+\gamma^*_{\min}}(\sqrt{|z|}+\sqrt{\lambda_m})}. \tag{22}$$

The main differences with respect to the general result in [10], under our new assumptions (8), are

- the coefficient in the exponential, depending on $1/(\gamma_{\min}+\gamma^*_{\min})$ instead of $1/\gamma_{\min}$,
- the multiplicative coefficient $1+\frac{(\gamma_{\min}+\gamma^*_{\min})^2}{\gamma_{\min}(2\sqrt{\lambda_1}+\gamma_{\min})}$:

since we have in mind examples where $\gamma_{\min}$ is small and $\gamma^*_{\min}$ is not small, the coefficient in the exponential will be of the order $1/\gamma^*_{\min}$ (hence not large), and the only large coefficient is the multiplicative one, of the order $1/\gamma_{\min}$ if $\lambda_1>0$.

Then we can proceed as in [10], taking into account these changes:

- We choose
$$T' := \min\{T, \frac{1}{(\gamma_{\min} + \gamma^*_{\min})^2}\}, \tag{23}$$
and
$$N' \geq 2 + \frac{\theta_3}{(\gamma_{\min} + \gamma^*_{\min})T'} \tag{24}$$
with a suitable $\theta_3$ (independent of $T > 0$ and of $m \geq 0$, and given in (29)).

- We consider $a_k := \frac{C_{N',T'}}{k^2}$ with $C_{N',T'} := \frac{T'}{2\sum_{k=N'}^{\infty} \frac{1}{k^2}}$, in order that we obtain $\sum_{k=N'}^{\infty} a_k = \frac{T'}{2}$, and the associated mollifier
$$P_{N',T'}(z) := e^{iz\frac{T'}{2}} \prod_{k=N'}^{\infty} \cos(a_k z). \tag{25}$$

Then we have the following

**Lemma 3.4.** ([10])

(a) *The regularity and the growth of $P_{N',T'}$ over $\mathbb{C}$: The function $P_{N',T'}$ is entire over $\mathbb{C}$ and satisfies*
$$\begin{cases} P_{N',T'}(0) = 1, \\ \forall z \in \mathbb{C} \text{ such that } \Im z \geq 0, \quad |P_{N',T'}(z)| \leq 1, \\ \forall z \in \mathbb{C}, \quad |e^{-iz\frac{T'}{2}} P_{N',T'}(z)| \leq e^{|z|\frac{T'}{2}}. \end{cases} \tag{26}$$

(b) *The behaviour of $P_{N',T'}$ over $\mathbb{R}$: there exist $\theta_0 > 0$, $\theta_1 > 0$, both independent of $N'$ and $T'$ such that $P_{N',T'}$ satisfies*
$$\begin{cases} \left(\frac{C_{N',T'}|x|}{\theta_0}\right)^{1/2} + 1 \geq N' \implies \ln|P_{N',T'}(x)| \leq -\frac{\theta_1}{2^3}\left(\frac{C_{N',T'}|x|}{\theta_0}\right)^{1/2}, \\ \left(\frac{C_{N',T'}|x|}{\theta_0}\right)^{1/2} + 1 \leq N' \implies \ln|P_{N',T'}(x)| \leq -\frac{\theta_1}{(N')^3}\left(\frac{C_{N',T'}|x|}{\theta_0}\right)^{2}. \end{cases} \tag{27}$$

(c) *The behaviour of $P_{N',T'}$ over $i\mathbb{R}_+$: there is some constant $\theta_2 > 0$, independent of $N'$ and $T'$, such that $P_{N',T'}$ satisfies*
$$\forall x \in \mathbb{R}_+, \quad P_{N',T'}(ix) \geq e^{-\theta_2\sqrt{C_{N',T'}x}}. \tag{28}$$

Using these parameters, we define
$$\theta_3 = \frac{2^{11}\theta_0\pi^2}{\theta_1^2}. \tag{29}$$

- Finally we consider
$$\forall m \geq 1, \forall z \in \mathbb{C}, \quad f_{m,N',T'}(z) := F_m(z)\frac{P_{N',T'}(-z)}{P_{N',T'}(i\lambda_m)}, \tag{30}$$
and we have the following

**Lemma 3.5.** *When $T'$ and $N'$ satisfy* (23) *and* (24)*, the functions $f_{m,N',T'}$ are entire and satisfy the following properties:*

(a) *for all $m, n \geq 1$, we have* $f_{m,N',T'}(-i\lambda_n) = \delta_{m,n}$; (31)

(b) *for all* $m \geq 1$*, for all* $\varepsilon > 0$*, there exists* $C_{m,\gamma_{min},\gamma^*_{min},N',T',\varepsilon} > 0$ *such that*

$$\forall z \in \mathbb{C}, \quad |f_{m,N',T'}(-z)e^{-iz\frac{T}{2}}| \leq C_{m,\gamma_{min},\gamma^*_{min},N',T',\varepsilon} e^{(\frac{T}{2}+\varepsilon)|z|}; \tag{32}$$

(c) *for all* $m \geq 1$*,* $f_{m,N',T'} \in L^2(\mathbb{R})$*.*

(The proof of Lemma 3.5 is directly adapted from the one of Lemma 4.4 of [10], taking into account the new estimate (22)).

### 3.5. End of the proof of Theorem 2.4: the resulting biorthogonal sequence

With our choices, the function $x \mapsto f_{m,N',T'}(-x)e^{ixT/2}$ is in $L^2(\mathbb{R})$, and we can consider its Fourier transform $\phi_{m,N',T'}$:

$$\phi_{m,N',T'}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} f_{m,N',T'}(-x)e^{-ix\frac{T}{2}} e^{-i\xi x}\, dx.$$

It is well-defined since $f_{m,N',T'} \in L^2(\mathbb{R})$, and the Paley-Wiener theorem ([41] p. 100) shows that $\phi_{m,N',T'}$ is compactly supported in $[-\frac{T}{2} - \varepsilon, \frac{T}{2} + \varepsilon]$ (thanks to (32)). Since this is true for all $\varepsilon > 0$, $\phi_{m,N',T'}$ is compactly supported in $[-\frac{T}{2}, \frac{T}{2}]$.

To obtain good results, we will choose $N'$ satisfying the stronger property:

$$2 + \frac{\theta_3}{(\gamma_{\min} + \gamma_{\min})^2 T'} \leq N' \leq 4 + \frac{\theta_3}{(\gamma_{\min} + \gamma_{\min})^2 T'}. \tag{33}$$

Then we have the following

**Lemma 3.6.** *Take* $T'$ *and* $N'$ *satisfying* (23) *and* (33)*, and consider*

$$\sigma^+_{m,N',T'}(t) := \phi_{m,N',T'}(\frac{T}{2} - t)e^{-\lambda_m T}. \tag{34}$$

*Then the family* $(\sigma^+_{m,N',T'})_{m\geq 0}$ *is biorthogonal to the family* $(e^{\lambda_n t})_{n\geq 0}$ *in* $L^2(0,T)$*:*

$$\forall m, n \geq 1, \quad \int_0^T \sigma^+_{m,N',T'}(t)e^{\lambda_n t}\, dt = \delta_{m,n}. \tag{35}$$

*Moreover, it satisfies: there is some universal constant* $C_u$ *independent of* $T$*,* $\gamma^*_{min}$*, and* $m$ *such that, for all* $m \geq 1$*, we have*

$$\|\sigma^+_{m,N',T'}\|^2_{L^2(0,T)} \leq C_u \Big(1 + \frac{(\gamma_{min} + \gamma^*_{min})^2}{\gamma_{min}(\sqrt{\lambda_1} + \gamma_{min})}\Big)^2 e^{-2\lambda_m T} e^{C_u \frac{\sqrt{\lambda_m}}{\gamma_{min}+\gamma^*_{min}}} B(T, \gamma_{min}, \gamma^*_{min}), \tag{36}$$

*where* $B(T, \gamma_{min}, \gamma^*_{min})$ *is given by* (11)*.*

**Proof.** Similar to the proof of Lemma 4.5 of [10], taking into account (22). □

## 4. Proof of Theorem 2.1

First we recall from [7] some useful properties, that we will need to use.

### 4.1. Useful properties ([7]): well-posedness, eigenvalues and eigenfunctions

#### 4.1.1. Functional setting and well-posedness ([7])

For $0 \le \alpha < 1$, we consider

$$H^1_\alpha(-1,1) := \Big\{ u \in L^2(-1,1) \ \mid \ u \text{ absolutely continuous in } [-1,1], \\ \int_{-1}^1 |x|^\alpha u_x^2 \, dx < \infty \text{ and } u(-1) = 0 = u(1) \Big\}. \tag{37}$$

$H^1_\alpha(-1,1)$ is endowed with the natural scalar product

$$\forall f, g \in H^1_\alpha(-1,1), \quad (f,g) = \int_{-1}^1 \big( |x|^\alpha f'(x) g'(x) + f(x) g(x) \big) \, dx.$$

Next, consider $H^2_\alpha(-1,1) := \Big\{ u \in H^1_\alpha(-1,1) \ \mid \int_{-1}^1 |(|x|^\alpha u'(x))'|^2 \, dx < \infty \Big\}$,

and the operator $A : D(A) \subset L^2(-1,1) \to L^2(-1,1)$ defined by

$$D(A) := H^2_\alpha(-1,1) \quad \text{ and } \quad \forall u \in D(A), \quad Au := (|x|^\alpha u_x)_x.$$

Then the following results hold:

**Proposition 4.1.** ([7]) *Given $\alpha \in [0,1)$, we have the following:*

(a) *$H^1_\alpha(-1,1)$ is a Hilbert space;*

(b) *$A : D(A) \subset L^2(-1,1) \to L^2(-1,1)$ is a self-adjoint negative operator with dense domain.*

Hence, $A$ is the infinitesimal generator of an analytic semigroup of contractions $e^{tA}$ on $L^2(-1,1)$. Given a source term $h$ in $L^2((-1,1) \times (0,T))$ and an initial condition $v_0 \in L^2(-1,1)$, consider the problem

$$\begin{cases} v_t - (|x|^\alpha v_x)_x = h(x,t), \\ v(-1,t) = 0 = v(1,t), \\ v(x,0) = v_0(x). \end{cases} \tag{38}$$

The function $v \in \mathcal{C}^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H^1_\alpha(-1,1))$ given by the variation of constant formula

$$v(\cdot,t) = e^{tA} v_0 + \int_0^t e^{(t-s)A} h(\cdot,s) \, ds$$

is called the mild solution of (38). We say that a function

$$v \in \mathcal{C}^0([0,T]; H^1_\alpha(-1,1)) \cap H^1(0,T; L^2(-1,1)) \cap L^2(0,T; D(A))$$

is a strict solution of (38) if $v$ satisfies $v_t - (|x|^\alpha v_x)_x = h(x,t)$ almost everywhere in $(-1,1) \times (0,T)$, and the initial and boundary conditions are fulfilled for all $t \in [0,T]$ and all $x \in [-1,1]$. And we have the following

**Proposition 4.2.** ([7]) *If $v_0 \in H^1_\alpha(-1,1)$, then the mild solution of* (38) *is the unique strict solution of* (38).

#### 4.1.2. Eigenvalues and eigenfunctions ([7])

Now we recall from [7] the eigenvalues and associated eigenfunctions of the degenerate diffusion operator $u \mapsto -(|x|^\alpha u')'$, i.e. the solutions $(\lambda, \Phi)$ of

$$\begin{cases} -(|x|^\alpha \Phi'(x))' = \lambda \Phi(x) & x \in (-1,1), \\ \Phi(-1) = 0 = \Phi(1). \end{cases} \tag{39}$$

Let us recall some notations:

- when $\alpha \in [0,1)$, let $\nu_\alpha := \dfrac{|\alpha-1|}{2-\alpha} = \dfrac{1-\alpha}{2-\alpha}$, $\kappa_\alpha := \dfrac{2-\alpha}{2}$,
- and given $\nu > 0$, we also denote $J_\nu$ the Bessel function of positive order $\nu$, $J_{-\nu}$ the Bessel function of negative order $-\nu$, $(j_{\nu,m})_{m\geq 1}$ the sequence of positive zeros of $J_\nu$ and $(j_{-\nu,m})_{m\geq 1}$ the sequence of positive zeros of $J_{-\nu}$ (see of course Watson [40], Lebedev [27], and the useful properties [10]).

Then we have the following description for (39), see Proposition 2.7 and also equation (25) in [7]: when $\alpha \in [0,1)$, we have exactly two sub-families of eigenvalues and associated eigenfunctions for problem (39), that is:

- the eigenvalues of the form $\kappa_\alpha^2 j_{\nu_\alpha,n}^2$, associated with the odd functions

$$\Phi_{\alpha,n}^{(o)}(x) = \begin{cases} x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}(j_{\nu_\alpha,n} x^{\kappa_\alpha}) & \text{if } x \in (0,1) \\ -|x|^{\frac{1-\alpha}{2}} J_{\nu_\alpha}(j_{\nu_\alpha,n} |x|^{\kappa_\alpha}) & \text{if } x \in (-1,0) \end{cases}, \tag{40}$$

- the eigenvalues of the form $\kappa_\alpha^2 j_{-\nu_\alpha,n}^2$, associated with the even functions

$$\Phi_{\alpha,n}^{(e)}(x) = \begin{cases} x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}(j_{-\nu_\alpha,n} x^{\kappa_\alpha}) & \text{if } x \in (0,1) \\ |x|^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}(j_{-\nu_\alpha,n} |x|^{\kappa_\alpha}) & \text{if } x \in (-1,0) \end{cases}. \tag{41}$$

Moreover, the family $\{\Phi_{\alpha,n}^{(o)}, \Phi_{\alpha,n}^{(e)}, n \geq 1\}$ forms an orthogonal basis of $L^2(-1,1)$.

It is easy and practical to order the eigenvalues: since $J_{\nu_\alpha}(0) = 0$ and the zeros of $J_{\nu_\alpha}$ and $J_{-\nu_\alpha}$ are interlaced (because of Sturm's theorems), we have

$$0 < j_{-\nu_\alpha,1} < j_{\nu_\alpha,1} < j_{-\nu_\alpha,2} < j_{\nu_\alpha,2} < \cdots,$$

hence it is natural to denote

$$\forall n \geq 1, \quad \lambda_{\alpha,2n-1} := \kappa_\alpha^2 j_{-\nu_\alpha,n}^2, \quad \lambda_{\alpha,2n} := \kappa_\alpha^2 j_{\nu_\alpha,n}^2, \tag{42}$$

hence in such a way that

$$0 < \lambda_{\alpha,1} < \lambda_{\alpha,2} < \lambda_{\alpha,3} < \lambda_{\alpha,4} < \cdots,$$

and the associated normalized eigenfunctions

$$\forall n \geq 1, \quad \tilde\Phi_{\alpha,2n-1} := \frac{\sqrt{\kappa_\alpha}}{|J'_{-\nu_\alpha}(j_{-\nu_\alpha,n})|} \Phi_{\alpha,n}^{(e)}, \quad \tilde\Phi_{\alpha,2n} := \frac{\sqrt{\kappa_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,n})|} \Phi_{\alpha,n}^{(o)} \tag{43}$$

form an orthonormal basis of $L^2(-1,1)$.

### 4.2. Upper estimate of the cost of controllability

We use the moment method. First we expand the initial condition $u_0 \in L^2(-1,1)$: there exists $(\mu^0_{\alpha,n})_{n\geq 1} \in \ell^2(\mathbb{N}^\star)$ such that

$$u_0(x) = \sum_{n\geq 1} \mu^0_{\alpha,n} \tilde{\Phi}_{\alpha,n}(x), \qquad x \in (-1,1),$$

and we see that $h$ is a control that drives the solution of (1) to 0 in time $T$ if and only if

$$\forall n \geq 1, \qquad \int_0^T \int_{-1}^1 h(x,t)\chi_{[a,b]}(x)\tilde{\Phi}_{\alpha,n}(x)e^{\lambda_{\alpha,n}t}dxdt = -\mu^0_{\alpha,n}. \tag{44}$$

And if the sequence $(\sigma^+_{\alpha,m})_{m\geq 1}$ is biorthogonal to the sequence $(e^{\lambda_{\alpha,n}t})_{n\geq 1}$ in $L^2(0,T)$ and satisfies suitable upper estimates, then the function defined by

$$h(x,t) := \sum_{m\geq 1} -\mu^0_{\alpha,m}\sigma^+_{\alpha,m}(t)\frac{\tilde{\Phi}_{\alpha,m}(x)}{\int_a^b \tilde{\Phi}^2_{\alpha,m}} \tag{45}$$

belongs to $L^2((-1,1)\times(0,T))$ and satisfies the moment problem (44) (see, e.g. [7, 12]).

We would like to use Theorem 2.4, so we need to check that (8) holds in our case. We prove the following:

**Lemma 4.3.** *There exist $m_1, m_2 > 0$ independent of $\alpha \in [0,1)$ and there exists $\alpha^* \in [0,1)$ such that*

$$\forall \alpha \in [\alpha^*,1), \forall n \geq 1, \quad \sqrt{\lambda_{\alpha,2n}} - \sqrt{\lambda_{\alpha,2n-1}} \geq m_1(1-\alpha) =: \gamma_{min}(\alpha), \tag{46}$$

*and*

$$\forall \alpha \in [\alpha^*,1), \forall n \geq 1, \quad \sqrt{\lambda_{\alpha,2n+1}} - \sqrt{\lambda_{\alpha,2n}} \geq m_2 =: \gamma^*_{min}(\alpha). \tag{47}$$

**Proof.** This follows from Lemma 6.3 in [7], and its consequence: using an integral formula:

$$\frac{dj_{\nu,n}}{d\nu} = 2j_{\nu,n}\int_0^{+\infty} K_0(2j_{\nu,n}\sinh t)e^{-2\nu t}\,dt,$$

(see Watson [40], p. 508), it is proved that there exist $m_*$, $M_*$ and $\alpha_*$ such that $0 < m_* < M_*$, $\alpha_* \in (0,1)$ and such that

$$\forall \alpha \in [\alpha_*,1), \forall n \geq 1, \quad m_*(1-\alpha) \leq j_{\nu_\alpha,n} - j_{-\nu_\alpha,n} \leq M_*(1-\alpha),$$

and as a consequence

$$\forall \alpha \in [\alpha_*,1), \forall n \geq 1, \quad m_*\kappa_\alpha(1-\alpha) \leq \sqrt{\lambda_{\alpha,2n}} - \sqrt{\lambda_{\alpha,2n-1}} \leq M_*\kappa_\alpha(1-\alpha).$$

The left part already gives (46). For the right part, it is sufficient to note that the gap $\sqrt{\lambda_{\alpha,2n}} - \sqrt{\lambda_{\alpha,2n-1}}$ (uniformly small, of the order $1-\alpha$) allows us to estimate the gap $\sqrt{\lambda_{\alpha,2n+1}} - \sqrt{\lambda_{\alpha,2n}}$: indeed,

$$\begin{aligned}\sqrt{\lambda_{\alpha,2n+1}} - \sqrt{\lambda_{\alpha,2n}} &= \Big(\sqrt{\lambda_{\alpha,2n+1}} - \sqrt{\lambda_{\alpha,2n-1}}\Big) - \Big(\sqrt{\lambda_{\alpha,2n}} - \sqrt{\lambda_{\alpha,2n-1}}\Big) \\ &\geq \kappa_\alpha\Big(j_{-\nu_\alpha,n+1} - j_{-\nu_\alpha,n}\Big) - M_*(1-\alpha);\end{aligned}$$

from Komornik-Loreti [24] p. 135, the sequence $(j_{-\nu_\alpha,n+1}-j_{-\nu_\alpha,n})_{n\geq 1}$ is nondecreasing, and converges to $\pi$, hence

$$j_{-\nu_\alpha,n+1}-j_{-\nu_\alpha,n}\geq j_{-\nu_\alpha,2}-j_{-\nu_\alpha,1},$$

and since the function $\alpha\mapsto j_{-\nu_\alpha,2}-j_{-\nu_\alpha,1}$ is continuous on $[0,1)$, positive and has a positive limit (equal to $j_{0,2}-j_{0,1}$) when $\alpha\to 1^-$, there exists $m_0>0$ such that

$$\forall\alpha\in[0,1),\forall n\geq 1,\quad j_{-\nu_\alpha,n+1}-j_{-\nu_\alpha,n}\geq j_{-\nu_\alpha,2}-j_{-\nu_\alpha,1}\geq m_0,$$

and then
$$\sqrt{\lambda_{\alpha,2n+1}}-\sqrt{\lambda_{\alpha,2n}}\geq\frac{1}{2}m_0-M_*(1-\alpha),$$

which completes the proof (taking eventually another $\alpha_*$ closer to 1). □

Now, Lemma 4.3 gives that the condition (8) is satisfied (with explicit values of $\gamma_{\min}$ and $\gamma^*_{\min}$), then Theorem 2.4 applies, and we can use the sequence $(\sigma^+_{\alpha,m})_{m\geq 1}$ constructed in Theorem 2.4: using (10), and the estimate (47) in [7]:

$$\exists m^*,\forall\alpha\in[0,1),\forall n\geq 1,\quad \int_a^b\tilde{\Phi}^2_{\alpha,m}\geq m^*,$$

in (45), we obtain

$$\begin{aligned}\|h\|^2_{L^2((-1,1)\times(0,T))}&=\sum_{m\geq 1}|\mu^0_{\alpha,m}|^2\|\sigma^+_{\alpha,m}\|^2_{L^2(0,T)}\frac{1}{\left(\int_a^b\tilde{\Phi}^2_{\alpha,m}\right)^2}\\
&\leq\Big(\sum_{m\geq 1}|\mu^0_{\alpha,m}|^2\Big)\sup_{m\geq 1}\Big(\|\sigma^+_{\alpha,m}\|^2_{L^2(0,T)}\frac{1}{\left(\int_a^b\tilde{\Phi}^2_{\alpha,m}\right)^2}\Big)\\
&\leq\|u_0\|^2_{L^2(-1,1)}\frac{C_u}{m^{*2}}\Big(1+\frac{(\gamma_{\min}(\alpha)+\gamma^*_{\min}(\alpha))^2}{\gamma_{\min}(\alpha)(\sqrt{\lambda_{\alpha,1}}+\gamma_{\min}(\alpha))}\Big)^2 e^{\frac{C_u}{(\gamma_{\min}(\alpha)+\gamma^*_{\min}(\alpha))^2T}}\cdot\\
&\quad\cdot B(T,\gamma_{\min}(\alpha),\gamma^*_{\min}(\alpha))\sup_{m\geq 1}\Big(e^{C_u\frac{\sqrt{\lambda_{\alpha,m}}}{\gamma_{\min}(\alpha)+\gamma^*_{\min}(\alpha)}}e^{-2\lambda_{\alpha,m}T}\Big).\end{aligned}$$

From classical estimates, see for example Lorch-Muldoon [30]:

$$\forall\nu\in[0,\frac{1}{2}],\forall n\geq 1,\quad \pi(n+\frac{\nu}{2}-\frac{1}{4})\leq j_{\nu,n}\leq\pi(n+\frac{\nu}{4}-\frac{1}{8}),\tag{48}$$

there exists $0<C_1<C_2$ independent of $\alpha\in[0,1)$ such that

$$\forall\alpha\in[0,1),\forall m\geq 1,\quad C_1m^2\leq\lambda_{\alpha,m}\leq C_2m^2;$$

hence we have

$$\forall\alpha\in[0,1),\forall m\geq 1,\quad e^{C_u\frac{\sqrt{\lambda_{\alpha,m}}}{\gamma_{\min}(\alpha)+\gamma^*_{\min}(\alpha)}}e^{-2\lambda_{\alpha,m}T}\leq e^{C_u\sqrt{C_2}m-2C_1Tm^2},$$

and studying the function $y\in[0,+\infty)\mapsto C_u\sqrt{C_2}y-2C_1Ty^2$, we see that

$$e^{C_u\sqrt{C_2}m-2C_1Tm^2}\leq e^{C_3/T}\quad\text{with } C_3=\frac{C_u^2C_2}{8C_1}.$$

Finally, since $\alpha \mapsto \lambda_{\alpha,1}$ is continuous on $[0,1)$, positive and with a positive limit $(\frac{1}{4}j_{0,1}^2)$ as $\alpha \to 1^-$, there exists some $m_*$ such that $\lambda_{\alpha,1} \geq m_*$. Hence

$$\|h\|^2_{L^2((-1,1)\times(0,T))} \\ \leq \|u_0\|^2_{L^2(-1,1)} \frac{C_u}{m^{*2}} \Big(1 + \frac{C_4}{\gamma_{\min}(\alpha)}\Big)^2 e^{(C_3 + \frac{C_u}{m_2^2})\frac{1}{T}} B(T, \gamma_{\min}(\alpha), \gamma^*_{\min}(\alpha)),$$

hence $C_{NC}(\alpha, T) \leq \dfrac{\sqrt{C_u}}{m^*} \Big(1 + \dfrac{C_4}{\gamma_{\min}(\alpha)}\Big) e^{(C_3 + \frac{C_u}{m_2^2})\frac{1}{2T}} \sqrt{B(T, \gamma_{\min}(\alpha), \gamma^*_{\min}(\alpha))}.$

This gives the expected behavior of the null controllability cost and completes the proof of Theorem 2.1. □

## 5. Proof of Theorem 2.2

### 5.1. The initial remark given by the moment method

The initial remark goes back to the relation given by the moment method: looking at (44), we see that if (1) is null-controllable in time $T$, then choosing $u_0 := -\Phi_{\alpha,1}$, any control $h_1$ that drives the initial condition $-\Phi_{\alpha,1}$ to 0 in time $T$ satisfies

$$\forall n \geq 1, \qquad \int_0^T \int_a^b h_1(x,t) \tilde{\Phi}_{\alpha,n}(x) e^{\lambda_{\alpha,n} t} dx dt = \delta_{n,1}. \tag{49}$$

Let us take now $u_0 := -\Phi_{\alpha,2}$, any control $h_2$ that drives the initial condition $-\Phi_{\alpha,2}$ to 0 in time $T$ satisfies

$$\forall n \geq 1, \qquad \int_0^T \int_a^b h_2(x,t) \tilde{\Phi}_{\alpha,n}(x) e^{\lambda_{\alpha,n} t} dx dt = \delta_{n,2}. \tag{50}$$

And then, if (1) is null-controllable in time $T$, choose any control $h_m$ that drives the initial condition $-\Phi_{\alpha,m}$ to 0 in time $T$, and the related sequence $(h_m)_{m\geq 1}$ is then biorthogonal to the sequence $(\Phi_{\alpha,n}(x)\, e^{\lambda_{\alpha,n} t})_{n\geq 1}$ in the space $L^2((a,b)\times(0,T))$. Hence it is natural to study biorthogonal sequences to the sequence $(\Phi_{\alpha,n}(x)\, e^{\lambda_{\alpha,n} t})_{n\geq 1}$ in the space $L^2((a,b)\times(0,T))$.

### 5.2. Classical general considerations

Define
$$\varepsilon_{\alpha,n} : t \in (0,T) \mapsto e^{-\lambda_{\alpha,n} t}, \tag{51}$$

and $E(\alpha, T)$ the smallest closed subspace of $L^2(0,T)$ containing all the functions $\varepsilon_{\alpha,n}$ with $n \geq 1$. Introduce also their $x$-dependent version: denote

$$\tilde{\varepsilon}_{\alpha,n} : (x,t) \in (a,b)\times(0,T) \mapsto e^{-\lambda_{\alpha,n} t} \Phi_{\alpha,n}(x), \tag{52}$$

and $E(\alpha, T, a, b)$ the smallest closed subspace of $L^2((a,b)\times(0,T))$ containing all the functions $\tilde{\varepsilon}_{\alpha,n}$ with $n \geq 1$.

We claim the following:

**Lemma 5.1.** *$E(\alpha, T, a, b)$ is a proper subspace of $L^2((a,b)\times(0,T))$.*

**Proof.** If $E(\alpha, T, a, b) = L^2((a,b)\times(0,T))$, then any element of $L^2((a,b)\times(0,T))$ is the limit of a sequence of linear combinations of $\tilde\varepsilon_{\alpha,n}$. In particular, if we choose $f \in L^2(0,T)$, then $f$ is the limit of such a sequence, and integrating with respect to $x\in(a,b)$, we obtain that $f$ is also the limit of a sequence of linear combinations of $\varepsilon_{\alpha,n}$, hence $L^2(0,T) = E(\alpha,T)$, which is known to be false (see, e.g., [36]) since

$$\sum_{n=1}^\infty \frac{1}{\lambda_{\alpha,n}} < \infty. \qquad \square$$

Now, given $m\geq 1$, denote $E(\alpha,T,a,b;m)$ the smallest closed subspace of $L^2((a,b)\times(0,T))$ containing all the functions $\tilde\varepsilon_{\alpha,n}$ with $n\geq 1$, and $n\neq m$. Then consider $\tilde p_{\alpha,T,a,b;m}$ the orthogonal projection of $\tilde\varepsilon_{\alpha,m}$ on $E(\alpha,T,a,b;m)$, and $d_{\alpha,T,a,b;m}$ the distance between $\tilde\varepsilon_{\alpha,m}$ and $E(\alpha,T,a,b;m)$: we have

$$\begin{aligned} d^2_{\alpha,T,a,b;m} &= \inf_{\tilde p\in E(\alpha,T,a,b;m)} \|\tilde\varepsilon_{\alpha,m} - \tilde p\|^2_{L^2((0,T)\times(a,b))} \\ &= \int_0^T \int_a^b (e^{-\lambda_{\alpha,m}s}\Phi_{\alpha,m}(x) - \tilde p_{\alpha,T,a,b;m}(x,s))^2\, dx\, ds. \end{aligned} \tag{53}$$

Then $\tilde\varepsilon_{\alpha,m} - \tilde p_{\alpha,T,a,b;m}$ is orthogonal to $E(\alpha,T,a,b;m)$, which implies that

$$\forall n\neq m, \quad \int_0^T\int_a^b (e^{-\lambda_{\alpha,m}s}\Phi_{\alpha,m}(x) - \tilde p_{\alpha,T,a,b;m}(x,s)) e^{-\lambda_{\alpha,n}s}\Phi_{\alpha,n}(x)\, dx\, ds = 0,$$

and

$$\begin{aligned} &\int_0^T (e^{-\lambda_{\alpha,m}s}\Phi_{\alpha,m}(x) - \tilde p_{\alpha,T,a,b;m}(x,s)) e^{-\lambda_{\alpha,m}s}\Phi_{\alpha,m}(x)\, dx\, ds \\ &= \int_0^T (e^{-\lambda_{\alpha,m}s}\Phi_{\alpha,m}(x) - \tilde p_{\alpha,T,a,b;m}(x,s))(e^{-\lambda_{\alpha,m}s}\Phi_{\alpha,m}(x) - \tilde p_{\alpha,T,a,b;m}(x,s))\, ds \\ &= d^2_{\alpha,T,a,b;m}. \end{aligned}$$

Hence consider
$$\tilde\sigma^-_{\alpha,T,a,b;m}(x,s) := \frac{e^{-\lambda_{\alpha,m}s}\Phi_{\alpha,m}(x) - \tilde p_{\alpha,T,a,b;m}(x,s)}{d^2_{\alpha,T,a,b;m}} : \tag{54}$$

the sequence of functions $(\tilde\sigma^-_{\alpha,T,a,b;m})_{m\geq 1}$ is a biorthogonal family for the set $(\tilde\varepsilon_{\alpha,n})_{n\geq1}$ in $L^2((a,b)\times(0,T))$.

Moreover it is optimal in the following sense: if $(\tilde\Sigma^-_m)_{m\geq1}$ is another biorthogonal family for the set $(\tilde\varepsilon_{\alpha,n})_{n\geq1}$ in $L^2((a,b)\times(0,T))$, then for all $m\geq 1$, $\tilde\Sigma^-_m - \tilde\sigma^-_{\alpha,T,a,b;m}$ is orthogonal to all $\tilde\varepsilon_{\alpha,n}$, hence to $E(\alpha,T,a,b)$, hence to $\tilde\sigma^-_{\alpha,T,a,b;m}$ since we have $\tilde\sigma^-_{\alpha,T,a,b;m}\in E(\alpha,T,a,b)$. Hence

$$\begin{aligned} \|\tilde\Sigma^-_m\|^2_{L^2((a,b)\times(0,T))} &= \|\tilde\sigma^-_{\alpha,T,a,b;m}\|^2_{L^2((a,b)\times(0,T))} + \|\tilde\Sigma^-_m - \tilde\sigma^-_{\alpha,T,a,b;m}\|^2_{L^2(0,T)} \\ &\geq \|\tilde\sigma^-_{\alpha,T,a,b;m}\|^2_{L^2((a,b)\times(0,T))}. \end{aligned}$$

Therefore
$$\|\tilde\Sigma^-_m\|_{L^2((a,b)\times(0,T))} \geq \|\sigma^-_{\alpha,T,a,b;m}\|_{L^2((a,b)\times(0,T))} = \frac{1}{d_{\alpha,T,a,b;m}}. \tag{55}$$

Hence $1/(d_{\alpha,T,a,b;m})$ is a lower bound of every biorthogonal sequence $(\tilde{\Sigma}_m^-)_{m\geq 1}$. So a bound from above for $d_{\alpha,T,a,b;m}$ gives a bound from below for every biorthogonal sequence.

At last, we note that if the sequence of functions $(\tilde{\Sigma}_m^+)_{m\geq 1}$ is a biorthogonal family for the set $(e^{\lambda_{\alpha,n}t}\Phi_{\alpha,n}(x))_{n\geq 1}$ in $L^2((a,b)\times(0,T))$, then

$$\int_0^T\int_a^b \tilde{\Sigma}_m^+(x,T-s)e^{\lambda_{\alpha,m}T}e^{-\lambda_{\alpha,n}s}\Phi_{\alpha,n}(x)\,dx\,ds = \delta_{m,n},$$

hence $(\tilde{\Sigma}_m^+(x,T-s)e^{\lambda_{\alpha,m}T})_m$ is biorthogonal for the set $(e^{-\lambda_{\alpha,n}t}\Phi_{\alpha,n}(x))_{n\geq 1}$ in the space $L^2((a,b)\times(0,T))$. This implies that

$$\|\tilde{\Sigma}_m^+\|_{L^2((a,b)\times(0,T))} \geq \frac{e^{-\lambda_{\alpha,m}T}}{d_{\alpha,T,a,b;m}}. \tag{56}$$

(And of course this lower bound is achieved for the optimal biorthogonal sequence and hence optimal.)

Now, comparing with what we noticed in section 5.1, we obtain that

$$\forall m\geq 1, \quad C_{NC}(\alpha,T) \geq \frac{e^{-\lambda_{\alpha,m}T}}{d_{\alpha,T,a,b;m}}. \tag{57}$$

In the $x$-independent version, we were able to produce precise lower bounds of the right-hand side, see [11, 12]. But unfortunately, in the present $x$-dependent case, we were not able to provide the expected property:

$$\exists c_u>0, \forall\alpha\in[0,1), \quad C_{NC}(\alpha,T) \geq \frac{c_u}{1-\alpha}e^{\frac{c_u}{T}}e^{-\frac{1}{c_u}T}.$$

However, in the following we provide independent estimates of the behavior of $C_{NC}(\alpha,T)$ with respect to $\alpha$ and with respect to $T$.

### 5.3. Optimal lower bound with respect to $\alpha$

Here the basic remark is that

$$\begin{aligned} d_{\alpha,T,a,b;1} &= \text{dist }(\tilde{\varepsilon}_{\alpha,1}, E_{\alpha,T,a,b;1}) = \text{dist }(\tilde{\varepsilon}_{\alpha,1}, \overline{\text{Vect }(\tilde{\varepsilon}_{\alpha,n}, n\geq 2)} \\ &\leq \text{dist }(\tilde{\varepsilon}_{\alpha,1},\tilde{\varepsilon}_{\alpha,2}) = \|\tilde{\varepsilon}_{\alpha,1}-\tilde{\varepsilon}_{\alpha,2}\|_{L^2((a,b)\times(0,T))}. \end{aligned}$$

From the expressions (52) and (40)-(43), we have for $t\in(0,T)$ and $x\in(a,b)$:

$$\tilde{\varepsilon}_{\alpha,1}(x,t) = e^{-\kappa_\alpha^2 j_{-\nu_\alpha,1}^2 t}\frac{\sqrt{\kappa_\alpha}}{|J'_{-\nu_\alpha}(j_{-\nu_\alpha,1})|}x^{\frac{1-\alpha}{2}}J_{-\nu_\alpha}(j_{-\nu_\alpha,1}x^{\kappa_\alpha}),$$

and
$$\tilde{\varepsilon}_{\alpha,2}(x,t) = e^{-\kappa_\alpha^2 j_{\nu_\alpha,1}^2 t}\frac{\sqrt{\kappa_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,1})|}x^{\frac{1-\alpha}{2}}J_{\nu_\alpha}(j_{\nu_\alpha,1}x^{\kappa_\alpha}).$$

These two expressions differ only by the order of the Bessel functions and related zeros: $-\nu_\alpha$ for $\tilde{\varepsilon}_{\alpha,1}$ and $\nu_\alpha$ for $\tilde{\varepsilon}_{\alpha,2}$. We recall that

$$\forall\nu\in[-\frac{1}{2},\frac{1}{2}], \forall y>0, \quad J_\nu(y) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}\left(\frac{y}{2}\right)^{2m+\nu}. \tag{58}$$

Then by standard regularity results (smoothness with respect to $\nu \in [-\frac{1}{2}, \frac{1}{2}]$ and $y \in (0, +\infty)$, implicit value theorem and smoothness of $\nu \mapsto j_{\nu,1}$ near $\nu = 0$), we directly have

$$\exists C_{a,b}, \quad \sup_{x\in[a,b]} \sup_{t\in[0,T]} |\tilde{\varepsilon}_{\alpha,1}(x,t) - \tilde{\varepsilon}_{\alpha,2}(x,t)| \leq C_{a,b}(1-\alpha).$$

Hence, $d_{\alpha,T,a,b;1} \leq C_{a,b}(1-\alpha)$, and $C_{NC}(\alpha, T) \geq \dfrac{e^{-\lambda_{\alpha,m}T}}{C_{a,b}(1-\alpha)}$.

This proves the first part of Theorem 2.2. (Note that this was already done in [7], but in a slightly more elementary approach that was not useful for the exponential behavior, that we prove in the following). □

### 5.4. Classical exponential lower bound in short time

#### 5.4.1. Adding a control region

Let us consider the symmetrised control region

$$\omega_{\text{sym}} := (-b, -a) \cup (a, b),$$

and the associated control problem:

$$\begin{cases} u_t - (|x|^\alpha u_x)_x = \tilde{h}(x,t)\chi_{\omega_{\text{sym}}}, & x \in (-1,1), t > 0, \\ u(-1,t) = 0 = u(1,t), & t > 0, \\ u(x,0) = u_0(x), & x \in (-1,1). \end{cases} \tag{59}$$

The associated cost of controllability is

$$C_{ctr}(\alpha, T; \omega_{\text{sym}}) := \sup_{\|u_0\|=1} \inf\{\|\tilde{h}\|_{L^2(\omega_{sym})}, u^{(h)}(T) = 0\}.$$

Of course, given $u_0$ of $L^2$-norm equal to 1, any control $h$ that drives the solution of (1) to 0 in time $T$ gives a control that drives the solution of (59) to 0 in time $T$, just choosing

$$\begin{cases} \tilde{h} = h & \text{on } (a,b), \\ \tilde{h} = 0 & \text{on } (-b,-a), \end{cases}$$

and with this choice we obviously have

$$\|\tilde{h}\|_{L^2(\omega_{\text{sym}})} = \|h\|_{L^2(\omega)}.$$

Hence it is cheaper to control (59) than (1), or, more rigorously:

$$C_{ctr}(\alpha, T; \omega_{\text{sym}}) \leq C_{ctr}(\alpha, T; \omega).$$

Hence, to bound from below $C_{ctr}(\alpha, T; \omega)$, it is sufficient to bound from below $C_{ctr}(\alpha, T; \omega_{\text{sym}})$.

### 5.4.2. A related boundary control problem

Now, consider a control $\tilde{h}$ that drives the solution $u^{\tilde{h}}$ of (59) to 0 in time $T$, and define

$$H_-^{[u_0]}(t) := u^{\tilde{h}}(-a,t), \quad H_+^{[u_0]}(t) := u^{\tilde{h}}(a,t).$$

As recalled in Subsection 4.1.1, we have $u \in L^2(0,T;H^1_\alpha(-1,1))$; but remember that $H^1_\alpha(-1,1) \subset C^0([-1,1])$, the injection being continous; in consequence we have $u \in L^2(0,T;C^0([-1,1]))$, which implies that $H_-^{[u_0]}, H_+^{[u_0]} \in L^2(0,T)$.

Define
$$\forall x \in (-a,a), \forall t \in (0,T), \quad v(x,t) := u^{\tilde{h}}(x,t). \tag{60}$$

Then $v$ satisfies
$$\begin{cases} v_t - (|x|^\alpha v_x)_x = 0, & x \in (-a,a), t > 0, \\ v(a,t) = H_+^{[u_0]}(t), & t > 0, \\ v(-a,t) = H_-^{[u_0]}(t), & t > 0, \\ v(x,0) = u_0(x), & x \in (-a,a), \end{cases} \tag{61}$$

and
$$v(T) = 0. \tag{62}$$

Hence, roughly speaking, $H_-^{[u_0]}$ and $H_+^{[u_0]}$ are boundary controls that drive the solution of (61) to 0 in time $T$. In the following we investigate the associated spectral problem.

### 5.4.3. The eigenvalue problem in $(-a,a)$

Consider the associated eigenvalue problem:

$$\begin{cases} -(|x|^\alpha \psi')' = \mu \psi, & x \in (-a,a), \\ \psi(-a) = 0 = \psi(a). \end{cases} \tag{63}$$

The solutions of this eigenvalue problem are given in the following:

**Lemma 5.2.** *When $\alpha \in [0,1)$ and $a \in (0,1)$, we have exactly two sub-families of eigenvalues and associated eigenfunctions for problem (63), that is:*

(a) *the eigenvalues of the form* $\dfrac{\kappa_\alpha^2}{a^{2\kappa_\alpha}} j^2_{\nu_\alpha,n}$ *associated with the odd function*

$$\psi^{(o)}_{\alpha,n}(x) = x^{\frac{1-\alpha}{2}} J_{\nu_\alpha}(j_{\nu_\alpha,n}(\frac{x}{a})^{\kappa_\alpha}) \quad \text{if } x \in (0,a), \tag{64}$$

(b) *the eigenvalues of the form* $\dfrac{\kappa_\alpha^2}{a^{2\kappa_\alpha}} j^2_{-\nu_\alpha,n}$ *associated with the even function*

$$\psi^{(e)}_{\alpha,n}(x) = x^{\frac{1-\alpha}{2}} J_{-\nu_\alpha}(j_{-\nu_\alpha,n}(\frac{x}{a})^{\kappa_\alpha}) \quad \text{if } x \in (0,a). \tag{65}$$

The proof is the same as the proof of Proposition 2.7 of [7], taking into account that here the space domain is $(-a,a)$.

#### 5.4.4. Gap properties of the square roots of the eigenvalues

It is practical to order the eigenvalues in the increasing order: as we did before, the increasing sequence of eigenvalues is $(\mu_{\alpha,n})_{n\geq 1}$, where

$$\forall n \geq 1, \quad \mu_{\alpha,2n-1} = \frac{\kappa_\alpha^2}{a^{2\kappa_\alpha}} j_{-\nu_\alpha,n}^2 \quad \text{and} \quad \mu_{\alpha,2n} = \frac{\kappa_\alpha^2}{a^{2\kappa_\alpha}} j_{\nu_\alpha,n}^2.$$

We note the following gap estimates:

- concerning consecutive eigenvalues of even order:

$$\sqrt{\mu_{\alpha,2n+2}} - \sqrt{\mu_{\alpha,2n}} = \frac{\kappa_\alpha}{a^{\kappa_\alpha}} (j_{\nu_\alpha,n+1} - j_{\nu_\alpha,n}),$$

  hence using Komornik-Loreti [24] p. 135:

$$\forall n \geq 1, \quad \sqrt{\mu_{\alpha,2n+2}} - \sqrt{\mu_{\alpha,2n}} \leq \frac{\kappa_\alpha}{a^{\kappa_\alpha}} \pi, \tag{66}$$

- concerning consecutive eigenvalues of odd order:

$$\sqrt{\mu_{\alpha,2n+1}} - \sqrt{\mu_{\alpha,2n-1}} = \frac{\kappa_\alpha}{a^{\kappa_\alpha}} (j_{-\nu_\alpha,n+1} - j_{-\nu_\alpha,n}),$$

  hence, in the same way ([24] p. 135):

$$\forall n \geq 1, \quad \sqrt{\mu_{\alpha,2n+1}} - \sqrt{\mu_{\alpha,2n-1}} \leq \frac{\kappa_\alpha}{a^{\kappa_\alpha}} \pi, \tag{67}$$

- concerning consecutive eigenvalues, we derive that
  - first $\sqrt{\mu_{\alpha,2n}} - \sqrt{\mu_{\alpha,2n-1}} \leq \sqrt{\mu_{\alpha,2n}} - \sqrt{\mu_{\alpha,2n-2}} \leq \frac{\kappa_\alpha}{a^{\kappa_\alpha}} \pi,$
  - and in the same way

$$\sqrt{\mu_{\alpha,2n+1}} - \sqrt{\mu_{\alpha,2n}} \leq \sqrt{\mu_{\alpha,2n+1}} - \sqrt{\mu_{\alpha,2n-1}} \leq \frac{\kappa_\alpha}{a^{\kappa_\alpha}} \pi.$$

These gap properties will be important in the following.

#### 5.4.5. Consequences of the moment method for (61)

We use the moment method to obtain useful estimates given by (61) and (62). First we denote

$$\forall n \geq 1, \quad \psi_{\alpha,2n-1}(x) := C_n^{(e)} \psi_{\alpha,n}^{(e)}(x), \quad \text{and} \quad \psi_{\alpha,2n}(x) := C_n^{(o)} \psi_{\alpha,n}^{(o)}(x)$$

in such a way that $\psi_{\alpha,n}$ is an eigenfunction associated to the eigenvalue $\mu_{\alpha,n}$, and where the constants $C_n^{(e)}$ and $C_n^{(o)}$ are chosen such that $\|\psi_{\alpha,n}\|_{L^2((-a,a))} = 1$ (hence the set $\{\psi_{\alpha,n}, n \geq 1\}$ forms an orthonormal basis of $L^2(-a,a)$). Next, we denote

$$w_{\alpha,n}(x,t) = \psi_{\alpha,n}(x) e^{\mu_{\alpha,n} t},$$

and we see that $w_{\alpha,n}$ is solution of the adjoint problem:

$$\begin{cases} w_t + (|x|^\alpha w_x)_x = 0 \quad \text{ on } (-a,a)\times(0,T), \\ w(-a,t) = 0 = w(a,t). \end{cases}$$

Then, we multiply the first equation in (61) by $w_{\alpha,n}$:

$$0 = \int_0^T \int_{-a}^a (v_t - (|x|^\alpha v_x)_x w_{\alpha,n},$$

and integrating by parts, we obtain

$$\langle u_0, \psi_{\alpha,n}\rangle_{L^2((-a,a))} \\ = a^\alpha \psi'_{\alpha,n}(a) \int_0^T H_+^{[u_0]}(t)\, e^{\mu_{\alpha,n} t}\, dt - a^\alpha \psi'_{\alpha,n}(-a) \int_0^T H_-^{[u_0]}(t)\, e^{\mu_{\alpha,n} t}\, dt.$$

Now, take specifically $u_0 := \psi_{\alpha,1}$. Then, choosing $p \geq 1$ and $n = 2p-1$, we have: for all $p \geq 1$,

$$a^\alpha \psi'_{\alpha,2p-1}(a) \int_0^T H_+^{[\psi_{\alpha,1}]}(t)\, e^{\mu_{\alpha,2p-1} t}\, dt \\ - a^\alpha \psi'_{\alpha,2p-1}(-a) \int_0^T H_-^{[\psi_{\alpha,1}]}(t)\, e^{\mu_{\alpha,2p-1} t}\, dt = \delta_{1,2p-1} = \delta_{1,p}.$$

Since $\psi_{\alpha,2p-1}$ is an even function, $\psi'_{\alpha,2p-1}(-a) = -\psi'_{\alpha,2p-1}(a)$, and from its formula we have also of course $\psi'_{\alpha,2p-1}(a) \neq 0$. Hence, for all $p \geq 1$, we have

$$\int_0^T H_+^{[\psi_{\alpha,1}]}(t)\, e^{\mu_{\alpha,2p-1} t}\, dt + \int_0^T H_-^{[\psi_{\alpha,1}]}(t)\, e^{\mu_{\alpha,2p-1} t}\, dt = \frac{\delta_{1,p}}{a^\alpha \psi'_{\alpha,2p-1}(a)} = \frac{\delta_{1,p}}{a^\alpha \psi'_{\alpha,1}(a)}.$$

This implies that

$$\forall p \geq 1, \quad \int_0^T \Big( a^\alpha \psi'_{\alpha,1}(a) (H_+^{[\psi_{\alpha,1}]}(t) + H_-^{[\psi_{\alpha,1}]}(t)) \Big)\, e^{\mu_{\alpha,2p-1} t}\, dt = \delta_{1,p}. \tag{68}$$

In the same way, choosing $k \geq 1$ and $u_0 := \psi_{\alpha,2k-1}$, we obtain

$$\forall k,p \geq 1, \quad \int_0^T \big( a^\alpha \psi'_{\alpha,2k-1}(a) (H_+^{[\psi_{\alpha,2k-1}]}(t) + H_-^{[\psi_{\alpha,2k-1}]}(t)) \big)\, e^{\mu_{\alpha,2p-1} t}\, dt = \delta_{k,p}.$$

This tells us that the sequence

$$\Big( a^\alpha \psi'_{\alpha,2k-1}(a) (H_+^{[\psi_{\alpha,2k-1}]} + H_-^{[\psi_{\alpha,2k-1}]}) \Big)_{k\geq 1}$$

is biorthogonal to the sequence $(e^{\mu_{\alpha,2p-1} t})_{p\geq 1}$ in $L^2(0,T)$, and thanks to (67) we are in position to apply Theorem 2.5 of [10], that gives a lower bound for any biorthogonal sequence to a set of exponentials satisfying some gap condition as (67). In our situation, since (67) is true, Theorem 2.5 of [10] gives us that there exists $c_u > 0$ independent of $T$, $\alpha$ and $k$ such that

$$\forall k \geq 1, \quad \|a^\alpha \psi'_{\alpha,2k-1}(a)(H_+^{[\psi_{\alpha,2k-1}]} + H_-^{[\psi_{\alpha,2k-1}]})\|^2_{L^2(0,T)} \geq e^{-2\mu_{\alpha,2k-1}T} e^{\frac{a^{2\kappa_\alpha}}{2\kappa_\alpha^2 \pi^2 T}} \beta(T,a,2k-1), \quad (69)$$

with some $\beta(T,a,2k-1)$ explicitly given in [10], that behaves in a rational way with respect to $T$ (and then the main behavior as $T \to 0$ is given by the exponential factor $e^{\frac{a^{2\kappa_\alpha}}{2\kappa_\alpha^2\pi^2 T}}$). In particular, in the case $k = 1$, we have

$$\beta(T,a,1) = \frac{c_u^2 a^{4\kappa_\alpha}}{T^3} \frac{1}{(\frac{T}{a^{2\kappa_\alpha}} + 1)^2}.$$

Moreover, since $|\psi'_{\alpha,1}(a)|$ is bounded from below by a positive constant (independent of $\alpha \in [0,1)$ but of course depending on $a$), we derive from (69) that

$$\|H_+^{[\psi_{\alpha,1}]} + H_-^{[\psi_{\alpha,1}]}\|_{L^2(0,T)} \geq e^{-\mu_{\alpha,1}T} e^{\frac{a^{2\kappa_\alpha}}{4\kappa_\alpha^2\pi^2 T}} \tilde{\beta}(\alpha,T,a) \quad (70)$$

with
$$\tilde{\beta}(\alpha,T,a) = \frac{c_u a^{2\kappa_\alpha}}{T^{3/2}} \frac{1}{\frac{T}{a^{2\kappa_\alpha}} + 1} \frac{1}{a^\alpha |\psi'_{\alpha,1}(a)|}.$$

In the following, we conclude using some energy estimates.

#### 5.4.6. Energy estimates

Now we can conclude, relating the locally distributed control (on $\omega_{\text{sym}}$) to the boundary controls acting at $-a$ and $a$, using energy methods, as we did in [12].

As proved in section 5.4.5, $H_+^{[\psi_{\alpha,1}]} + H_+^{[\psi_{\alpha,1}]} = u^{\tilde{h}}(a,\cdot) + u^{\tilde{h}}(-a,\cdot)$ has to be exponentially large when $T \to 0$, and this will force $\tilde{h}$ to be also exponentially large. Indeed, first we have for all $y \geq a$

$$-u(y,t) = \int_y^1 u_x(x,t)\,dx,$$

hence for all $y \geq a$

$$u(y,t)^2 = \Big(\int_y^1 u_x(x,t)\,dx\Big)^2 \leq \Big(\int_y^1 x^\alpha u_x^2(x,t)\,dx\Big)\Big(\int_y^1 x^{-\alpha}\,dx\Big)$$
$$\leq \frac{1}{a^\alpha}\int_y^1 x^\alpha u_x^2(x,t)\,dx,$$

and in the same way for all $y \leq -a$,

$$u(y,t)^2 \leq \frac{1}{a^\alpha}\int_{-1}^y |x|^\alpha u_x^2(x,t)\,dx.$$

Then, multiplying the first equation of (59) by $u$, we have

$$\int_0^T \int_{-1}^1 u\tilde{h}\chi_{\omega_{\mathrm{sym}}} = \int_0^T \int_{-1}^1 u(u_t - (|x|^\alpha u_x)_x) = -\frac{1}{2}\int_{-1}^1 u_0^2 + \int_0^T \int_{-1}^1 |x|^\alpha u_x^2,$$

hence

$$\begin{aligned}
&\int_0^T \int_{-1}^1 |x|^\alpha u_x^2 = \frac{1}{2}\int_{-1}^1 u_0^2 + \int_0^T \int_a^b u\tilde{h} + \int_0^T \int_{-b}^{-a} u\tilde{h} \\
&\le \frac{1}{2} + \int_0^T \int_a^b \Big(\frac{1}{a^\alpha}\int_0^1 x^\alpha u_x^2(x,t)\,dx\Big)^{1/2} |\tilde{h}| + \int_0^T \int_{-b}^{-a} \Big(\frac{1}{a^\alpha}\int_{-1}^0 |x|^\alpha u_x^2(x,t)\,dx\Big)^{1/2} |\tilde{h}| \\
&\le \frac{1}{2} + \frac{1}{2}\int_0^T \int_{-1}^1 |x|^\alpha u_x^2(x,t)\,dx\,dt + \frac{1}{2a^\alpha}\int_0^T \int_{\omega_{\mathrm{sym}}} \tilde{h}(x,t)^2\,dx\,dt.
\end{aligned}$$

We obtain that

$$\int_0^T \int_{-1}^1 |x|^\alpha u_x^2\,dx\,dt \le 1 + \frac{1}{a^\alpha}\int_0^T \int_{\omega_{\mathrm{sym}}} \tilde{h}(x,t)^2\,dx\,dt,$$

hence

$$\begin{aligned}
\int_0^T u(a,t)^2\,dt + \int_0^T u(-a,t)^2\,dt &\le \frac{1}{a^\alpha}\int_0^T \int_{-1}^1 |x|^\alpha u_x^2\,dx\,dt \\
&\le \frac{1}{a^\alpha} + \frac{1}{a^{2\alpha}}\int_0^T \int_{\omega_{\mathrm{sym}}} \tilde{h}(x,t)^2\,dx\,dt.
\end{aligned}$$

Hence

$$\int_0^T \int_{\omega_{\mathrm{sym}}} \tilde{h}(x,t)^2\,dx\,dt \ge a^{2\alpha}[\|u(a,\cdot)\|^2_{L^2(0,T)} + \|u(-a,\cdot)\|^2_{L^2(0,T)}] - a^\alpha,$$

and the lower bound (70) implies that

$$\|\tilde{h}\|^2_{L^2(\tilde{\omega}\times(0,T))} \ge \frac{a^{2\alpha}}{4} e^{-2\mu_{\alpha,1}T} e^{\frac{a^{2\kappa_\alpha}}{2\kappa_\alpha^2\pi^2 T}} \tilde{\beta}(\alpha,T,a)^2 - a^\alpha.$$

Then the null controllability cost for (59) blows up at least exponentially fast when $T \to 0^+$, and as a consequence also for (1), as stated in Theorem 2.2. □

### 5.5. Some remarks

We were not able to use (53) and (57) to prove (7). However, (53) and (57) contain interesting informations, and in the following we give two limit cases.

#### 5.5.1. The limit case $a = -1$, $b = 1$

Consider the case of the globally distributed control $a = -1$, $b = 1$: then

$$d^2_{\alpha,T,-1,1;m} = \inf_{\tilde{p}\in E(\alpha,T,-1,1;m)} \|\tilde{\varepsilon}_{\alpha,m} - \tilde{p}\|^2_{L^2((-1,1)\times(0,T))} = \|\tilde{\varepsilon}_{\alpha,m}\|^2_{L^2((-1,1)\times(0,T))}$$

since $\tilde{\varepsilon}_{\alpha,m}$ and $E(\alpha,T,-1,1;m)$ are orthogonal.

Then we immediately have

$$d^2_{\alpha,T,-1,1;m} = \int_0^T e^{-\lambda_{\alpha,m} t}\, dt = \frac{1 - e^{-\lambda_{\alpha,m} T}}{\lambda_{\alpha,m}} \leq T,$$

which gives that the null controllability cost blows up at most as $\frac{1}{T}$, and not exponentially. Of course this can be proved directly, but the goal of this remark was to obtain this information through (53) and (57).

### 5.5.2. The limit case $b$ "close" to $a$

Let us look to a formula "close" to the one of $d^2_{\alpha,T,-1,1;m}$, changing $\Phi_{\alpha,n}(x)$ by $\Phi_{\alpha,n}(a)$ on the interval $(a, b)$. (Of course this would only have a sense if $b = a$, which would bring other problems.) Then, if $\Phi_{\alpha,n}(a) \neq 0$ for all $n$, we would obtain that

$$\begin{aligned} &\text{dist }^2_{L^2((a,b)\times(0,T))}(\varepsilon_{\alpha,1}\Phi_{\alpha,1}(a), \overline{\text{Vect }\{\varepsilon_{\alpha,n}\Phi_{\alpha,n}(a), n \geq 2\}}) \\ &\qquad = (b-a)\Phi_{\alpha,1}(a)^2 \text{ dist }^2_{L^2(0,T)}(\varepsilon_{\alpha,1}, \overline{\text{Vect }\{\varepsilon_{\alpha,n}, n \geq 2\}}), \end{aligned}$$

and then Theorem 2.5 of [10], already applied before in section 5.4.5, says that

$$\text{dist }_{L^2(0,T)}(\varepsilon_{\alpha,1}, \overline{\text{Vect }\{\varepsilon_{\alpha,n}, n \geq 2\}}) \leq (1-\alpha)e^{-\frac{c_u}{T}}$$

(using Lemmas 5.1 and 5.2 of [10]). That would give (7) in this "limit case".

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