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A Non-Local Regularization of the Short Pulse Equation

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The short pulse equation provides a model for the propagation of ultra-short light pulses in silica optical fibers. In this paper, we consider a nonlocal regularization of that equation and prove its well-posedness.

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1. Introduction

The following evolution equation

$$\partial_x \left(\partial_t u + q\partial_x u^3\right) = bu, \qquad q,\, b \in \mathbb{R}, \tag{1}$$

where $u(t,x)$ is the electric field amplitude, t is the time, and x is the distance propagated in the laser cavity [45], is known as the short pulse equation.

It was introduced by Kozlov and Sazonov [34] as a model equation describing the nonlinear propagation of optical pulses of the duration of a few oscillations in dielectric media, and by Schäfer and Wayne [45] as a model equation describing the propagation of ultra-short light pulses in silica optical fibers.

In [3, 4, 18, 36, 37, 38], the authors show that (1) is also a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons. Instead, [5, 13, 42, 44] show that (1) is a particular Rabelo equation which describes pseudospherical surfaces.

Equation (1) is also deduced in [50] to describe the short pulse propagation in nonlinear metamaterials characterized by a weak Kerr-type nonlinearity in their dielectric response.

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We remind that equation (1) was proposed earlier in [39] in the context of plasma physic and radiating gases [35, 46]. Moreover, [19, 31, 32, 33] show that (1) is also a model for ultrafast pulse propagation in a mode-locked laser cavity in the few-femtosecond pulse regime. Finally, an interpretation of (1) in the context of Maxwell equations is given in [41].

Recently, wellposedness results for the Cauchy problem of (1) are proven in the context of energy spaces (see [29, 40, 49]). A similar result is proven in [10, 15, 22, 30] for entropy solution, while, in [12, 21, 23, 43], the wellposedness of the homogeneous initial boundary value problem is studied. Finally, the convergence of a finite difference numerical scheme is studied in [26].

In this paper, we consider the following nonlocal regularization of (1)

$$\begin{cases} \partial_t u + q\partial_x v = bP, & t > 0,\ x \in \mathbb{R}, \\ \partial_x P = u, & t > 0,\ x \in \mathbb{R}, \\ \alpha\partial_x^2 v + \beta\partial_x v + \gamma v = \kappa u^3, & t > 0,\ x \in \mathbb{R}, \\ P(t, -\infty) = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2}$$

where q, b, α, β, γ, $\kappa \in \mathbb{R}$.

Conservation laws with non-local fluxes can be found in the context of traffic flow modeling [1, 7, 27, 28], in the context of sedimentation dynamic modeling [6], in the context of slow erosion modeling [2, 25, 47] and in the context of the linearly polarized continuum spectrum pulses in optical waveguides modeling [17, 20, 48].

On the initial datum, we assume that

$$u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R}), \qquad \int_{\mathbb{R}} u_0(x)dx = 0. \tag{3}$$

Following [8, 11, 9, 10, 15, 16], on the function

$$P_0(x) = \int_{-\infty}^{x} u_0(y)dy, \tag{4}$$

we assume that

$$\begin{aligned} \int_{\mathbb{R}} P_0(x)dx &= \int_{\mathbb{R}} \left(\int_{-\infty}^{x} u_0(y)dy \right) dx = 0, \\ \|P_0\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left(\int_{-\infty}^{x} u_0(y)dy \right)^2 dx < \infty. \end{aligned} \tag{5}$$

Moreover, on the constants q, b, α, β, γ, κ, we assume that

$$\frac{q\beta}{\kappa} \geq 0, \qquad b = \frac{2q\kappa}{\gamma}, \qquad \alpha,\ \beta,\ \kappa,\ \gamma \neq 0, \tag{6}$$

or

$$b = \frac{2q\kappa}{\gamma}, \qquad \alpha = -\gamma, \qquad \beta = 0, \qquad \gamma \neq 0. \tag{7}$$

Since in both cases $\alpha \neq 0$ we may set it equal to 1 and work with only three constants. Moreover, (6) and (7) are necessary to keep the solutions of (2) in the energy space.

The main result of this paper is the following theorem.

Theorem 1.1. *Assume* (3), (4), (5), (6) *or* (7). *Fix* $T > 0$, *there exists an unique solution* (u, v, P) *of* (2) *such that*

$$\begin{aligned} & u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^2(\mathbb{R})),\\ & v \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^4(\mathbb{R})) \cap W^{1,\infty}((0,T)\times\mathbb{R}),\\ & \partial_{tx}^2 v \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}), \quad \text{for every } 0 \le t \le T,\\ & P \in L^\infty(0,T;H^3(\mathbb{R})). \end{aligned} \tag{8}$$

In particular, we have that
$$\int_{\mathbb{R}} u(t,x)dx = 0, \qquad t \ge 0. \tag{9}$$

Moreover, if (u_1, v_1, P_1) *and* (u_2, v_2, P_2) *are two solutions of* (2), *we have that*

$$\begin{aligned} \|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R})} \le & e^{C(T)t}\|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})},\\ \|v_1(t,\cdot) - v_2(t,\cdot)\|_{H^2(\mathbb{R})} \le & e^{C(T)t}\|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})},\\ \|P_1(t,\cdot) - P_2(t,\cdot)\|_{H^1(\mathbb{R})} \le & e^{C(T)t}\|P_{1,0} - P_{2,0}\|_{H^1(\mathbb{R})}, \end{aligned} \tag{10}$$

where
$$P_{1,0}(x) = \int_{-\infty}^x u_{1,0}(y)dy, \qquad P_{2,0}(x) = \int_{-\infty}^x u_{2,0}(y)dy, \tag{11}$$

for some suitable $C(T) > 0$, *and every* $0 \le t \le T$.

The arguments of this paper are inspired by those of [20], where the well-posedness of

$$\begin{cases} \partial_t u + q\partial_x(uv) = bP, & t > 0,\ x \in \mathbb{R},\\ \partial_x P = u, & t > 0,\ x \in \mathbb{R},\\ \alpha\partial_x^2 v + \beta\partial_x v + \gamma v = \kappa u^2, & t > 0,\ x \in \mathbb{R},\\ P(t,-\infty) = 0, & t > 0,\\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{12}$$

is proven under the assumption $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$.

The paper is organized as follows. In Section 2, we prove several a priori estimates on a vanishing viscosity approximation of (2). These results play a key role in the proof of our main result, that is given in Section 3.

2. Vanishing viscosity approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (2).

Fix a small number $0 < \varepsilon < 1$ and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following mixed problem [14, 24]:

$$\begin{cases} \partial_t u_\varepsilon + q\partial_x v_\varepsilon = bP_\varepsilon + \varepsilon\partial_x^2 u_\varepsilon, & t > 0,\ x \in \mathbb{R},\\ \partial_x P_\varepsilon = u_\varepsilon, & t > 0,\ x \in \mathbb{R},\\ \alpha\partial_x^2 v_\varepsilon + \beta\partial_x v_\varepsilon + \gamma v_\varepsilon = \kappa u_\varepsilon^3, & t > 0,\ x \in \mathbb{R},\\ P_\varepsilon(t,-\infty) = 0, & t > 0,\\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \tag{13}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned}\|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})}, \quad \|u_{\varepsilon,0}\|_{H^2(\mathbb{R})} \le \|u_0\|_{H^2(\mathbb{R})}, \quad \int_{\mathbb{R}} u_{\varepsilon,0}(x)dx = 0,\\ \|P_{\varepsilon,0}\|_{L^2(\mathbb{R})} \le \|P_0\|_{L^2(\mathbb{R})}, \quad \int_{\mathbb{R}} P_{\varepsilon,0}(x)dx = 0,\end{aligned} \tag{14}$$

for every $\varepsilon > 0$. Let us prove some a priori estimates on u_ε, P_ε and v_ε. We denote with C_0 the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on T.

Arguing as in [20, Lemmas 2.2 and 2.5], we have the following result.

Lemma 2.1. *For each $t \ge 0$,*

$$P_\varepsilon(t,\infty) = 0, \tag{15}$$

$$\int_0^{-\infty} P_\varepsilon(t,x)dx = -\frac{1}{b}\partial_t P_\varepsilon(t,0) - \frac{q}{b}v_\varepsilon(t,0) + \frac{\varepsilon}{b}\partial_x u_\varepsilon(t,0), \tag{16}$$

$$\int_0^{\infty} P_\varepsilon(t,x)dx = -\frac{1}{b}\partial_t P_\varepsilon(t,0) - \frac{q}{b}v_\varepsilon(t,0) + \frac{\varepsilon}{b}\partial_x u_\varepsilon(t,0). \tag{17}$$

In particular, we have that

$$\int_{\mathbb{R}} u_\varepsilon(t,x)dx = \int_{\mathbb{R}} P_\varepsilon(t,x)dx = 0, \quad t \ge 0. \tag{18}$$

Lemma 2.2. *We have that*

$$\int_{\mathbb{R}} u_\varepsilon^3 \partial_x v_\varepsilon dx = \begin{cases} \dfrac{\beta}{\kappa}\|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}, & \text{if (6) holds,}\\ 0, & \text{if (7) holds.}\end{cases} \tag{19}$$

Proof. We begin by assuming (6). Multiplying the third equation of (13) by $2\partial_x v_\varepsilon$, we have

$$2\alpha\partial_x v_\varepsilon \partial_x^2 v_\varepsilon + 2\beta(\partial_x v_\varepsilon)^2 + 2\gamma v_\varepsilon \partial_x v_\varepsilon = 2\kappa u_\varepsilon^3 \partial_x v_\varepsilon. \tag{20}$$

Since

$$\begin{aligned} 2\alpha\int_{\mathbb{R}} \partial_x v_\varepsilon \partial_x^2 v_\varepsilon dx =& \alpha\int_{\mathbb{R}} \partial_x((\partial_x v_\varepsilon)^2)dx = 0,\\ 2\gamma\int_{\mathbb{R}} v_\varepsilon \partial_x v_\varepsilon dx =& \gamma\int_{\mathbb{R}} \partial_x(v_\varepsilon^2)dx = 0.\end{aligned} \tag{21}$$

It follows from (21) and an integration on $\mathbb{R}$ of (20) that

$$\kappa\int_{\mathbb{R}} u_\varepsilon^3 \partial_x v_\varepsilon dx = \beta\|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}, \tag{22}$$

which gives (19).

Finally, we observe that, if (7) holds, by (22), we have (19). □

Lemma 2.3. *Let $T > 0$. Assume (6), or (7). There exists a constant $C(T) > 0$, independent on ε such that,*

$$\|u_\varepsilon(t,\cdot)\|_{L^4(\mathbb{R})} \le C(T), \quad \|P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le C(T), \quad \|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le C(T),$$
$$\varepsilon \int_0^t \|u_\varepsilon(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{23}$$
$$\varepsilon \int_0^t \|u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \le C(T), \quad \varepsilon \int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \le C(T).$$

For every $0 \le t \le T$. In particular, if (6) holds, we get

$$\int_0^t \|\partial_x v_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \le C(T). \tag{24}$$

Moreover, we have that
$$\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{25}$$

Proof. Assume (6). Let $0 \le t \le T$. Multiplying the first equation of (13) by $4u_\varepsilon^3$, we have

$$4u_\varepsilon^3 \partial_t u_\varepsilon = 4bP_\varepsilon u_\varepsilon^3 - 4qv_\varepsilon u_\varepsilon^3 + 4\varepsilon u_\varepsilon^3 \partial_x^2 u_\varepsilon. \tag{26}$$

Since
$$4\int_{\mathbb{R}} u_\varepsilon^3 \partial_t u_\varepsilon dx = \frac{d}{dt} \|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})},$$
$$4\varepsilon \int_{\mathbb{R}} u_\varepsilon^3 \partial_x^2 u_\varepsilon dx = -12\varepsilon \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

integrating (26) on $\mathbb{R}$, thanks to (6) and (19), we get

$$\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + 12\varepsilon \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{4q\beta}{\kappa}\|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} =$$
$$= 4b\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx. \tag{27}$$

Thanks to (16), we can consider the following function:

$$F_\varepsilon(t,x) = \int_{-\infty}^x P_\varepsilon(t,y)dy. \tag{28}$$

Observe that integrating the second equation of (13) on $(-\infty, x)$, we get

$$P_\varepsilon(t,x) = \int_{-\infty}^x u_\varepsilon(t,y)dy. \tag{29}$$

Differentiating (29) with respect to t, we have

$$\partial_t P_\varepsilon(t,x) = \frac{d}{dt}\int_{-\infty}^x u_\varepsilon(t,y)dy = \int_{-\infty}^x \partial_t u_\varepsilon(t,y)dy. \tag{30}$$

Integrating the first equation of (13) on $(-\infty, x)$, by (28) and (30), we get

$$\partial_t P_\varepsilon(t,x) = bF_\varepsilon(t,x) + \varepsilon\partial_x u_\varepsilon(t,x) - qv_\varepsilon(t,x). \tag{31}$$

Observe that, from (13) and (15),

$$2\varepsilon\int_{\mathbb{R}} P_\varepsilon\partial_x u_\varepsilon dx = -2\varepsilon\int_{\mathbb{R}}\partial_x P_\varepsilon u_\varepsilon dx = -2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \tag{32}$$

Moreover, by (18) and (28),

$$2b\int_{\mathbb{R}} P_\varepsilon F_\varepsilon dx = 2b\int_{\mathbb{R}} F_\varepsilon\partial_x F_\varepsilon dx = bF^2_\varepsilon(t,\infty) = b\left(\int_{\mathbb{R}} P_\varepsilon(t,x)dx\right)^2 = 0. \tag{33}$$

Multiplying (31) by $2P_\varepsilon$, an integration on $\mathbb{R}$, (32) and (33) give

$$\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = -2q\int_{\mathbb{R}} P_\varepsilon v_\varepsilon dx. \tag{34}$$

Multiplying the third equation of (13) by P_ε, an integration on $\mathbb{R}$ gives

$$\int_{\mathbb{R}} P_\varepsilon v_\varepsilon dx = \frac{\kappa}{\gamma}\int_{\mathbb{R}} u^3_\varepsilon P_\varepsilon dx - \frac{\beta}{\gamma}\int_{\mathbb{R}} P_\varepsilon\partial_x v_\varepsilon dx - \frac{\alpha}{\gamma}\int_{\mathbb{R}} P_\varepsilon\partial^2_x v_\varepsilon dx. \tag{35}$$

It follows from (34) and (35) that

$$\begin{aligned}\frac{d}{dt}&\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =\\ &= -\frac{2q\kappa}{\gamma}\int_{\mathbb{R}} u^3_\varepsilon P_\varepsilon dx + \frac{2q\beta}{\gamma}\int_{\mathbb{R}} P_\varepsilon\partial_x v_\varepsilon dx + \frac{2q\alpha}{\gamma}\int_{\mathbb{R}} P_\varepsilon\partial^2_x v_\varepsilon dx.\end{aligned} \tag{36}$$

Adding (27) and (36), thanks to (6), we have that

$$\begin{aligned}\frac{d}{dt}&\Big(\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\Big) + 12\varepsilon\left\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} +\\ &\qquad + \frac{4q\beta}{\kappa}\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =\\ &= \frac{2q\beta}{\gamma}\int_{\mathbb{R}} P_\varepsilon\partial_x v_\varepsilon dx + \frac{2q\alpha}{\gamma}\int_{\mathbb{R}} P_\varepsilon\partial^2_x v_\varepsilon dx.\end{aligned} \tag{37}$$

Multiplying the first equation of (13) by $2u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &= 2\int_{\mathbb{R}} u_\varepsilon\partial_t u_\varepsilon dx\\ &= 2b\int_{\mathbb{R}} P_\varepsilon u_\varepsilon dx - 2q\int_{\mathbb{R}}\partial_x v_\varepsilon u_\varepsilon dx + 2\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial^2_x u_\varepsilon dx\\ &= 2b\int_{\mathbb{R}} P_\varepsilon u_\varepsilon dx - 2q\int_{\mathbb{R}}\partial_x v_\varepsilon u_\varepsilon dx - 2\varepsilon\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{38}$$

Observe that by (13) and (15),

$$2b\int_{\mathbb{R}} P_\varepsilon u_\varepsilon dx = 2b\int_{\mathbb{R}} P_\varepsilon\partial_x P_\varepsilon dx = P^2_\varepsilon(t,\infty) = 0.$$

Consequently, from (38), we have that

$$\frac{d}{dt}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=-2q\int_{\mathbb{R}}\partial_x v_\varepsilon u_\varepsilon dx \tag{39}$$

Adding (37) and (39), we get

$$\begin{aligned}\frac{d}{dt}&\Big(\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\Big)+\\&+12\varepsilon\left\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{4q\beta}{\kappa}\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\\&+2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=\\=&\frac{2q\beta}{\gamma}\int_{\mathbb{R}}P_\varepsilon\partial_x v_\varepsilon dx+\frac{2q\alpha}{\gamma}\int_{\mathbb{R}}P_\varepsilon\partial_x^2 v_\varepsilon dx-2q\int_{\mathbb{R}}\partial_x v_\varepsilon u_\varepsilon dx.\end{aligned} \tag{40}$$

Observe that, by (13) and (15),

$$\frac{2q\alpha}{\gamma}\int_{\mathbb{R}}P_\varepsilon\partial_x^2 v_\varepsilon dx=-\frac{2q\alpha}{\gamma}\int_{\mathbb{R}}\partial_x v_\varepsilon\partial_x P_\varepsilon dx=-\frac{2q\alpha}{\gamma}\int_{\mathbb{R}}\partial_x v_\varepsilon u_\varepsilon dx.$$

It follows from (40) that

$$\begin{aligned}\frac{d}{dt}&\Big(\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})}+\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\Big)+\\&+12\varepsilon\left\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{4q\beta}{\kappa}\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\\&+2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=\\=&\frac{2q\beta}{\gamma}\int_{\mathbb{R}}P_\varepsilon\partial_x v_\varepsilon dx-2q\left(1+\frac{\alpha}{\gamma}\right)\int_{\mathbb{R}}\partial_x v_\varepsilon u_\varepsilon dx.\end{aligned} \tag{41}$$

Due to (6) and the Young inequality,

$$\begin{aligned}2\left|\frac{q\beta}{\gamma}\right|\int_{\mathbb{R}}|P_\varepsilon||\partial_x v_\varepsilon|dx&=2\int_{\mathbb{R}}\left|\sqrt{\frac{|\kappa|}{|q\beta|}}\frac{q\beta P_\varepsilon}{\gamma}\right|\left|\sqrt{\frac{|q\beta|}{|\kappa|}}\partial_x v_\varepsilon\right|dx\\&\le\left|\frac{\kappa q\beta}{\gamma}\right|\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{q\beta}{\kappa}\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\2\left|q\left(1+\frac{\alpha}{\gamma}\right)\right|\int_{\mathbb{R}}|\partial_x v_\varepsilon||u_\varepsilon|dx&=2\int_{\mathbb{R}}\left|\sqrt{\frac{|q\beta|}{|\kappa|}}\partial_x v_\varepsilon\right|\left|\sqrt{\frac{|\kappa|}{|q\beta|}}q\left(1+\frac{\alpha}{\gamma}\right)u_\varepsilon\right|dx\\&\le\frac{q\beta}{\kappa}\left\|\partial_x v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left|\frac{q\kappa}{\beta}\right|\left(1+\frac{\alpha}{\gamma}\right)^2\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, by (41), we obtain that

$$\begin{aligned}
&\frac{d}{dt}\Big(\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + \|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \Big)+\\
&\qquad + 12\varepsilon \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{2q\beta}{\kappa} \|\partial_x v_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} +\\
&\qquad + 2\varepsilon \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le \left|\frac{\kappa q\beta}{\gamma}\right| \|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \left|\frac{q\kappa}{\beta}\right| \left(1+\frac{\alpha}{\gamma}\right)^2 \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C_0 \left(\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + \|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right).
\end{aligned}$$

The Gronwall Lemma and (14) give

$$\begin{aligned}
&\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + \|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} +\\
&\quad + 12\varepsilon e^{C_0 t}\int_0^t e^{-C_0 s} \|u_\varepsilon(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds+\\
&\quad + \frac{2q\beta}{\kappa} e^{C_0 t}\int_0^t e^{-C_0 s} \|\partial_x v_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds+\\
&\quad + 2\varepsilon e^{C_0 t}\int_0^t e^{-C_0 s} \|u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds + 2\varepsilon e^{C_0 t}\int_0^t e^{-C_0 s} \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\\
&\le C_0 e^{C_0 t} \le C(T),
\end{aligned}$$

which gives (23) and (24). Assume (7). Since $\beta = 0$, from (37) we have

$$\begin{aligned}
&\frac{d}{dt}\Big(\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + \|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \Big) + 12\varepsilon \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} +\\
&\qquad +2\varepsilon \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} = \frac{2q\alpha}{\gamma}\int_{\mathbb{R}} P_\varepsilon \partial_x^2 v_\varepsilon dx.
\end{aligned} \tag{42}$$

Adding (39) and (42), we get

$$\begin{aligned}
&\frac{d}{dt}\Big(\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + \|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \Big)+\\
&\qquad + 12\varepsilon \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\
&= \frac{2q\alpha}{\gamma}\int_{\mathbb{R}} P_\varepsilon \partial_x^2 v_\varepsilon dx - 2q\int_{\mathbb{R}} \partial_x v_\varepsilon u_\varepsilon dx.
\end{aligned} \tag{43}$$

Observe that, by (7), (13) and (15),

$$\frac{2q\alpha}{\gamma}\int_{\mathbb{R}} P_\varepsilon \partial_x^2 v_\varepsilon dx - 2q\int_{\mathbb{R}} \partial_x v_\varepsilon u_\varepsilon dx = -2q\left(\frac{\alpha}{\gamma}+1\right)\int_{\mathbb{R}} u_\varepsilon \partial_x v_\varepsilon dx = 0.$$

It follows from (43) that

$$\begin{aligned}
&\frac{d}{dt}\Big(\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + \|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \Big)+\\
&+12\varepsilon \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} = 0.
\end{aligned}$$

Integrating on $(0,t)$, by (14), we have (23).

Finally, we prove (25). Thanks to (13), (23) and the Hölder inequality,

$$\begin{aligned} P_\varepsilon^2(t,x) &= 2\int_{-\infty}^x P_\varepsilon \partial_x P_\varepsilon dy \leq 2\int_{\mathbb{R}} |P_\varepsilon||u_\varepsilon| dx \\ &\leq 2\left\|P_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq C(T). \end{aligned}$$

Therefore,

$$\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} \leq C(T), \tag{44}$$

which gives (25). □

Lemma 2.4. *Assume* (6) *or* (7). *Let* $T > 0$. *There exists a constant* $C(T) > 0$, *independent on* ε, *such that, for every* $0 \leq t \leq T$,

$$\begin{aligned} \left\|\partial_x v_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}, \left\|\partial_x v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} &\leq C(T), \\ \left\|v_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}, \left\|v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} &\leq C(T). \end{aligned} \tag{45}$$

Proof. Let $0 \leq t \leq T$. We begin by observing that, thanks to (23) and the Young inequality, we have that

$$\kappa u_\varepsilon^3(t,\cdot) \in L^1(\mathbb{R}), \quad 0 \leq t \leq T. \tag{46}$$

Therefore, by [20, Lemma 2.1], (45) holds. □

Arguing as in [18, Lemma 2.6] and [20, Lemma 2.8], we have the following result.

Lemma 2.5. *Assume* (6) *or* (7). *Fix* $T > 0$. *Then, there exists a constant* $C(T) > 0$, *independent on* ε, *such that, for every* $0 \leq t \leq T$,

$$\left\|u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T), \tag{47}$$

$$\left\|\partial_x^2 v_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T), \tag{48}$$

$$\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq C(T). \tag{49}$$

Lemma 2.6. *Assume* (6) *or* (7). *Fix* $T > 0$. *Then, there exists a constant* $C(T) > 0$, *independent on* ε, *such that*

$$\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon e^t \int_0^t e^{-s}\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T), \tag{50}$$

for every $0 \leq t \leq T$. *In particular, we have*

$$\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq C(T). \tag{51}$$

Proof. Let $0 \leq t \leq T$. Multiplying the first equation of (13) by $-2\partial_x^2 u_\varepsilon$, integration on $\mathbb{R}$ gives

$$\frac{d}{dt}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 2b\int_{\mathbb{R}} P_\varepsilon \partial_x^2 u_\varepsilon dx - 2q\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x v_\varepsilon dx. \tag{52}$$

Observe that, by (13) and (15),

$$2b\int_{\mathbb{R}} P_\varepsilon \partial_x^2 u_\varepsilon dx = -2b\int_{\mathbb{R}} \partial_x P_\varepsilon \partial_x u_\varepsilon dx = -2b\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon dx = 0,$$
$$-2q\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x v_\varepsilon dx = 2q\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 v_\varepsilon dx.$$

Consequently, by (49), (52) and the Young inequality,

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&= 2q\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 v_\varepsilon dx \le 2|q|\int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^2 v_\varepsilon| dx\\
&\le \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + q^2\left\|\partial_x^2 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T).
\end{aligned}$$

The Gronwall Lemma and (14) give

$$\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon e^t\int_0^t e^{-s}\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C_0 + C(T)e^t\int_0^t e^{-s} ds \le C(T),$$

which gives (50). Finally, arguing as in [20, Lemma 2.9], we have (51). □

Lemma 2.7. *Assume* (6) *or* (7). *Fix* $T>0$. *Then, there exists a constant* $C(T)>0$, *independent on* ε, *such that*

$$\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon e^t\int_0^t e^{-s}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{53}$$

for every $0\le t\le T$. *In particular, we have that*

$$\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{54}$$

Proof. Let $0\le t\le T$. Multiplying the first equation of (13) by $2\partial_x^4 u_\varepsilon$, it follows from integration on $\mathbb{R}$ that

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad = 2b\int_{\mathbb{R}} P_\varepsilon \partial_x^4 u_\varepsilon dx - 2q\int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_x v_\varepsilon dx.
\end{aligned} \tag{55}$$

Observe that, by (13) and (15),

$$2b\int_{\mathbb{R}} P_\varepsilon \partial_x^4 u_\varepsilon dx = -2b\int_{\mathbb{R}} \partial_x P_\varepsilon \partial_x^3 u_\varepsilon dx = -2b\int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon dx = 2b\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx = 0,$$
$$-2q\int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_x v_\varepsilon dx = 2q\int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^2 v_\varepsilon dx = -2q\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 v_\varepsilon dx.$$

It follows from (51), (55) and the Young inequality that

$$\begin{aligned}
\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &= -2q\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 v_\varepsilon dx \le 2|q|\int_{\mathbb{R}} |\partial_x^2 u_\varepsilon||\partial_x^3 v_\varepsilon| dx\\
&\le \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + q^2\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T).
\end{aligned}$$

The Gronwall Lemma and (14) give

$$\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon e^t \int_0^t e^{-s}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C_0 + C(T)e^t \int_0^t e^{-s} ds \le C(T),$$

which gives (53).

Finally, we prove (54). Thanks to (50), (53) and the Hölder inequality,

$$\begin{aligned}(\partial_x u_\varepsilon(t,x))^2 =& 2\int_{-\infty}^x \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy \le 2\int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| dx\\ \le& 2\left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \le C(T),\end{aligned}$$

which gives (54). □

Lemma 2.8. *Assume* (6) *or* (7). *Fix* $T>0$. *Then, there exists a constant* $C(T)>0$, *independent on* ε, *such that*

$$\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \le C(T), \tag{56}$$

for every $0\le t\le T$. *In particular, we have that*

$$\left\|\partial_x^3 v_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{57}$$

Proof. Let $0\le t\le T$. Differentiating two times the third equation of (13) with respect to x, we get

$$\alpha\partial_x^4 v_\varepsilon + \beta\partial_x^3 v_\varepsilon + \gamma\partial_x^2 v_\varepsilon = 6\kappa u_\varepsilon(\partial_x u_\varepsilon)^2 + 3\kappa u_\varepsilon^2\partial_x^2 u_\varepsilon. \tag{58}$$

Since

$$\partial_x^3 v_\varepsilon(t,\pm\infty) = \partial_x^2 v_\varepsilon(t,\pm\infty) = u_\varepsilon(t,\pm\infty) = \partial_x u_\varepsilon(t,\pm\infty) = \partial_x^2 u_\varepsilon(t,\pm\infty) = 0, \tag{59}$$

it follows from (58) that

$$\partial_x^4 v_\varepsilon(t,\pm\infty) = 0. \tag{60}$$

Multiplying (58) by $2\alpha\partial_x^4 v_\varepsilon$, (59), (60) and an integration on $\mathbb{R}$ give

$$\begin{aligned}2\alpha^2\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& 12\alpha\kappa\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^4 v_\varepsilon dx +\\ &+6\alpha\kappa\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^4 v_\varepsilon dx + 2\gamma\alpha\left\|\partial_x^3 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{61}$$

Due to (23), (47), (53), (54) and the Young inequality,

$$\begin{aligned}12|\alpha||\kappa|\int_{\mathbb{R}}|u_\varepsilon(\partial_x u_\varepsilon)^2||\partial_x^4 v_\varepsilon| dx =& \int_{\mathbb{R}}|12\kappa u_\varepsilon(\partial_x u_\varepsilon)^2||\alpha\partial_x^4 v_\varepsilon| dx\\ \le& 72\kappa^2\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^4 dx + \frac{\alpha^2}{2}\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \le& 72\kappa^2\left\|\partial_x u_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\alpha^2}{2}\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ \le& C(T) + \frac{\alpha^2}{2}\left\|\partial_x^4 v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\end{aligned}$$

$$
\begin{aligned}
6|\alpha||\kappa|\int_{\mathbb{R}}|u_\varepsilon^2\partial_x^2u_\varepsilon||\partial_x^4v_\varepsilon|dx &= \int_{\mathbb{R}}|6\kappa u_\varepsilon^2\partial_x^2u_\varepsilon||\alpha\partial_x^4v_\varepsilon|dx\\
&\le 18\kappa^2\int_{\mathbb{R}}u_\varepsilon^4(\partial_x^2u_\varepsilon)^2dx+\frac{\alpha}{2}\left\|\partial_x^4v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le 18\kappa^2\left\|u_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\alpha}{2}\left\|\partial_x^4v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T)+\frac{\alpha^2}{2}\left\|\partial_x^4v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (51) and (61) that $\alpha^2\left\|\partial_x^4v_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T)$, which gives (56).

Finally, we prove (57). Due to (51), (56), (60) and the Hölder inequality,

$$
\begin{aligned}
(\partial_x^3v_\varepsilon(t,x))^2 =& 2\int_{-\infty}^{x}\partial_x^3v_\varepsilon\partial_x^4v_\varepsilon dy\le 2\int_{\mathbb{R}}|\partial_x^3v_\varepsilon||\partial_x^4v_\varepsilon|dx\\
\le& 2\left\|\partial_x^3v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^4v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\le C(T),
\end{aligned}
$$

which gives (57). □

Lemma 2.9. *Assume* (6) *or* (7). *Fix* $T>0$. *Then, there exists a constant* $C(T)>0$, *independent on* ε, *such that, for every* $0\le t\le T$,

$$
\left\|\partial_tu_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\le C(T). \tag{62}
$$

Proof. Let $0\le t\le T$. Multiplying the first equation of (13) by $2\partial_tu_\varepsilon$, an integration on $\mathbb{R}$ gives

$$
2\left\|\partial_tu_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=2b\int_{\mathbb{R}}\partial_tu_\varepsilon P_\varepsilon dx+2\varepsilon\int_{\mathbb{R}}\partial_tu_\varepsilon\partial_x^2u_\varepsilon dx-2q\int_{\mathbb{R}}\partial_tu_\varepsilon\partial_xv_\varepsilon dx. \tag{63}
$$

Since $0<\varepsilon<1$, thanks to (23), (45), (50) and the Young inequality,

$$
\begin{aligned}
2|b|\int_{\mathbb{R}}|\partial_tu_\varepsilon||P_\varepsilon|dx &= \int_{\mathbb{R}}|\partial_tu_\varepsilon||2bP_\varepsilon|dx\\
&\le\frac{1}{2}\left\|\partial_tu_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2b^2\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le\frac{1}{2}\left\|\partial_tu_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T),\\
2\varepsilon\int_{\mathbb{R}}|\partial_tu_\varepsilon||\partial_x^2u_\varepsilon|dx &= \int_{\mathbb{R}}|\partial_tu_\varepsilon||2\varepsilon\partial_x^2u_\varepsilon|dx\\
&\le\frac{1}{2}\left\|\partial_tu_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\varepsilon^2}{2}\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le\frac{1}{2}\left\|\partial_tu_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T),\\
2|q|\int_{\mathbb{R}}|\partial_tu_\varepsilon||\partial_xv_\varepsilon|dx &= \int_{\mathbb{R}}|\partial_tu_\varepsilon||2q\partial_xv_\varepsilon|dx\\
&\le\frac{1}{2}\left\|\partial_tu_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2q^2\left\|\partial_xv_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le\frac{1}{2}\left\|\partial_tu_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T).
\end{aligned}
$$

Therefore, by (63), we have that

$$
\frac{1}{2}\left\|\partial_tu_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C(T), \tag{64}
$$

which gives (62). □

Lemma 2.10. *Assume* (6) *or* (7)*. Let* $T > 0$*. There exists a constant* $C(T) > 0$*, independent on* ε*, such that*

$$\begin{aligned}\left\|\partial_{tx}^2 v_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}, \left\|\partial_{tx}^2 v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq& C(T),\\ \left\|\partial_t v_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}, \left\|\partial_t v_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq& C(T),\end{aligned} \tag{65}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Differentiating the third equation of (13) with respect to t, we have that

$$\alpha\partial_{tx}^3 v_\varepsilon + \beta\partial_{tx}^2 v_\varepsilon + \gamma\partial_t v_\varepsilon = 3\kappa u_\varepsilon^2 \partial_t u_\varepsilon.$$

We begin by observing that, thanks to (23), (62) and the Young inequality, we have

$$\left\|3\kappa u_\varepsilon^2(t,\cdot)\partial_t u_\varepsilon(t,\cdot)\right\|_{L^1(\mathbb{R})} \leq C(T), \quad 0 \leq t \leq T. \tag{66}$$

Therefore, by [20, Lemma 2.1], (65) holds. □

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Using the Sobolev Immersion Theorem, we begin by proving the following result.

Lemma 3.1. *Fix* $T > 0$*. There exist a subsequence* $\{(u_{\varepsilon_k}, v_{\varepsilon_k}, P_{\varepsilon_k})\}_{k\in\mathbb{N}}$ *of* $\{(u_\varepsilon, v_\varepsilon, P_\varepsilon)\}_{\varepsilon>0}$ *and a limit triplet* (u, v, P) *which satisfies* (8) *and* (9) *such that*

$$\begin{aligned}&u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0,T)\times\mathbb{R}),\ 1 \leq p < \infty,\\ &u_{\varepsilon_k} \rightharpoonup u \text{ in } H^1((0,T)\times\mathbb{R}),\\ &v_{\varepsilon_k} \to v \text{ a.e. and in } L^p_{loc}((0,T)\times\mathbb{R}),\ 1 \leq p < \infty,\\ &v_{\varepsilon_k} \rightharpoonup v \text{ in } H^1((0,T)\times\mathbb{R}),\\ &P_{\varepsilon_k} \rightharpoonup P \text{ in } L^2((0,T)\times\mathbb{R}).\end{aligned} \tag{67}$$

Moreover, (u, v, P) *is solution of* (2).

Proof. Let $0 \leq t \leq T$. We begin by observing that, thanks to Lemmas 2.3, 2.6 and 2.9,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } H^1((0,T)\times\mathbb{R}). \tag{68}$$

Lemmas 2.4 and 2.10 say that

$$\{v_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } H^1((0,T)\times\mathbb{R}). \tag{69}$$

Instead, by Lemma 2.3, we have that

$$\{P_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2((0,T)\times\mathbb{R}). \tag{70}$$

(68), (69) and (70) give (67). Observe that, thanks to Lemmas 2.3, 2.6 and the second equation of (13), we have $P \in L^\infty(0,T;H^3(\mathbb{R}))$.

Lemmas 2.3, 2.6 and 2.7 say that $u \in L^\infty(0,T;H^2(\mathbb{R}))$.

Instead, thanks to Lemmas 2.4, 2.5, 2.6, 2.8 and 2.10, we get

$$v \in L^\infty(0, T; H^4(\mathbb{R})) \cap W^{1,\infty}((0, T) \times \mathbb{R}).$$

Moreover, Lemmas 2.10 says also that $\partial^2_{tx} v \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, for every $0 \le t \le T$. Therefore, (8) holds and (u, v, P) is solution of (2).

Finally, (9) follows from (18) and (67). □

We are finally ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. Lemma 3.1 gives the existence of a solution of (2) such that (8) holds.

Let (u_1, v_1, P_1) and (u_2, v_2, P_2) be two solutions of (2), which verify (8). Arguing as in [20, Theorem 1.1], we have (10). □

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