Minimax Theory and its Applications
Volume 9 (2024), No. 1, 85–115

Nonconvex Homogeneous Optimization: a General Framework and Optimality Conditions of First and Second-Order

Fabián Flores-Bazán, Adrian Carrillo-Galvez
Departamento de Ingeniería Matemática, Universidad de Concepción, Concepción, Chile
fflores@ing-mat.udec.cl

Received: September 9, 2023
Accepted: October 12, 2023

This work discusses and analyzes a class of nonconvex homogeneous optimization problems, in which the objective function is a positively homogeneous function with a certain degree, and the constraints set is determined by a single homogeneous function with another degree, and a geometric set which is a (not necessarily convex) closed cone. Once a Lagrangian dual problem is associated, it is provided various characterizations for the validity of strong duality property: one of them is related to the convexity of a certain image of the geometric set involving both homogeneous functions, so revealing a hidden convexity. We also derive a suitable S-lemma. In the case where both functions are of the same degree of homogeneity, a copositive reformulation of the original problem is established. It is also established zero-, first- and second-order optimality conditions; KKT (local or global) optimality, giving rise to the notion of L-eigenvalues with applications to symmetric tensors eigenvalues analysis.

Keywords: Nonconvex optimization, homogeneous functions, copositivity, hidden convexity, strong duality, S-lemma.

2020 Mathematics Subject Classification: 90C20, 90C26, 90C34, 90C46, 49N15.

1. Introduction

This paper discusses and analyzes what we call as the "generalized minimum eigenvalue problem":

$$\mu_0 := \inf \Big\{ f(x) : \ g(x) = 1, \ x \in C \Big\}, \tag{1}$$

where $C \subseteq \mathbb{R}^n$ is a (not necessarily convex) closed cone and f, g are positively homogeneous functions on C with degree p and q, respectively, such that $g(x) > 0$ for all $x \in C$, $x \neq 0$. Such a formulation, spite its simplicity, encompasses several important problems in many different areas, as we briefly describe below. One method to approximate μ_0 is via the optimal value of a "dual problem", which is not uniquely determined. We propose, as done in previous works by one of the authors, and others, the following (Lagrangian) dual problem:

$$\nu := \sup_{\lambda \in \mathbb{R}} \inf_{x \in C} \Big\{ f(x) + \lambda(g(x) - 1) \Big\}. \tag{2}$$

The research for the first author was supported in part by ANID-Chile through FONDECYT 1212004, ACE210010 and Basal FB210005. The second author was supported by ANID-Chile through scholarship PFCHA/Doctorado Nacional/2017-21170239.

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

Obviously ν is always a lower bound for μ_0 (weak duality). It worth-while to mention that the dual problem admits the unique solution $-\mu_0 \, \frac{p}{q}$ (Theorem 3.4).

One of the main purposes of this work is to discuss the validity of the equality $\mu_0 = \nu$ (zero duality gap), and the strong duality property, which means that $\mu_0 = \nu$ and problem (2) has solution: these topics are analyzed in Section 3. Another important issue to be developed in this paper concerns first- or second-order optimality conditions, including KKT-type optimality, which are established in Section 4.

In many practical models, C is described by inequalities and/or equalities, and so, one can talk about a standard dual problem, whose associated Lagrangian involves all the constraint functions, and so, the infimum in (2) is taken over $\mathbb{R}^n$. With respect to this standard dual problem, we will see that a zero duality gap, or strong duality property occurs under very strong assumptions.

More precisely, we establish various characterizations for the fulfillment of a strong duality property (see Theorem 3.4). In particular, the theorem holds if, and only if $\overline{(g, f)(C) + \mathbb{R}_+(0, 1)}$ is convex. This is satisfied if either $\mu_0 = 0$ or $p = q$ (this holds for some of the models below), and in the latter situation, the copositive formulation holds (Theorem 3.4 and Corollary 3.6):

$$\mu_0 = \nu = \sup\Big\{\lambda : f - \lambda g \text{ is copositive on } C\Big\} = \inf\Big\{f(x) - \mu_0(g(x) - 1) : x \in C\Big\},$$

where by copositivity on C of any function h, we mean that $h(x) \geq 0$ for all $x \in C$. Furthermore, we introduce the notion of L(agrange)-eigenvalue (see Subsection 4.2) as a Lagrange multiplier associated to the existence of a KKT-point (different from the zero vector) suitably defined, which is related to a necessary optimality condition for the problem

$$\inf\Big\{f(x) - \mu_0 \frac{p}{q}(g(x) - 1) : \ x \in C\Big\}.$$

In some sense, our approach provides a variational scheme to the analysis of eigenvalues to certain mappings; in particular, to real symmetric tensors (see Model 2). For other perspective we refer to [31].

Before describing some of the main models where our approach applies, we list just a few concrete applications. For instance, a class of quadratic homogeneous optimization problems arises in telecommunications and robust control as reported in [43, 55]; minimum eigenvalue of a symmetric matrix; minimization of a homogeneous polynomial over spheres or hyperspheres (for instance [39, 47], giving rise to the sum of squares (SOS) relaxation as an aproximation method); several important classes of quadratic programming problems lying in matrix theory; special relativity [18]; trust region problems [28, 57] (for recent advances on this matter, we refer to [61]).

Model 1: Minimizing a quadratic form subject to two homogeneous quadratic forms over the unit sphere

This problem is taken from [46]:

$$\mu := \inf\Big\{x^\top Ax : \ x^\top B_1 x = 0, \ x^\top B_2 x = 0, \ : \ x^\top x = 1\}, \tag{3}$$

where A, B_1, B_2 are symmetric matrices with real entries.

The problem has as a dual problem (according to (2)):

$$\nu := \sup_{\lambda\in\mathbb{R}} \inf_{x\in C} \Big\{x^\top Ax + \lambda(x^\top x - 1)\Big\}, \tag{4}$$

where $C := \{x \in \mathbb{R}^n : x^\top B_1 x = 0,\ x^\top B_2 x = 0\}$. This problem satisfies $p = q = 2$, and therefore strong duality holds (Theorem 3.4): $\mu = \nu$ and problem (4) has solution. In particular, a copositive reformulation can be obtained (see Corollary 3.6). This problem was studied in [46] under the SDP relaxation approach (following [1]), and so yielding tightness, which in turns implies that standard strong duality holds. Our approach considers situations where standard strong duality may fail.

Model 2: Tensors eigenvalues analysis

The analysis will be developed in detail in Section 5 and it refers to the problem

$$\mu_k := \min \Big\{\mathcal{A}x^m :\ \|x\|_k^m = 1,\ x \in C\Big\}, \tag{5}$$

where: $\mathcal{A}$ is an m-order n-dimensional real symmetric tensor, and so $\mathcal{A}x^m$ defines a homogeneous polynomial of degree m; $\|x\|_k := (|x_1|^k + \cdots + |x_n|^k)^{1/k}$ denotes the l^k-norm, and C is a closed convex cone in $\mathbb{R}^n$. It will be showed that μ_k is the least L(agrange)-eigenvalue (to be introduced in Subsection 4.2) associated to problem (5). By particularizing $k = 2$ or $k = m$ and either $C = \mathbb{R}^n$ or $C = \mathbb{R}^n_+$, we recover the notion of Z-eigenvalue (eigenvector) or H-eigenvalue (eigenvector), see [53, 51, 40, 56], among others. Hence, we provide a unified approach in tensor analysis, and produce new results about the copositivity meaning in this context (Proposition 5.4).

Model 3: Approaching linear complementarity problems

A quadratic programming approach to linear complementary problems (LCP) is to consider the bilinear program (see, for instance [2])

$$\min\{z^\top w :\ -Mz + w = q,\ z \geq 0,\ w \geq 0\}.$$

This problem can be written, in an equivalent way as:

$$\min\{x^\top Ax :\ Hx = q,\ x \geq 0\}, \tag{6}$$

where $$x = \begin{pmatrix} z \\ w \end{pmatrix}, \qquad A = \left(\begin{array}{c|c} 0 & \frac{1}{2}\, I \\ \hline \frac{1}{2}\, I & 0 \end{array}\right), \qquad H = \begin{pmatrix} -M \mid I \end{pmatrix},$$

and 0, I stand for the null and identity matrices of order $n \times n$, respectively, and A is indefinite but copositive on $\mathbb{R}^{2n}_+$, i.e., $x^\top Ax \geq 0$ for all $x \in \mathbb{R}^{2n}_+$.

We consider an example from [16] to illustrate how to re-formulate (6) in the form (1). In such an example, we have

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}, \qquad q = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}.$$

By setting $f(x) := x^\top Ax$, $g(x) := \frac{1}{3}(x_1+x_2+x_3+x_6)$, $g_1(x) := -x_1-x_2-x_3+3x_4-x_6$, $g_2(x) = 4x_1 + x_2 + x_3 + 3x_5 + x_6$, and choosing as C to be the closed convex cone $C := \{x \in \mathbb{R}^6 : x \geq 0, g_1(x) = 0, g_2(x) = 0\}$, we have that f and g are positively homogeneous functions of degrees $p = 2$ and $q = 1$, respectively. Moreover, $g(x) > 0$ for all $x \in C$, $x \neq 0$. Certainly, f is nonconvex and $f(x) \geq 0$ for all $x \geq 0$. We refer to [22] for a great account about linear complementarity problems.

Model 4: Extensions of the standard quadratic and portfolio optimization problems

We consider now two problems with the following general structure

$$\min\Big\{f(x):\ e^\top x = 1,\ x \in C\Big\},$$

where $C \subseteq \mathbb{R}^n$ is a pointed, closed, convex cone having non-empty interior, and $e \in \operatorname{int} C^*$. Here, C^* is the non-negative polar (or, simply polar) cone of C.

For $f(x) = \frac{1}{2}\, x^\top Ax$, with A being a real symmetric matrix of order n, problem

$$\mu_q \doteq \min\Big\{\frac{1}{2}x^\top Ax:\ e^\top x = 1,\ x \in C\Big\}, \tag{7}$$

is an extension of the standard quadratic optimization problem (StQOP). This model generalizes the problem introduced by Bomze in [8] (where $C = \mathbb{R}^n_+$, $e := (1, \ldots, 1)$), which models: quadratic allocation problems [33]; (classical mean-variance) portfolio optimization problems [41, 42]; the maximum weight clique problem [29, 44]; the indefinite quadratic knapsacks problem [48], see also [43, 55], among others. Due to the structure of the feasible set, it is not restrictive to consider homogeneous functions, since

$$\frac{1}{2}x^\top Ax + a^\top x = \frac{1}{2}x^\top(A + ae^\top + ea^\top)x.$$

The StQOP was introduced in [8] and further developed in [9, 10, 11, 12, 13] and references therein.

According to our Theorem 3.4, problem (7) has strong duality if, and only if A is copositive, since $p = 2 > q = 1$. On the other hand, we can find a copositive formulation of (7), since it can be written equivalently as

$$\mu_q := \min\Big\{\frac{1}{2}x^\top Ax:\ x^\top ee^\top x = 1,\ x \in C\Big\},$$

whose dual problem is

$$\nu_q := \sup_{\lambda\in\mathbb{R}} \inf_{x\in C}\Big\{\frac{1}{2}x^\top Ax + \lambda(x^\top ee^\top x - 1)\Big\}.$$

By particularizing Corollary 3.6 ($p = q = 2$), one gets

$$\mu_q = \nu_q = \sup_{\lambda\in\mathbb{R}}\Big\{-\lambda:\ A + 2\lambda ee^\top \text{ is copositive on } C\Big\}.$$

The case $C = \mathbb{R}^n_+$, allows us to introduce the standard dual problem, as we said at the beginning. In this situation, standard strong duality is satisfied if, and only if A is positive semidefinite; and therefore, in practice such a dual is not of much interest.

On the other hand, a different model to the mean-variance portfolio optimization problem (7), is that known as the "standard deviation premium" considered in [37], where the variance is replaced by the standard deviation, that is, f takes the form $f(x) = a^\top x + \rho\sqrt{x^\top Vx}$, so the problem is

$$\mu_p := \min\Big\{a^\top x + \rho\sqrt{x^\top Vx} : \ e^\top x = 1, \ x \in C\Big\}.$$

We refer to [38] for a further discussion. Here, V is only required to be copositive on C. Since $p = q = 1$, strong duality holds because of Theorem 3.4, and by Corollary 3.6, the copositive formulation ($g(x) = e^\top x$) is

$$\mu_p = \max\{-\lambda : \ f + \lambda g \text{ is copositive on } C\}.$$

If, instead, one considers the equivalent constraint $g(x) := x^\top ee^\top x = 1$, then we have $p = 1 < q = 2$, and so by the same theorem, strong duality is satisfied provided $f(\bar{x}) < 0$ for some $\bar{x} \in C$.

Several other problems, after some mathematical manipulations like in Model 3, can also be formulated as in (1).

This paper is organized as follows. Section 2 serves to introduce some basic definitions and preliminaries, as well as to revisit the Lagrangian duality scheme. Sections 3 and 4 contain our main results. Section 3 establishes various new characterizations of the validity of: strong duality for (1) (revealing convexity as Theorem 3.4 shows); the S-lemma (Lemma 3.7), a copositive formulation for (1) when $p = q$, as Corollary 3.6 shows. In Section 4, it is discussed: zero-order optimality conditions; KKT optimality and L-eigenvalues.

Finally, in Section 5 the case of a real m-order n-dimensional supersymmetric tensor is discussed. In particular, some relationships linking our notion of L-eigenvalue and those of Z-eigenvalue or H-eigenvalue, are presented.

2. Some notation, basic definition and preliminaries

Throughout this paper, we will work on a finite dimensional space, say $\mathbb{R}^m$. Given any nonempty set M in $\mathbb{R}^m$, its closure, topological interior, convex hull, closed convex hull, are denoted, respectively, by $\overline{M}$, int M, co M, $\overline{\text{co}}$ M. In addition, by aff M, span M, ri M and bd M we denote the affine set of M, span of M, relative interior of M and the boundary of M, respectively. Moreover, we define $\alpha M := \{\alpha m : m \in M\}$, for all $\alpha \in \mathbb{R}$; the cone M is the smallest cone containing M, i.e., cone $M = \bigcup_{t\geq 0} tM$; cone$_+$ $M = \bigcup_{t>0} tM$; the polar cone of M is defined by

$$M^* := \{z \in \mathbb{R}^m : \ \langle z, y\rangle \geq 0 \ \ \forall\, y \in M\}.$$

Here, $\langle z, y\rangle = z^\top y$ stands for the scalar product between two vectors z and y in $\mathbb{R}^m$, where $z^\top$ means the transpose of the vector z, which is considered a column vector.

More generally, if A is a real matrix in $\mathbb{R}^{m\times n}$, $A^\top$ is the transpose of A belonging to $\mathbb{R}^{n\times m}$. Furthermore, given $x \in \mathbb{R}^n$, $x \neq 0$, $x^\perp$ denotes the orthogonal hyperplane to x through the origin; for sets M, N in $\mathbb{R}^n$, $M + N$ stands for the Minkowski sum given by $M + N := \{m + n : \ m \in M, n \in N\}$, so $M - N = M + (-N)$.

Let $h : \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$, $\overline{h}$ and $\overline{\text{co}}\ h$ stand for the greatest lower semicontinuous function not larger than h and for the greatest convex and lower semicontinuous function not larger than h, respectively. Just for convenience, we need the following definition of epigraph of a function: epi $h := \{(y, t) \in \mathbb{R}^m \times \mathbb{R} : \ h(y) \leq t\}$. It is known that

$$\text{epi } \overline{h} = \overline{\text{epi } h}; \quad \overline{\text{co}}(\text{epi } h) = \text{epi } \overline{\text{co}}\ h.$$

Moreover, $$\overline{\text{co}}\ h(y) > -\infty \ \ \forall\ y \in \mathbb{R}^m \Longrightarrow \overline{\text{co}}\ h(y) = h^{**}(y) \ \ \forall\ y \in \mathbb{R}^m,$$

where $h^{**} = (h^*)^*$ is the bipolar or biconjugate of h, that is, the conjugate (or polar) of h^* defined by

$$h^*(z) := \sup_{y\in\mathbb{R}^m} \{\langle z, y\rangle - h(y)\}.$$

In addition, δ_M stands for the indicator function of the set M, defined by 0 on M, and $+\infty$ on the complementary of M.

There are examples showing that the assumption $\overline{\text{co}}\ h(y) > -\infty$ for all $y \in \mathbb{R}^m$ is necessary to get the equality $h^{**} = \overline{\text{co}}\ h$. In general we have $h^{**} \leq \overline{\text{co}}\ h \leq h$. For details see [52].

In case $h : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$, the subdifferential of h at $\bar{y} \in \mathbb{R}^m$ is denoted by

$$\partial h(\bar{y}) := \{\xi \in \mathbb{R}^m : \ h(y) \geq h(\bar{y}) + \langle \xi, y - \bar{y}\rangle, \ \forall\ y \in \mathbb{R}^m\},$$

if $\bar{y} \in$ dom h, and $\partial h(\bar{y}) = \emptyset$ elsewhere.

In the subsequent sections, we set $\mathbb{R}_+ := [0, +\infty[$; $\mathbb{R}_{++} := \ ]0, +\infty[$. Given a vector $a \in \mathbb{R}^m \setminus \{0\}$, $\mathbb{R}_+ a$ stands for the ray starting from the origin and direction a; and $a^\perp$ is the orthogonal subspace to a, which is a hyperplane.

In the remaining part of this section, we present a duality scheme for a minimization problem under one single equality constraint and a geometric constraint set, mainly taken from Section 3 in [20]. Let $f, g_0 : \mathbb{R}^n \to \mathbb{R}$ be any finite-valued functions, and let $C \subseteq \mathbb{R}^n$ be any nonempty set. Let us consider the problem

$$\mu := \inf\{f(x) : \quad g_0(x) = 0, \ x \in C\}, \tag{8}$$

whose (Lagrangian) dual problem is defined by

$$\nu := \sup_{\lambda\in\mathbb{R}} \inf_{x\in C} [f(x) + \lambda g_0(x)]. \tag{9}$$

We say that there is no duality gap, or the duality gap is zero, between (8) and (9) if $\nu = \mu$. It is said that (8) has the strong duality property with respect to (9), or simply that strong duality holds for (8) if $\mu = \nu$ and problem (9) admits a solution. One infers immediately that $\nu \leq \mu$. Thus, if $\mu = -\infty$ then there is no duality gap, and we conclude that any element in $\mathbb{R}$ is a solution for the problem (9). Hence, we always have strong duality for (8) whenever $\mu = -\infty$. So, we suppose from now on that $\mu \in \mathbb{R}$, which means, in particular, that the feasible set to (8) is nonempty. Set $F(x) := \big(g_0(x), f(x)\big)$. Notice that $F = (f, g_0)$ was used in [21] instead. Assuming that $\mu \in \mathbb{R}$, we obtain

$$(F(C) - \mu(0,1)) \cap -(\{0\} \times \mathbb{R}_{++}) = \emptyset. \tag{10}$$

We will show, next, that strong duality can be characterized by reinforcing (10). The optimal value function $\psi : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ to problem (8) is defined by

$$\psi(a) = \begin{cases} \inf\{f(x) : \ x \in K(a)\} & \text{if} \ \ K(a) \neq \emptyset; \\ +\infty & \text{otherwise,} \end{cases}$$

where $K(a) := \{x \in C : \ g_0(x) = a\}$.

Notice that $K = K(0)$, and $K(a) \neq \emptyset$ if and only if $a \in g_0(C)$, that is,

$$\text{dom}\ \psi := \{a \in \mathbb{R} : \ \psi(a) < +\infty\} = g_0(C).$$

The sets $\quad \mathcal{F} := F(C) + \mathbb{R}_+(0,1), \ \mathcal{E}_\rho := \mathcal{F} - \rho(0,1) \ \ (\rho \in \mathbb{R}),$

will play an important role in our analysis.

Remark 2.1. (a) By definition, strong duality holds if and only if there exists $\lambda_0 \in \mathbb{R}$ such that

$$f(x) + \lambda_0 g_0(x) \geq \mu, \ \forall\ x \in C,$$

or equivalently, $\mathcal{L}_{SD} \neq \emptyset$, where $\mathcal{L}_{SD} := \{\lambda_0 \in \mathbb{R} : (\lambda_0, 1) \in (\mathcal{E}_\mu)^*\}$.

Hence, $\mathcal{L}_{SD} \subseteq \mathcal{S}_D$ with $\mathcal{S}_D$ being the solution set to the dual problem (9). Moreover, $\mathcal{L}_{SD} = \mathcal{S}_D$ whenever zero duality gap holds.

(b) The following chain of inclusions shows useful and well-known properties of the optimal value function ψ (see [21] for instance):

$$F(C) + \mathbb{R}_+(0,1) \subseteq \text{epi}\ \psi \subseteq \overline{F(C) + \mathbb{R}_+(0,1)}. \tag{11}$$

Consequently, $\quad \overline{\mathcal{E}_\mu} = \overline{\text{epi}\ \psi} - \mu(0,1) = \text{epi}\ \overline{\psi} - \mu(0,1);$

$$\overline{\text{co}}\ \mathcal{E}_\mu = \overline{\text{co}}(\text{epi}\ \psi) - \mu(0,1) = \text{epi}(\overline{\text{co}}\ \psi) - \mu(0,1).$$

The next result states various equivalences, of topological or geometric nature, to the validity of strong duality.

Proposition 2.2. ([21, Theorem 4.2]) *Assume that $\mu = \psi(0)$ is finite. The following assertions are equivalent:*

(a) *Strong duality holds for* (8)*;*

(b) $\overline{\text{cone}}(\text{co}\ \mathcal{E}_\mu) \cap (-\{0\} \times \mathbb{R}_{++}) = \emptyset$;

(c) $\partial\psi(0) \neq \emptyset$*;*

(d) $\overline{\text{cone}}(\mathcal{E}_\mu) \cap (-\{0\} \times \mathbb{R}_{++}) = \emptyset$ *and* $\overline{\text{cone}}(\mathcal{E}_\mu)$ *is convex.*

Hence, under any of the above conditions, one gets

$$\partial\psi(0) = \{-\lambda_0 \in \mathbb{R} : \ \lambda_0 \in \mathcal{S}_D\}.$$

We must point out that a more detailed description of the disjointness appearing in (d), is presented in [14, Theorem 3]. On the other hand, one can see from the previous proposition that the convexity of $\overline{\text{cone}}(\mathcal{E}_\mu)$ arises in a natural way under strong duality no matter the functions f and g are. The equivalence between the validity of strong duality and the convexity of $\overline{\text{cone}}(F(C)+\mathbb{R}_+(0,1)-\mu(0,1))$ was proved in [14, 21] under a Slater-type condition. In case $C=\mathbb{R}^n$ and f,g are quadratic functions, the authors in [26] established (under a Slater condition) that strong duality holds if and only if $F(\mathbb{R}^n)+\mathbb{R}_+(0,1)$ is convex. When C is a pointed closed convex cone, and f is a quadratic form and g linear (see Model 4), such an equivalence with the convexity of $F(C)+\mathbb{R}_+(0,1)$, was established in [20].

In addition, we have to mention that in case we have more than one constraint, one may proceed by including all the constraints, except one, in the geometric constraint set C, as it is suggested by Model 3, for instance. Among the recent results about the convexity of images of quadratic functions, we mention [5, 34, 58, 26, 20].

3. Formulation of the problem: characterizing strong duality and S-lemma; copositive reformulation

Let us go back to our original problem (1) formulated in Section 1 whose feasible set is denoted by K (which is supposed to be nonempty), and where $C\subseteq\mathbb{R}^n$ is a closed cone, and $f,g:\mathbb{R}^n\to\mathbb{R}$ satisfy the following assumption

Assumption (A):
Let p,q be positive real numbers. The functions f and g are lower semicontinuous (lsc, in short) such that:

(i) $f(tx)=t^pf(x)$ for all $t>0$ and all $x\in C$;

(ii) $g(tx)=t^qg(x)$ for all $t>0$ and all $x\in C$;

(iii) $g(x)>0$ for all $x\in C$, $x\neq 0$; as a consequence, $x\in C$, $g(x)=0$ if, and only if $x=0$.

Consequently, as shows below, $\mu_0\in\mathbb{R}$. Some remarks are in order.

Remark 3.1. Let C be a closed cone.

(a) Let $h:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ be a lsc function and positively homogeneous with degree p. Then, $h(0)=0$; the set $C\cap\text{dom }h$ is a cone, and $t(C\setminus\text{dom }h)\subseteq C\setminus\text{dom }h$ for all $t>0$. Moreover, in case h is differentiable around $x\in C\cap\text{dom }h$, the so-called Euler identity holds:

$$\nabla h(x)^\top x=ph(x).$$

In addition, ∇h is positively homogeneous with degree $p-1$.

(b) We observe that under Assumption (A), one gets $\mu_0\in\mathbb{R}$. Indeed, under (ii), (iii) and lsc on g, the set $\{x\in C:\ g(x)\leq\gamma\}$ is bounded (so, compact) for all $\gamma\geq 0$. Hence, since f is lsc, we get $\mu_0\in\mathbb{R}$.

(c) In most applications C is, in addition, convex and pointed, and $g(x)=e^\top x$ with $e\in\text{int }C^*$; thus g satisfies (iii). Another useful specialization is

$$g(x)=(\|x\|_m)^m\doteq|x_1|^m+|x_2|^m+\cdots+|x_m|^m$$

and $C=\mathbb{R}^m_+$.

The fact that the cone C is not necessarily convex makes our model to be very versatile, as described at the introduction section. On the other hand, by virtue of Assumption (A) and since K is nonempty, problem (1) always fulfills the Slater condition: there exist $x_1, x_2 \in C$ such that $g(x_1) < 1 < g(x_2)$.

It is said that $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is *copositive* on a cone $P \subseteq \mathbb{R}^n$ if $h(x) \geq 0$ for all $x \in P$; it is strictly copositive on P if $h(x) > 0$ for all $x \in P$, $x \neq 0$. By extension, a (real) symmetric matrix A is said to be (resp. strictly) copositive on P, if the function $h(x) = x^\top Ax \geq 0$ (resp. > 0) for all $x \in P$ (resp. for all $x \in P, x \neq 0$).

By assumption, one gets
$$C = \bigcup_{t\geq 0} tK,$$
with K being a bounded set; it will be compact if g is continuous. Thus, it is easy to check that

- $\mu_0 \geq 0 \Longleftrightarrow f$ is copositive on C;
- $\mu_0 = 0 \Longleftrightarrow f$ is copositive but not strictly copositive on C;
- $\mu_0 > 0 \Longleftrightarrow f$ is strictly copositive on C.

The following lemma shows some intrinsic properties of problem (1), and relationships with the problems:
$$\mu_+ \doteq \inf\Big\{f(x) : \; g(x) \geq 1, \; x \in C\Big\}; \; \mu_- \doteq \inf\Big\{f(x) : \; g(x) \leq 1, \; x \in C\Big\}.$$
Set $K_+ \doteq \{x \in C : \; g(x) \geq 1\}$ and $K_- \doteq \{x \in C : \; g(x) \leq 1\}$.

Lemma 3.2. *Let C be a closed cone, and f, g satisfy Assumption* (A) (*so* $\mu_0 \in \mathbb{R}$). *The following assertions hold.*

(a) *Assume that $\mu_0 \geq 0$. Then,*

(a1) $x \in C, \; f(x) < \mu_0 \Longrightarrow g(x) < 1$.

(a2) $\mu_0 = \mu_+$ *and* $\operatorname*{argmin}_K f \subseteq \operatorname*{argmin}_{K_+} f$.

(a3) *If* $\mu_0 > 0$ *then* $\operatorname*{argmin}_K f = \operatorname*{argmin}_{K_+} f$, *and* $[x \in C, \; f(x) = \mu_0 \Longrightarrow g(x) \leq 1]$.

(b) *Assume that $\mu_0 \leq 0$. Then,*

(b1) $x \in C, \; f(x) < \mu_0 \Longrightarrow g(x) > 1$.

(b2) $\mu_0 = \mu_-$ *and* $\operatorname*{argmin}_K f \subseteq \operatorname*{argmin}_{K_-} f$.

(b3) *If* $\mu_0 < 0$ *then* $\operatorname*{argmin}_K f = \operatorname*{argmin}_{K_-} f$, *and* $[x \in C, \; f(x) = \mu_0 \Longrightarrow g(x) \geq 1]$.

Proof. We only prove (a), being the other entirely similar.

(a1): Take any $x \in C$ such that $f(x) < \mu_0$. Obviously the implication holds if $x = 0$. Suppose now that $g(x) > 1$ (the case $g(x) = 1$ is excluded), and write $x = ty$ for some $t > 0$ and $y \in K$. Thus $t^q > 1$ and so $t > 1$. Moreover, we obtain $\mu_0 > f(x) = t^p f(y) \geq t^p \mu_0$, which implies $\mu_0 < 0$, and this contradicts our assumption.

(a2): Clearly $\mu_+ \leq \mu_0$. Suppose that $\mu_+ < \mu_0$ and choose $y \in K_+$ satisfying $f(y) < \mu_0$. By (*a*1), $g(y) < 1$, a contradiction.

(a3) The first part follows easily, and the second one is similar to (a2). □

Denote, given $a \in \mathbb{R}$, $\quad K(a) \doteq \{x \in C : \ g(x) = 1 + a\}$.

It is worth noticing that a complete study in the case when $p = 2,\ q = 1$ was carried out in [20], including some necessary or sufficient conditions for local or global optimality.

The following proposition, which is new, collects some useful facts on the optimal value function

$$\psi(a) \doteq \inf_{x \in K(a)} f(x).$$

In order to make compatible problem (1) with that of (8) we introduce the function $g_0(x) \doteq g(x) - 1$, and so set $F \doteq (g_0, f)$.

Proposition 3.3. *Let C be a closed cone, and f, g satisfy Assumption* (A)*. The following assertions hold.*

(a) *$K(a) \neq \emptyset$ if, and only if $a \geq -1$.*

(b) *Let $a > -1$. Then, $x \in K$ if, and only if $(a+1)^{1/q} x \in K(a)$.*

(c) *The optimal value function is given by*

$$\psi(a) = \begin{cases} \mu_0 (1+a)^{p/q} & \text{if} \quad a \geq -1; \\ +\infty & \text{if} \quad a < -1. \end{cases}$$

(d) *Assume that f and g are continuous. Then $F(C)$ and $F(C) + \mathbb{R}_+(0,1)$ are closed, so* epi $\psi = F(C) + \mathbb{R}_+(0,1)$.

Proof. (a) is obvious.

(b) Let $x \in K$. Then $g((a+1)^{1/q} x) = (a+1) g(x) = a + 1$. Thus $(a+1)^{1/q} x \in K(a)$, and so, by symmetry, the result follows.

(c) is a consequence of (b).

(d) We check the closedness of $F(C) + \mathbb{R}_+(0,1)$. The same argument also shows that $F(C)$ is closed. Let $(a, r) \in \overline{F(C) + \mathbb{R}_+(0,1)}$. Then, there exist sequences $x_k \in C$, $q_k \geq 0$ satisfying $f(x_k) + q_k \to r$ and $g(x_k) - 1 \to a$. By assumption, the second relation implies the boundedness of $\|x_k\|$. Thus, up to a subsequence, $x_k \to \bar{x} \in C$, implying that $q_k = f(x_k) + q_k - f(x_k) \to r - f(\bar{x})$. Setting $q \doteq r - f(\bar{x})$, we get $q \geq 0$, and so $(a, r) = (g(\bar{x}) - 1, f(\bar{x}) + q) \in F(C) + \mathbb{R}_+(0,1)$.

The last part follows from Remark 2.1(b) applied to $g_0(x) = g(x) - 1$. $\square$

By Proposition 3.3(c) and (11), it is not difficult to prove that

$$\overline{\text{cone}}(F(C) + \mathbb{R}_+(0,1) - \mu_0(0,1)) = \overline{\text{cone}}\Big(\overline{F(C) + \mathbb{R}_+(0,1)} - \mu_0(0,1)\Big)$$

$$= \begin{cases} \{(u,v) \in \mathbb{R}^2 : v \geq \dfrac{\mu_0 p}{q} u\} \text{ if } [0 < q \leq p,\ \mu_0 \geq 0] \text{ or } [0 < p \leq q,\ \mu_0 \leq 0]; \\ \{(u,v) \in \mathbb{R}^2 : v \geq \mu_0 u\} \cup (\mathbb{R}_+ \times \mathbb{R}) \text{ if } 0 < q < p,\ \mu_0 < 0; \\ \{(u,v) \in \mathbb{R}^2 : v \geq \mu_0 u\} \cup (\mathbb{R} \times \mathbb{R}_+) \text{ if } 0 < p < q,\ \mu_0 > 0. \end{cases} \tag{12}$$

From this it follows that

$$\overline{\text{cone}}\Big(F(C)+\mathbb{R}_+(0,1)-\mu_0(0,1)\Big) \text{ is convex } \iff \begin{cases} 0<q\le p,\ \ \mu_0\ge 0, \\ \text{or} \\ 0<p\le q,\ \ \mu_0\le 0. \end{cases} \tag{13}$$

We now establish how the fulfillment of strong duality property reveals the hidden convexity of some image set. To the best of our knowledge, this result is new and the first one (in a non-local sense) for functions beyond the quadratic world.

Theorem 3.4. *Let C be a closed cone, and f, g satisfy Assumption* (A). *Then, the following are equivalent:*

(a) *strong duality holds.*

(b) $\overline{F(C)+\mathbb{R}_+(0,1)}$ *is convex.*

(c) *Exactly one of the following assertions is satisfied:*

(c1) *f is copositive but not strictly copositive on C* ($\mu_0=0$);

(c2) *f is strictly copositive on C* ($\mu_0>0$) *and $p\ge q>0$;*

(c3) *f is not copositive on C* ($\mu_0<0$) *and $q\ge p>0$.*

Consequently, under any of conditions (a), (b) *or* (c), *the unique solution to the dual problem* (2) *is $-\frac{p}{q}\mu_0$, and so*

$$\mu_0=\inf_{x\in C}[f(x)-\mu_0\frac{p}{q}(g(x)-1)]. \tag{14}$$

Proof. By (11) one gets epi $\psi=\overline{F(C)+\mathbb{R}_+(0,1)}$, so the equivalence between (b) and (c) follows from Proposition 3.3(c).

(b)$\Rightarrow$(a): we obtain the convexity of $\overline{\text{cone}}(F(C)+\mathbb{R}_+(0,1)-\mu_0(0,1))$, and because of (12), the first condition in (*d*) of Proposition 2.2 holds. Then strong duality is satisfied.

(a)$\Rightarrow$(b): By applying Proposition 2.2, (d) holds.
In particular, $\overline{\text{cone}}(F(C)+\mathbb{R}_+(0,1)-\mu_0(0,1))$ is convex, and because of (13), ψ is convex. Consequently $\overline{F(C)+\mathbb{R}_+(0,1)}$ is convex as well, and therefore (b) holds.

The remaining part also follows from Proposition 2.2. □

Remark 3.5. Observe that in case f and g are continuous, because of Proposition 3.3, the set $F(C)+\mathbb{R}_+(0,1)$ is closed. Model 3 and the specializations of Model 4 satisfy the continuity assumptions.

One of the interpretations of Theorem 3.4 follows. In case $0<q\le p$, Theorem 3.4 characterizes the strict copositivity on C of every function f that is positively homogeneous with degree p on C, by means of the convexity of $\overline{F(C)+\mathbb{R}_+(0,1)}$ for some (any) function g positively homogeneous with degree q on C.

From Theorem 3.4, we realize that when $p=q$ it is possible to establish a copositive reformulation of the dual problem, and so of the primal one. Such a formulation extends Lemma 3.2 and (main) Theorem 3.5 in [50], which considers the quadratic (homogeneous) case with $C=\mathbb{R}^n_+$.

Corollary 3.6. (Copositive formulation) *Let $p = q > 0$. Assume that f and g satisfy Assumption* (A). *Then,*

$$\mu_0 = \nu = \max\{\lambda : \ f - \lambda g \text{ is copositive on } C\}, \quad \textit{and}$$

- *$f - \mu_0 g$ is copositive but not strictly copositive on C, that is, there exists $\bar{x} \in C$, $\bar{x} \neq 0$, such that*
$$\mu_0 = \min_{\substack{x \in C \\ x \neq 0}} \frac{f(x)}{g(x)} = \frac{f(\bar{x})}{g(\bar{x})};$$
- *$\forall \gamma < \mu_0$, $f - \gamma g$ is strictly copositive on C;*
- *$\forall \gamma > \mu_0$, $f - \gamma g$ is not copositive on C.*

Proof. By the previous theorem we get strong duality, and so

$$\begin{aligned} \mu_0 = \nu &= \sup_{\lambda \in \mathbb{R}} \inf_{x \in C} [f(x) - \lambda(g(x) - 1)] = \sup_{\lambda \in \mathbb{R}} \{\lambda + \inf_{x \in C} [f(x) - \lambda g(x)]\} \\ &= \sup_{\lambda \in \mathbb{R}} \{\lambda : \ \inf_{x \in C} [f(x) - \lambda g(x)] = 0\} \\ &= \max\{\lambda : \ f - \lambda g \text{ is copositive on } C\}. \end{aligned}$$

The remaining assertions are straightforward. □

We utilize Corollary 3.6 and the bisection algorithm to propose an algorithm to determine μ_0 or an approximation of it once a tolerance $\varepsilon > 0$ is prescribed. Here, f, g and C are as above.

Algorithm

1. Select γ_- and γ_+ such that $f - \gamma_- g$ is strictly copositive (so $\gamma_- < \mu_0$) and $f - \gamma_+ g$ is not strictly copositive on C (so $\mu_0 < \gamma_+$).

2. Set $\gamma \doteq (\gamma_- + \gamma_+)/2$.

3. If $f - \gamma g$ is strictly copositive, set $\gamma_- \doteq \gamma$. Otherwise, $\gamma_+ \doteq \gamma$.

4. If $f - \gamma_+ g$ is copositive or if $\gamma_+ - \gamma_- < \varepsilon$, then $\mu_0 = \gamma_+$ and stop. Otherwise, go to step **2**.

This algorithm, when f and g are quadratic forms and $C = \mathbb{R}^n_+$, was discussed in more detail in [50].

We are now ready to establish the S-lemma suitable for the problem (1). Such a lemma, which will be formulated in its strict version, asks for the equivalence between (15) and (16):

$$x \in C, \ g(x) - 1 = 0 \Longrightarrow f(x) > 0; \tag{15}$$

$$\exists \ \lambda \neq 0, \ f(x) + \lambda(g(x) - 1) > 0 \ \forall \ x \in C. \tag{16}$$

Results of this kind goes back to the works by Yakubovich in [59, 60] (or even earlier by Finsler in [19]), who considered f, g to be quadratic forms. To be more precise, his version reads as follows: the next two statements are equivalent provided there exists $\bar{x}$ satisfying $\bar{x}^\top B \bar{x} < 0$,

$$x \in \mathbb{R}^n, x^\top B x \leq 0 \Longrightarrow x^\top A x \geq 0.$$

$$\exists \ \lambda \in \mathbb{R} : A + \lambda B \text{ is positive semidefinite.}$$

Its inhomogeneous version was studied in [58, 26]. A further development when C is a convex cone is presented in [20]. A nice survey (until 2007) on the S-lemma in the quadratic world is [49]. More recent results may be found in [58, 27].

Lemma 3.7. (S-lemma) *Let C be a closed cone, g be continuous satisfying* (ii) *and* (iii) *of Assumption* (A). *The following assertions are equivalent:*

(a) *for all lsc function f that is positively homogeneous with degree $p > 0$ on C, one has*

$$[x \in C,\ g(x) = 1 \Longrightarrow f(x) > 0] \Longleftrightarrow \exists\ \lambda \neq 0,\ f(x) + \lambda(g(x) - 1) > 0\ \forall\, x \in C.$$

(b) $p \geq q$.

Proof. (a)⇒(b): Take $f(x) = \|x\|_p^p := |x_1|^p + \cdots + |x_n|^p$, then it holds (15) because of the assumptions on g. Thus (16) is satisfied if (a) holds. In particular, if $x = 0$ in (16) then $\lambda < 0$. Taking this fact into account, and again by taking $x = tx_0$ for $t > 0$ and any fixed $x_0 \in C \setminus \{0\}$ in (16), one infers, after dividing by t^p and letting $t \to +\infty$, that $p \geq q$.

(b)⇒(a): From (15) it follows that $\mu_0 := \min\{f(x) : g(x) = 1,\ x \in C\} > 0$ since K is compact. As $p \geq q$, strong duality holds for these data by Theorem 3.4. This means that $f(x) - \mu_0 \frac{p}{q}(g(x) - 1) \geq \mu_0 > 0$ for all $x \in C$, proving the desired result. □

4. Characterizing optimality conditions

We are now interested in obtaining necessary and/or sufficient optimality conditions of order zero, one or two, for local or global optimality for problem (1). All the results established in this section are new. In particular, in case f is a quadratic form and g the square of the Euclidean norm, Corollary 4.8 below enhances Proposition 3 of [54] since the convexity on C is not required. We point out that for quadratic optimization problems on a polyhedron, some optimality conditions were established in [10].

We refer to [6] for a method locating some particular local minima; some copositivity-based escape procedures for the StQO problem on the simplex are analyzed in [7].

4.1. Zero-order optimality conditions

We now provide a relationship between the minima of the original objective function and those of the Lagrangian, under strong duality and free of derivative.

Recall that $L(\lambda, x) := f(x) + \lambda(g(x) - 1)$.

Theorem 4.1. *Let C be a closed cone, and f, g satisfy Assumption* (A). *Then,*

(a) *Strong duality holds and* $\bar{x} \in \operatorname*{argmin}_K f$

$$\Longleftrightarrow\ \bar{x} \in K \ \ \textit{and}\ \ \bar{x} \in \operatorname*{argmin}_C L\left(-\frac{p}{q} f(\bar{x}), \cdot\right).$$

(b) *Strong duality holds and* $0 \in \operatorname*{argmin}_C L\left(-\frac{p}{q}\mu_0, \cdot\right) \Longleftrightarrow \mu_0(p - q) = 0.$

Proof. (a), $\Rightarrow$: This follows from (14):

$$L(-\frac{p}{q}\,\mu_0,\bar{x}) = f(\bar{x}) = \inf_{x\in C}\ L(-\frac{p}{q}\,\mu_0,x). \tag{17}$$

$\Leftarrow$: We have

$$\begin{aligned} \mu_0 \le f(\bar{x}) &= L(-\frac{p}{q}f(\bar{x}),\bar{x}) = \inf_{x\in C} L(-\frac{p}{q}f(\bar{x}),x) \\ &\le \inf_{x\in K} L(-\frac{p}{q}f(\bar{x}),x) = \inf_{x\in K} f(x) = \mu_0. \end{aligned}$$

Thus the proof of (a) is complete once one notices that

$$L(-\frac{p}{q}\mu_0,\bar{x}) = \sup_{\lambda\in\mathbb{R}}\ \inf_{x\in C}\ L(\lambda,x).$$

(b): It is a consequence of (14):

$$\mu_0 = f(0) - \mu_0\,\frac{p}{q}\,(g(0)-1) = \mu_0\,\frac{p}{q}.$$

□

When the assumption on strong duality is more precise, the following result is obtained.

Corollary 4.2. *Let C be a closed cone, and f, g satisfy Assumption* (A)*.*

(a) *Assume that $\mu_0(p-q) > 0$. Then*

$$\bar{x} \in \operatorname*{argmin}_{C}\ L(-\frac{p}{q}\mu_0,\cdot) \Longleftrightarrow \bar{x} \in \operatorname*{argmin}_{K}\ f.$$

(b) *Assume that $p = q > 0$. Then,*

$$0 \ne \bar{x} \in \operatorname*{argmin}_{C}\ L(-\mu_0,\cdot) \implies \frac{1}{(g(\bar{x}))^{1/p}}\bar{x} \in \operatorname*{argmin}_{K}\ f.$$

Proof. (a): By assumption on p, q, μ_0, strong duality holds. Thus, we need only to prove that $\bar{x} \in K$, and then the result follows from the previous theorem. We consider the case $\mu_0 > 0$ and $p > q > 0$ since the other is entirely similar. Suppose that $\bar{x} \notin K$; then $\bar{x} \ne 0$ due to the previous theorem, and so we can write set $\bar{x} = t\bar{y}$ for some $t > 0$ and $\bar{y} \in K$. By (17), we get

$$\begin{aligned} \mu_0 &= \min_{x\in C} L(-\frac{p}{q}\mu_0,x) = f(\bar{x}) - \frac{p}{q}\mu_0(g(\bar{x})-1)) = t^p f(\bar{y}) - \frac{p}{q}\mu_0(t^q g(\bar{y})-1) \\ &\ge t^p\mu_0 - \frac{p}{q}\mu_0(t^q-1). \end{aligned}$$

By defining $\varphi(\xi) \doteq \frac{\xi^q}{q} - \frac{\xi^p}{p}$, we obtain

$$\varphi(t) = \frac{t^q}{q} - \frac{t^p}{p} \ge \varphi(1) > 0. \tag{18}$$

On the other hand, we have $\varphi'(\xi) < 0$ for all $\xi > 1$; $\varphi'(\xi) > 0$ for all $\xi \in\ ]0,1[$. Simple arguments show that the only possible value for t satisfying (18) is $t = 1$, which finally yields $\bar{x} \in K$.

(b): Simply use (14). □

4.2. KKT-points, L(agrange)-eigenvalues and second-order optimality conditions

We now derive first- and second-order sufficient and/or necessary conditions for local or global optimality.

As usual, the notion of contingent cone will be needed. Given a set $M \subseteq \mathbb{R}^n$ and $x \in M$, the contingent cone of M at x, denoted by $T(M;x)$, is the set of vectors $v \in \mathbb{R}^n$ such that there exist $t_k > 0$, $x_k \in M$, $x_k \to x$, satisfying $t_k(x_k - x) \to v$. For a great account of its properties, we refer the book [3]. In general, we obtain

$$T(M;x) \subseteq \overline{\text{cone}}(M - x), \;\; x \in M.$$

The equality is satisfied whenever M is convex.

It is known that the notion of KKT-point plays a crucial role for optimality. Thus, we assume that the functions f and g are differentiable in a neighborhood of a reference point in C.

Following [14] for instance, a point $x \in C$, $x \neq 0$, is said to be a KKT-point for problem (1) if there exists (Lagrangian multiplier) $\lambda \in \mathbb{R}$ such that

$$\nabla f(x) - \lambda \frac{p}{q} \nabla g(x) \in (T(C;x))^*. \tag{19}$$

Let us denote the set of such λ associated to x by $\mathcal{L}(x)$.

Actually $\mathcal{L}(x) = \{\frac{f(x)}{g(x)}\}$ whenever x is a KKT-point because of $\pm x \in T(C;x)$ and Euler's identity.

Motivated by the previous remark, some notions are introduced:

Definition 4.3. We say that $\lambda \in \mathbb{R}$ is an (f,g)-*eigenvalue* (or simply, eigenvalue) if there exists $x \in C$, $x \neq 0$, such that $f(x) = \lambda g(x)$. The dependence of f and g will be omitted when no confusion arises. We say that $\lambda \in \mathbb{R}$ is an *L(agrange)-eigenvalue* if there exists $x \in C$, $x \neq 0$ such that $\lambda \in \mathcal{L}(x)$. The set of those x is denoted by $\mathcal{K}(\lambda)$, and so, λ is an L-eigenvalue if and only if $\mathcal{K}(\lambda) \neq \emptyset$. Every x in $\mathcal{K}(\lambda)$ is called L-eigenvector associated to λ. A pair $(\lambda, x) \in \mathbb{R} \times (C \setminus \{0\})$ with $x \in \mathcal{K}(\lambda)$ (or, equivalently, $\lambda \in \mathcal{L}(x)$) is called *L-eigenpair*.

Remark 4.4. Let us consider $x \in C$, $x \neq 0$. If, in addition, C is convex, then

$$\nabla f(x) - \lambda \frac{p}{q} \nabla g(x) \in (T(C;x))^* \Longleftrightarrow \begin{cases} \nabla f(x) - \lambda \frac{p}{q} \nabla g(x) \in C^*; \\ f(x) = \lambda g(x). \end{cases}$$

Thus, in this case, the L-eigenvalue problem reduces to the homogeneous complementarity problem:

$$\nabla f(x) - \lambda \frac{p}{q} \nabla g(x) \in C^*, \;\; (\nabla f(x) - \lambda \frac{p}{q} \nabla g(x))^\top x = 0, \;\; x \in C, \;\; x \neq 0.$$

From which we infer:

$$P_C(x - \nabla f(x) + \lambda \frac{p}{q} \nabla g(x)) = x; \;\; P_{C^0}(x - \nabla f(x) + \lambda \frac{p}{q} \nabla g(x)) = -\nabla f(x) + \lambda \frac{p}{q} \nabla g(x),$$

where $C^0 = -C^*$, $P_M(v)$ stands for the orthogonal projection of v onto M. This is certainly the basis for some proximal-point algorithms.

The next remark lists some basic facts from standard convex analysis.

Remark 4.5. Let $\emptyset \neq C \subseteq \mathbb{R}^n$ be a cone. We obtain:

(i) $C - C \subseteq \text{span } C = \text{aff } C$; $(C + \mathbb{R}x)^* = C^* \cap x^\perp$ for all $x \in \mathbb{R}^n$;

(ii) if C is convex and $x \in C$ then $\text{cone}_+(C - x) = C + \mathbb{R}x$, and therefore $[\overline{\text{cone}}(C - x)]^* = C^* \cap x^\perp$;

(iii) if $x \in \text{ri } C$ then $\text{cone}_+(C - \bar{x})$ is a subspace and $\text{aff } C \subseteq \text{cone}_+(C - x) \subseteq C - C$ and so $(\text{span } C)^\perp = [\text{cone}(C - x)]^\perp = (C - C)^\perp = C^\perp = (C - x)^\perp$.

We start with the following new second-order necessary optimality condition for local optimality. This result asserts that every local optimal solution to problem (1) is a KKT-point provided either C is convex or $\bar{x} \in \text{ri } C$, and so, $f(\bar{x})$ is L-eigenvalue. Theorem 1 in [54] is a special case of our result, when, besides the convexity of C, f is a quadratic form and g is the square of the Euclidean norm. We refer to [24] for first- and second-order optimality conditions for quadratically constrained quadratic programming problems.

Theorem 4.6. *Let C be a closed cone, and f, g satisfy Assumption* (A) *with both functions being twice differentiable in a neighborhood of $\bar{x}$, where $\bar{x}$ is any local solution to problem* (1). *If either C is convex or $\bar{x} \in \text{ri } C$, then the following assertions hold:*

(a) $\nabla f(\bar{x}) - \frac{p}{q} f(\bar{x}) \nabla g(\bar{x}) \in [\overline{\text{cone}}(C - \bar{x})]^*$. *As a consequence, $(f(\bar{x}), \bar{x})$ is an L-eigenpair. In other words, $\bar{x}$ is a KKT-point having $f(\bar{x})$ as Lagrange multiplier.*

(b) $\nabla^2 f(\bar{x}) - \frac{p}{q} f(\bar{x}) \nabla^2 g(\bar{x}) - \frac{p}{q}\Big(\frac{p}{q} - 1\Big) f(\bar{x}) \nabla g(\bar{x}) \nabla g(\bar{x})^\top$ *is copositive on $\mathcal{D}(\bar{x})$, where* $\mathcal{D}(\bar{x}) \doteq \overline{[\nabla f(\bar{x}) - \frac{p}{q} f(\bar{x}) \nabla g(\bar{x})]^\perp \cap (\text{cone}(C - \bar{x}))}$. *In case $\bar{x} \in \text{ri } C$, the previous expression reduces to $\mathcal{D}(\bar{x}) = \overline{\text{cone}}(C - \bar{x})$.*

Proof. Let $\bar{x}$ be a local solution to problem (1), that is, $f(\bar{x}) \leq f(x)$ for all $x \in C \cap U_0$ satisfying $g(x) = 1$, for some open neighborhood, U_0, of $\bar{x}$.

(a): It is known that in case C is convex, given any $v \in C - \bar{x}$, we can choose $\varepsilon \in\ ]0, 1[$ such that

$$\frac{\bar{x} + tv}{(g(\bar{x} + tv))^{1/q}} \in C \cap U_0, \quad \forall\, t \in\ ]0, \varepsilon[.$$

Set
$$\phi(t) := f\Big(\frac{\bar{x} + tv}{(g(\bar{x} + tv))^{1/q}}\Big) = \frac{1}{(g(\bar{x} + tv))^{p/q}} f(\bar{x} + tv).$$

By assumption, $\phi(0) \leq \phi(t)$ for all $t \in\ ]0, \varepsilon[$. This implies that the right-derivative of ϕ at 0 is nonnegative:

$$0 \leq \phi'(0) = \Big(\nabla f(\bar{x}) - \frac{p}{q} f(\bar{x}) \nabla g(\bar{x})\Big)^\top v.$$

It is valid for every $v \in C - \bar{x}$, and so the conclusion follows provided C is convex.

We now consider the case when $\bar{x} \in \text{ri}\ C$, meaning that $U_0 \cap \text{aff}\ C = U_0 \cap (C - C) \subseteq C$ for some open neighborhood U_0 of $\bar{x}$. From this, we derive the same result as in the convex case for the function (simply put $-v$ instead of v)

$$\phi(t) := f\Big(\frac{\bar{x} - tv}{(g(\bar{x} - tv))^{1/q}}\Big)$$

since $\text{cone}(C - \bar{x})$ is a vector subspace.

(b): Let $v \in [\nabla f(\bar{x}) - \frac{p}{q} f(\bar{x}) \nabla g(\bar{x})]^{\perp} \cap (C - \bar{x})$. We use the Maclaurin expansion for the function ϕ:

$$\phi(t) = \phi(0) + t\phi'(0) + \frac{1}{2}\phi''(0)t^2 + t^2 o(t),$$

where $o(t) \to 0$ as $t \to 0^+$. Then,

$$0 \leq \phi''(0) = v^{\top}\Big(\nabla^2 f(\bar{x}) - \frac{p}{q} f(\bar{x}) \nabla^2 g(\bar{x}) - \frac{p}{q}\Big(\frac{p}{q} - 1\Big) f(\bar{x}) \nabla g(\bar{x}) \nabla g(\bar{x})^{\top}\Big) v.$$

From which, the desired result is obtained.

Let us see the last statement of (b): if $\bar{x} \in \text{ri}\ C$ then, by (a),

$$\nabla f(\bar{x}) - \frac{p}{q} f(\bar{x}) \nabla g(\bar{x}) \in [\text{cone}(C - \bar{x})]^{\perp},$$

implying the simplified expression for $\mathcal{D}(\bar{x})$. □

A remark must be emphasized.

Remark 4.7. As was pointed out in Theorem 4.6, every local solution, $\bar{x}$, is not necessarily a KKT-point, even being global, except in the cases when either C is convex or $\bar{x} \in \text{ri}\ C$. Every solution will be a KKT-point under strong duality, as (b) of Theorem 4.15 below shows. When no extra-conditions are imposed on the data, a geometric characterization of KKT-points is established in (a) of Theorem 4.15.

An important consequence of Theorem 4.6 concerns the particular case when f and g are quadratic forms. The next corollary does not require convexity on C as Proposition 3 in [54] does.

Corollary 4.8. *Let C be a closed cone, and $f(x) = x^{\top}Ax$, $g(x) = x^{\top}x$ with $A = A^{\top}$. Assume that $\bar{x} \in \text{ri}\ C$ is a local solution to problem* (1). *Then $\bar{x}$ is a global solution.*

Proof. For every $x \in C$, we have

$$\begin{aligned}
&L(-f(\bar{x}), x) - L(-f(\bar{x}), \bar{x}) \\
&= \nabla_x L(-f(\bar{x}), \bar{x})^{\top}(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^{\top} \nabla_x^2 L(-f(\bar{x}), \bar{x})(x - \bar{x}) \\
&= \frac{1}{2}(x - \bar{x})^{\top} \nabla_x^2 L(-f(\bar{x}), \bar{x})(x - \bar{x}).
\end{aligned}$$

We now apply Theorem 4.6 together with the above expansion to conclude that $\bar{x} \in \underset{C}{\text{argmin}}\ L(-f(\bar{x}), \cdot)$. The result is obtained in consequence of Theorem 4.1(a). □

Results of the following kind are standard, but you can see how the structure of our problem takes place.

Recall that $L(\lambda, x) \doteq f(x) + \lambda(g(x) - 1)$ and $\nabla_x L(\lambda, x) \doteq \nabla f(x) + \lambda \nabla g(x)$.

Theorem 4.9. *Let C be a closed cone, and f, g satisfy Assumption* (A) *with both functions being twice differentiable in a neighborhood of $\bar{x} \in K$. If*

$$\nabla_x L(-\frac{p}{q} f(\bar{x}), \bar{x}) \in [\overline{\text{cone}}(C - \bar{x})]^*,$$

and the matrix $\nabla_x^2 L(-\frac{p}{q} f(\bar{x}), \bar{x})$ is strictly copositive on $\overline{\text{cone}}(C - \bar{x})$, then $\bar{x}$ is a strict local minimum of $L(-\frac{p}{q} f(\bar{x}), \cdot)$ on C.

For the sake of completeness, we provide a rather classical result (for the notions of pseudoconvexity or quasiconvexity, the reader may consult [4] for instance).

Theorem 4.10. *Let C be a convex closed cone, and f, g satisfy Assumption* (A) *with both functions being twice differentiable in a neighborhood of $\bar{x} \in K$. Assume that*

$$\nabla_x L(-\frac{p}{q} f(\bar{x}), \bar{x}) \in [\overline{\text{cone}}(C - \bar{x})]^*.$$

(a) *If $L(-\frac{p}{q} f(\bar{x}), \cdot)$ is pseudoconvex then $\bar{x}$ is a solution to problem* (1);

(b) *If f is pseudoconvex and the function $x \mapsto -\frac{p}{q} f(\bar{x}) g(x)$ is quasiconvex, then $\bar{x}$ is a solution to problem* (1).

In what follows we consider the case $C = \mathbb{R}^n_+$.

Set $I := \{1, \ldots, n\}$. Given $J \subseteq I$, any $x \in \mathbb{R}^n$ is written as $x = (x_J, x_{-J})$ where $x_J := (x_i)_{i \in J}$, that is, x_J is the vector with components whose indexes belong to J; and x_{-J} is the vector with the remaining components. Thus $0_{-J} \in \mathbb{R}^{n-|J|}$ is the vector with all its components being zero and indexes in $I \setminus J$, where $|J|$ means the cardinality of J.

The next result is expected.

Proposition 4.11. *Let f, g be functions that are differentiable in a neighborhood of the reference point satisfying Assumption* (A) *with $C = \mathbb{R}^n_+$, and $\lambda \in \mathbb{R}$. Then*

$$\lambda \textit{ is } L - \textit{eigenvalue} \iff \begin{cases} \exists\, \emptyset \neq J \subseteq I, \; \exists\, \bar{y} \in \mathbb{R}^{|J|}_{++} : \\ \nabla_J f(\bar{y}, 0_{-J}) - \lambda \frac{p}{q} \nabla_J g(\bar{y}, 0_{-J}) = 0; \\ \nabla_{-J} f(\bar{y}, 0_{-J}) - \lambda \frac{p}{q} \nabla_{-J} g(\bar{y}, 0_{-J}) \in \mathbb{R}^{n-|J|}_+. \end{cases}$$

Here, $\nabla_J f(x)$ (resp. $\nabla_{-J} f(x)$) stands for the vector whose components are the partial derivatives of f at $x \in \mathbb{R}^n$ with respect to the indexes belonging to J (resp. $I \setminus J$).

Proof. $\Rightarrow$: Let $\bar{x} \in \mathcal{K}(\lambda)$. Then $\bar{x} \in \mathbb{R}^n_+ \setminus \{0\}$, and set $J \doteq \{i \in I : \bar{x}_i > 0\}$. Thus $\bar{x} = (\bar{x}_J, 0_{-J})$ and therefore

$$\nabla f(\bar{x}) - \lambda \frac{p}{q} \nabla g(\bar{x}) \in [T(\mathbb{R}^n_+; \bar{x})]^* = \{0_J\} \times \mathbb{R}^{n-|J|}_+,$$

since
$$T(\mathbb{R}^n_+; \bar{x}) = T(\mathbb{R}^{|J|}_+; \bar{x}_J) \times T(\mathbb{R}^{n-|J|}_+; 0_{-J}).$$

By taking $\bar{y} = \bar{x}_J \in \mathbb{R}^{|J|}_{++}$, the desired implication is proved.

$\Leftarrow$: By setting $\bar{x} = (\bar{y}_J, 0_{-J})$, we obtain

$$\nabla f(\bar{x}) - \lambda \frac{p}{q} \nabla g(\bar{x}) \in \{0_J\} \times \mathbb{R}^{n-|J|}_+ = [T(\mathbb{R}^n_+; \bar{x})]^*.$$

This means that λ is L-eigenvalue. □

Since ∇h is positively homogeneous with degree $p-1$, the next assertions are easy to check.

- $x \in \mathcal{K}(\lambda) \Longleftrightarrow \lambda \in \mathcal{L}(x)$;
- $x \in K$, $\mu_0 \in \mathcal{L}(x) \Longrightarrow x \in \operatorname*{argmin}_K f \Longrightarrow \mathcal{L}(x) \subseteq \{\mu_0\}$;
- given any $x \in C$, $x \neq 0$, it holds ($p \neq q$):

$$T(C; x) = T(C; tx) \ \ \forall\, t > 0; \ \lambda \in \mathcal{L}(x) \Longleftrightarrow \lambda t^{p-q} \in \mathcal{L}(tx) \ \ \forall\, t > 0. \tag{20}$$

The next result establishes a relationship between both types of eigenvalues under strong duality.

Proposition 4.12. *Let C be a closed cone, and f, g be any functions that are differentiable in a neighborhood of the reference point satisfying Assumption* (A). *Then,*

(a) $\mu_0 = \min\ \{\lambda \in \mathbb{R} :\ \lambda$ *is eigenvalue*$\}$;

(b) $\mu_0 = \min\ \{\lambda \in \mathbb{R} :\ \lambda$ *is $L-$eigenvalue*$\}$, *provided strong duality holds* (*see Theorem* 3.4).

Proof. (a) is straightforward.

(b): By (a), we only need to check the inequality "$\geq$". Take any $\bar{x} \in \operatorname*{argmin}_K f$. The usual necessary optimality condition along with Theorem 4.1 allow us to infer that $\nabla f(\bar{x}) - \mu_0 \frac{p}{q} \nabla g(\bar{x}) \in [T(C; \bar{x})]^*$, which shows that μ_0 is L-eigenvalue, and so the proof is complete. □

The next remark refers to eigenvalues analysis of symmetric (supersymmetric) tensors.

Remark 4.13. One important implication from Proposition 4.12 concerns the existence of L-eigenvalues of a real m-order n-dimensional symmetric tensor. See Section 5 for details.

Notice that, in principle, a KKT-point is defined also for infeasible points. In order to characterize the validity of the KKT optimality conditions, (19), for problem (1), we need to consider the linearized approximation problem defined, given $\bar{x} \in C$, by

$$\mu_L := \inf_{v \in G_0(\bar{x})} \nabla f(\bar{x})^\top v, \tag{21}$$

where
$$G_0(\bar{x}) := \Big\{ v \in T(C;\bar{x}) : \ \nabla g(\bar{x})^\top v = 0 \Big\}. \tag{22}$$

In our model, we have $\nabla g(x) \neq 0$ for all $x \in C$, $x \neq 0$ because of the Euler identity. Set $F_L(v) := (\nabla g(\bar{x})^\top v, \nabla f(\bar{x})^\top v)$. It is obvious that $\mu_L \in \{-\infty, 0\}$, and

$$\mu_L = 0 \Longleftrightarrow [v \in T(C;\bar{x}),\ \nabla f(\bar{x})^\top v < 0 \Longrightarrow \nabla g(\bar{x})^\top v \neq 0] \tag{23}$$

$$\Longleftrightarrow F_L(T(C;\bar{x})) \cap -(\{0\} \times \mathbb{R}_{++}) = \emptyset$$

$$\Longleftrightarrow [F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1)] \cap (-\{0\} \times \mathbb{R}_{++}) = \emptyset. \tag{24}$$

Some important facts on the set $F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1)$ are collected in the next remark. Such facts will be considered in Theorem 4.15.

Remark 4.14. With the above data and notation, we obtain the following

(i) $\lambda \in \mathcal{L}(\bar{x}) \Longleftrightarrow (-\lambda\frac{p}{q}, 1) \in [F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1)]^* \Longleftrightarrow (\lambda, \bar{x})$ is L-eigenpair.

Indeed, by (19),

$$\lambda \in \mathcal{L}(\bar{x}) \Longleftrightarrow \nabla f(\bar{x})^\top v - \lambda\frac{p}{q}\nabla g(\bar{x})^\top v \geq 0 \ \forall\ v \in T(C;\bar{x})$$

$$\Longleftrightarrow (-\lambda\frac{p}{q}, 1) \in [F_L(T(C;\bar{x}))]^* \Longleftrightarrow (-\lambda\frac{p}{q}, 1) \in [F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1)]^*.$$

The remaining equivalence is straightforward.

(ii) Assume that $F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1)$ is convex. Then, either

1. $F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1) = \mathbb{R}^2$, or
2. $F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1) = \Big\{(v,w) :\ w \geq \frac{p}{q}\frac{f(\bar{x})}{g(\bar{x})} v\Big\}.$

   Indeed, since $\pm\bar{x} \in T(C;\bar{x})$, we get

$$F_L(\pm\bar{x}) = \pm(\nabla g(\bar{x})^\top \bar{x}, \nabla f(\bar{x})^\top \bar{x}) = \pm(qg(\bar{x}), pf(\bar{x})),$$

   and so, $\mathbb{R}\Big(1, \frac{p}{q}\frac{f(\bar{x})}{g(\bar{x})}\Big) \subseteq F_L(T(C;\bar{x}))$.

   This implies $\mathbb{R}\Big(1, \frac{p}{q}\frac{f(\bar{x})}{g(\bar{x})}\Big) + \mathbb{R}_+(0,1) \subseteq F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1)$.

From which the conclusion follows by the convexity assumption.

We are now ready to describe, firstly, a new necessary and sufficient condition for a non-zero vector to be a KKT-point; this shows, looking at carefully its proof, that a Fritz John point is indeed a KKT-point, thanks to the structure of our model. Secondly, under strong duality, it establishes that every minimizer is a KKT-point. Thus, next theorem supplements the results stated in Theorem 4.6.

Theorem 4.15. *Let C be a closed cone, and f, g be functions that are differentiable in a neighborhood of $\bar{x}$, satisfying Assumption* (A). *Assume that $\bar{x} \in C$, $\bar{x} \neq 0$. The following assertions hold:*

(a) *$\bar{x}$ is a KKT-point $\Longleftrightarrow$ $F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1)$ is different from $\mathbb{R}^2$ and convex. In such a case we have* $F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1) = \Big\{(v,w) : w \geq \frac{p}{q}\frac{f(\bar{x})}{g(\bar{x})}v\Big\}$.

(b) *if strong duality holds, then, for $\bar{x} \in K$,*

$$\bar{x} \in \operatorname*{argmin}_K f \Longleftrightarrow \mu_0 \in \mathcal{L}(\bar{x})(= \{\mu_0\}) \Longleftrightarrow \bar{x} \in \mathcal{K}(\mu_0).$$

Proof. (a), $\Rightarrow$: $\bar{x}$ is a KKT-point if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\inf_{v \in T(C;\bar{x})} \Big\langle \nabla f(\bar{x}) - \lambda \frac{p}{q} \nabla g(\bar{x}), v \Big\rangle \geq 0 \geq \inf_{v \in G_0(\bar{x})} \nabla f(\bar{x})^\top v.$$

Thus, $\bar{x} \in$ is a KKT-point if and only if $\mu_L = 0$ and (strong duality for (21) holds)

$$\inf_{v \in T(C;\bar{x})} \Big\langle \nabla f(\bar{x}) - \lambda \frac{p}{q} \nabla g(\bar{x}), v \Big\rangle = \inf_{v \in G_0(\bar{x})} \nabla f(\bar{x})^\top v = \mu_L.$$

We have already noticed that $\mu_L = 0$ is equivalent to (23); which means that $F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1) \neq \mathbb{R}^2$. The convexity of $F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1)$ follows from Proposition 5.1 in [25], when applying to $A = F_L(T(C;\bar{x}))$ and $P = \mathbb{R}_+(0,1)$.

$\Leftarrow$: By Remark 4.14,

$$\mathbb{R}\Big(1, \frac{p}{q}\frac{f(\bar{x})}{g(\bar{x})}\Big) + \mathbb{R}_+(0,1) = F_L(T(C;\bar{x})) + \mathbb{R}_+(0,1),$$

which yields (24). By applying a separation result on convex sets to (24), we have the existence of $\alpha \in \mathbb{R}$, $\beta \geq 0$, $(\alpha,\beta) \neq (0,0)$ such that $\beta \nabla f(\bar{x}) + \alpha \nabla g(\bar{x}) \in [T(C;\bar{x})]^*$. By the Euler identity and $\pm x \in T(C;\bar{x})$, one gets $\beta p f(\bar{x}) + \alpha q g(\bar{x}) = 0$. Thus, if $\beta = 0$ then $\alpha = 0$ since $g(\bar{x}) \neq 0$, and so $\beta > 0$. Hence, the result is obtained.

(b): It follows from Theorem 4.1 and the standard first-order necessary optimality condition for $x \in \operatorname*{argmin}_C L\Big(-\mu_0 \frac{p}{q}, \cdot\Big)$. $\square$

The next example shows an application of (*a*) in Theorem 4.15 without having strong duality, and so (*b*) of the same theorem is not applicable.

Example 4.16. (Strong duality fails, and so (b) of Theorem 4.15 is not applicable, but (a) is) Let α and β be any positive real numbers. We will analyze the problem

$$\mu_0(\alpha,\beta) := \min\Big\{f(x) : \ g_{\alpha,\beta}(x) = 1, \ x \in \mathbb{R}^2\Big\},$$

with the data $f(x_1,x_2) = x_1 + 2x_2$ and $g_{\alpha,\beta}(x) = \sqrt[4]{\alpha x_1^2 + \beta x_2^2}$ and $C = \mathbb{R}^2$. In this case, $p = 1$, $q = 1/2$ and $T(C;x) = \mathbb{R}^2$ for all $x \in \mathbb{R}^2$. By virtue of (20), we search KKT-points only in the feasible set K.

Thus, from (19), we get $1-\lambda\alpha x_1 = 0 = 2-\lambda\beta x_2$, implying $\lambda \neq 0$, $x_1 \neq 0 \neq x_2$, and therefore $x_2 = \frac{2\alpha}{\beta}x_1$. For such points x and $(v_1, v_2) \in \mathbb{R}^2$, we obtain

$$F_L(v) = \Big(\frac{\alpha}{2}x_1v_2 + \frac{\beta}{2}x_2v_2, v_1+2v_2\Big) = \Big(\frac{\alpha}{2}x_1, 1\Big)\Big(v_1+2v_2\Big).$$

Then $F_L(T(C;x)) = \Big\{(w_1,w_2) : w_2 = \frac{2}{\alpha x_1}w_1, w_1 \in \mathbb{R}\Big\}$, implying the convexity of $F_L(T(C;x)) + \mathbb{R}_+(0,1)$ and (24). Hence, by Theorem 4.15, $(x_1,x_2) = x_1\big(1, \frac{2\alpha}{\beta}\big)$ is a KKT-point with $g_{\alpha,\beta}(x_1,x_2) = 1$. That is,

$$\Big\{x \in K :\ \mathcal{L}(x) \neq \emptyset\Big\} = \Big\{\pm\sqrt{\frac{\beta}{\alpha\beta+4\alpha^2}}\,\Big(1, \frac{2\alpha}{\beta}\Big)\Big\}.$$

It is easy to check that one is the minimizer (and the other the maximizer) with minimum value

$$\mu_0 = \mu_0(\alpha,\beta) = -\sqrt{\frac{\beta+4\alpha}{\alpha\beta}}.$$

We analyze the previous facts via Theorems 4.6 and 4.10. By applying the second-order optimality condition from Theorem 4.6, we discard those points which are not local minima. For this example, we get

$$\begin{aligned}&\nabla^2 f(\bar{x}) - \tfrac{p}{q}f(\bar{x})\nabla^2 g(\bar{x}) - \tfrac{p}{q}\Big(\tfrac{p}{q}-1\Big)f(\bar{x})\nabla g(\bar{x})\nabla g(\bar{x})^\top \\ &= -2f(\bar{x})\left(\nabla^2 g(\bar{x}) + \nabla g(\bar{x})\nabla g(\bar{x})^\top\right) = -2f(\bar{x})\begin{pmatrix} \dfrac{2\alpha^2}{\beta+4\alpha} & -\dfrac{\alpha\beta}{\beta+4\alpha} \\ -\dfrac{\alpha\beta}{\beta+4\alpha} & \dfrac{\beta^2}{2(\beta+4\alpha)}\end{pmatrix}.\end{aligned}$$

This is positive semidefinite if and only if $f(\bar{x}) \leq 0$, since the matrix

$$\begin{pmatrix} \dfrac{2\alpha^2}{\beta+4\alpha} & -\dfrac{\alpha\beta}{\beta+4\alpha} \\ -\dfrac{\alpha\beta}{\beta+4\alpha} & \dfrac{\beta^2}{2(\beta+4\alpha)}\end{pmatrix}$$

is positive semidefinite, having as eigenvalues 0 and $\frac{4\alpha^2+\beta^2}{2(\beta+4\alpha)}$. This second-order condition is satisfied only for the point

$$\bar{x} = -\sqrt{\frac{\beta}{\alpha\beta+4\alpha^2}}\,\Big(1, \frac{2\alpha}{\beta}\Big).$$

Finally, by using Theorem 4.10, we conclude that such a point is, in fact, a minimizer.

Let us establish a necessary and sufficient condition for $\bar{x}$ to be a KKT-point when it is a minimizer of f on K. By virtue of Lemma 3.2, we split our discussion into two cases: $\mu_0 < 0$ and $\mu_0 > 0$. The remaining case $\mu_0 = 0$, which implies strong duality, can be dealt with (b) of Theorem 4.15.

We need the following notation:

$$F_L^-(v) := (-\nabla g(\bar{x})^\top v, \nabla f(\bar{x})^\top v),$$

and the condition

$$[v_k \in T(C;\bar{x}), \nabla g(\bar{x})^\top v_k \to 0, \nabla f(\bar{x})^\top v_k < 0] \Longrightarrow \limsup_k \nabla f(\bar{x})^\top v_k = 0. \tag{25}$$

Notice that Example 4.16 also shows that condition (25) cannot be substituted by (23) in the following theorem.

Theorem 4.17. *Let C be a closed cone, and f, g be functions that are differentiable in a neighborhood of $\bar{x} \in K$, satisfying Assumption* (A)*.*

(a) *Assume that $\mu_0 < 0$ and $\bar{x} \in \operatorname*{argmin}_K f$. Then,*

$$\bar{x} \textit{ is a KKT} - \textit{point} \iff [F_L(T(C;\bar{x})) + \mathbb{R}^2_+ \textit{ is convex and } (25) \textit{ is satisfied}].$$

(b) *Assume that $\mu_0 > 0$ and $\bar{x} \in \operatorname*{argmin}_K f$. Then,*

$$\bar{x} \textit{ is a KKT} - \textit{point} \iff [F_L^-(T(C;\bar{x})) + \mathbb{R}^2_+ \textit{ is convex and } (25) \textit{ is satisfied}].$$

Proof. See Corollary 5.5 in [23] and Lemma 3.2. □

A more verifiable condition than (25) is given next:

Proposition 4.18. *With the data as above, we have*

$$(26) \Longrightarrow (25) \Longrightarrow \mu_L = 0,$$

where $$[v_k \in T(C;\bar{x}), v_k \to v \neq 0, \nabla f(\bar{x})^\top v_k < 0] \Longrightarrow \nabla g(\bar{x})^\top v \neq 0. \tag{26}$$

Proof. Firstly, we easily obtain that (25) implies $\mu_L = 0$ (simply take the constant sequence $v_k = v$ to get $\mu_L = 0$).

(26) ⇒ (25): Let $v_k \in T(C;\bar{x})$ such that

$$\|v_k\| \to +\infty,\ \nabla g(\bar{x})^\top v_k \to 0,\ \nabla f(\bar{x})^\top v_k < 0.$$

In case $v_k \to 0$, we get (25) obviously holds. Now, two possibilities arise:

$$\sup_{k\in\mathbb{N}} \|v_k\| < +\infty \text{ with } v_k \not\to 0 \text{ and } \sup_{k\in\mathbb{N}} \|v_k\| = +\infty.$$

Under the first possibility, up to a subsequence, we get $v_k \to v \neq 0$. In such a case, since $\nabla f(\bar{x})^\top v_k < 0$, by (26) we obtain $\nabla g(\bar{x})^\top v > 0$, yielding a contradiction.

If the second possibility holds, up to a subsequence, we may suppose that we have $\frac{v_k}{\|v_k\|} \to v_0 \neq 0$. Since $\nabla f(\bar{x})^\top \frac{v_k}{\|v_k\|} < 0$ by (26) we get $\nabla g(\bar{x})^\top v_0 \neq 0$. Moreover, since $\nabla g(\bar{x})^\top v_k \to 0$, we have $\nabla g(\bar{x})^\top \frac{v_k}{\|v_k\|} \to 0 = \nabla g(\bar{x})^\top v_0$, which contradicts the fact that $\nabla g(\bar{x})^\top v_0 \neq 0$. This proves that under (26), the conditions in the left-hand side of (25) are fulfilled only if $v_k \to 0$, so that (25) holds. □

5. L-eigenvalues as an extension of H and Z-eigenvalues in real symmetric tensors problems

Let $m, n \in \mathbb{N}$ with $m \geq 2$, $n \geq 2$. A real m-order n-dimensional tensor $\mathcal{A}$ consists of n^m entries in $\mathbb{R}$, and it is denoted by

$$\mathcal{A} = (\mathcal{A}_{i_1 i_2 \ldots i_m}),\ i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, n\}.$$

We say a tensor $\mathcal{A}$ is symmetric (the term supersymmetric is also used by some authors) if its entries $\mathcal{A}_{i_1 i_2 \ldots i_m}$ are invariant under any permutation of the indices $(i_1 i_2 \ldots i_m)$. Furthermore, given any $x \in \mathbb{R}^n$, it is defined

$$\mathcal{A}x^m := \sum_{i_1, \cdots, i_m = 1}^{n} \mathcal{A}_{i_1 i_2 \cdots i_m} x_{i_1} \ldots x_{i_m},$$

which is an mth-degree homogeneous polynomial whenever $\mathcal{A}$ is symmetric. Throughout this section the tensor $\mathcal{A}$ will be symmetric.

Given $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$: $\|x\|_k := (|x_1|^k + \cdots + |x_n|^k)^{1/k}$; $\mathcal{A}x^{m-1}$ is the vector in $\mathbb{R}^n$ whose i-th component is,

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2, \cdots, i_m = 1}^{n} \mathcal{A}_{i\ i_2 \cdots i_m} x_{i_2} \ldots x_{i_m} \quad \text{for } i = 1, \ldots, n,$$

and set $x^{[k]} := (x_1^k, \ldots, x_n^k)$ and $x^{[0]} \doteq (1, \ldots, 1)$.

We now consider the following constrained optimization problem:

$$\mu_k := \min \Big\{ \mathcal{A}x^m : \ \|x\|_k^m = 1, \quad x \in C \Big\}, \tag{27}$$

where C is a closed cone in $\mathbb{R}^n$. Clearly, under symmetry on $\mathcal{A}$, problem (27) is a particular model of (1), where

$$f(x) = \mathcal{A}x^m, \quad g(x) = \|x\|_k^m. \tag{28}$$

Both functions f and g have the same degree of homogeneity, m, and satisfy Assumption (A). Thus (Theorem 3.4)

$$\mu_k = \nu_k := \sup_{\lambda \in \mathbb{R}} \inf_{x \in C} \Big\{ \mathcal{A}x^m + \lambda(\|x\|_k^m - 1) \Big\} = \inf_{x \in C} \Big\{ \mathcal{A}x^m - \mu_k(\|x\|_k^m - 1) \Big\}.$$

By Theorem 4.1, $\bar{x}$ is a solution to (27) if and only if

$$\|\bar{x}\|_k^m = 1 \ \text{ and } \ \bar{x} \in \operatorname*{argmin}_C L(-f(\bar{x}), \cdot).$$

Hence, in order to obtain first- and second-order necessary optimality conditions, we can use Theorem 4.6; whereas Theorem 4.9 provides sufficient optimality conditions for strict local optimality. Here, we need the following computation, given any $x \in C$, $x \neq 0$,

$$\nabla f(x) = m\mathcal{A}x^{m-1},\ \nabla^2 f(x) = m(m-1)\mathcal{A}x^{m-2}.$$

The symbol $\mathcal{A}x^{m-2}$ denotes the $(m-2)$-times product of the tensor $\mathcal{A}$ with the vector x, which is defined as the matrix of $\mathbb{R}^{n\times n}$ whose entries are

$$(\mathcal{A}x^{m-2})_{i_1 i_2} = \sum_{i_3,\dots,i_m}^{n} A_{i_1 i_2 i_3 \dots i_m} x_{i_3} \cdots x_{i_m}.$$

Moreover, $\quad \nabla g(x) = m\|x\|_k^{m-k}\varphi_k(x), \quad \nabla^2 g(x) = \Big(\nabla^2 g(x)_{ij}\Big),$

where $\varphi_k(x) := (x_1|x_1|^{k-2}, x_2|x_2|^{k-2}, \dots, x_n|x_n|^{k-2})$ and

$$\nabla^2 g(x)_{ij} = \begin{cases} m(m-k)\|x\|_k^{m-2k}(\varphi_k(x)_i)^2 + m\|x\|_k^{m-k}(k-1)|x_i|^{k-2}, & \text{if } i = j; \\ m(m-k)\|x\|_k^{m-2k}\varphi_k(x)_i\varphi_k(x)_j, & \text{if } i \neq j. \end{cases}$$

Here, $\varphi_k(x)_i$ stands for the i-th component of the vector $\varphi_k(x)$.

Observe that in case $C \subseteq \mathbb{R}^n_+$, one gets $\nabla g(x) = m\|x\|_k^{m-k}x^{[k-1]}$ and

$$\nabla^2 g(x)_{ij} = \begin{cases} m(m-k)\|x\|_k^{m-2k}(x_i^{k-1})^2 + m\|x\|_k^{m-k}(k-1)x_i^{k-2}, & \text{if } i = j; \\ m(m-k)\|x\|_k^{m-2k}x_i^{k-1}x_j^{k-1}, & \text{if } i \neq j. \end{cases}$$

In matrix notation, by introducing the diagonal matrix, $X_{k-2}(x)$, whose entries are the components of vector $x^{[k-2]}$, that is, $X_{k-2}(x) := \operatorname{diag}(x^{[k-2]})$, we obtain

$$\nabla^2 g(x) = m(m-k)\|x\|_k^{m-2k}x^{[k-1]}(x^{[k-1]})^\top + m(k-1)\|x\|_k^{m-k}X_{k-2}(x).$$

Consequently,

$$\begin{aligned} \nabla^2 g(x) &= m(m-1)X_{m-2}(x), \quad \text{if } m = k; \\ \nabla^2 g(x) &= m(m-2)\|x\|_2^{m-4}xx^\top + m\|x\|_2^{m-2}I, \quad \text{if } k = 2. \end{aligned}$$

The case when C is convex

We now assume that C is additionally convex. By Definition 4.3 and Remark 4.4, any L-eigenvalue λ ensures the existence of $x \in \mathbb{R}^n$ such that (see Subsection 4.2)

$$\begin{cases} \mathcal{A}x^{m-1} - \lambda\|x\|_k^{m-k}\varphi_k(x) \in C^*, \\ \mathcal{A}x^m - \lambda\|x\|_k^m = 0, \\ x \in C \setminus \{0\}. \end{cases} \tag{29}$$

A pair (λ, x) satisfying (29) is termed *L-eigenpair* (see Definition 4.3 in Subsection 4.2). In other words, $x \in C$, $x \neq 0$, is a KKT-point of (27) if (λ, x) is a L-eigenpair for some $\lambda \in \mathbb{R}$. According to the choice of m, k and C, such a pair takes different names, see the remarks below.

The system (29) is a class of homogeneous complementarity problem, since

$$\langle x, \mathcal{A}x^{m-1} - \lambda\|x\|_k^{m-k}\varphi_k(x)\rangle = 0 \Longleftrightarrow \mathcal{A}x^m - \lambda\|x\|_k^m = 0.$$

Remark 5.1. (Z-eigenvalues/eigenvectors, $C = \mathbb{R}^n$)
The case $k = 2$ and $C = \mathbb{R}^n$ was discussed in [51], where the term Z-eigenvalues (eigenvectors) is employed. The authors in [40] use the name l^2-eigenvalues (eigenvectors). Here, (29) reduces to

$$\begin{cases} \mathcal{A}x^{m-1} - \lambda\|x\|_2^{m-2}x = 0, \\ \mathcal{A}x^m - \lambda\|x\|_2^m = 0, \\ x \neq 0, \end{cases}$$

since $\varphi_2(x) = x$ and $T(\mathbb{R}^n; x) = \mathbb{R}^n$.

Remark 5.2. (H-eigenvalues/eigenvectors, $C = \mathbb{R}^n$)
The case $k = m$ and $C = \mathbb{R}^n$ was also analyzed in [51], which gives rise to H-eigenvalues (eigenvectors). It coincides with the notion of l^m-eigenvalues (eigenvectors), introduced in [40]. The homogeneous complementarity problem (29) becomes

$$\begin{cases} \mathcal{A}x^{m-1} - \lambda\varphi_m(x) = 0, \\ \mathcal{A}x^m - \lambda\|x\|_m^m = 0, \\ x \neq 0. \end{cases}$$

Observe that when m is even, $\varphi_m(x) = x^{[m-1]}$, and if m is odd, one gets

$$\varphi_m(x)_i = \begin{cases} x_i^{m-1} & \text{if } x_i \geq 0; \\ -x_i^{m-1} & \text{if } x_i < 0. \end{cases}$$

Remark 5.3. The cases $C = \mathbb{R}^n_+$ and either $k = 2$ or $k = m$ were studied in [56]. In view of the additional nonnegative constraint, $\varphi_k(x) = x^{[k-1]}$ for all $k \in \mathbb{N}$. Here, problem (29) takes the form

$$\begin{cases} \mathcal{A}x^{m-1} - \lambda\|x\|_k^{m-k}x^{[k-1]} \geq 0, \\ \mathcal{A}x^m - \lambda\|x\|_k^m = 0, \\ x \geq 0, \;\; x \neq 0. \end{cases} \tag{30}$$

From the previous remarks and together with Proposition 4.12 and Theorem 3.4, the following result extends those appeared in [51, 40, 56], where the cases $C = \mathbb{R}^n_+$ or $C = \mathbb{R}^n$ are only considered. In what follows f and g are as in (28).

Proposition 5.4. *Let $\mathcal{A}$ be an m-order n-dimensional symmetric tensor and C be a convex closed cone in $\mathbb{R}^n$. Then*

(a) (*Set* $C = \mathbb{R}^n$) *For each case, the set of all Z-eigenvalues and that of all H-eigenvalues coincide with our notion of L-eigenvalues, as well as with that of simply eigenvalues* (*see Definition* 4.3).

(b) *One has, for some $\bar{x} \in C$, $\bar{x} \neq 0$,*

$$\begin{aligned} \mu_k &= \min\{\lambda \in \mathbb{R} : \lambda \textit{ is } L-\textit{eigenvalue}\} = \inf\{\mathcal{A}x^m - \mu_k(\|x\|_k^m - 1) : \; x \in C\} \\ &= \min\{\lambda \in \mathbb{R} : \lambda \textit{ is eigenvalue}\} = \min_{\substack{x \in C \\ x \neq 0}} \frac{\mathcal{A}x^m}{\|x\|_k^m} = \frac{\mathcal{A}\bar{x}^m}{\|\bar{x}\|_k^m}. \end{aligned}$$

(c) $\mu_k = \max\{\lambda : \; f - \lambda g \text{ is copositive on } C\}$.

Remark 5.5. Among those KKT-points for problem (27), one can identify a solution, $\bar{x}$, to (27) simply by using Theorem 4.10, once we impose conditions implying the pseudoconvexity of f and the quasiconvexiy of the function $x \mapsto -f(\bar{x})g(x)$. Notice that g is always quasiconvex, and so we must take care on the sign of $f(\bar{x})$.

The case $C := \mathbb{R}^n_+$ is of particular interest. In this situation, one can also consider the standard dual problem to (27):

$$\nu_S := \sup_{\substack{\lambda_1 \geq 0 \\ \lambda_2 \in \mathbb{R}^n_+}} \inf_{x \in \mathbb{R}^n} \left\{ \mathcal{A}x^m + \lambda_1(\|x\|_k^m - 1) - \lambda_2^\top x \right\}.$$

We recall that, in this setting, the notions of copositivity or strict copositivity of $\mathcal{A}$ on C are referred to f on C.

Theorem 5.6. *Set $C := \mathbb{R}^n_+$. Let $\mathcal{A}$ be a m-order, n-dimensional symmetric tensor, that is copositive on C but not copositive in $\mathbb{R}^n$. Then,*

$$\nu_S < 0 \leq \mu_k = \nu_k.$$

As a consequence, standard strong duality does not hold.

Proof. By assumption, $\mu_k \geq 0$ and $\mu_k = \nu_k$ (strong duality holds).

We choose $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) < 0$.

Set $\bar{\lambda}_1 \doteq -\dfrac{f(\bar{x})}{g(\bar{x})} > 0$; it follows that

$$f(\bar{x}) + \lambda g(\bar{x}) < 0 \text{ for all } \lambda < \bar{\lambda}_1 \text{ and } f(\bar{x}) + \lambda g(\bar{x}) > 0 \text{ for all } \lambda > \bar{\lambda}_1.$$

Since for every $t > 0$

$$\begin{aligned} \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \lambda_1 g(x) - \lambda_2^\top x) \right\} &\leq f(t\bar{x}) + \lambda_1 g(t\bar{x}) - t\lambda_2^\top \bar{x} \\ &= t^m \left[f(\bar{x}) + \lambda_1 g(\bar{x}) - \frac{1}{t^{m-1}} \lambda_2^\top \bar{x} \right], \end{aligned}$$

we obtain

$$\inf_{x \in \mathbb{R}^n} \left\{ f(x) + \lambda_1 g(x) - \lambda_2^\top x) \right\} = -\infty, \quad \forall\ \lambda_1 < \bar{\lambda}_1,\ \forall\ \lambda_2 \in \mathbb{R}^n_+.$$

Moreover,

$$\inf_{x \in \mathbb{R}^n} \left\{ f(x) + \lambda_1 (g(x) - 1) - \lambda_2^\top x) \right\} \leq -\lambda_1 \quad \forall\ \lambda_1 \geq \bar{\lambda}_1,\ \forall\ \lambda_2 \in \mathbb{R}^n_+.$$

Hence

$$\nu_S := \sup_{\substack{\lambda_1 \in \mathbb{R} \\ \lambda_2 \in \mathbb{R}^n_+}} \inf_{x \in \mathbb{R}^n} \left\{ f(x) + \lambda_1 (g(x) - 1) - \lambda_2^\top x \right\} \leq -\bar{\lambda}_1 < 0 \leq \mu_k = \nu_k,$$

and the proof is complete. □

We end this section by making some comments about numerical approaches on computing eigenvalues of symmetric tensors. When only the smallest or the largest eigenvalue of a tensor is needed, one can use: the NQZ method [15] or an iterative one as proposed in [45] (for irreducible nonnegative tensors); an unconstrained optimization approach [30] for even order tensors; a SOS polynomial optimization scheme as discussed in [32] (for essentially nonnegative tensors), or the S-HOPM or SS-HOPM methods as presented in [35] and [36], respectively. For computing all the real eigenvalues of a symmetric tensor, an approach based on the Jacobian SDP relaxation method is introduced in [17].

6. Conclusions

We have analyzed an important class of positively homogeneous optimization problems without convexity assumptions. This class subsummes various models: from classical portfolio problems to some of its variants; quadratic optimization with two quadratic constraints over the unit sphere. More particularly, a new perspective to tensors eigenvalues analysis is outlined, and it seems to be a promising method to be developed.

Acknowledgements. The authors want to express their gratitude to the Referees for the carefully reading, criticisms and for pointing out many innacuracies in the first version. All of them were taken into account to reach the present improved version.

References

[1] W. Ai, S. Zhang: *Strong duality for the CDT subproblem: a necessary and sufficient condition*, SIAM J. Optim. 19 (2009) 1735–1756.

[2] F. A. Al-Khayyal: *Linear, quadratic, and bilinear programming approaches to the linear complementarity problem*, Eur. J. Oper. Res. 24 (1986) 216–227.

[3] J. P. Aubin, H. Frankowska: *Set-Valued Analysis*, Birkhäuser, Boston (1990).

[4] M. S. Bazaraa, H. D. Sherali, C. M. Shetty: *Nonlinear Programming: Theory and Algorithms*, John Wiley & Sons, Hoboken (2006).

[5] A. Beck: *On the convexity of a class of quadratic mappings and its application to the problem of finding the smallest ball enclosing a given intersection of balls*, J. Global Optim. 39 (2007) 113–126.

[6] M. J. Best, B. Ding: *Global and local quadratic minimization*, J. Global Optim. 10 (1997) 77–90.

[7] I. M. Bomze: *Global escape strategies for maximizing quadratic forms over a simplex*, J. Global Optim. 11 (1997) 325–338.

[8] I. M. Bomze: *On standard quadratic optimization problems*, J. Global Optim. 13 (1998) 369–387.

[9] I. M. Bomze: *Copositivity relaxation beats Lagrangian dual bounds in quadratically and linearly constrained quadratic optimization problems*, SIAM J. Optim. 25 (2015) 1249–1275.

[10] I. M. Bomze: *Copositivity for second-order optimality conditions in general smooth optimization problems*, Optimization 65 (2016) 779–795.

[11] I. M. Bomze, E. De Klerk: *Solving standard quadratic optimization problems via linear, semidefinite and copositive programming*, J. Global Optim. 24 (2002) 163–185.

[12] I. M. Bomze, M. Dür, E. De Klerk, C. Roos, A. J. Quist, T. Terlaky: *On copositive programming and standard quadratic optimization problems*, J. Global Optim. 18 (2000) 301–320.

[13] I. M. Bomze, M. Locatelli, F. Tardella: *New and old bounds for standard quadratic optimization: dominance, equivalence and incomparability*, Math. Program. Ser. A 115 (2008) 31–64.

[14] G. Cárcamo, F. Flores-Bazán: *Strong duality and KKT conditions in nonconvex optimization with a single equality constraint and geometric constraint*, Math. Program. 168 (2018) 369–400.

[15] K. C. Chang, K. J. Pearson, T. Zhang: *Primitivity, the convergence of the NQZ method, and the largest eigenvalue for nonnegative tensors*, SIAM J. Matrix Analysis Appl. 32 (2011) 806–819.

[16] R. W. Cottle, G. B. Dantzig: *Complementary pivot theory of mathematical programming*, Linear Algebra Appl. 1 (1968) 103–125.

[17] C. F. Cui, Y. H. Dai, J. Nie: *All real eigenvalues of symmetric tensors*, SIAM J. Matrix Anal. Appl. 35 (2014) 1582–1601.

[18] J. H. Elton: *Indefinite quadratic forms and the invariance of the interval in special relativity*, Amer. Math. Monthly 117 (2010) 540–547.

[19] P. Finsler: *Über das Vorkommen definiter und semi-definiter Formen in Scharen quadratischer Formen*, Comm. Mat. Helvetici 9 (1937) 188–192.

[20] F. Flores-Bazán, G. Cárcamo, S. Caro: *Extension of the standard quadratic optimization problem: strong duality, optimality, hidden convexity and S-lemma*, Appl. Math. Optim. 81 (2020) 383–408.

[21] F. Flores-Bazán, W. Echegaray, Fernando Flores-Bazán, E. Ocaña: *Primal or dual strong-duality in nonconvex optimization and a class of quasiconvex problems having zero duality gap*, J. Global Optim. 69 (2017) 823–845.

[22] F. Flores-Bazán, R. López: *The linear complementarity problem under asymptotic analysis*, Math. Oper. Res. 30 (2005) 73–90.

[23] F. Flores-Bazán, G. Mastroeni: *Characterizing FJ and KKT conditions in nonconvex mathematical programming with applications*, SIAM J. Optim. 25 (2015) 647–676.

[24] F. Flores-Bazán, G. Mastroeni: *First- and second-order optimality conditions for quadratically constrained quadratic programming problems*, J. Optim. Theory Appl. 193 (2022) 118–138.

[25] F. Flores-Bazán, G. Mastroeni, C. Vera: *Proper or weak efficiency via saddle point conditions in cone-constrained nonconvex vector optimization*, J. Optim. Theory Appl. 181 (2019) 787–816.

[26] F. Flores-Bazán, F. Opazo: *Characterizing the convexity of joint-range for a pair of inhomogeneous quadratic functions and strong duality*, Minimax Theory Appl. 1 (2016) 257–290.

[27] F. Flores-Bazán, F. Opazo: *Characterizing convexity of images for quadratic-linear mappings with applications in nonconvex quadratic optimization*, SIAM J. Optim. 31 (2021) 1774–1796.

[28] D. M. Gay: *Computing optimal locally constrained steps*, SIAM J. Sci. Statist. Comput. 2 (1981) 186–197.

[29] L. E. Gibbons, D. W. Hearn, P. M. Pardalos, M. V. Ramana: *Continuous characterizations of the maximum clique problem*, Math. Oper. Res. 22 (1997) 754–768.

[30] L. Han: *An unconstrained optimization approach for finding real eigenvalues of even order symmetric tensors*, Numer. Algebra Control Optim. 3 (2013) 583–599.

[31] J. B. Hiriart-Urruty, A. Seeger: *A variational approach to copositive matrices*, SIAM Review 52 (2010) 593–629.

[32] S. Hu, G. Li, L. Qi, Y. Song: *Finding the maximum eigenvalue of essentially nonnegative symmetric tensors via sum of squares programming*, J. Optim. Theory Appl. 158 (2013) 717–738.

[33] T. Ibaraki, N. Katoh: *Resource Allocations Problems: Algorithm Approaches*, MIT Press, Cambridge (1988).

[34] V. Jeyakumar, G. Y. Li: *Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization*, Math. Program. A 147 (2014) 171–206.

[35] E. Kofidis, P. A. Regalia: *On the best rank-1 approximation of higher-order supersymmetric tensors*, SIAM J. Matrix Analysis Appl. 26 (2013) 863–884.

[36] T. G. Kolda, J. R. Mayo: *Shifted power method for computing tensor eigenpairs*, SIAM J. Matrix Analysis Appl. 32 (2011) 1095–1124.

[37] Z. Landsman: *Minimization of the root of a quadratic funcional under a system of affine equality constraints with application to portfolio management*, J. Comput. Appl. Math. 220 (2008) 739–748.

[38] Z. Landsman, U. Makov: *Translation-invariant and positive-homogeneous risk measures and optimal portfolio management*, Eur. J. Finance 17 (2011) 307–320.

[39] J. Lasserre: *Global optimization with polynomials and the problem of moments*, SIAM J. Optim. 11 (2001) 796–817.

[40] L. H. Lim: *Singular values and eigenvalues of tensors: a variational approach*, in: Proc. 1st IEEE Int. Workshop on Computational Advances in Multi-Sensor Adaptive Processing, Mexico (2005) 129–132.

[41] H. M. Markowitz: *Portfolio selection*, J. Finance 7 (1952) 77–91.

[42] H. M. Markowitz: *The general mean-variance portfolio selection problem*, in: *Mathematical Models in Finance*, S. D. Howison et al. (eds.), Chapman and Hall, London (1995) 93–98.

[43] E. Matskani, N. D. Sidiropoulos, Z. Q. Luo, L. Tassiulas: *Convex approximations techniques for joint multiuser downlink beamforming and admission control*, IEEE Trans. Wireless Commun. 7 (2008) 2682–2693.

[44] T. S. Motzkin, E. G. Strauss: *Maxima for graphs and a new proof of a theorem of Turán*, Can. J. Math. 17 (1965) 533–540.

[45] M. Ng, L. Qi, G. Zhou: *Finding the largest eigenvalue of a nonnegative tensor*, SIAM J. Matrix Analysis Appl. 31 (2009) 1090–1099.

[46] V. B. Nguyen, T. N. Nguyen, R. L. Sheu: *Strong duality in minimizing a quadratic form subject to two homogeneous quadratic inequalities over the unit sphere*, J. Global Optim. 76 (2020) 121–135.

[47] J. Nie: *Sum of squares methods for minimizing polynomial forms over spheres and hypersurfaces*, Front. Math. China 7 (2012) 321–346.

[48] P. M. Pardalos, Y. Ye, C. G. Han: *Algorithms for the solution of quadratic knapsack problems*, Linear Algebra Appl. 152 (1991) 69–91.

[49] I. Pólik, T. Terlaky: *A survey of the S-Lemma*, SIAM Review 49 (2007) 371–418.

[50] J. C. Preisig: *Copositivity and the minimization of quadratic functions with nonnegativity and quadratic equality constraints*, SIAM J. Control Optim. 34 (1996) 1135–1150.

[51] L. Qi: *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Computation 40 (2005) 1302–1324.

[52] R. T. Rockafellar: *Conjugate Duality and Optimization*, CBMS-NSF Regional Conference Series in Applied Mathematics Vol. 16, SIAM, Philadelphia (1974).

[53] A. Seeger: *Eigenvalue analysis of equilibrium processes defined by linear complementarity conditions*, Linear Algebra Appl. 292 (1999) 1–14.

[54] A. Seeger, M. Torki: *Local minima of quadratic forms on convex cones*, J. Glob. Optim. 44 (2009) 1–28.

[55] N. D. Sidiropoulos, T. N. Davidson, Z. Q. Lou: *Transmit beamforming for physical-layer multicasting*, IEEE Trans. Signal Process. Wireless Communications 54 (2006) 2239–2251.

[56] Y. Song, L. Qi: *Eigenvalue analysis of constrained minimization problem for homogeneous polynomial*, J. Global Optim. 64 (2016) 563–575.

[57] D. C. Sorensen: *Newton's method with a model trust region modification*, SIAM Numer. Analysis 19 (1982) 409–426.

[58] Y. Xia, S. Wang, R. L. Sheu: *S-Lemma with equality and its applications*, Math. Prog. Ser A 156 (2016) 513–547.

[59] V. A. Yakubovich: *S-procedure in nonlinear control theory (Russian)*, Vestnik Leningrad. Univ. 1 (1971) 62–77.

[60] V. A. Yakubovich: *S-procedure in nonlinear control theory*, Vestnik Leningrad. Univ. 4 (1977) 73–93; English translation of [59].

[61] Y. Yuan: *Recent advances in trust region algorithms*, Math. Program. 151 (2015) 249–281.