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Some Remarks on the Mixed Problem of Elastostatics in Exterior Domains

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We consider a mixed boundary problem for inhomogeneous linear elastostatics in a three-dimensional exterior Lipschitz domain. We prove that it has a unique variational solution $\boldsymbol{u}$ vanishing at infinity with a rate depending on the elasticity tensor and/or suitable assumptions on the boundary data.

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1. Introduction

Consider a linearly elastic body $\mathscr{B}$ identified with the exterior Lipschitz domain

$$\Omega = \mathbb{R}^3 \setminus \bigcup_{i=1}^{m} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \varnothing\,,$$

where Ω_i, $i = 1, \dots m$ are m bounded domains of $\mathbb{R}^3$ with connected boundaries, and assume that it occupies an assigned stress free reference configuration. The elastic material properties of $\mathscr{B}$ are expressed by the elasticity tensor (see Section 2 for notation), i.e., a map

$$\mathbb{C} : \Omega \times \mathrm{Sym} \to \mathrm{Sym}$$

linear in Sym and such that

$$\mu_0 |\boldsymbol{E}|^2 \le \boldsymbol{E} \cdot \mathbb{C}[\boldsymbol{E}] \le \mu_e |\boldsymbol{E}|^2, \quad \forall\, \boldsymbol{E} \in \mathrm{Sym}, \ \text{ a.e. in } \Omega\,, \tag{1}$$

for some positive scalars μ_0 and μ_e (elastic moduli) independent of $\boldsymbol{E}$. If the points of the surface

$$\Sigma = \bigcup_{i=1}^{k} \partial\Omega_i, \quad k \le m,$$

of $\partial\Omega$ are constrained to assume an assigned displacement $\hat{\boldsymbol{u}}$ and the remaining portion of $\partial\Omega$ is subject to a load $\hat{\boldsymbol{s}}$, $\mathscr{B}$ deforms in a new equilibrium configuration

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$\{x + \boldsymbol{u}(x)\}$, where $\boldsymbol{u}(x)$ is a solution to the boundary value problem [7]

$$\left\{ \begin{array}{ll} \operatorname{div} \mathbb{C}[\nabla \boldsymbol{u}] = \boldsymbol{0} \quad \text{in } \Omega\,, & \boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \Sigma, \\ \boldsymbol{s}(\boldsymbol{u}) = \hat{\boldsymbol{s}} \quad \text{on } \partial\Omega \setminus \Sigma, & \lim\limits_{|x| \to +\infty} \boldsymbol{u}(x) = \boldsymbol{0}\,, \end{array} \right. \tag{2}$$

where, in indicial notation, $(\operatorname{div} \mathbb{C}[\nabla \boldsymbol{u}])_i = \partial_j(\mathsf{C}_{ijhk} \partial_k u_h)$ and $\boldsymbol{s}(\boldsymbol{u})$ is the traction on the boundary defined by

$$\boldsymbol{s}(\boldsymbol{u}) = \mathbb{C}[\nabla \boldsymbol{u}]\boldsymbol{n}.$$

If $D^{1,q}(\Omega)$, $q \geq 1$, denote the completion of $C_0^\infty(\bar{\Omega})$ with respect to the norm $\|\nabla \boldsymbol{u}\|_{L^q(\Omega)}$, let

$$\widehat{D}^{1,q}(\Omega) = \Big\{ \boldsymbol{v} \in D^{1,q}(\Omega) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Sigma \Big\}.$$

Assume that $$\hat{\boldsymbol{s}} \in W^{-1/2,2}(\partial\Omega \setminus \Sigma), \quad \hat{\boldsymbol{u}} \in W^{1/2,2}(\Sigma)\,,$$

and denote by $\boldsymbol{u}_0$ an extension of $\hat{\boldsymbol{u}}$ in $D^{1,2}(\Omega)$ vanishing on $\partial\Omega \setminus \Sigma$. A variational solution to $(2)_{1,2,3}$ is a field $\boldsymbol{u}$ such that $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_0 \in \widehat{D}^{1,2}(\Omega)$ satisfies

$$\int\limits_\Omega \mathbb{C}[\nabla \boldsymbol{v}] \cdot \nabla \boldsymbol{\varphi} = \overset{*}{\int\limits_{\partial\Omega\setminus\Sigma}} \boldsymbol{\varphi} \cdot \hat{\boldsymbol{s}} - \int\limits_\Omega \nabla \boldsymbol{\varphi} \cdot \mathbb{C}[\nabla \boldsymbol{u}_0], \quad \forall \boldsymbol{\varphi} \in \widehat{D}^{1,2}(\Omega). \tag{3}$$

where by abuse of notation, the integral with the asterisk means the value of the functional $\hat{\boldsymbol{s}}$ at $\boldsymbol{\varphi}$.

By classical arguments (see Section 2), we deduce that $(2)_{1,2,3}$ has a unique variational solution $\boldsymbol{u}$. As fas as more regularity of $\boldsymbol{u}$ is concerned, a classical results of N. G. Meyers [9] ensures that $\boldsymbol{u} \in W^{1,q}_{\text{loc}}(\Omega)$, for some $q > 2$. In [1], it has been shown that if $\mathbb{C} \in VMO_{\text{loc}}(\Omega)$, then $\boldsymbol{u}$ is locally Hölder continuous in Ω. Moreover, if $\Omega, \mathbb{C}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{s}}$ are sufficiently regular, then $\boldsymbol{u}$ is a classical solution to $(2)_{1,2,3}$ (see, e.g., [5]).

As regards $(2)_4$, standard arguments (see Section 2) show that such variational solution $\boldsymbol{u}$ satisfies $(2)_4$ in the sense of the convergence in the mean of order two, that is, if S denote the unit ball centered at the origin, then

$$\lim_{R \to +\infty} \left(R \int\limits_{\partial S} |\boldsymbol{u}|^2(R)\, d\sigma \right) = 0 \;\Leftrightarrow\; \int\limits_{\partial S} |\boldsymbol{u}|^2(R)\, d\sigma = o(R^{-1}). \tag{4}$$

In [2], it is proved that condition (4) can be improved, in the sense that there is $q < 2$, depending only on μ_0 and μ_e, such that

$$\int\limits_{\partial S} |\boldsymbol{u}|^q(R)\, d\sigma = o(R^{q-3}). \tag{5}$$

A famous counter-example of De Giorgi [3] on the regularity of weak solution to elliptic systems illustrates that (5) cannot be improved: in Section 2 we construct a sequence of elliptic systems of the type introduced in [3], having variational solutions satisfying (5) with $q = q(\mu_0, \mu_e)$ such that, as μ_0 goes to zero then q tends to 2, so that (5) reduces to (4).

As regards the pointwise attainment of condition $(2)_4$, in [14] Souček, inspired by the mentioned counter-example of De Giorgi, constructed an elliptic system with a solution which is discontinuous on a dense countable set. Hence it follows that we cannot expect pointwise decay at infinity under the only hypothesis of boundedness of $\mathbb{C}$ at large distance.

Observe that, in general, a variational solution $\boldsymbol{u}$ having zero resulting tractions on the boundary

$$\int_{\partial\Omega}^{*} \boldsymbol{s}(\boldsymbol{u}) = \boldsymbol{0}\,, \tag{6}$$

shows a better decay at infinity. For instance, if $\mathbb{C}$ is homogeneous (say), then $\boldsymbol{u} = O(r^{-1})$, but if, in addition, $\boldsymbol{u}$ satisfies (6), then $\boldsymbol{u} = O(r^{-2})$. Of course, unless $\Sigma = \varnothing$, condition (6) cannot be a hypothesis of the problem, but must be consequence of suitable assumptions on $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{s}}$.

In this perspective, the purpose of the present paper is to prove a decay behavior faster than the canonical one (4), with rates connected to the elasticity tensor and to a suitable choice of boundary data. In particular, we will require L^2-orthogonality to the traction forces associated to the solutions of some auxiliary problems. More precisely, let $\mathfrak{C}$ be the linear space spanned by the variational solutions $\boldsymbol{h}$ to the equations

$$\begin{aligned} \operatorname{div}\mathbb{C}[\nabla\boldsymbol{h}] &= \boldsymbol{0} \quad \text{in } \Omega,\\ \boldsymbol{h} &= \boldsymbol{e}_i \quad \text{on } \Sigma,\\ \boldsymbol{s}(\boldsymbol{h}) &= \boldsymbol{0} \quad \text{on } \partial\Omega\setminus\Sigma. \end{aligned} \tag{7}$$

where $\boldsymbol{e}_i$, for $i = 1, 2, 3$, denote the vectors of the standard euclidean orthonormal basis, then the following result holds.

Theorem 1.1. *Let $\boldsymbol{u}$ be the variational solution to* (2) *and let $\mathfrak{C}$ be the linear space spanned by the variational solutions $\boldsymbol{h}$ to* (7)*. If*

$$\int_{\partial\Omega\setminus\Sigma} \hat{\boldsymbol{s}} = \boldsymbol{0} \tag{8}$$

and

$$\int_{\Sigma} \hat{\boldsymbol{u}}\cdot\boldsymbol{s}(\boldsymbol{h}) = \boldsymbol{0}, \quad \text{for all } \boldsymbol{h}\in\mathfrak{C}, \tag{9}$$

then

$$\int_{\partial S} |\boldsymbol{u}|^2(R)\, d\sigma = O(R^{-1-\gamma}), \tag{10}$$

where

$$\gamma = \frac{\sqrt{2}\,\mu_0}{2\mu_e + \mu_0}\,. \tag{11}$$

The decay expressed by (10) is linked to the Boussinesq formulation of the Saint-Venant principle for elastic bodies of general form (see Section 54 of [7]). In particular, it is a confirmation in nonhomogeneous elasticity that to obtain a deformation which is negligible at large spatial distance, it is not necessary to require that the net torque acting on the body vanishes.

For the Dirichlet problem in $\mathbb{R}^2$ and in $\mathbb{R}^3$, such kind of problem has been studied in [4] and [2] respectively. Similar results have been proved for Stokes system in [13] (see also [11], [15] and [12] where the boundary value problems with singular data or singular geometry are examined).

The paper is organized as follows: in Section 2, we introduce some notation and we discuss the existence of a unique variational solution satisfying (4) and (5). Section 3 is devoted to the proof of Theorem 1.1.

2. Notation and preliminary results

Let Ω be a domain in $\mathbb{R}^3$. We will say that Ω is an exterior Lipshitz domain if its complementary set $\complement\Omega$ is a bounded closed set of $\mathbb{R}^3$ and the boundary is of class $C^{0,1}$. By $(o, \{\boldsymbol{e}_i\}_{i=1,2,3})$, with the origin $o \in \complement\Omega$, we denote the standard orthonormal reference frame, which allows us to identify each vector $\boldsymbol{v} \in \mathbb{R}^3$ with its components $\boldsymbol{v} \equiv [v_i]$. For $x \in \mathbb{R}^3$, we set $r = |\boldsymbol{x}| = |x - o|$. Denote by S_R the ball centered at the origin, having radius R, $S_R = \{x \in \mathbb{R}^3 : r = |\boldsymbol{x}| < R\}$ (when $R = 1$ we will write only S) and set $T_R = S_{2R} \setminus S_R$. Let $\boldsymbol{e}_R$ be the unit outward normal to ∂S_R.

Unless otherwise specified, we will essentially use the notation of the classical monograph [7] of M. E. Gurtin. Lin is the space of second-order tensors (linear maps from $\mathbb{R}^3$ into itself) and Sym, Skw are the spaces of the symmetric and skew elements of Lin respectively. The summation convention is assumed. As is customary, if $\boldsymbol{E} \in \text{Lin}$, by $\boldsymbol{E} \equiv [E_{ij}]$ we intend that, if $\boldsymbol{v} \in \mathbb{R}^3$, then $\boldsymbol{E}\boldsymbol{v} \equiv [E_{ij}v_j]$ is the vector image of $\boldsymbol{v}$ by $\boldsymbol{E}$. By $\boldsymbol{E} \cdot \boldsymbol{F}$ we denote the standard inner product of tensors, that is the scalar $\boldsymbol{E} \cdot \boldsymbol{F} = E_{ij}F_{ij}$ and we will write $|\boldsymbol{E}|^2 = \boldsymbol{E} \cdot \boldsymbol{E}$.

If f and g are nonnegative functions defined in a neighborhood of infinity, we will write $f = O(g)$ if f/g is bounded. We will write $f = o(g)$ as r goes to $+\infty$, if $\lim_{r\to+\infty} f/g = 0$. Finally, by c we intend a constant which possibly depends on the data of the problem and which can change from line to line.

Now let us recall standard arguments to show that $(2)_{1,2,3}$ has a unique variational solution $\boldsymbol{u}$. Indeed, by the well known Korn inequality (see [7] Section 13), if $\widehat{\nabla}\boldsymbol{v}$ denotes the symmetric part of $\nabla\boldsymbol{v}$, there exists a positive constant c such that

$$\int_\Omega |\nabla\boldsymbol{v}|^q \leq c\int_\Omega |\widehat{\nabla}\boldsymbol{v}|^q,$$

for $q \in (1, +\infty)$, then the left-hand side of (3) defines a coercive bilinear form in $\widehat{D}^{1,2}(\Omega) \times \widehat{D}^{1,2}(\Omega)$.

On the other hand, by the Schwarz inequality, the trace theorem and (1), the right hand side of (3) defines a linear and continuous functional on $\widehat{D}^{1,2}(\Omega)$, since

$$\left|\int_{\partial\Omega\setminus\Sigma} \boldsymbol{\varphi}\cdot\hat{\boldsymbol{s}}\right| \le \|\hat{\boldsymbol{s}}\|_{W^{-1/2,2}(\partial\Omega\setminus\Sigma)}\|\boldsymbol{\varphi}\|_{D^{1,2}\Omega)},$$

$$\left|\int_{\Omega} \nabla\boldsymbol{\varphi}\cdot\mathbb{C}[\nabla\boldsymbol{u}_0]\right| \le \mu_e\|\nabla\boldsymbol{u}_0\|_{L^2(\Omega)}\|\nabla\boldsymbol{\varphi}\|_{L^2(\Omega)}.$$

Hence by Lax-Milgram lemma $(2)_{1,2,3}$ has a unique variational solution.

If $d\sigma$ denotes the surface element of ∂S and $\boldsymbol{u}(R)=\boldsymbol{u}(R\boldsymbol{x})$, for $\boldsymbol{x}\in\partial S$, from the trace theorem

$$\int_{\partial S} |\boldsymbol{u}|^2(R)\,d\sigma \le \frac{c}{R^3}\int_{T_R}|\boldsymbol{u}|^2+\frac{c}{R}\int_{T_R}|\nabla\boldsymbol{u}|^2 \tag{12}$$

and the well known Hardy's inequality (see, e.g., [8])

$$\int_{\Omega}\frac{|\boldsymbol{u}|^q}{|\boldsymbol{x}|^q}\le c(\Omega,q)\int_{\Omega}|\nabla\boldsymbol{u}|^q \qquad q\in(1,3)\,, \tag{13}$$

one sees that such variational solution $\boldsymbol{u}$ satisfies $(2)_4$ in the sense of the convergence in the mean of order two (4).

The following example (see [10]), inspired by a famous one of E. De Giorgi [3] on the regularity of weak solution to elliptic systems, tells us that condition (5) cannot be improved.

Let $\tilde{\mathbb{C}}$ be the elasticity tensor defined by

$$\tilde{\mathbb{C}}[\boldsymbol{L}]=(\boldsymbol{B}\otimes\boldsymbol{B})+\xi^2\,\boldsymbol{L},\quad \xi\in\mathbb{R},\quad \forall\boldsymbol{L}\in\mathrm{Sym},$$

where
$$B_{ij}=\delta_{ij}+3\frac{x_ix_j}{|\boldsymbol{x}|^2}\,,$$

and, in indicial notation, $((\boldsymbol{B}\otimes\boldsymbol{B}))_i=B_{ij}B_{hk}L_{hk}$. Observe that $\tilde{\mathbb{C}}$ is bounded on $\mathbb{R}^3$ and real analytic outside S_R for every positive R. Since

$$\boldsymbol{L}\cdot\tilde{\mathbb{C}}(\boldsymbol{L})=(\boldsymbol{B}\cdot\boldsymbol{L})^2+\xi^2\boldsymbol{L}^2\,,\quad \forall\,L\in\mathrm{Sym},$$

then $\tilde{\mathbb{C}}$ satisfies (1) with $\mu_0=\xi^2$ and $\mu_e=18+\xi^2$, since

$$|\boldsymbol{B}|^2=\left(\delta_{ij}+3\frac{x_ix_j}{|x|}\right)^2=18\,.$$

Consider the following system

$$\mathrm{div}\,\tilde{\mathbb{C}}[\nabla\boldsymbol{u}]=\mathrm{div}\,[\boldsymbol{B}\otimes\boldsymbol{B}\,](\nabla\boldsymbol{u})+\tfrac{1}{2}\xi^2\,[\Delta\boldsymbol{u}+\nabla\mathrm{div}\,\boldsymbol{u}\,]=\boldsymbol{0}\,. \tag{14}$$

If $\boldsymbol{u}=|\boldsymbol{x}|^\alpha\boldsymbol{x}$, then, an easy computation shows that

$$(\nabla\boldsymbol{u})_{ij}=\delta_{ij}|\boldsymbol{x}|^\alpha+\alpha|\boldsymbol{x}|^{\alpha-2}x_ix_j\,,\qquad \mathrm{div}\,\tilde{\mathbb{C}}[\nabla\boldsymbol{u}]=(6+4\alpha)^2|\boldsymbol{x}|^{\alpha-2}\boldsymbol{x}\,,$$

$$\Delta\boldsymbol{u}=\nabla\mathrm{div}\,\boldsymbol{u}=(3\alpha+\alpha^2)|\boldsymbol{x}|^{\alpha-2}\boldsymbol{x}\,.$$

Then, (14) admits the solution

$$\boldsymbol{u} = (c_1|\boldsymbol{x}|^{\alpha_1} + c_2|\boldsymbol{x}|^{\alpha_2})\boldsymbol{x}\,, \tag{15}$$

for every pair of scalars c_1, c_2, with

$$\alpha_1 = \frac{3}{2}\left(\frac{|\xi|}{\sqrt{16+\xi^2}} - 1\right) > -\frac{3}{2}\,,$$
$$\alpha_2 = -\frac{3}{2}\left(\frac{|\xi|}{\sqrt{16+\xi^2}} + 1\right) < -\frac{3}{2}.$$

Of course, if $r = |\boldsymbol{x}|$,

$$\boldsymbol{u}_1 = c_1 r^{\alpha_1}\boldsymbol{x} = O(r^{\frac{3\epsilon-1}{2}}) \quad \text{and} \quad \boldsymbol{u}_2 = c_2 r^{\alpha_2}\boldsymbol{x} = O(r^{-\frac{3\epsilon+1}{2}}),$$

as $r \to +\infty$, where $\displaystyle \epsilon = \sqrt{\frac{\mu_0}{\mu_e - 2}} = \frac{|\xi|}{\sqrt{16+\xi^2}}$

If $c_1 = 0$ and $c_2 = 1$, the function $\boldsymbol{u}$ defined in (15) is a variational solution outside a ball S_R. Moreover, $\boldsymbol{u}$ belongs to $D^{1,q}(\complement S_R)$, for $q > \frac{2}{1+\epsilon}$. As a consequence of trace theorem (12) and Hardy's inequality (13) with Ω replaced by $\complement S_R$, (5) holds for any $q > \frac{2}{1+\epsilon}$. As $\epsilon \to 0$ when $\xi \to 0$, this shows that (5) cannot be improved.

3. Proof of Theorem 1.1

Let us, first, collect the main tools we need to prove Theorem 1.1.

First of all, let us recall two classical and well known inequalities, like the Wirtinger inequality with optimal constant in $\mathbb{R}^3$ ([6]), which asserts that,

$$\text{if } \boldsymbol{u}_{\partial S_R} = \frac{1}{|\partial S_R|}\int_{\partial S_R}\boldsymbol{u}, \quad \text{then } \|\boldsymbol{u} - \boldsymbol{u}_{\partial S_R}\|_{L^2(\partial S_R)} \le \frac{R}{\sqrt{2}}\|\nabla \boldsymbol{u}\|_{L^2(S_R)}\,, \tag{16}$$

and the Poincaré inequality, which asserts that, if $\boldsymbol{u}_{S_R} = \frac{1}{|S_R|}\int_{S_R}\boldsymbol{u}$, there exists a positive constant c such that

$$\int_{S_R}|\boldsymbol{u} - \boldsymbol{u}_{S_R}|^2\,dx \le cR^2\int_{S_R}|\nabla\boldsymbol{u}|^2\,. \tag{17}$$

Lemma 3.1. *Let $\bar{R} > 0$ such that $\partial\Omega \subset S_{\bar{R}}$. If $\boldsymbol{u} \in D^{1,2}(\complement S_{\bar{R}})$ is a solution to the equation* $(2)_1$ *such that*

$$\int^{*}_{\partial S_{\bar{R}}} \boldsymbol{s}(\boldsymbol{u}) = \boldsymbol{0}\,, \tag{18}$$

then, for any $\bar{R} \le \rho \le R$, we have

$$\int_{\complement S_R}|\nabla\boldsymbol{u}|^2 \le c\left(\frac{\rho}{R}\right)^{\gamma}\int_{\complement S_\rho}|\nabla\boldsymbol{u}|^2,$$

where γ is defined in (11).

Proof. Let us prove the result by assuming that $\boldsymbol{u}$ is sufficiently regular and then the thesis will follow by approximation. If $\varrho \geq \bar{R}$, let consider the cut-off function

$$g_\varrho(r) = \begin{cases} 0, & r > 2\varrho, \\ 1, & r < \varrho, \\ 2 - \frac{r}{\varrho}, & r \in [\varrho, 2\varrho]\,. \end{cases}$$

Recalling that, if $\boldsymbol{u}$ is a smooth vector field and $\tilde{\nabla}\boldsymbol{u}$ denotes the skew part of $\nabla\boldsymbol{u}$, then the following identity holds

$$|\hat{\nabla}\boldsymbol{u}|^2 - |\tilde{\nabla}\boldsymbol{u}|^2 = \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^T = \operatorname{div}\,[(\nabla\boldsymbol{u})\boldsymbol{u} - (\operatorname{div}\boldsymbol{u})\boldsymbol{u}] + |\operatorname{div}\,\boldsymbol{u}|^2\,;$$

integration by part and (1) imply

$$\begin{aligned} \mu_0 \int_{\complement S_R} & g_\varrho(r)\left(|\nabla\boldsymbol{u}|^2 + |\operatorname{div}\boldsymbol{u}|^2\right) \\ &\leq \int_{T_\varrho} \nabla g_\varrho(r)\cdot[\mu_0((\nabla\boldsymbol{u})\boldsymbol{u} - (\operatorname{div}\boldsymbol{u})\boldsymbol{u}) - 2\mathbb{C}[\nabla\boldsymbol{u}]\boldsymbol{u}] \\ &+ \int_{\partial S_R} \boldsymbol{e}_R\cdot[2\mathbb{C}[\nabla\boldsymbol{u}]\boldsymbol{u} - \mu_0((\nabla\boldsymbol{u})\boldsymbol{u} - (\operatorname{div}\boldsymbol{u})\boldsymbol{u})]\;. \end{aligned} \tag{19}$$

Since $\boldsymbol{u}$ satisfies (18) and it is a solution to $(2)_1$, then

$$\overset{*}{\int\limits_{\partial S_\varrho}} \boldsymbol{s}(\boldsymbol{u}) = \boldsymbol{0}\,, \qquad \text{for any } \varrho \geq R\,,$$

and then, for any $\boldsymbol{a} \in \mathbb{R}^3$, we get

$$\int_{T_\varrho} \boldsymbol{e}_r\cdot\mathbb{C}[\nabla\boldsymbol{u}]\boldsymbol{a} = 0\,.$$

If we choose $\boldsymbol{a} = \boldsymbol{u}_{T_\varrho}$, taking into account the expression of g_ϱ and Schwarz inequality we get

$$\begin{aligned} \left|\int_{T_\varrho} \nabla g_\varrho(r)\cdot\mathbb{C}[\nabla\boldsymbol{u}]\boldsymbol{u}\right| &= \left|\int_{T_\varrho} \nabla g_\varrho(r)\cdot\mathbb{C}[\nabla\boldsymbol{u}](\boldsymbol{u} - \boldsymbol{u}_{T_\varrho})\right| \\ &\leq \frac{\mu_e}{\varrho}\|\nabla\boldsymbol{u}\|_{L^2(T_\varrho)}\|\boldsymbol{u} - \boldsymbol{u}_{T_\varrho}\|_{L^2(T_\varrho)}\,, \end{aligned} \tag{20}$$

which together with Poincaré inequality (17), gives

$$\lim_{\varrho\to\infty} \int_{T_\varrho} \nabla g_\varrho(r)\cdot\mathbb{C}[\nabla\boldsymbol{u}]\boldsymbol{u} = 0\,. \tag{21}$$

Similar arguments can be adopted to show that also

$$\lim_{\varrho\to\infty} \int_{T_\varrho} \nabla g_\varrho(r)\cdot[(\nabla\boldsymbol{u})\boldsymbol{u} - (\operatorname{div}\boldsymbol{u})\boldsymbol{u}] = 0\,. \tag{22}$$

Analogously, by the Schwarz inequality and the Wirtinger inequality (16), we get

$$\begin{aligned}\left|2\int_{\partial S_R} \boldsymbol{e}_R\cdot\mathbb{C}[\nabla\boldsymbol{u}]\boldsymbol{u}\right| &= \left|2\int_{\partial S_R} \boldsymbol{e}_R\cdot\mathbb{C}[\nabla\boldsymbol{u}](\boldsymbol{u}-\boldsymbol{u}_{\partial S_R})\right|\\ &\le 2\mu_e\frac{R}{\sqrt{2}}\int_{\partial S_R}|\nabla\boldsymbol{u}|^2\,. \end{aligned}\tag{23}$$

Finally, since $|\nabla\boldsymbol{u}-(\operatorname{div}\boldsymbol{u})\mathbf{1}|^2=\left(\dfrac{\partial u_i}{\partial x_j}-\dfrac{\partial u_k}{\partial x_k}\delta_{ij}\right)^2=|\nabla\boldsymbol{u}|^2+|\operatorname{div}\boldsymbol{u}|^2$, by similar arguments we get

$$\begin{aligned}&\left|\mu_0\int_{\partial S_R}\boldsymbol{e}_R\cdot[(\nabla\boldsymbol{u})\boldsymbol{u}-(\operatorname{div}\boldsymbol{u})\boldsymbol{u}]\right|\\ &\qquad=\left|\mu_0\int_{\partial S_R}\boldsymbol{e}_R\cdot[(\nabla\boldsymbol{u})-(\operatorname{div}\boldsymbol{u})\mathbf{1}]\,(\boldsymbol{u}-\boldsymbol{u}_{\partial S_R})\right|\\ &\qquad\le\mu_0\frac{R}{\sqrt{2}}\int_{\partial S_R}(|\nabla\boldsymbol{u}|^2+|\operatorname{div}\boldsymbol{u}|^2)\,.\end{aligned}\tag{24}$$

Collecting (19)–(24), we get that, if γ is the number defined in (11), then

$$\int_{\complement S_R}(|\nabla\boldsymbol{u}|^2+|\operatorname{div}\boldsymbol{u}|^2)\le\frac{1}{\gamma}R\int_{\partial S_R}(|\nabla\boldsymbol{u}|^2+|\operatorname{div}\boldsymbol{u}|^2)\,.\tag{25}$$

If
$$G(R)=\int_{\complement S_R}(|\nabla\boldsymbol{u}|^2+|\operatorname{div}\boldsymbol{u}|^2)$$

inequality (25) can be equivalently written as

$$\gamma G(R)\le -RG'(R)\,.\tag{26}$$

By (26) we get that the function $R^\gamma G(R)$ is a decreasing function, that is for any $\rho\le R$

$$\int_{\complement S_R}(|\nabla\boldsymbol{u}|^2+|\operatorname{div}\boldsymbol{u}|^2)\le\left(\frac{\rho}{R}\right)^\gamma\int_{\complement S_\rho}(|\nabla\boldsymbol{u}|^2+|\operatorname{div}\boldsymbol{u}|^2)\,,$$

and then the thesis easy follows recalling that $|\operatorname{div}\boldsymbol{u}|^2\le c|\nabla\boldsymbol{u}|^2$. □

Lemma 3.2. *If $\boldsymbol{u}$ is the variational solution to* (2) *and $\mathfrak{C}$ is the linear space spanned by the variational solutions $\boldsymbol{h}$ to* (7)*, then*

$$\int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{s}(\boldsymbol{h})=\int_{\partial\Omega}\boldsymbol{h}\cdot\boldsymbol{s}(\boldsymbol{u}),\quad\forall\boldsymbol{h}\in\mathfrak{C}.$$

Proof. If $\boldsymbol{u}$ is the variational solution to (2), then an integration by parts yields

$$\int_{\partial\Omega}\boldsymbol{u}\cdot\boldsymbol{s}(\boldsymbol{h})=\int_{\partial\Omega}\boldsymbol{h}\cdot\boldsymbol{s}(\boldsymbol{u})+\int_{\Omega}\Big(\boldsymbol{u}\cdot\mathbb{C}[\nabla\boldsymbol{h}]\nabla g_R-\boldsymbol{h}\cdot\mathbb{C}[\nabla\boldsymbol{u}]\nabla g_R\Big).\tag{27}$$

By Schwarz' inequality

$$\left|\int_{\Omega} \boldsymbol{u}\cdot\mathbb{C}[\nabla\boldsymbol{h}]\nabla g_R\right| = \frac{1}{R}\left|\int_{S_{2R}\setminus S_R} \boldsymbol{u}\cdot\mathbb{C}[\nabla\boldsymbol{h}]\boldsymbol{e}_r\right|$$

$$\leq c\left\{\int_{S_{2R}\setminus S_R} \frac{|\boldsymbol{u}|^2}{|\boldsymbol{x}|^2}\right\}^{1/2}\left\{\int_{S_{2R}\setminus S_R} |\nabla\boldsymbol{h}|^2\right\}^{1/2},$$

$$\left|\int_{\Omega} \boldsymbol{h}\cdot\mathbb{C}[\nabla\boldsymbol{u}]\nabla g_R\right| = \frac{1}{R}\left|\int_{S_{2R}\setminus S_R} \boldsymbol{h}\cdot\mathbb{C}[\nabla\boldsymbol{u}]\boldsymbol{e}_r\right|$$

$$\leq c\left\{\int_{S_{2R}\setminus S_R} \frac{|\boldsymbol{h}|^2}{|\boldsymbol{x}|^2}\right\}^{1/2}\left\{\int_{S_{2R}\setminus S_R} |\nabla\boldsymbol{u}|^2\right\}^{1/2}.$$

Letting $R \to +\infty$ in (27) and using Hardy's inequality (13) we get the thesis. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\boldsymbol{u}$ be a varational solution to (2). Assumptions (8), (9) and Lemma 3.2 yield

$$\overset{*}{\int_{\partial\Omega}} \boldsymbol{s}(\boldsymbol{u}) = 0\,. \tag{28}$$

Let $\bar{R} > 0$, such that $\partial\Omega \subset S_{\bar{R}}$. By (28) and $(2)_1$, we deduce that (18) holds. We can then apply Lemma 3.1 to obtain that, for any $R > \bar{R}$,

$$\int_{T_R} |\nabla\boldsymbol{u}|^2 \leq \int_{\complement S_R} |\nabla\boldsymbol{u}|^2 \leq c\left(\frac{\bar{R}}{R}\right)^{\gamma}\int_{\complement S_{\bar{R}}} |\nabla\boldsymbol{u}|^2 = \frac{c}{R^{\gamma}}\,. \tag{29}$$

Collecting trace theorem (12), Hardy's inequality (13) and (29), we get

$$\int_{\partial S} |\boldsymbol{u}|^2(R)\,d\sigma \leq \frac{c}{R^{\gamma+1}}\,,$$

which concludes the proof. □

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