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Brézis-Pseudomonotone Mixed Equilibrium Problems Involving a Set-Valued Mapping with Application

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We study the existence of solutions for quasi mixed equilibrium problems involving a set-valued mapping in topological spaces. In case of Banach spaces, we find its strong solutions using (η, g, f)-pseudomonotone mappings. The approach developed in this paper is completely different from most of the techniques used in literature for the study of similar problems, it is based on the notion of pseudomonotonicity in the sense of Brézis for bifunctions in addition to standard use of finite intersection property of compact sets and fixed point theorems. A recent paper by D. Steck [*Brezis pseudomonotonicity is strictly weaker than Ky-Fan hemicontinuity*, J. Optim. Theory Appl. 181 (2019) 318–323] has applied this notion and showed that it is strictly weaker than the notion called *Ky Fan hemicontinuity*, which has been used in many recent works in the literature related to the problem studied in this paper. The results obtained in this paper are new and they improve considerably many existing results in the literature. As an application, we study the existence of solutions of a generalized nonlinear hemivariational inequality problem involving a set-valued operator.

Keywords: Set-valued variational inequalities, equilibrium problems, set-valued mapping, pseudomonotonicity, hemivariational inequalities.

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1. Introduction

The equilibrium problems play key roles in studying various kinds of mathematical problems like optimization, variational inequalities, complementarity problems among others. Although the formulation of the equilibrium problems started implicitly in the paper by Nikaido and Isoda [22] in 1955 with a view to characterizing the Nash equilibrium, the first set of existence results on equilibrium problems was given by Ky Fan [14] in 1972. But the term *equilibrium problems* was coined by

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Blum and Oettli in their seminal paper [6], where they highlighted the unifying aspects of this new field of investigation and gave some fundamental results as well as new concepts. On the other hand Mosco [21] in 1975, has considered a general formulation of some mathematical problems like variational and quasi-variational inequalities, fixed points, Nash equilibrium, minimization problems and they called it *implicit variational problem*. This general formulation is an inequality problem described by the sum of two bifunctions. Later, it is called *mixed equilibrium problems*. Most of the techniques used in the study the existence of solutions for equilibrium problems are based on the KKM principle and arguments from generalized monotonicity/convexity, see [4, 5, 6]. In 2016, Chadli et al. [9] have used a new approach which is based on the notions of maximal monotonicity for bifunctions, initiated by Blum and Oettli in [6] and Hadjisavvas and Khatibzadeh in [18], as well as the concept of pseudomonotonicity in the sense of Brézis initiated by Gwinner [15] to study the existence of solutions for mixed equilibrium problems.

Inspired and motivated by these works, in this paper we study the existence of solutions for mixed equilibrium problems involving a set-valued mapping in topological spaces. This problem has been considered recently by Liu, Migórski and Zeng [20] in the setting of reflexive Banach spaces by using the concept of (f, g, h)-quasimonotonicity and KKM techniques. We use a different approach, based on the notion of pseudomonotonicity in the sense of Brézis for bifunctions, the finite intersection property of compact sets, and a fixed point theorem. The approach developed in this paper is intersecting in the following sense: (1) we present a study in a general setting of vector spaces in duality, (2) we improve considerably some recent results in literature by avoiding some very restrictive assumptions, (3) The results obtained can be used to study many problems in applied mathematics when for instance one of the bifunction involved in the formulation of the problem is described by a Leary-Lions operator or a Navier-Stokes operator. As application, we study the existence of solutions of a generalized nonlinear hemivariational inequality problem involving a set-valued operator. Our result extend and generalize some previous results in the literature, mainly those obtained in the papers [12, 20, 30].

This paper is organized as follows. In Section 2, we provide some preliminary concepts and results along with the problem considered in this paper. In Section 3, we study the existence of solutions for mixed equilibrium problems in a general setting of vector spaces in duality. Section 4 is devoted to study the existence of strong solutions for mixed equilibrium problems associated to a set-valued mapping in the setting of Banach spaces. As an application, in Section 5 we study the existence of solutions of a generalized nonlinear hemivariational inequality problem involving a set-valued operator.

2. Preliminaries

Let X be a topological vector space over $\mathbb{R}$ and Y be a real vector space. Let $\langle\cdot,\cdot\rangle : Y \times X \to \mathbb{R}$ be a bilinear form and C be a nonempty closed and convex subset of X. For a subset A of X, we shall denote by $co(A)$ the convex hull of A, by $cl(A)$ the closure of A in X, by $int(A)$ the interior of A and by $\mathcal{F}(A)$ the family of all finite subsets of A. Let $f, g : C \times C \to \mathbb{R}$ be two real-valued bifunctions, $\eta : C \times C \to X$ be an operator and $T : C \to 2^Y$ be a multi-valued operator.

In this paper, we study in the general setting of vector spaces in duality the following mixed equilibrium problem involving a multi-valued operator:

$$\begin{cases} \text{Find } \bar{x} \in C \text{ and } \bar{x}^* \in T(\bar{x}) \text{ such that} \\ \langle \bar{x}^*, \eta(\bar{x}, y)\rangle + f(\bar{x}, y) + g(\bar{x}, y) \geq 0, \text{ for all } y \in C. \end{cases} \tag{1}$$

Problem (1) has been considered recently by Liu, Migorski and Zeng [20] in the setting of reflexive Banach spaces. Below, we give some particular forms of the problem (1) considered in the literature:

1. In the setting of finite dimensional spaces, Parida, Sahoo and Kumar [24] introduced the following problem: Find $x \in C$ such that

$$\langle T(x), \eta(y, x)\rangle \geq 0, \text{ for all } y \in C, \tag{2}$$

where C is a nonempty, closed, convex subset of $\mathbb{R}^n$, and $T : C \to \mathbb{R}^n$, $\eta : C \times C \to \mathbb{R}^n$ are two continuous operators. Problem (2) is a particular form of (1) and is called *variational-like inequality problem.* It appears in the optimality conditions of some mathematical programming problems. Indeed, Hanson [19] introduced a class of differentiable functions f from $\mathbb{R}^n$ into $\mathbb{R}$ for which there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(x) - f(y) \geq \langle \nabla f(y), \eta(x, y)\rangle, \quad \text{for all } x, y \in \mathbb{R}^n,$$

where ∇ denotes the gradient, known as η-invex functions. Optimality conditions for the minimization problem: $\min f(x)$ subject to $x \in C$, where C is a closed and convex subset of $\mathbb{R}^n$ and f is a differential η-invex function, can be obtained by solving problem (2) with $T(x) = \nabla f(x)$.

2. In the particular case where X is the bidual space Z^{**} of a Banach space Z (not necessarily reflexive), $Y := Z^*$ the dual space of Z and the bilinear form $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ is nothing else than the duality paring between X and Y, $f \equiv 0$, and $g(x, y) = \phi(y) - \phi(x)$ where $\phi : X \to \mathbb{R}$ is a proper and lower semicontinous function, then problem (1) becomes the following problem studied by Costea, Ion and Lupu [12]: Find $x \in C$ and $x^* \in T(x)$ such that

$$\langle x^*, \eta(x, y)\rangle + \phi(y) - \phi(x) \geq 0, \text{ for all } y \in C.$$

3. When X is a reflexive Banach space and $Y := X^*$ is the dual space of X, Wangkeeree and Preechasip [28] considered the following hemivariational inequality governed by a multivalued operator and perturbed by a nonlinear term: Find $x \in C$ and $x^* \in T(x)$ such that

$$\langle x^*, y - x\rangle + f(x, y) + J^0(\hat{x}; \hat{y} - \hat{x}) \geq 0, \text{ for all } y \in K. \tag{3}$$

This problem is a particular form of (1) for which

$$\eta(x, y) = y - x \text{ and } g(x, y) = J^0(\hat{x}; \hat{y} - \hat{x}).$$

If $T \equiv 0$ and $g(x, y) = J^0(x; y - x)$, problem (1) reduces to an hemivariational inequality problem studied by Costea and Rădulescu [13].

Different concepts of solutions of the problem (1) are given in the following definition.

Definition 2.1. An element $\bar{x} \in C$ is called,

(i) *strong solution* of the problem (1) if,
$$\langle x^*, \eta(\bar{x}, y)\rangle + f(\bar{x}, y) + g(\bar{x}, y) \geq 0, \text{ for all } y \in C \text{ and all } x^* \in T(\bar{x});$$

(ii) *solution* of the problem (1) if, there exists $\bar{x}^* \in T(\bar{x})$ such that
$$\langle \bar{x}^*, \eta(\bar{x}, y)\rangle + f(\bar{x}, y) + g(\bar{x}, y) \geq 0, \text{ for all } y \in C;$$

(iii) *weak solution* of the problem (1) if, for each $y \in C$, there exists $x_y^* \in T(\bar{x})$ such that $\langle x_y^*, \eta(\bar{x}, y)\rangle + f(\bar{x}, y) + g(\bar{x}, y) \geq 0$.

Definition 2.2. A multi-valued operator $T : X \to 2^Y$ is said to be

(a) *monotone* if, for any $x, z \in \mathcal{D}(T)$,
$$\langle x^* - z^*, x - z\rangle \geq 0, \quad \text{for all } x^* \in T(x) \text{ and } z^* \in T(z),$$
where $\mathcal{D}(T)$ represents the domain of T;

(b) *maximal monotone* if, $\langle x^* - z^*, x - z\rangle \geq 0$ for all $(z, z^*) \in \mathcal{G}(T)$ implies $x \in \mathcal{D}(T)$ and $x^* \in T(x)$, where $\mathcal{G}(T)$ represents the graph of T.

Definition 2.3. [7] A single-valued mapping $T : X \to Y$ is said to be *pseudomonotone* in the sense of Brézis, for short *B-pseudomonotone*, if for any generalized sequence $\{x_\alpha\}_{\alpha\in I}$ satisfying $\{x_\alpha\}_{\alpha\in I}$ stays in a compact set and converges to $\bar{x}$ and $\limsup\langle T(x_\alpha), x_\alpha - \bar{x}\rangle \leq 0$, its limit $\bar{x}$ satisfies
$$\langle T(\bar{x}), \bar{x} - z\rangle \leq \liminf\langle T(x_\alpha), x_\alpha - z\rangle, \text{ for all } z \in X.$$

Definition 2.4. A single-valued mapping $T : X \to Y$ is said to be *hemicontinuous* (resp., *upper hemicontinuous*) if for all $x, y, z \in X$, the functional $t \mapsto \langle T(x + ty), z\rangle$ is continuous (respectively, upper semicontinuous) on $[0, 1]$.

We recall some concepts for real-valued bifunctions inspired from similar concepts defined for operators acting from a topological vector space to its dual space, see, for instance, [6, 16].

Definition 2.5. A bifunction $f : X \times X \to \mathbb{R}$ is said to be

(a) *monotone* if $f(x, z) + f(z, x) \leq 0$ for all $x, z \in X$;

(d) *hemicontinuous* (respectively, *upper hemicontinuous*) if for all $x, y \in X$, the functional $t \mapsto f(tx + (1 - t)y, x)$ is continuous (respectively, upper semicontinuous) on $[0, 1]$.

Definition 2.6. Let C be a nonempty, closed and convex subset of X. A bifunction $f : X \times X \to \mathbb{R}$ is said to be *pseudomonotone* in the sense of Brézis, for short *B-pseudomonotone*, if for any generalized sequence $\{x_\alpha\}_{\alpha\in I} \subset C$ satisfying $\{x_\alpha\}_{\alpha\in I}$ stays in a compact set, converges to $\bar{x} \in C$ and $\liminf f(x_\alpha, \bar{x}) \geq 0$, its limit $\bar{x}$ satisfies $f(\bar{x}, z) \geq \limsup f(x_\alpha, z)$, for all $z \in C$.

Remark 2.7. The convergence of $\{x_\alpha\}_{\alpha\in I}$ to $\bar{x} \in C$ in Definition 2.6 is considered with respect to the topology of X. For instance, in the case where X is a reflexive Banach space endowed with it weak topology $\sigma(X, X^*)$, the generalized sequence $\{x_\alpha\}_{\alpha\in I}$ will be replaced by a sequence $\{x_n\}_{n\in\mathbb{N}}$ and the convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $\bar{x} \in C$ will be considered in the weak sense, i.e. with respect to the weak topology $\sigma(X, X^*)$ of X.

Remark 2.8. The concept of B-pseudomonone bifunctions has been initiated by Gwinner [15] as an extension to bifunctions of the notion of pseudomonotone operators in the sense of Brézis. In the general setting of Hausdorff topological vector space, this concept has been considered by Aubin [3, page 412] and Gwinner [16, 17] with an aim to relax the continuity properties in the study of many problems related to minimax formulations in game theory as well as fixed points problems. The attractive property of Brézis pseudomonotonicity (which should not be confused with pseudomonotonicity in the sense of Karamardian) is that it provides a unified approach to both monotonicity arguments and compactness arguments, since for instance if T_1 is a monotone and hemicontinuous operator and T_2 is a strongly continuous operator, then $T = T_1 + T_2$ is a pseudomonotone operator in the sense of Brézis, see [29, Proposition 27.6]. We have the following properties:

(a) If $T : X \to Y$ is a B-pseudomonotone operator, then the bifunction f defined by $f(x, y) := \langle T(x), y - x\rangle$ is B-pseudomonotone.

(b) If a bifunction $f : C \times C \to \mathbb{R}$ is (weakly) upper semicontinuous with respect to the first argument, then it is B-pseudomonone. The converse is not true, this has been shown recently by Steck [27] in a counter example to a claim in a recent paper by Sadeqi and Paydar [26] on the equivalence of the two properties for operators.

(c) If $f, g : C \times C \to \mathbb{R}$ are two real-valued B-pseudomonotone bifunctions such that $f(x, x) \leq 0$ and $g(x, x) \leq 0$ for all $x \in C$, then $f+g$ is B-pseudomonotone, see [10, Proposition 2.1]. Hence, the B-pseudomonotonicity notion enjoys the stability under the sum.

Definition 2.9. Let E, F be two Hausdorff topological vector spaces. A multi-valued operator $T : E \to 2^F$ is said to be

(i) *lower semicontinuous* at a point $x_0 \in E$ (for short, l.s.c. at x_0), if and only if, for any open set $\mathcal{O} \subset F$ such that $T(x_0) \cap \mathcal{O} \neq \emptyset$, there exists a neighborhood U of x_0 such that $T(x) \cap \mathcal{O} \neq \emptyset$ for every $x \in U$. We say that T is lower semicontinuous (for short, l.s.c.) if T is l.s.c. for every $x_0 \in E$;

(ii) *upper semicontinuous* at a point $x_0 \in E$ (for short, u.s.c. at x_0), if and only if, for any open set $\mathcal{O} \subset F$ such that $T(x_0) \subset \mathcal{O}$, there exists a neighborhood U of x_0 such that $T(x) \subset \mathcal{O}$ for every $x \in U$. We say that T is upper semicontinous (for short, u.s.c.) if T is u.s.c. for every $x_0 \in E$;

(iii) *closed*, if and only if, for every net $\{x_\alpha\}_{\alpha\in I} \subset E$ converging to x and any $\{y_\alpha\}_{\alpha\in I} \subset F$ converging to y such that $y_\alpha \in T(x_\alpha)$ for all $\alpha \in I$, we have $y \in T(x)$, i.e. $\mathcal{G}(T)$ is closed.

The following proposition gives a characterization of the previous definition in terms of generalized sequences, see e.g. Propositions 5.6.4 and 5.6.5 in [23].

Proposition 2.10. *Let E, F be two Hausdorff topological vector spaces and let $T : E \to 2^F$ be a multi-valued operator. Then,*

(i) *T is l.s.c. if and only if, for any pair $(x, y) \in \mathcal{G}(T)$ and any net $\{x_\alpha\}_{\alpha\in I} \subset E$ converging to x, we can find, for each $\alpha \in I$, an element $y_\alpha \in T(x_\alpha)$ such that $y_\alpha \to y$;*

(ii) *Let M be a subset of X such that $F(x)$ is closed for all $x \in M$. If F is u.s.c. and M is closed, then $\mathcal{G}(F)$ is closed. If $cl(F(M))$ is compact and M is closed, then F is u.s.c. if and only if $\mathcal{G}(F)$ is closed;*

(iii) *If $M \subset X$ is compact, F is u.s.c. and $F(x)$ is compact for all $x \in M$, then $F(M)$ is compact.*

(iv) *If $\{x_\alpha\}_{\alpha\in I} \subset E$ is a net, $x_\alpha \to x$ and $V \subset F$ is open with $F(x) \subset V$, then there exists an index $\alpha_0 \in I$ such that $F(x_\alpha) \subset V$ for all $\alpha \geq \alpha_0$.*

The following result will be needed in the sequel.

Lemma 2.11. [6] *Let D be a convex and compact set, and let K be a convex set. Let $p : D \times K \to \mathbb{R}$ be convex and lower semicontinuous in the first argument, and concave in the second argument. Assume that*

$$\min_{x\in D} p(x, y) \leq 0, \text{ for all } y \in K.$$

Then, there exists $\bar{x} \in D$ such that $p(\bar{x}, y) \leq 0$ for all $y \in K$.

Remark 2.12. It is noteworthy that Lemma 2.11, which is the basic tool in the analysis of variational inequalities with set-valued operators, follows from the classical Kneser minimax theorem and can be directly proved by the separation theorem in finite dimensional spaces, see the comments to Lemma 4.1 in [17]. □

We end this section by the following fixed point theorem for set-valued maps due to Q. H. Ansari and J.-C. Yao [2] that will be needed in the proof of the existence of solutions for the problem (1).

Theorem 2.13. *Let K be a nonempty, closed, and convex subset of a Hausdorff topological vector space E and let $S, W : K \to 2^K$ be two set-valued maps. Assume that*

(i) *For each $x \in K$, $co(S(x)) \subset W(x)$ and $S(x)$ is nonempty;*

(ii) $K = \bigcup\{int_K(S^{-1}(y)) : y \in K\}$;

(iii) *If K is not compact, there exists a nonempty compact convex subset K_0 of K and a nonempty compact subset K_1 of K such that for each $x \in K \setminus K_1$, there exists $\tilde{y} \in K_0$ satisfying $x \in int_K(S^{-1}(\tilde{y}))$.*

Then there exists $\bar{x} \in K$ such that $\bar{x} \in W(\bar{x})$.

For the convenience of the reader, we recall some basic tools from nonsmooth analysis that will be needed in the sequel.

Definition 2.14. [11] Let $\psi : X \to \mathbb{R}$ be a locally Lipschitz function, i.e. for every $u \in X$ there exists a constant $l_u > 0$ and a neighbourhood $\mathcal{O}$ of u such that $|\psi(v) - \psi(w)| \leq l_u \|v - w\|_X$ for all $v, w \in \mathcal{O}$. The *Clarke's generalized directional derivative* of ψ at the point u in the direction v is defined by

$$\psi^0(u; v) := \limsup_{w\to u,\ t\downarrow 0} \frac{\psi(w + tv) - \psi(w)}{t}.$$

The *generalized Clarke gradient* $\partial\psi : X \to 2^{X^*}$ of $\psi : X \to \mathbb{R}$ at $u \in X$ is defined by

$$\partial\psi(u) := \{u^* \in X^* : \psi^0(u; v) \geq \langle u^*, v\rangle,\ \forall v \in X\}.$$

The next two propositions provide important properties of the Clarke's generalized directional derivative and the generalized gradient.

Proposition 2.15. [11, Proposition 2.1.1] *Let $\psi : X \to \mathbb{R}$ be a locally Lipschitz functional of constant l_u near the point $u \in X$. Then,*

(i) *The function $v \in X \mapsto \psi^0(u; v)$ is finite, positively homogeneous, sub-additive and satisfies $|\psi^0(u; v)| \leq l_u \; \|v\|_X$;*

(ii) *$\psi^0(u; v)$ is upper semicontinuous as a function of (u, v).*

Proposition 2.16. [11, Proposition 2.1.2] *Let $\psi : X \to \mathbb{R}$ be a locally Lipschitz functional of constant l_u near the point $u \in X$. Then,*

(i) *$\partial\psi(u)$ is a convex, weak* compact subset of X^* and*

$$\|u^*\|_{X^*} \leq l_u \quad \text{for all } u^* \in \partial\psi(u);$$

(ii) *For each $v \in X$, one has*

$$\psi^0(u; v) = \max\{\langle u^*, v\rangle : u^* \in \partial\psi(u)\}.$$

3. Existence of solutions in Topological vector spaces in duality

Let X be a topological vector space over $\mathbb{R}$ and Y be a real vector space. We assume that X and Y are in duality defined by a bilinear form $\langle\cdot,\cdot\rangle : Y \times X \to \mathbb{R}$, and the vector space Y is endowed with the topology $\sigma(Y, X)$ generated by the family $\{V(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$ as a basis of the neighbourhood system at $\mathbf{0}$, where $V(x, \varepsilon) := \{y \in Y : |\langle y, x\rangle| < \varepsilon\}$. Note that the vector space Y endowed with the topology $\sigma(Y, X)$ is a Hausdorff topological vector space since the family of linear functions $\{\langle\cdot, x\rangle\}_{x\in X}$ separates the points of Y, see [1, page 48].

Let C be a nonempty closed and convex subset of X, $\eta : C \times C \to X$ be a mapping, $T : C \to 2^Y$ be a multi-valued operator and $f, g : C \times C \to \mathbb{R}$ be two real-valued bifunctions. The problem studied in this section is as the following:

$$\begin{cases} \text{Find } \bar{x} \in C \text{ and } \bar{x}^* \in T(\bar{x}) \text{ such that} \\ \langle\bar{x}^*, \eta(\bar{x}, y)\rangle + f(\bar{x}, y) + g(\bar{x}, y) \geq 0, \text{ for all } y \in C. \end{cases} \tag{4}$$

We consider the following assumptions:

$[A_T^1]$ $T : C \to 2^Y$ is closed with respect to Y endowed with the topology $\sigma(Y, X)$ and quasi-$\sigma(Y, X)$-compact, i.e. for any relatively compact set $M \subset X$, $T(M)$ is relatively compact with respect to the topology $\sigma(Y, X)$ of Y.

$[A_\eta^1]$ $\eta : C \times C \to X$ is a map such that

(i) For any $y \in C$, the map $x \in C \mapsto \eta(x, y)$ is continuous;

(ii) For all $x \in C$, $m \in \mathbb{N}$ and $0 \leq t_i \leq 1$ $(i = 1, \dots, m)$ satisfying $\sum_{i=1}^m t_i = 1$, we have for $y_i \in C$ $(i = 1, \cdots, m)$

$$\eta(x, \sum_{i=1}^m t_i y_i) = \sum_{i=1}^m t_i \eta(x, y_i);$$

(iii) For any $x, y, z \in C$, $\eta(x, y) = \eta(x, z) + \eta(z, y)$.

First, we start by proving the following result.

Lemma 3.1. *Let X be a Hausdorff topological vector space, C be a nonempty closed and convex subset of X, and Y be a vector space endowed with the topology $\sigma(Y,X)$ such that for each $y \in Y$, the function $x \mapsto \langle y, x\rangle$ is continuous. Let $h : C \times C \to \mathbb{R}$ be a real-valued bifunction such that $h(x,x) \geq 0$ for all $x \in C$, $\eta : C \times C \to X$ be a map, and $T : C \to 2^Y$ be a multi-valued operator such that $T(x)$ is convex for each $x \in C$. Suppose that $[\mathrm{A}_T^1]$ and $[\mathrm{A}_\eta^1]$ are satisfied. Furthermore, suppose that*

(i) *For each $N \in \mathcal{F}(C)$ and any $y \in C$, if $\{x_i\}_{i\in I} \subset co(N)$ such that $x_i \to x$, we have*
$$\liminf h(x_i, y) \leq h(x, y);$$

(ii) *For all $x \in C$, the function $y \in C \mapsto h(x, y)$ is convex.*

Then, for each $Z \in \mathcal{F}(C)$ there exists $x \in co(Z)$ and $x^* \in T(x)$ such that
$$\langle x^*, \eta(x, y)\rangle + h(x, y) \geq 0, \qquad \text{for all } y \in co(Z). \tag{5}$$

Proof. We consider the set-valued map $\mathbb{Q} : co(Z) \to 2^{co(Z)}$ defined for $y \in co(Z)$ by
$$\mathbb{Q}(y) := \{x \in co(Z) : \sup_{x^*\in T(x)} \langle x^*, \eta(x, y)\rangle + h(x, y) \geq 0\}.$$

From $[\mathrm{A}_\eta^1]$ (iii), we have $\eta(y, y) = 0$, and hence $\mathbb{Q}(y) \neq \emptyset$ since $y \in \mathbb{Q}(y)$. Let us verify that $\mathbb{Q}(y)$ is closed for each $y \in co(Z)$. To this aim, let $\{x_i\}_{i\in I} \subset \mathbb{Q}(y)$ be a net converging to $x \in co(Z)$. Since $x_i \in \mathbb{Q}(y)$, it follows that there exists $x_i^* \in T(x_i)$ such that
$$\langle x_i^*, \eta(x_i, y)\rangle + h(x_i, y) \geq 0, \qquad \text{for this } y \in co(Z). \tag{6}$$
Since $co(Z)$ is compact, we deduce from $[\mathrm{A}_T^1]$ that there exists $x^* \in Y$ and a subnet $\{x_j^*\}_{j\in J}$ of $\{x_i^*\}_{i\in I}$ such that $x_j^* \to x^*$ for the $\sigma(Y,X)$-topology and $x^* \in T(x)$. On the other hand, by using $[\mathrm{A}_\eta^1]$(iii), we get

$$\begin{aligned} |\langle x_j^*, \eta(x_j, y)\rangle - \langle x^*, \eta(x, y)\rangle| &= |\langle x_j^* - x^*, \eta(x_j, y)\rangle + \langle x^*, \eta(x_j, x)\rangle| \\ &= |\langle x_j^* - x^*, \eta(x_j, x)\rangle + \langle x_j^* - x^*, \eta(x, y)\rangle + \langle x^*, \eta(x_j, x)\rangle| \\ &\leq |\langle x_j^* - x^*, \eta(x_j, x)\rangle| + |\langle x_j^* - x^*, \eta(x, y)\rangle| + |\langle x^*, \eta(x_j, x)\rangle|. \end{aligned} \tag{7}$$

Since $x_j^* \to x^*$ for the $\sigma(Y,X)$-topology of Y, it follows that
$$\langle x_j^* - x^*, \eta(x, y)\rangle \to 0. \tag{8}$$

Since $x_j \to x$ and for each $z \in Y$ the function $x \mapsto \langle x, z\rangle$ is continuous, it follows, thanks to $[\mathrm{A}_\eta^1]$(i), that
$$\langle x^*, \eta(x_j, x)\rangle \to \langle x^*, \eta(x, x)\rangle = 0, \tag{9}$$

by $[\mathrm{A}_\eta^1]$(iii). As $co(Z)$ is compact, then it is of second category. It follows from the Banach-Steinhaus theorem (see [25, Theorem 2.5]) that the family of functions $\{\langle x_j^* - x^*, \cdot\rangle\}_{j\in J}$ is equicontinuous on $co(Z)$. Hence,
$$\langle x_j^* - x^*, \eta(x_j, x)\rangle \to 0. \tag{10}$$

Therefore, by taking account of relations (8)-(10), we deduce from (7) that

$$\langle x_j^*, \eta(x_j, y)\rangle \to \langle x^*, \eta(x, y)\rangle.$$

Consequently, by considering the lower limit in relation (6) and by taking account of assumption (i), we deduce that

$$\langle x^*, \eta(x, y)\rangle + h(x, y) \geq 0.$$

Hence, $x \in \mathbb{Q}(y)$ and therefore $\mathbb{Q}(y)$ is closed for each $y \in co(Z)$.

Now, let us verify that $\bigcap_{y\in co(Z)} \mathbb{Q}(y) \neq \emptyset$.

Suppose the contrary to this holds, then the family $\{[\mathbb{Q}(y)]^c\}_{y\in co(Z)}$, where $[\mathbb{Q}(y)]^c$ is the complementary of $\mathbb{Q}(y)$, is an open covering of the compact set $co(Z)$. It follows that there exists $\{y_1, \cdots, y_N\} \subset co(Z)$ such that $co(Z) \subset \bigcup_{k=1}^N [\mathbb{Q}(y_k)]^c$.

For $k \in \{1, \cdots, N\}$, let $d_k(x)$ be the distance of x to $T(y_k)$ and let us consider the function $\gamma_k : co(Z) \to \mathbb{R}$ defined by $\gamma_k(x) = d_k(x)/\sum_{j=1}^N d_j(x)$. For each $k \in \{1, \cdots, N\}$, γ_k is a Lipschitz function such that $\gamma_k(x) = 0$ for all $x \in \mathbb{Q}(y_k)$.

Furthermore, for $k = 1, \cdots, N$ we have $0 \leq \gamma_k(x) \leq 1$ and $\sum_{k=1}^N \gamma_k(x) = 1$ for all $x \in co(Z)$. Let us now consider the function $\rho : co(Z) \to co(Z)$ defined by $\varrho(x) = \sum_{k=1}^N \gamma_k(x) y_k$. The function ϱ is continuous on the compact set $co(Z)$.

Therefore, by the Brouwer fixed point theorem, we deduce that there exists some $x_0 \in co(Z)$ such that $\varrho(x_0) = x_0$. We consider the function $\pi : co(Z) \to \mathbb{R}$ defined by

$$\pi(x) = \sup_{\varpi\in T(x)} \langle x^*, \eta(x, \rho(x))\rangle + h(x, \rho(x)).$$

By using assumptions (ii) and $[\mathrm{A}_\eta^1]$(ii), we obtain

$$\pi(x) \leq \sum_{k=1}^N \gamma_k(x) \Big[\sup_{x^*\in T(x)} \langle x^*, \eta(x, y_k)\rangle + h(x, y_k) \Big]. \tag{11}$$

On the other hand, for any $x \in co(Z)$ we have two possibilities:

- If $x \in [\mathbb{Q}(y_k)]^c$, then $\gamma_k(x) > 0$ and $\sup_{x^*\in T(x)} \langle x^*, \eta(x, y_k)\rangle + h(x, y_k) < 0$;
- If $x \in \mathbb{Q}(y_k)$, then $\gamma_k(x) = 0$ and $\sup_{x^*\in T(x)} \langle x^*, \eta(x, y_k)\rangle + h(x, y_k) \geq 0$.

Since $co(Z) \subset \bigcup_{k=1}^N [\mathbb{Q}(y_k)]^c$, it follows that for any $x \in co(Z)$ there exists at least $k_0 \in \{1, \cdots, N\}$ such that $x \in [\mathbb{Q}(y_{k_0})]^c$ and hence from relation (11) and what precedes we have $\pi(x) < 0$. Therefore, we have shown that $\pi(x) < 0$ for every $x \in co(Z)$. This leads to a contradiction since $\pi(x_0) \geq 0$.

Therefore, $\bigcap_{y\in co(Z)} \mathbb{Q}(y) \neq \emptyset$. Let $\bar{x} \in \bigcap_{y\in co(Z)} \mathbb{Q}(y)$, then

$$\sup_{x^*\in T(\bar{x})} \langle x^*, \eta(\bar{x}, y)\rangle + h(\bar{x}, y) \geq 0, \text{ for all } y \in co(Z). \tag{12}$$

Let us consider the function $p : T(\bar{x}) \times co(Z) \to \mathbb{R}$ defined for $(x^*, y) \in T(\bar{x}) \times co(Z)$ by $\quad p(x^*, y) = -[\langle x^*, \eta(\bar{x}, y)\rangle + h(\bar{x}, y)]$.

Note that the function $p(\cdot, z)$ is convex and lower semicontinuous, and that the function $p(x^*, \cdot)$ is concave.

Furthermore, from relation (12) we deduce that for each $y \in co(Z)$ there exists $x^* = x^*(y) \in T(\bar{x})$ such that $p(x^*, y) \leq 0$. As $T(\bar{x})$ is convex and $co(Z)$ is compact, then by Lemma 2.11 we deduce that there exists $\bar{x}^* \in T(\bar{x})$ such that $p(\bar{x}^*, y) \leq 0$ for all $y \in co(Z)$. Which completes the proof of the lemma. □

Remark 3.2. (1) Condition (i) in Lemma 3.1 is satisfied if for instance we suppose that for each $y \in C$, the function $x \in C \mapsto h(x, y)$ is upper semicontinuous on $co(N)$ for each $N \in \mathcal{F}(C)$.

(2) The proof of Lemma 3.1 has been inspired by a contradiction argument used by F. E. Browder in his seminal work on nonlinear variational inequalities, see [8].

Our main result of this section is the following.

Theorem 3.3. *Let X be a Hausdorff topological vector space, C be a nonempty closed and convex subset of X, and Y be a vector space endowed with the topology $\sigma(Y, X)$ such that for each $y \in Y$, the function $x \mapsto \langle y, x\rangle$ is continuous. Let $T : C \to 2^Y$ be a multi-valued operator, $\eta : C \times C \to X$ be a map and $f, g : C \times C \to \mathbb{R}$ be two real-valued bifunctions such that $f(x, x) = g(x, x) = 0$ for all $x \in C$. Suppose that $[\mathrm{A}^1_T]$ and $[\mathrm{A}^1_\eta]$ are satisfied. Furthermore, suppose that*

(i) *f is monotone;*

(ii) *g is B-pseudomonotone;*

(iii) *$f(x, \cdot)$ and $g(x, \cdot)$ are convex functions;*

(iv) *$f(x, \cdot)$ is lower semicontinuous;*

(v) *For each $y \in C$ and $N \in \mathcal{F}(C)$, $g(\cdot, y)$ is upper semicontinuous on $co(N)$ and $f(\cdot, y)$ is continuous on $co(N)$;*

(vi) *There exists a nonempty compact set $D \subset C$ and a nonempty compact convex subset $K \subset C$, such that for each $x \in C \setminus D$, there exists $z \in K$ satisfying*
$$\langle x^*, \eta(x, z)\rangle + g(x, z) < f(z, x), \text{ for each } x^* \in T(x).$$

Then the problem (1) *has at least a weak solution $\bar{x} \in C$. Moreover, if $T(\bar{x})$ is convex, then $\bar{x}$ is a solution of* (1).

Proof. We proceed in two steps.

Step 1: We suppose that C is compact. From Lemma 3.1, we have that for each $Z \in \mathcal{F}(C)$ there exists $x \in co(Z)$ and $x^* \in T(x)$ such that

$$\langle x^*, \eta(x, y)\rangle + f(x, y) + g(x, y) \geq 0, \qquad \text{for all } y \in co(Z).$$

Since f is monotone, it follows that

$$\langle x^*, \eta(x, y)\rangle + g(x, y) \geq f(y, x), \qquad \text{for all } y \in co(Z). \tag{13}$$

For each $Z \in \mathcal{F}(C)$, let us consider the following set

$$\mathbb{W}_Z := \{x \in C : \exists x^* \in T(x) \text{ such that } \langle x^*, \eta(x, z)\rangle + g(x, z) \geq f(z, x), \ \forall z \in co(Z)\}.$$

From (13), we have that $\mathbb{W}_Z \neq \emptyset$ for each $Z \in \mathcal{F}(C)$. Furthermore, $cl(\mathbb{W}_Z)$ is a compact set since C is assumed to be a compact set. Let us show that $\bigcap_{Z\in\mathcal{F}(C)} cl(\mathbb{W}_Z) \neq \emptyset$. To this aim, it suffices to verify that the family of sets $\{cl(\mathbb{W}_Z)\}_{Z\in\mathcal{F}(C)}$ has the finite intersection property.

Let $Z_1,\ Z_2,\cdots,Z_p$ be a finite number of elements in $\mathcal{F}(C)$. Set $M=\cup_{i=1}^{p}Z_i\in\mathcal{F}(C)$. One can easily verify that $\mathbb{W}_M\subset\cap_{i=1}^{p}\mathbb{W}_{Z_i}$. It follows that $\cap_{i=1}^{p}cl(\mathbb{W}_{Z_i})\neq\emptyset$. Therefore, $\bigcap_{Z\in\mathcal{F}(C)}cl(\mathbb{W}_Z)\neq\emptyset$. Let $\bar{x}\in\bigcap_{Z\in\mathcal{F}(C)}cl(\mathbb{W}_Z)$ and let $y\in C$ be an arbitrary element. Let us set $E=\{\bar{x},y\}\in\mathcal{F}(C)$. Since $\bar{x}\in\bigcap_{Z\in\mathcal{F}(C)}cl(\mathbb{W}_Z)$, it follows that $\bar{x}\in cl(\mathbb{W}_E)$. Hence, there exists $\{x_\alpha\}_{\alpha\in I}\subset\mathbb{W}_E$ such that $x_\alpha\to\bar{x}$, with x_α depending on y. Since $x_\alpha\in\mathbb{W}_E$, there exists $x_\alpha^*=x_\alpha^*(y)\in Y$ such that $x_\alpha^*\in T(x_\alpha)$ and

$$\langle x_\alpha^*,\eta(x_\alpha,z)\rangle+g(x_\alpha,z)\geq f(z,x_\alpha),\qquad\text{for all } z\in co(\{y,\bar{x}\}).\tag{14}$$

Since T is closed with respect to Y endowed with the topology $\sigma(Y,X)$ and it is quasi-$\sigma(Y,X)$-compact, it follows that there exists a subnet $\{x_\beta^*\}_{\beta\in J}$ of $\{x_\alpha^*\}_{\alpha\in I}$ such that $x_\beta^*\to x^*=x^*(y)\in Y$ and $x^*\in T(\bar{x})$. By considering relation (14) for $z=\bar{x}$, we get for each $\beta\in J$

$$\langle x_\beta^*,\eta(x_\beta,\bar{x})\rangle+g(x_\beta,\bar{x})\geq f(\bar{x},x_\beta).\tag{15}$$

Hence, by taking the lower limit in the previous relation, we obtain

$$\lim\langle x_\beta^*,\eta(x_\beta,\bar{x})\rangle+\liminf g(x_\beta,\bar{x})\geq\liminf f(\bar{x},x_\beta).$$

Since $\lim\langle x_\beta^*,\eta(x_\beta,\bar{x})\rangle=0$ and $\liminf f(\bar{x},x_\beta)\geq f(\bar{x},\bar{x})=0$, we deduce that

$$\liminf g(x_\beta,\bar{x})\geq 0.$$

Therefore, since g is B-pseudomonotone we deduce that

$$\limsup g(x_\beta,z)\leq g(\bar{x},z),\quad\text{for all } z\in C.$$

For $z=y$ in the previous inequality, we get

$$\limsup g(x_\beta,y)\leq g(\bar{x},y).\tag{16}$$

On the other hand, let us take $z=y$ in relation (14), we obtain

$$\langle x_\beta^*,\eta(x_\beta,y)\rangle+g(x_\beta,y)\geq f(y,x_\beta).\tag{17}$$

From the fact that $\langle x_\beta^*,\eta(x_\beta,y)\rangle\to\langle x^*,\eta(\bar{x},y)\rangle$ and that $f(y,\cdot)$ is lower semicontinuous, we deduce, after taking the lower limit in (17) and using relation (16),

$$\langle x^*,\eta(\bar{x},y)\rangle+g(\bar{x},y)\geq f(y,\bar{x}).$$

Therefore, we have proved the following: There exists $\bar{x}\in C$ such that for each $y\in C$ there exists $x^*\in T(\bar{x})$ satisfying

$$\langle x^*,\eta(\bar{x},y)\rangle+g(\bar{x},y)\geq f(y,\bar{x}).\tag{18}$$

Step 2: We suppose that C is not necessarily compact. Let $N\in\mathcal{F}(C)$ and let us set $\widetilde{C}:=co(N\cup K)$. Note that $\widetilde{C}$ is a compact, convex subset of X. From the first step, there exists $\tilde{x}\in\widetilde{C}$ such that for any $z\in\widetilde{C}$ we have $\langle x^*,\eta(\bar{x},z)\rangle+g(\bar{x},z)\geq f(z,\bar{x})$

for some $x^* \in T(\tilde{x})$. Moreover, from the coercivity assumption (vi), we deduce that $\tilde{x} \in D$. Now, let us consider for each $N \in \mathcal{F}(C)$ the following set

$$\widetilde{\mathbb{W}}_N := \left\{ x \in D : \begin{array}{l} \forall z \in co(K \cup N),\ \exists x^* \in T(x) \text{ such} \\ \text{that } \langle x^*, \eta(x,z)\rangle + g(x,z) \geq f(z,x) \end{array} \right\}.$$

From what precedes, we have that $\widetilde{\mathbb{W}}_N \neq \emptyset$ for each $N \in \mathcal{F}(C)$. On the other hand, one can easily verify that the family of sets $\{\widetilde{\mathbb{W}}_N\}_{N\in\mathcal{F}(C)}$ has the finite intersection property. Since D is compact, it follows that $\bigcap_{N\in\mathcal{F}(C)} cl(\widetilde{\mathbb{W}}_N) \neq \emptyset$. Let $\bar{x} \in \bigcap_{N\in\mathcal{F}(C)} cl(\widetilde{\mathbb{W}}_N)$ and y be an arbitrary element in C. We set $L = \{y, \bar{x}\} \in \mathcal{F}(C)$. Since $\bar{x} \in cl(\widetilde{\mathbb{W}}_L)$, there exists a net $\{\tilde{x}_\alpha\}_{\alpha\in I} \subset \widetilde{\mathbb{W}}_L$ such that $\tilde{x}_\alpha \to \bar{x}$. From $x_\alpha \in \widetilde{\mathbb{W}}_L$, we deduce that for any $z \in co(K \cup \{y, \bar{x}\})$ there exists $x^*_\alpha \in T(\tilde{x}_\alpha)$ with

$$\langle x^*_\alpha, \eta(x_\alpha, z)\rangle + g(x_\alpha, z) \geq f(z, x_\alpha). \tag{19}$$

By proceeding similarly as in Step 1, we deduce from (19) that there exists $x^* \in T(\bar{x})$ such that

$$\langle x^*, \eta(\bar{x}, y)\rangle + g(\bar{x}, y) \geq f(y, \bar{x}). \tag{20}$$

Now, let $\{t_i\}_{i\in I} \subset]0,1[$ such that $t_i \to 0$ and let $y_i := t_i z + (1-t_i)\bar{x} \in C$, where $z \in C$ is an arbitrary element. From (20), we have for each $i \in I$, there exists $x^*_i \in T(\bar{x})$ such that

$$\langle x^*_i, \eta(\bar{x}, y_i)\rangle + g(\bar{x}, y_i) \geq f(y_i, \bar{x}). \tag{21}$$

On the other hand, we have that

$$0 = f(y_i, y_i) \leq t_i f(y_i, z) + (1-t_i) f(y_i, \bar{x}).$$

As a result, by using (21), we obtain

$$t_i[f(y_i,\bar{x}) - f(y_i,z)] \leq f(y_i,\bar{x}) \leq \langle x^*_i, \eta(\bar{x}, y_i)\rangle + g(\bar{x}, y_i)$$

Hence, from $[\mathrm{A}^1_\eta]$(ii) and assumption (iii), we get

$$\begin{aligned} &t_i[f(y_i,\bar{x}) - f(y_i,z)] \\ &\leq t_i\langle x^*_i, \eta(\bar{x}, z)\rangle + (1-t_i)\langle x^*_i, \eta(\bar{x},\bar{x})\rangle + t_i g(\bar{x}, z) + (1-t_i) g(\bar{x},\bar{x}). \end{aligned} \tag{22}$$

Since $\eta(\bar{x},\bar{x}) = \mathbf{0}$, $g(\bar{x},\bar{x}) = 0$ and $t_i > 0$, we deduce from (22) that

$$f(y_i,\bar{x}) \leq \langle x^*_i, \eta(\bar{x}, z)\rangle + g(\bar{x}, z) + f(y_i, z). \tag{23}$$

On the other hand, we have that $\{x^*_i\}_{i\in I} \subset T(\bar{x})$. Hence, from $[\mathrm{A}^1_T]$ and without loss of generality, we deduce that $x^*_i \to x^* \in T(\bar{x})$. By taking the limit in (23) and using assumption (v) and the fact that $f(\bar{x},\bar{x}) = 0$, we obtain

$$\langle x^*, \eta(\bar{x}, z)\rangle + g(\bar{x}, z) + f(\bar{x}, z) \geq 0, \tag{24}$$

and hence $\bar{x}$ is a weak solution of the problem (1).

Now, let us suppose that $T(\bar{x})$ is convex. We shall show that $\bar{x}$ is a strong solution of the problem (1). To this aim, let us consider the function $p : T(\bar{x}) \times C \to \mathbb{R}$ defined for $(x^*, z) \in T(\bar{x}) \times C$ by $p(x^*, z) = -[\langle x^*, \eta(\bar{x}, z)\rangle + g(\bar{x}, z) + f(\bar{x}, z)]$. Note that the function $p(\cdot, z)$ is convex and lower semicontinuous, and that the function $p(x^*, \cdot)$ is concave. Furthermore, from (24) we have that for each $z \in C$ there exists $x^* \in T(\bar{x})$ such that $p(x^*, z) \leq 0$. Hence, by using Lemma 2.11, there exists $\bar{x}^* \in T(\bar{x})$ such that $p(\bar{x}^*, z) \leq 0$ for all $z \in C$. Which implies that $\bar{x}$ is a strong solution of the problem (1). □

Remark 3.4. From the proof of Theorem 3.3, we can see that the result still true if the assumption of B-pseudomonotonicity on the bifunction g is replaced by the following weaker assumption:

(ii)′ If for any generalized sequence $\{x_\alpha\}_{\alpha \in I}$ satisfying $\{x_\alpha\}_{\alpha \in I}$ stays in a compact set and converges to $\bar{x}$ and $\liminf g(x_\alpha, \bar{x}) \geq 0$, its limit $\bar{x}$ satisfies

$$g(\bar{x}, z) \geq \liminf g(x_\alpha, z), \text{ for all } z \in C.$$

We end this section by the following remark in which we give a comparison with recent existing results in literature on the problem (1).

Remark 3.5. If in Theorem 3.3 we suppose that X is a Banach space and $Y := X^*$ is the topological dual space of X, and that the bilinear form $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ is noting else than the duality pairing between X^* and X, then Theorem 3.3 improves [20, Theorem 3.1].

4. Strong solutions in Banach spaces

Let X be a Banach space with X^* its topological dual space. We denote by $\langle x^*, x\rangle$ the duality pairing between $x^* \in X^*$ and $x \in X$. The norms of X and X^* are denoted by $\|\cdot\|$. We use the standard notation $x_n \to x$ to denote the strong convergence of a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ to x and $x_n \rightharpoonup x$ to denote the weak convergence of a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ to x with respect to the weak topology $\sigma(X, X^*)$ of X. Let C be a nonempty closed and convex subset of X.

In this section, we consider the following assumptions:

$[A_T^2]$ $T : C \to 2^{X^*}$ is a multi-valued mapping such that for any $N \in \mathcal{F}(C)$, the restriction of T to $co(N)$ is l.s.c. with respect to the weak*-topology of X^*.

$[A_\eta^2]$ $\eta : C \times C \to X$ is a map such that

- (i) For any $y \in C$, the map $x \in C \mapsto \eta(x, y)$ is weakly continuous;
- (ii) For any $x, z \in C$ and all $z^* \in T(z)$, $\langle z^*, \eta(y, y)\rangle \geq 0$ and the function $y \in C \mapsto \langle z^*, \eta(x, y)\rangle$ is convex.

Next, we recall the concept of (η, g, f)-pseudomonotonicity which will be useful for the sequel. For examples and more details on this concept and its extensions, we refer for instance to [20, 28] and the references therein.

Definition 4.1. Let $T : C \to 2^{X^*}$ be a multi-valued operator, $\eta : C \times C \to X$ be a mapping and $g, f : C \times C \to \mathbb{R}$ be two bifunctions. Then T is said to be

(i) (η, g, f)-*pseudomonotone*, if for each $x, y \in C$

$$\langle x^*, \eta(x, y)\rangle + g(x, y) \geq f(y, x) \Longrightarrow \langle y^*, \eta(x, y)\rangle + g(x, y) \geq f(y, x),$$

for all $x^* \in T(x)$ and $y^* \in T(y)$.

(ii) stably (η, g, f)-*pseudomonotone* with respect to a set $V^* \subset X^*$, if $T(\cdot) - v^*$ is (η, g, f)-pseudomonotone for every $v^* \in V^*$.

First, we start by showing the following Lemma.

Lemma 4.2. *Let C be a nonempty, closed and convex subset of a Banach space X, and let $h : C \times C \to \mathbb{R}$ be a real-valued bifunction such that $h(x, x) \geq 0$ for all $x \in C$. Let $T : C \to 2^{X^*}$ be a multi-valued mapping and $\eta : C \times C \to X$ be a map. Suppose that* $[\mathrm{A}_T^2]$ *and* $[\mathrm{A}_\eta^2]$ *hold. Furthermore, suppose that*

(i) *For each $N \in \mathcal{F}(C)$ and any $y \in C$, the function $x \in C \mapsto h(x, y)$ is upper semicontinuous on $co(N)$;*

(ii) *For each $x \in C$, the function $y \in C \mapsto h(x, y)$ is convex.*

Then, for any $L \in \mathcal{F}(C)$ there exists $x \in co(L)$ such that for all $x^* \in T(x)$, we have

$$\langle x^*, \eta(x, y)\rangle + h(x, y) \geq 0, \ \textit{for all } y \in co(L).$$

Proof. We proceed by contradiction. Suppose that there exists $L \in \mathcal{F}(C)$ such that for all $x \in co(L)$, there exists $x^* \in T(x)$ and $y = y(x, x^*) \in co(L)$ satisfying

$$\langle x^*, \eta(x, y)\rangle + h(x, y) < 0. \tag{25}$$

We consider the multi-valued mapping $\Xi : co(L) \to 2^{co(L)}$ defined for $x \in co(L)$ by

$$\Xi(x) = \{y \in co(L) : \langle x^*, \eta(x, y)\rangle + h(x, y) < 0\},$$

where $x^* \in T(x)$ is given in (25). From relation (25), we deduce that $\Xi(x) \neq \emptyset$ for any $x \in co(L)$. Furthermore, for each $x \in co(L)$, $\Xi(x)$ is convex. Indeed, let $y_1, y_2 \in \Xi(x)$ and $y_t := ty_1 + (1 - t)y_2$ for $t \in [0, 1]$. By using assumption (ii) and $[\mathrm{A}_\eta^2]$(ii), we deduce that

$$\langle x^*, \eta(x, y_t)\rangle + h(x, y_t) \leq t[\langle x^*, \eta(x, y_1)\rangle + h(x, y_1)] + (1-t)[\langle x^*, \eta(x, y_2)\rangle + h(x, y_2)] < 0.$$

Now let us verify that for each $y \in co(L)$, $\Xi^{-1}(y) = \{x \in co(L) : y \in \Xi(x)\}$ is open in $co(L)$. To this aim, we verify that for each $y \in co(L)$, the set $[\Xi^{-1}(y)]^c$ is closed in $co(L)$, where the complementary is considered with respect to $co(L)$. Let $\{x_n\}_{n\in\mathbb{N}} \subset [\Xi^{-1}(y)]^c$ such that $x_n \to x \in co(L)$. We verify that $x \in [\Xi^{-1}(y)]^c$. Note that on $co(L)$ the weak and strong topologies coincides. Since

$$\Xi^{-1}(y) = \{x \in co(L) : \exists x^* \in T(x) \text{ such that } \langle x^*, \eta(x, y)\rangle + h(x, y) < 0\}$$

and $$[\Xi^{-1}(y)]^c = \{x \in co(L) : \langle x^*, \eta(x, y)\rangle + h(x, y) \geq 0, \forall x^* \in T(x)\},$$

it follows that for any $n \in \mathbb{N}$ we have

$$\langle x_n^*, \eta(x_n, y)\rangle + h(x_n, y) \geq 0, \quad \forall x_n^* \in T(x_n). \tag{26}$$

From the weak continuity of $\eta(\cdot, y)$, we deduce that $\eta(x_n, y) \rightharpoonup \eta(x, y)$. Let $x^* \in T(x)$ be an arbitrary element. As the restriction of T to $co(L)$ is l.s.c. with respect to the weak*-topology of X^*, then there exists $\xi_n^* \in T(x_n)$ such that $\xi_n^* \rightharpoonup x^*$.

Hence, from (26) we deduce, by taking account of assumption (i), that

$$\langle x^*, \eta(x,y)\rangle + h(x,y) \geq 0, \quad \forall x^* \in T(x).$$

Thus, $x \in [\Xi^{-1}(y)]^c$. Therefore, $\Xi^{-1}(y)$ is open for each $y \in co(L)$.

We proceed by verifying that $co(L) = \bigcup_{y\in co(L)} \Xi^{-1}(y)$. Since, $\Xi^{-1}(y) \subset co(L)$ for each $y \in co(L)$ it follows that $\bigcup_{y\in co(L)} \Xi^{-1}(y) \subset co(L)$. Now, let $x \in co(L)$. By using relation (25), we deduce that there exists $y \in co(L)$ such that $x \in \Xi^{-1}(y)$. Hence, $co(L) \subset \bigcup_{y\in co(L)} \Xi^{-1}(y)$ and therefore $co(L) = \bigcup_{y\in co(L)} \Xi^{-1}(y)$. Consequently, we have shown that all the conditions of Theorem 2.13 are satisfied with $S = W := \Xi$ and $K := co(L)$ which is a compact set. Thus, there exists $\bar{x} \in co(L)$ such that $\bar{x} \in \Xi(\bar{x})$. Hence, $0 \leq \langle x^*, \eta(\bar{x},\bar{x})\rangle + h(\bar{x},\bar{\bar{x}}) < 0$, which is impossible. □

Next, we provide the existence of strong solutions of the problem (1).

Theorem 4.3. *Let X be a reflexive Banach space and C be a nonempty, closed and convex subset of X. Let $f, g : C \times C \to \mathbb{R}$ be two bifunctions satisfying $f(x,x) = g(x,x) = 0$ for all $x \in C$, $\eta : C \times C \to X$ be a map such that $\eta(x,x) = \boldsymbol{0}$ for all $x \in C$, and $T : C \to 2^{X^*}$ be a multi-valued operator. Suppose that $[A_T^2]$ and $[A_\eta^2]$ are satisfied, and that*

- (i) *f is monotone;*
- (ii) *g is B-pseudomonotone with respect to the weak topology of X;*
- (iii) *For each $x \in C$, the functions $f(x,\cdot)$ and $g(x,\cdot)$ are convex;*
- (iv) *For each $y \in C$ and $N \in \mathcal{F}(C)$, $g(\cdot,y)$ is upper semicontinuous on $co(N)$ and $f(\cdot,y)$ is continuous on $co(N)$;*
- (v) *For each $x \in C$, the function $f(x,\cdot)$ is lower semicontinous;*
- (vi) *T is (η,Ψ,Φ)-pseudomonotone;*
- (vii) *(Coercivity) There exists a nonempty weakly compact subset D and a weakly compact convex subset B of C such that for each $x \in C \setminus D$, there exists $x^* \in T(x)$ and $y \in B$ satisfying*

$$\langle x^*, \eta(x,y)\rangle + g(x,y) < f(y,x).$$

Then problem (1) *has at least one strong solution.*

Proof. We proceed in two steps.

Step 1. First, we suppose that C is weakly compact. Let $L \in \mathcal{F}(C)$ and let us consider the following set

$$\mathbb{M}_L := \{x \in C : \forall x^* \in T(x),\ \langle x^*, \eta(x,y)\rangle + g(x,y) \geq f(y,x), \quad \forall y \in co(L)\}.$$

From Lemma 4.2 and assumption (i), $\mathbb{M}_L$ is a nonempty subset of C. Let us verify that the family of sets $\{\mathbb{M}_L\}_{L\in\mathcal{F}(C)}$ has the finite intersection property. To this aim, let $L_1, \cdots, L_p \in \mathcal{F}(C)$ and let us set $M = L_1 \cup \cdots \cup L_p$. One can easily verify that $\mathbb{M}_M \subset \bigcap_{i=1}^p \mathbb{M}_{L_i}$. Since $\mathbb{M}_M \neq \emptyset$, it follows that $\bigcap_{i=1}^p \mathbb{M}_{L_i} \neq \emptyset$ and hence $\bigcap_{i=1}^p cl^w(\mathbb{M}_{L_i}) \neq \emptyset$, where the closure is considered with respect to the weak topology. Therefore, $\bigcap_{L\in\mathcal{F}(C)} cl^w(\mathbb{M}_L) \neq \emptyset$ since C is weakly compact.

Let $\bar{x} \in \bigcap_{L\in\mathcal{F}(C)} cl^w(\mathbb{M}_L)$ and $y \in C$ be an arbitrary element. Let $N = \{y, \bar{x}\} \in \mathcal{F}(C)$.

Since $\bar{x} \in cl^w(\mathbb{M}_N)$, it follows that there exists $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{M}_N$ such that $x_n \rightharpoonup \bar{x}$. Hence, for each $n \in \mathbb{N}$ we have

$$\forall x_n^* \in T(x_n),\ \langle x_n^*, \eta(x_n, z)\rangle + g(x_n, z) \geq f(z, x_n), \quad \forall z \in co(\{y, \bar{x}\}). \tag{27}$$

By considering $z = \bar{x}$ in (27), we obtain

$$\langle x_n^*, \eta(x_n, \bar{x})\rangle + g(x_n, \bar{x}) \geq f(\bar{x}, x_n), \quad \forall x_n^* \in T(x_n). \tag{28}$$

From condition (vi), we deduce that

$$\langle x^*, \eta(x_n, \bar{x})\rangle + g(x_n, \bar{x}) \geq f(\bar{x}, x_n), \quad \forall x^* \in T(\bar{x}). \tag{29}$$

By considering the limit inferior in the previous inequality and by taking account of assumption (v) and the fact that $f(\bar{x}, \bar{x}) = 0$, we obtain

$$\liminf g(x_n, \bar{x}) \geq 0.$$

Since g is B-pseudomonotone, it follows that

$$\limsup g(x_n, z) \leq g(\bar{x}, z), \quad \text{for all } z \in C. \tag{30}$$

In particular by taking $z = y$ in (30), we get

$$\limsup g(x_n, y) \leq g(\bar{x}, y). \tag{31}$$

On the other hand, by considering $z = y$ in (27), we obtain

$$\langle x_n^*, \eta(x_n, y)\rangle + g(x_n, y) \geq f(y, x_n), \quad \forall x_n^* \in T(x_n).$$

Since T is (η, g, f)-pseudomonotone, it follows

$$\langle y^*, \eta(x_n, y)\rangle + g(x_n, y) \geq f(y, x_n), \quad \forall y^* \in T(y). \tag{32}$$

By taking the upper limit in (32), we obtain by using relation (31) and $[\mathrm{A}_\eta^2]$(i) that

$$\langle y^*, \eta(\bar{x}, y)\rangle + g(\bar{x}, y) \geq f(y, \bar{x}), \quad \text{for all } y^* \in T(y) \text{ and any } y \in C. \tag{33}$$

Let z be an arbitrary element in C and let us set $y_n := \frac{1}{n}z + (1 - \frac{1}{n})\bar{x} \in C$, where $n \in \mathbb{N}^*$. Note that $y_n \to \bar{x}$. From (33), we get

$$\langle y_n^*, \eta(\bar{x}, y_n)\rangle + g(\bar{x}, y_n) \geq f(y_n, \bar{x}), \quad \text{for all } y_n^* \in T(y_n). \tag{34}$$

By using $[\mathrm{A}_\eta^2]$(ii) and assumption (iii), we deduce that

$$\langle y_n^*, \eta(\bar{x}, z)\rangle + g(\bar{x}, z) \geq n f(y_n, \bar{x}), \quad \text{for all } y_n^* \in T(y_n) \text{ and } n \in \mathbb{N}^*. \tag{35}$$

On the other hand, since f is convex with respect to the second argument, we have

$$\begin{aligned}[0 = f(y_n, y_n) &\leq \frac{1}{n} f(y_n, z) + (1 - \frac{1}{n}) f(y_n, \bar{x}) \\ &\leq f(y_n, \bar{x}) + \frac{1}{n}[f(y_n, z) - f(y_n, \bar{x})]\end{aligned}$$

It follows that $n f(y_n, \bar{x}) \geq f(y_n, \bar{x}) - f(y_n, z)$.

Consequently, from (35) we deduce that

$$\langle y_n^*, \eta(\bar{x}, z)\rangle + f(\bar{x}, z) + f(y_n, z) \geq f(y_n, \bar{x}), \quad \text{for all } y_n^* \in T(y_n) \text{ and } n \in \mathbb{N}^*. \tag{36}$$

Now, consider $x^* \in T(\bar{x})$. Since $y_n \to \bar{x}$ and T is l.s.c. on $co(L)$ for any $L \in \mathcal{F}(C)$ into the weak*-topology of X^*, it follows that there exists $\xi_n^* \in T(y_n)$ such that $\xi_n^* \rightharpoonup x^*$. Hence, from (36) we get

$$\langle \xi_n^*, \eta(\bar{x}, z)\rangle + g(\bar{x}, z) + f(y_n, z) \geq f(y_n, \bar{x}), \quad \text{for all } n \in \mathbb{N}^*. \tag{37}$$

By considering the limit in the previous inequality and by taking account of assumption (iv), we obtain

$$\langle x^*, \eta(\bar{x}, z)\rangle + g(\bar{x}, z) + f(\bar{x}, z) \geq f(\bar{x}, \bar{x}) = 0,$$

for any $x^* \in T(\bar{x})$ and any $z \in C$. Hence, $\bar{x}$ is a strong solution of problem (1).

Step 2. We suppose that C is not necessarily weakly compact. Let $L \in \mathcal{F}(C)$ and let us set $\widetilde{C} := co(B \cup L)$ which is a compact and convex subset of X. From the first step, there exists $\tilde{x} \in \widetilde{C}$ such that for any $x^* \in T(\tilde{x})$, we have

$$\langle x^*, \eta(\tilde{x}, z)\rangle + g(\tilde{x}, z) + f(\tilde{x}, z) \geq 0, \text{ for all } y \in \widetilde{C}.$$

Since f is monotone, it follows that

$$\langle x^*, \eta(\tilde{x}, z)\rangle + g(\tilde{x}, z) \geq f(z, \tilde{x}), \text{ for all } y \in \widetilde{C}.$$

By using assumption (vii), we deduce that $\tilde{x} \in D$. For $N \in \mathcal{F}(C)$, let us consider

$$\widetilde{\mathbb{M}}_N := \{x \in D : \forall x^* \in T(x), \langle x^*, \eta(x, z)\rangle + g(x, z) \geq f(z, x), \forall z \in co(B \cup N)\}.$$

By what precedes, we have that $\tilde{\mathbb{M}}_N \neq \emptyset$ for any $N \in \mathcal{F}(C)$. On the other hand, one can easily verify that the family of sets $\{\widetilde{\mathbb{M}}_N\}_{N\in\mathcal{F}(C)}$ has the finite intersection property. Since D is compact, it follows that $\bigcap_{N\in\mathcal{F}(C)} cl^w(\widetilde{\mathbb{M}}_N) \neq \emptyset$, where the closure is considered with respect to the weak topology. Let $\bar{x} \in \bigcap_{N\in\mathcal{F}(C)} cl^w(\widetilde{\mathbb{M}}_N)$ and y be an arbitrary element in C. Let us set $E = \{y, \bar{x}\} \in \mathcal{F}(C)$. Since we have $\bar{x} \in cl^w(\widetilde{\mathbb{M}}_E)$, there exists a sequence $\{\widetilde{x}_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathbb{M}}_E$ such that $\widetilde{x}_n \rightharpoonup \bar{x}$. Since $x_n \in \widetilde{\mathbb{M}}_E$ for each $n \in \mathbb{N}$, we obtain

$$\forall x_n^* \in T(x_n),\ \langle x_n^*, \eta(x_n, z)\rangle + g(x_n, z) \geq f(z, x_n), \quad \forall z \in co(B \cup \{y, \bar{x}\}). \tag{38}$$

Repeating the same demarche used in Step 1, by considering respectively $z = \bar{x}$ and $z = y$ in (38), we obtain

$$\langle y^*, \eta(\bar{x}, y)\rangle + g(\bar{x}, y) \geq f(y, \bar{x}), \quad \text{for all } y^* \in T(y) \text{ and any } y \in C. \tag{39}$$

We proceed as in Step 1 to show that

$$\forall x^* \in T(\bar{x}),\ \langle x^*, \eta(\bar{x}, z)\rangle + g(\bar{x}, z) + f(\bar{x}, z) \geq 0, \text{ for all } y \in C.$$

Hence, $\bar{x}$ is a strong solution of the problem (1). Which completes the proof. □

We end this section by comparing our results with some recent results in literature.

Remark 4.4. Our Theorem 4.3 improves Theorem 3.3 of Liu, Migórski and Zeng [20] by avoiding the use of any antimonotonicity assumption on the data of the problem. On the other hand, if we take $X := Z^{**}$, where Z^{**} is the bidual of a Banach space Z, $Y = Z^*$ the dual space of Z and the bilinear form $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ is traduced by the duality paring between X and Y, $g \equiv 0$, and $f(x, y) = \phi(y) - \phi(x)$, then Theorem 4.3 reduces to the Theorem 3.1 of Costea, Ion and Lupu [12]. Hence, Theorem 4.3 generalizes and improves Theorem 3.1 of Costea, Ion and Lupu [12].

5. Application

In this section, we apply the approach developed in this paper to study the existence of solutions of a generalized nonlinear hemivariational inequality problem studied in a recent paper by Zeng, Migórski and Khan [30].

Let V be a reflexive Banach space with topological dual V^*, and E and F be two Banach spaces. Let C be a nonempty, closed and convex subset of V, $T : C \to 2^{V^*}$ be a multi-valued operator, $\psi : V \times V \to \mathbb{R}$ be a bifunction, $J : E \to \mathbb{R}$ be a locally Lipschitz functional, $\gamma : V \to E$ and $\pi : V \to F$ be two operators, and $l \in F^*$, where F^* is the topological dual of F.

We consider the following generalized nonlinear hemivariational inequality problem:

$$\text{Find } u \in C \text{ and } u^* \in T(u) \text{ such that}$$
$$\langle u^*, v - u \rangle_V + \psi(u, v) + J^0(\gamma(u); \gamma(v - u)) \geq \langle l, \pi(v - u) \rangle_F \text{ for all } v \in C, \quad (40)$$

where $J^0(\gamma(u); \gamma(v - u))$ is the Clarke's generalized directional derivative of J at $\gamma(u)$ in the direction $\gamma(v - u)$.

We consider the following assumptions on the data of the problem:

[$\mathbf{A}_C$] C is a nonempty, closed and convex subset of V.

[$\mathbf{A}_T$] $T : V \to 2^{V^*}$ is bounded, upper semicontinuous, closed and convex valued.

[$\mathbf{A}_J$] $J : E \to \mathbb{R}$ is a locally Lipschitz functional.

[$\mathbf{A}_\psi$] $\psi : V \times V \to \mathbb{R}$ is a bifunction satisfying $\psi(v, v) = 0$, for all $v \in V$, and

- (i) ψ is B-pseudomonotone bifunction with respect to the weak topology of V;
- (ii) for each $u \in V$, the function $v \in V \mapsto \psi(u, v)$ is convex and lower semicontinuous;
- (iii) for each $v \in V$ and $N \in \mathcal{F}(C)$, the function $u \in V \mapsto \psi(u, v)$ is upper semicontinuous on $co(N)$.

[$\mathbf{A}_\gamma$] $\gamma : V \to E$ is a linear and bounded operator.

[$\mathbf{A}_\pi$] $\pi : V \to F$ is a linear and bounded operator.

[$\mathbf{A}_l$] $l \in F^*$.

[$\mathbf{A}_{coer}$] Coercivity condition: If C is unbounded, there exists $v_0 \in C$ such that

$$\lim_{\substack{\|u\| \to +\infty \\ u \in C}} \frac{\inf_{u^* \in T(u)} \langle u^*, u - v_0 \rangle_V + \inf_{\varpi \in \partial J(\gamma(u))} \langle \varpi, \gamma(u - v_0) \rangle_E - \psi(u, v_0)}{\|u\|} = +\infty. \quad (41)$$

We have the following result on the existence of solutions of the problem (40).

Theorem 5.1. *Assume that* $[\mathbf{A}_C]$, $[\mathbf{A}_T]$, $[\mathbf{A}_J]$, $[\mathbf{A}_\psi]$, $[\mathbf{A}_\gamma]$, $[\mathbf{A}_\pi]$, $[\mathbf{A}_l]$ *and* $[\mathbf{A}_{coer}]$ *are satisfied. Then the solution set of problem (40) is nonempty and bounded.*

Proof. We shall apply Theorem 3.3 with $X := V$ is endowed with the weak topology $\sigma(V, V^*)$ and $Y := V^*$, where the bilinear form $\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{R}$ is nothing else than the duality pairing $\langle \cdot, \cdot \rangle_V$ between V^* and V. The mapping $\eta : C \times C \to V$ is defined by $\eta(u, v) = v - u$ and the bifunctions $f, g : C \times C \to \mathbb{R}$ are defined, respectively, by

$$f(u, v) := \langle l, \pi(u - v)\rangle_F, \quad g(u, v) := \psi(u, v) + J^0(\gamma(u); \gamma(v - u)).$$

All the topological properties of Theorem 3.3 that we will need to show will be with respect to the weak topology $\sigma(V, V^*)$ of V. For instance the B-pseudomonotonicity condition (ii) of Theorem 3.3 will be considered with respect to the weak topology, as mentioned in Remark 2.7.

Since the operator π is linear, we derive that $f(u, v) + f(v, u) = 0$ for all $u, v \in C$. This implies that f is monotone and hence condition (i) of Theorem 3.3 holds.

On the other hand, $[\mathbf{A}_\pi]$ implies that f is convex and lower semicontinuous with respect to the second argument. Moreover, from Proposition 2.15(i) and assumption $[\mathbf{A}_\gamma]$ we deduce that the function $v \in V \mapsto J^0(\gamma(u); \gamma(v - u))$ is convex. It follows from $[\mathbf{A}_\psi]$ that the function $v \in V \mapsto g(u, v)$ is convex as sum of two convex functions. Therefore, conditions (iii) and (iv) of Theorem 3.3 are satisfied.

The bifunction g is B-pseudomonotone with respect to the weak topology of V. Indeed, thanks to Proposition 2.15(ii) and assumption $[\mathbf{A}_\gamma]$ we deduce that the function $u \in V \mapsto J^0(\gamma(u); \gamma(v - u))$ is weakly upper semicontinuous. Hence, from Remark 2.8 (b) we derive that the bifunction $h(u, v) := J^0(\gamma(u); \gamma(v - u))$ is B-pseudomonotone with respect to the weak topology of V. Therefore, from Remark 2.8 (c) (see also [10, Proposition 2.1]) we obtain that g is B-pseudomonotone as the sum of two B-pseudomonotone bifunctions. Thus condition (ii) of Theorem 3.3 holds.

Now, let us verify that the condition (vi) of Theorem 3.3 is satisfied. Note that from $[\mathbf{A}_{coer}]$, we get

$$\lim_{\substack{\|u\| \to +\infty \\ u \in C}} \frac{\sup_{u^* \in T(u)} \langle u^*, v_0 - u\rangle_V + \sup_{\varpi \in \partial J(\gamma(u))} \langle \varpi, \gamma(v_0 - u)\rangle_E + \psi(u, v_0)}{\|u\|} = -\infty. \tag{42}$$

On the other hand, we have

$$\begin{aligned} &\sup_{u^* \in T(u)} \langle u^*, v_0 - u\rangle_V + \sup_{\varpi \in \partial J(\gamma(u))} \langle \varpi, \gamma(v_0 - u)\rangle_E + \psi(u, v_0) - f(v_0, u) \\ &\leq \sup_{u^* \in T(u)} \langle u^*, v_0 - u\rangle_V + \sup_{\varpi \in \partial J(\gamma(u))} \langle \varpi, \gamma(v_0 - u)\rangle_E \\ &\qquad\qquad\qquad + \psi(u, v_0) + \|f\|_{F^*}\|\pi\| \, \|u - v_0\|. \end{aligned}$$

Hence, by using (42) we deduce that

$$\lim_{\substack{\|u\|\to+\infty\\ u\in C}} \frac{\sup_{u^*\in T(u)}\langle u^*, v_0-u\rangle_V+\sup_{\varpi\in\partial J(\gamma(u))}\langle\varpi,\gamma(v_0-u)\rangle_E+\psi(u,v_0)-f(v_0,u)}{\|u-v_0\|} = -\infty. \quad (43)$$

It follows that there exists $R_0 > 0$ such that for each $u \in C$ satisfying $\|u-v_0\| > R_0$, one has

$$\sup_{u^*\in T(u)} \langle u^*, v_0 - u\rangle_V + \sup_{\varpi\in\partial J(\gamma(u))} \langle\varpi, \gamma(v_0 - u)\rangle_E + \psi(u, v_0) < f(v_0, u). \quad (44)$$

Note that by Proposition 2.16 (ii) we have

$$J^0(\gamma(u); \gamma(v_0 - u)) = \max_{\varpi\in\partial J(\gamma(u))} \langle\varpi, \gamma(v_0 - u)\rangle_E.$$

Thus, from (44) we obtain that for all $u \in C$ satisfying $\|u - v_0\| > R_0$, one has

$$\langle u^*, v_0 - u\rangle_V + g(u, v_0) < f(v_0, u) \text{ for all } u^* \in T(u). \quad (45)$$

This implies that condition (vi) of Theorem 3.3 is satisfied with $D = K := \bar{\mathbb{B}}_V(v_0, R_0)$ and $z = v_0$, where $\bar{\mathbb{B}}_V(v_0, R_0)$ is the closed ball in V of center v_0 and radius R_0. Therefore, all the conditions of Theorem 3.3 are satisfied, and hence problem (40) has at least one solution. Furthermore, from (45) we deduce that if $u \in C$ is a solution of problem (40) then $u \in \bar{\mathbb{B}}_V(v_0, R_0)$, which implies that the solution set $\mathbb{S}$ of problem (40) is bounded. Which completes the proof of the theorem. □

Remark 5.2. Comparing [30, Theorem 3.4] with our Theorem 5.1 we see that our result provides an existence result for the generalized nonlinear hemivariational inequality problem (40) without assuming that the operator $T(\cdot) + \gamma^*\partial J(\gamma(\cdot))$ is (φ, h)-stably pseudomonotone with respect to $\{\pi^*l\}$ (see hypothesis (H_T) page 1250 in [30]), where $\varphi : V \times V \to \mathbb{R}$ is the bifunction defined by $\varphi(v, u) = \psi(u, v)$. Moreover, by our approach we relax considerably the assumptions on the bifunction ψ in comparison with those considered on the bifunction φ in [30, Theorem 3.4], where $\varphi(v, u) = \psi(u, v)$).

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