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On Resonant $(p, 2)$-Equations

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We consider a nonlinear nonhomogeneous elliptic equation driven by the sum of a p-Laplacian and a Laplacian ($(p, 2)$-equation) with a Carathéodory reaction which at $\pm\infty$ is resonant with respect to the principal eigenvalue of the negative Dirichlet p-Laplacian. Using minimax methods based on the critical point theory, together with truncation techniques and Morse theory, we obtain multiplicity results producing three or four solutions with sign information (constant sign solutions and nodal solutions).

Keywords: Minimax characterization, nonlinear regularity, nonlinear maximum principle, nodal solutions, resonant equation.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear nonhomogeneous elliptic equation:

$$-\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \text{ in } \Omega, \; u_{|\partial\Omega} = 0 \quad (2 < p < \infty). \tag{1}$$

Here Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = div(|Du|^{p-2} Du) \;\text{ for all }\; u \in W_0^{1,p}(\Omega).$$

Also, $f(x, z)$ is a Carathéodory reaction (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous), which is resonant with respect to $\hat{\lambda}_1(p) > 0$, the principal eigenvalue of the negative Dirichlet p-Laplacian $(-\Delta_p, W_0^{1,p}(\Omega))$. The resonance occurs from the left of $\hat{\lambda}_1(p) > 0$ and

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this makes the energy functional of the problem coercive. Coercive problems driven by the p-Laplacian, were investigated by Carl-Perera [9], Liu [21], Liu-Liu [22] (Dirichlet problems) and by Kyritsi-Papageorgiou [18] (Neumann problems). In all the aforementioned works, the authors deal with nonresonant equations and prove multiplicity theorems producing three nontrivial solutions. More recently, Aizicovici-Papageorgiou-Staicu [2], [3], Papageorgiou-Smyrlis [28] and Sun [31] studied nonlinear, nonhomogeneous equations. Papageorgiou-Smyrlis [28] and Sun [31] deal with equations driven by the $(p, 2)$-Laplacian as in problem (1). Only Papageorgiou-Smyrlis [28] produce one nodal solution under different conditions on the reaction. Aizicovici-Papageorgiou-Staicu [2], [3] consider equation driven by a more general nonlinear, nonhomogeneous differential operator and prove a three solutions theorem with one solution being nodal. We mention that $(p, 2)$-equations arise in mathematical physics (see Cherfils-Ilyasov [11] and the references therein).

Here, under less restrictive conditions on the reaction $f(z, x)$, we prove multiplicity results producing nodal solutions under conditions of resonance. Our approach uses minimax methods based on the critical point theory, together with suitable truncation techniques and Morse theory.

2. Mathematical Background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). We say that $\varphi \in C^1(X)$ satisfies the Palais-Smale condition (the $PS-$condition, for short), if the following is true:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ s.t. $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence."

This compactness-type condition on φ compensates for the fact that the ambient space X need not to be locally compact (when X is infinite dimensional). It leads to a deformation theorem from which we can derive the mini-max theory of the critical values of φ. Prominent in this theory is the so-called "mountain pass theorem" due to Ambrosetti-Rabinowitz [4].

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X)$ and satisfies the $PS-$condition, u_0, $u_1 \in X$, $\|u_1 - u_0\| > \rho > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf[\varphi(u) : \|u - u_0\| = \rho] = m_\rho, \quad \textit{and}$$

$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$, where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \geq m_\rho$ and c is a critical value of φ.

In the study of problem (1), in addition to the Sobolev spaces $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$ (recall that $W_0^{1,p}(\Omega) \subseteq H_0^1(\Omega)$, since $2 < p < \infty$), we will also use the Banach space $C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u_{|\partial\Omega} = 0\}$. This is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\},$$

which has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n} < 0 \text{ for all } z \in \partial\Omega \right\}.$$

Here $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

Suppose $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with subcritical growth in $x \in \mathbb{R}$, that is

$$|f_0(z,x)| \leq a(z)(1 + |x|^{r-1}) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+$, $1 < r < p^* = \begin{cases} \frac{Np}{N-p} & , \text{if } N > p \\ +\infty & , \text{if } N \leq p \end{cases}$.

We set $F_0(z,x) = \int_0^x f_0(z,s)\, ds$ and consider the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_0(z,u(z))dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

The next proposition is a particular case of a more general result proved by Aizicovici-Papageorgiou-Staicu [2], which is based on the nonlinear regularity theory of Ladyzhenskaya-Uraltseva [19] and Liebermann [20].

Proposition 2.2. *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\bar{\Omega})$-minimizer of φ_0, that is there exists $\rho_0 > 0$ s.t.*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \textit{ for all } h \in C_0^1(\bar{\Omega}), \textit{ with } \|h\|_{C_0^1(\bar{\Omega})} \leq \rho_0,$$

then $u_0 \in C_0^{1,\alpha}(\bar{\Omega})$ with $\alpha \in (0,1)$ and it is a local $W_0^{1,p}(\Omega)-$minimizer of φ_0, that is there exists $\rho_1 > 0$ s.t.

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \textit{ for all } h \in W_0^{1,p}(\Omega), \textit{ with } \|h\| \leq \rho_1.$$

Let $h, g \in L^\infty(\Omega)$. We write $h \prec g$ if for every compact $K \subseteq \Omega$, we can find $\varepsilon = \varepsilon(K) > 0$ s.t. $h(z) + \varepsilon \leq g(z)$ for a.a. $z \in K$. The next proposition can be found in a more general form in Gasinski-Papageorgiou [16] and extends an earlier result of Arcoya-Ruiz [5] (see Proposition 2.6).

Proposition 2.3. *If $c > 0$, $h, g \in L^\infty(\Omega)$ with $h \prec g$ and $u, v \in C_0^1(\bar{\Omega})$ with $v \in \operatorname{int} C_+$ are solutions of*

$$-\Delta_p u(z) - \Delta u(z) + c|u(z)|^{p-2}u(z) = h(z) \textit{ in } \Omega,$$

$$-\Delta_p v(z) - \Delta v(z) + cv(z)^{p-1} = g(z) \textit{ in } \Omega,$$

then $v - u \in \operatorname{int} C_+$.

Consider the following nonlinear eigenvalue problem

$$-\Delta_p u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) \text{ in } \Omega,\ u_{|\partial\Omega} = 0 \quad (1 < p < \infty). \tag{2}$$

We say that $\hat{\lambda}$ is an eigenvalue $(-\Delta_p, W_0^{1,p}(\Omega))$, if problem (2) admits a nontrivial solution $\hat{u} \in W_0^{1,p}(\Omega)$, which is an eigenfunction corresponding to the eigenvalue $\hat{\lambda}$. There exists a smallest eigenvalue $\hat{\lambda}_1(p) > 0$, which is simple and isolated and admits the following variational characterization:

$$\hat{\lambda}_1(p) = \inf\left[\frac{\|Du\|_p^p}{\|u\|_p^p}:\ u \in W_0^{1,p}(\Omega),\ u \neq 0\right]. \tag{3}$$

The infimum in (3) is realized at the corresponding one dimensional eigenspace. It is clear from (3) that the elements of this eigenspace do not change sign. By $\hat{u}_1(p)$ we denote the L^p−normalized positive eigenfunction corresponding to $\hat{\lambda}_1(p)$. The nonlinear regularity theory (see Liebermann [20]) and the nonlinear maximum principle (see Pucci-Serrin [30]), imply that $\hat{u}_1(p) \in \operatorname{int} C_+$. The Ljusternik-Schnirelmann theory implies that $(-\Delta_p, W_0^{1,p}(\Omega))$ has a whole strictly increasing sequence of eigenvalues $\{\widehat{\lambda}_k(p)\}_{k\geq 1}$ such that $\widehat{\lambda}_k(p) \to +\infty$ as $k \to +\infty$. However, we do not know if this sequence exhausts the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. This is the case if $p = 2$ (linear eigenvalue problem, see below) or if $N = 1$ (ordinary differential equations). The second eigenvalue $\widehat{\lambda}_2(p)$ is well defined by

$$\widehat{\lambda}_2(p) = \inf\left[\widehat{\lambda}:\ \widehat{\lambda} \text{ is an eigenvalue of (2) and } \widehat{\lambda}_1(p) < \lambda\right].$$

It admits the following minimax characterization (see [12]).

Proposition 2.4. $\widehat{\lambda}_2(p) = \inf_{\widehat{\gamma}\in\widehat{\Gamma}} \max_{-1\leq t\leq 1} \|D\widehat{\gamma}(t)\|_2^2$, *where*

$$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1,1], M):\ \widehat{\gamma}(-1) = -\widehat{u}_1(p),\ \widehat{\gamma}(1) = \widehat{u}_1(p)\}$$

and $M = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p}$, *where* $\partial B_1^{L^p} = \{u \in L^p(\Omega):\ \|u\|_p = 1\}$.

For $p = 2$ we have a complete description of the eigenvalues of $(-\Delta, H_0^1(\Omega))$. They form a sequence $\{\widehat{\lambda}_k(2)\}_{k\geq 1} \subseteq (0, +\infty)$ such that $\widehat{\lambda}_k(2) \to +\infty$ as $k \to +\infty$. By $E(\widehat{\lambda}_k(2))$ we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k(2)$. We have

$$H_0^1(\Omega) = \overline{\oplus_{k\geq 1} E(\widehat{\lambda}_k(2))}$$

and the eigenspaces $E(\widehat{\lambda}_k(2))$ exhibit the unique continuation property (UCP for short), which says that, if $u \in E(\widehat{\lambda}_k(2))$ vanishes on a set of positive Lebesgue measure, then $u \equiv 0$. We have the following variational characterizations for the eigenvalues $\{\widehat{\lambda}_k(2)\}_{k\geq 1}$

$$\widehat{\lambda}_1(2) = \inf\left[\frac{\|Du\|_2^2}{\|u\|_2^2}:\ u \in H_0^1(\Omega),\ u \neq 0\right], \tag{4}$$

$$\begin{aligned}\widehat{\lambda}_k(2) &= \inf\left[\frac{\|Du\|_2^2}{\|u\|_2^2}:\ u \in \overline{\oplus_{n\geq k} E(\widehat{\lambda}_n(2))},\ u \neq 0\right] \\ &= \sup\left[\frac{\|Du\|_2^2}{\|u\|_2^2}:\ u \in \oplus_{n=1}^{k} E(\widehat{\lambda}_n(2)),\ u \neq 0\right] \text{ for } k \geq 2.\end{aligned} \tag{5}$$

The infimum and the supremum in (4) and (5) are realized on $E(\widehat{\lambda}_k(2))$. We will also use a weighted version of (2) when $p = 2$. So, let $m \in L^\infty(\Omega)$, $m \geq 0$, $m \neq 0$ and consider the following weighted eigenvalue problem

$$-\Delta u(z) = \tilde{\lambda} m(z) u(z) \text{ in } \Omega,\ u_{|\partial\Omega} = 0. \tag{6}$$

Problem (6) has also a sequence of eigenvalues $\{\hat{\lambda}_k(2,m)\}_{k\geq 1}$ such that $\widehat{\lambda}_k(2,m) \to +\infty$ as $k \to +\infty$. They have variational expressions analogous to (4) and (5) using the Rayleigh quotient $\frac{\|Du\|_2^2}{\int_\Omega mu^2dx}$ for all $u \in H_0^1(\Omega)$, $u \neq 0$ and the corresponding eigenspaces have the UCP. In fact as a consequence of the UCP, we have

$$m(z) \leq \hat{m}(z) \text{ a.e. in } \Omega,\ m \neq \hat{m} \Rightarrow \widehat{\lambda}_n(2,\hat{m}) < \widehat{\lambda}_n(2,m) \text{ for all } n \geq 1\,. \tag{7}$$

Finally, we mention that for all the above eigenvalue problems (linear end nonlinear alike), only the first eigenvalue has eigenfunctions of constant sign. All the other eigenvalues, have nodal eigenfunctions. Next, we briefly recall some basic facts about critical groups (Morse theory). So, let $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets

$$\varphi^c = \{u \in X : \varphi(u) \leq c\},\ K_\varphi = \{u \in X : \varphi'(u) = 0\},\ K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

Let (Y_1, Y_2) be a topological pair s.t. $Y_2 \subseteq Y_1 \subseteq X$, and $k \geq 0$ an integer. By $H_k(Y_1,\, Y_2)$ we denote the $k^{th}-$relative singular homology group for the topological pair (Y_1, Y_2) with integer coefficients. The critical groups of φ at an isolated $u \in K_\varphi^c$ are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap \mathcal{U},\, \varphi^c \cap \mathcal{U} \setminus \{u\}),\ \text{for all } k \geq 0,$$

with $\mathcal{U}$ being a neighborhood of u such that $K_\varphi \cap \varphi^c \cap \mathcal{U} = \{u\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the choice of the neighborhood $\mathcal{U}$ of u.

Suppose that φ satisfies the $PS-$condition and $-\infty < \inf \varphi(K_\varphi)$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \geq 0.$$

The second deformation theorem (see, for example Gasinski-Papageorgiou [15], (p.628)), implies that the above definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that $\varphi \in C^1(X)$ satisfies the PS-condition and that K_φ is finite. We set:

$$M(t,u) = \sum_{k\geq 0} \operatorname{rank} C_k(\varphi,u)t^k \text{ for all } t \in \mathbb{R} \ \text{ and all } u \in K_\varphi,$$

$$P(t,\infty) = \sum_{k\geq 0} \operatorname{rank} C_k(\varphi,\infty)t^k \text{ for all } t \in \mathbb{R}\,.$$

The Morse relation says that

$$\sum_{u\in K_\varphi} M(t,u) = P(t,\infty) + (1+t)Q(t)\,, \tag{8}$$

where $Q(t) = \sum_{k\geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$, with nonnegative integer coefficients. In the sequel for $1 < p < \infty$ and $A_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ (where $\frac{1}{p} + \frac{1}{p'} = 1$ and $W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))^*$), we denote the nonlinear map defined by

$$\langle A_p(u), y\rangle = \int_\Omega |Du|^{p-2}(Du, Dy)_{\mathbb{R}^N} dz \quad \text{for all } u, y \in W_0^{1,p}(\Omega)\,. \tag{9}$$

We have (see [27]):

Proposition 2.5. *If $1 < p < \infty$ and $A_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is defined by (9), then A_p is of type $(S)_+$, that is, if $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n\to+\infty} \langle A_p(u_n), u_n - u\rangle \le 0\,,$$

then $u_n \to u$ in $W_0^{1,p}(\Omega)$.

If $x \in \mathbb{R}$, then we set $x^\pm = \max\{\pm x, 0\}$ and for every $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u = u^+ - u^-,\ \ |u| = u^+ + u^- \text{ and } u^\pm \in W_0^{1,p}(\Omega)\,.$$

Given a measurable function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega)\,.$$

(the Nemytskii operator corresponding to h). By $|\cdot|_N$ and $|\cdot|$ we denote respectively the Lebesgue measure and the norm on $\mathbb{R}^N$, while $\|\cdot\|$ stands for the norm on $W_0^{1,p}(\Omega)$. By virtue of the Poincaré inequality, we have

$$\|u\| = \|Du\|_p \quad \text{for all } \ u \in W_0^{1,p}(\Omega)\,.$$

3. Solutions of Constant Sign

In this section we produce solutions of constant sign for problem (1). To this end, we introduce the following hypotheses on the reaction $f(z,x)$.

H_1: The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function s.t. $f(z,0) = 0$ for a.a. $z \in \Omega$, and

(i) $|f(z,x)| \le a(z)(1 + |x|^{p-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$;

(ii) $\limsup_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \le \hat\lambda_1(p)$ uniformly for a.a. $z \in \Omega$ and there exists $\xi_0 > 0$ s.t.

$$f(z,x)x - pF(z,x) \ge -\xi_0 \quad \text{for a.a } z \in \Omega, \ \text{all } x \in \mathbb{R};$$

(iii) there exists a function $\eta \in L^\infty(\Omega)_+$, s.t.

$$\hat\lambda_1(2) \le \eta(z) \text{ for a.a. } z \in \Omega,\ \ \eta \ne \hat\lambda_1(2)\,,$$

$$\eta(z) \le \liminf_{x\to 0} \frac{f(z,x)}{x} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) for a.a. $z \in \Omega$, $f(z,\cdot)$ is locally lower Lipschitz, that is for all $K \subseteq \mathbb{R}$ compact, there exists $c_K > 0$ s.t.

$$f(z,u) - f(z,v) \ge -c_K(u - v) \text{ for all } u, v \in K,\ \ u > v\,.$$

Remark 3.1. Hypothesis H_1(ii) implies that we can have resonance with respect to the principal eigenvalue $\hat{\lambda}_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$. The possibility of resonance, necessitates the second part of hypothesis H_1(ii). In fact, for equations driven by the p-Laplacian (see Jiu-Su [17] and Kyritsi-Papageorgiou [18]), this extra condition has the stronger form

$$\lim_{x\to\pm\infty}[f(z,x)x - pF(z,x)] = +\infty \text{ uniformly for a.a. } z \in \Omega\,.$$

Hypothesis H_1(iii) says that at zero we have nonuniform nonresonance with respect to $\hat{\lambda}_1(2) > 0$. In fact the hypothesis permits also nonlinearities with "concave" terms, namely terms which are $(\tau - 1)-$sublinear with $\tau \in (1, 2)$. Finally note that hypothesis H_1(iv) is satisfied if, for example, for a.a. $z \in \Omega$, $f(z, \cdot) \in C^1(\mathbb{R})$ and $f'_x(z, \cdot)$ is bounded on bounded sets.

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z, u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Evidently $\varphi \in C^1(W_0^{1,p}(\Omega))$. Also, we consider the positive and negative truncations of $f(z, \cdot)$, namely the Carathéodory functions

$$f_\pm(z, x) = f(z, \pm x^\pm).$$

We set

$$F_\pm(z, x) = \textstyle\int_0^x f_\pm(z, s)\,ds$$

and consider the C^1-functionals $\varphi^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi^\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_\pm(z, u(z))\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Proposition 3.2. *If hypotheses H_1 hold, then the functionals φ and $\varphi^\pm$ are coercive.*

Proof. We present the proof for the functional φ, the proofs for $\varphi^\pm$ being similar. We argue by contradiction. So, suppose that φ is not coercive. Then we can find $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ and $M_1 > 0$ s.t.

$$\|u_n\| \to \infty \text{ and } \varphi(u_n) \leq M_1 \text{ for all } n \geq 1\,. \tag{10}$$

We have

$$\varphi(u_n) = \frac{1}{p}\|Du_n\|_p^p + \frac{1}{2}\|Du_n\|_2^2 - \int_\Omega F(z, u_n(z))dz \leq M_1 \text{ for all } n \geq 1\,. \tag{11}$$

Let $y_n = \frac{u_n}{\|u_n\|}$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so by passing to a suitable subsequence if necessary, we may assume that

$$y_n \rightharpoonup y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega)\,. \tag{12}$$

From (11) we have

$$\frac{1}{p}\|Dy_n\|_p^p + \frac{\|Dy_n\|_2^2}{2\|u_n\|^{p-2}} - \int_\Omega \frac{F(z,u_n(z))}{\|u_n\|^p}dz \le \frac{M_1}{\|u_n\|^p} \text{ for all } n \ge 1\,. \tag{13}$$

Hypothesis H_1(i) implies that

$$|F(z,x)| \le \hat{a}(z)(1+|x|^p) \text{ for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R},\ \text{with } \hat{a} \in L^\infty(\Omega)_+,$$

so that $\left\{\frac{F(\cdot,u_n(\cdot))}{\|u_n\|^p}\right\}_{n\ge 1} \subseteq L^1(\Omega)$ is uniformly integrable.

By the Dunford-Pettis theorem, we may assume that

$$\frac{F(\cdot,u_n(\cdot))}{\|u_n\|^p} \rightharpoonup g \text{ in } L^1(\Omega)\,. \tag{14}$$

Hypothesis H_1(ii) implies that

$$\limsup_{x\to\pm\infty} \frac{F(z,x)}{|x|^p} \le \frac{1}{p}\hat{\lambda}_1(p) \text{ uniformly for a.a } z \in \Omega\,. \tag{15}$$

Using (15) and reasoning as in the proof of Proposition 16 in Aizicovici-Papageorgiou-Staicu [1], we show that

$$g = \frac{1}{p}\vartheta|y|^p \text{ with } \vartheta \in L^\infty(\Omega),\ \vartheta(z) \le \hat{\lambda}_1(p) \text{ a.e. in } \Omega\,. \tag{16}$$

So, if in (13) we pass to the limit as $n \to \infty$ and use (12), (14), (16) and the fact that $2 < p$, we obtain

$$\frac{1}{p}\|Dy\|_p^p \le \frac{1}{p}\int_\Omega \vartheta(z)|y|^p dz\,. \tag{17}$$

If $\vartheta \ne \hat{\lambda}_1(p)$, then from (17) and Lemma 5.1.3 (p.356) of [27] we have

$$c_0\|y\|^p \le 0,\ \text{hence } y = 0\,.$$

From (13) it follows that $y_n \to 0$ in $W_0^{1,p}(\Omega)$, which contradicts the fact that $\|y_n\| = 1$ for all $n \ge 1$.

Next we assume that $\vartheta(z) = \hat{\lambda}_1(p)$ for a.a. $z \in \Omega$. From (17) and (3), we have

$$\|Dy\|_p^p = \hat{\lambda}_1(p)\|y\|_p^p,\ \text{namely } y = \xi\hat{u}_1(p) \text{ with } \xi \in \mathbb{R}.$$

If $\xi = 0$, then $y = 0$ and arguing as above we obtain a contradiction. If $\xi \ne 0$, then without loss of generality, we may assume that $\xi > 0$ (the analysis is similar if we assume that $\xi < 0$). Since $\hat{u}_1(p) \in \operatorname{int} C_+$, then also $y \in \operatorname{int} C_+$ and so we have

$$u_n(z) \to +\infty \text{ for a.a } z \in \Omega\,. \tag{18}$$

For a.a. $z \in \Omega$ and all $s \neq 0$, we have

$$\frac{d}{ds}\frac{F(z,s)}{|s|^p} = \frac{f(z,s)|s|^p - p|s|^{p-2}sF(z,s)}{|s|^{2p}} = \frac{|s|^{p-2}s[f(z,s)s - pF(z,s)]}{|s|^{2p}}$$
$$= \frac{f(z,s)s - pF(z,s)}{|s|^p s}.$$

For $s > 0$, using hypothesis H_1(ii), we have

$$\frac{d}{ds}\frac{F(z,s)}{s^p} = \frac{f(z,s)s - pF(z,s)}{s^{p+1}} \geq \frac{-\xi_0}{s^{p+1}} \quad \text{for a.a } z \in \Omega, \text{ so}$$

$$\frac{F(z,y)}{y^p} - \frac{F(z,x)}{x^p} \geq \frac{\xi_0}{p}\left[\frac{1}{y^p} - \frac{1}{x^p}\right] \quad \text{for a.a } z \in \Omega, \text{ all } y \geq x \geq 0.$$

Let $y \to \infty$. Using (15), we obtain

$$\frac{\hat{\lambda}_1(p)}{p} - \frac{F(z,x)}{x^p} \geq -\frac{\xi_0}{p}\frac{1}{x^p} \quad \text{for a.a } z \in \Omega, \text{ all } x \geq 0,$$

that is

$$\hat{\lambda}_1(p)x^p - pF(z,x) \geq -\xi_0 \quad \text{for a.a } z \in \Omega, \text{ all } x \geq 0, \tag{19}$$

For $s < 0$ using hypothesis H_1(ii), we have

$$\frac{d}{ds}\frac{F(z,s)}{|s|^p} = \frac{f(z,s)s - pF(z,s)}{|s|^p s} \leq \frac{-\xi_0}{|s|^p s} \quad \text{for a.a } z \in \Omega, \text{ so}$$

$$\frac{F(z,x)}{|x|^p} - \frac{F(z,y)}{|y|^p} \leq \frac{-\xi_0}{p}\left[\frac{1}{|y|^p} - \frac{1}{|x|^p}\right] \text{ for a.a } z \in \Omega, \text{ all } y \leq x \leq 0.$$

Let $y \to -\infty$. Using (15), we obtain

$$\frac{F(z,x)}{|x|^p} - \frac{\hat{\lambda}_1(p)}{p} \leq \frac{\xi_0}{p}\frac{1}{|x|^p} \quad \text{for a.a } z \in \Omega, \text{ all } x \leq 0,$$

that is

$$\hat{\lambda}_1(p)|x|^p - pF(z,x) \geq -\xi_0 \quad \text{for a.a } z \in \Omega, \text{ all } x \leq 0. \tag{20}$$

From (11) we have

$$\|Du_n\|_p^p + \frac{p}{2}\|Du_n\|_2^2 - \int_\Omega pF(z,u_n(z))\,dz \leq pM_1 \quad \text{for all } n \geq 1,$$

and from (3)

$$\int_\Omega \left[\hat{\lambda}_1(p)(u_n^+)^p - pF(z,(u_n)^+)\right] dz + \int_\Omega \left[\hat{\lambda}_1(p)(u_n^-)^p - pF(z,(u_n)^-)\right] dz$$
$$+\frac{p}{2}\|Du_n\|_2^2 \leq pM_1 \quad \text{for all } n \geq 1,$$

then by (4), (19) and (20), we obtain

$$\int_\Omega u_n^2 dz \leq M_2, \quad \text{for some } M_2 > 0, \text{ all } n \geq 1\,. \tag{21}$$

But from (18) and Fatou's lemma we have

$$\int_\Omega u_n^2 dz \to +\infty, \text{ as } n \to \infty\,. \tag{22}$$

Comparing (21) and (22), we reach a contradiction. So, we have proved that the functional φ is coercive. Similarly we show the coercivity of the functionals $\varphi^\pm$. $\square$

Remark 3.3. From (19) and (20) we see that at $\pm\infty$ the resonance with respect to $\hat{\lambda}_1(p)$ occurs from the left of the principal eigenvalue and this leads to a coercive energy functional.

Now we are ready to produce two nontrivial constant sign solutions for problem (1).

Proposition 3.4. *If hypotheses H_1 hold, then problem* (1) *has at least two nontrivial solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ both local minimizers of the functional φ.*

Proof. From Proposition 3.2 we know that the functional φ^+ is coercive. Also, using the Sobolev embedding theorem, we can check that φ^+ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ s.t.

$$\varphi^+(u_0) = \min[\varphi^+(u) : \ u \in W_0^{1,p}(\Omega)]. \tag{23}$$

By virtue of hypothesis H_1(iii), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ s.t.

$$F(z,x) \geq \frac{1}{2}(\eta(z) - \varepsilon)x^2 \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta\,. \tag{24}$$

Let $t \in (0,1)$ be small s.t. $t\hat{u}_1(2)(z) \in [0,\delta]$ for all $z \in \bar{\Omega}$. Then using (24) and the fact that $\|\hat{u}_1(2)\|_2 = 1$

$$\begin{aligned}\varphi^+(t\hat{u}_1(2)) &= \frac{t^p}{p}\|D\hat{u}_1(2)\|_p^p + \frac{t^2}{2}\hat{\lambda}_1(2) - \int_\Omega F(z, t\hat{u}_1(2))dz \\ &\leq \frac{t^p}{p}\|D\hat{u}_1(2)\|_p^p + \frac{t^2}{2}\left[\int_\Omega (\hat{\lambda}_1(2) - \eta(z))\hat{u}_1(2)(z)^2 dz + \varepsilon\right]. \end{aligned} \tag{25}$$

Since $\hat{u}_1(2) \in \operatorname{int} C_+$, from the hypothesis on η in H_1(iii), we have

$$\hat{\xi} = \int_\Omega [\eta(z) - \hat{\lambda}_1(2)]\hat{u}_1(2)^2 dx > 0\,.$$

So, if we choose $\varepsilon \in (0,\hat{\xi})$, then from (25) we have

$$\varphi^+(t\hat{u}_1(2)) \le \frac{t^p}{p}\|D\hat{u}_1(2)\|_p^p - \frac{t^2}{2}c_1\,, \tag{26}$$

with $c_1 = \hat{\xi} - \varepsilon > 0$. Since $2 < p$, from (26) we see that by choosing $t \in (0,1)$ even smaller if necessary, we have

$$\varphi^+(t\hat{u}_1(2)) < 0, \text{ so, from (23) it follows}$$

$\varphi^+(u_0) < 0 = \varphi^+(0)$, so $u_0 \neq 0$. From (23), we have $(\varphi^+)'(u_0) = 0$, that is

$$A_p(u_0) + A(u_0) = N_{f_+}(u_0)\,. \tag{27}$$

On (27) we act with $-u_0^- \in W_0^{1,p}(\Omega)$. Then

$$\|Du_0^-\|_p^p + \|Du_0^-\|_2^2 = 0, \text{ hence } u_0 \ge 0,\ u_0 \ne 0.$$

Therefore equation (27) becomes

$$A_p(u_0) + A(u_0) = N_f(u_0)\,, \text{ that is}$$

$$-\Delta_p u_0(z) - \Delta u_0(z) = f(z,u_0(z)) \text{ a.e. in } \Omega,\ u_{0|\partial\Omega} = 0\,. \tag{28}$$

From Ladyzhenskaya-Uraltseva [19] (p. 286), we have that $u_0 \in L^\infty(\Omega)$. Then we can apply Theorem 1 of Lieberman [20] and infer that $u_0 \in C_+ \setminus \{0\}$.

Let $a : \mathbb{R}^N \to \mathbb{R}$ be the C^1-map defined by

$$a(y) = |y|^{p-2}y + y \text{ for all } y \in \mathbb{R}^N,\ p > 2\,.$$

We have

$$\nabla a(y) = |y|^{p-2}\left(I + (p-2)\frac{y\otimes y}{|y|^2}\right) + I \text{ for all } y \in \mathbb{R}^N\,.$$

Then

$$(\nabla a(y)\xi,\xi)_{\mathbb{R}^N} \ge |\xi|^2 \text{ for all } y,\xi \in \mathbb{R}^N\,.$$

So, we can apply the tangency principle of Pucci-Serrin [30] (p.35) and obtain that

$$u_0(z) > 0 \text{ for all } z \in \Omega\,. \tag{29}$$

From hypotheses H_1(i), (iii), we see that

$$f(z,x)x \ge c_2x^2 - c_3|x|^p \text{ for a.a } z \in \Omega,\ \text{all } x \in \mathbb{R},\ \text{some } c_2,\, c_3 > 0\,,\ \text{that is}$$

$$f(z,x)x + c_3|x|^p \ge c_2x^2 \ge 0 \text{ for a.a } z \in \Omega,\ \text{all } x \in \mathbb{R}\,. \tag{30}$$

Then from (28) and (30) we have

$$-\Delta_p u_0(z) - \Delta u_0(z) + c_3u_0(z)|^{p-1} = f(z,u_0(z)) + c_3u_0(z)^{p-1} \ge 0 \text{ for a.a. } z \in \Omega\,,$$

so
$$\Delta_p u_0(z) + \Delta u_0(z) \le c_3u_0(z)^{p-1} \text{ a.e. in } \Omega\,.$$

From this and (29), we see that we can apply the boundary point theorem of Pucci-Serrin [30] (p.120) and have $u_0 \in \operatorname{int} C_+$.

Since $\varphi_{|C_+} = \varphi^+_{|C_+}$ it follows that $u_0 \in \operatorname{int} C_+$ is a local $C^1_0(\bar{\Omega})-$minimizer of φ, hence by Proposition 2.2 u_0 is also a local $W^{1,p}_0(\Omega)-$minimizer of φ. Similarly, using this time the functional φ^-, we obtain another constant sign solution $v_0 \in -\operatorname{int} C_+$ which is a local minimizer of φ. $\square$

In fact by strengthening the conditions on $f(z,\cdot)$ near zero, we can generate extremal constant sign solutions for problem (1), that is the smallest positive and the biggest negative solutions of (1). These extremal constant sign solutions will be used in Section 4, in order to produce nodal solutions.

The new stronger conditions on $f(z,x)$ are the following:

H_2: The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function s.t. $f(z,0) = 0$ for a.a. $z \in \Omega$, hypotheses H_2 (i), (ii), (iv) are the same as the corresponding hypotheses H_1 (i), (ii), (iv) and

(iii) there exist functions η, $\hat{\eta} \in L^\infty(\Omega)_+$, s.t.

$$\hat{\lambda}_1(2) \leq \eta(z) \leq \hat{\eta}(z) \text{ a.e. in } \Omega,\ \eta \neq \hat{\lambda}_1(2).$$

$$\eta(z) \leq \liminf_{x\to 0} \frac{f(z,x)}{x} \leq \limsup_{x\to 0} \frac{f(z,x)}{x} \leq \hat{\eta}(z) \text{ uniformly for a.a. } z \in \Omega.$$

Remark 3.5. Now we require that the reaction $f(z,\cdot)$ is linear near zero.

Let S_+ (respectively S_-) be the set of positive (respectively, negative) solutions of (1). From Proposition 3.4 and its proof, we have that

$$\emptyset \neq S_+ \subseteq \operatorname{int} C_+ \text{ and } \emptyset \neq S_- \subseteq -\operatorname{int} C_+\,.$$

Moreover, as in [14] we have that S_+ is downward directed (that is, if $u_1,\ u_2 \in S_+$ then there exists $u \in S_+$ s.t. $u \leq u_1$, $u \leq u_2$) and S_- is upward directed (that is, if $v_1,\ v_2 \in S_-$ then there exists $v \in S_-$ s.t. $v_1 \leq v$, $v_2 \leq v$).

Proposition 3.6. *If hypotheses H_2 hold, then problem (1) has a smallest positive solution $u_* \in \operatorname{int} C_+$ and a biggest negative solution $v_* \in -\operatorname{int} C_+$.*

Proof. Since we are looking for the smallest positive solution and the set S_+ is downward directed, without loss of generality, we may assume that

$$\|u\|_\infty \leq M_3 \text{ for some } M_3 > 0,\ \text{all } u \in S_+\,. \tag{31}$$

From Dunford-Schwartz [13] (p.336), we know that we can find $\{u_n\}_{n\geq 1} \subseteq S_+$ s.t. $\inf S_+ = \inf_{n\geq 1} u_n$. We have

$$A_p(u_n) + A(u_n) = N_f(u_n) \text{ for all } n \geq 1. \tag{32}$$

Acting with $u_n \in \operatorname{int} C_+$ and using (31), we infer that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}_0(\Omega)$ is bounded. So, we may assume that

$$u_n \rightharpoonup u_* \text{ in } W^{1,p}_0(\Omega) \text{ and } u_n \to u_* \text{ in } L^p(\Omega). \tag{33}$$

On (32), we act with $u_n - u_* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (33). Then

$$\lim_{n\to\infty} [\langle A_p(u_n), u_n - u_* \rangle + \langle A(u_n), u_n - u_* \rangle] = 0,$$

and using the monotonicty of A we deduce the inequality

$$\limsup_{n\to\infty} [\langle A_p(u_n), u_n - u_* \rangle + \langle A(u), u_n - u_* \rangle] \le 0$$

that leads to

$$\limsup_{n\to\infty} \langle A_p(u_n), u_n - u_* \rangle \le 0\,.$$

So, by Proposition 2.5,

$$u_n \to u_* \in W_0^{1,p}(\Omega),\ \ u_* \ge 0\,. \tag{34}$$

So, if in (32) we pass to the limit as $n \to \infty$ and use (34), then $A_p(u_*) + A(u_*) = N_f(u_*)$, and $u_* \in S_+ \cup \{0\}$. We need to show that $u_* \neq 0$. We argue indirectly. So, suppose that $u_* = 0$. Then, from (34) we have

$$u_n \to 0 \in W_0^{1,p}(\Omega)\,. \tag{35}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \ge 1$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $n \ge 1$ and so we may assume that

$$y_n \rightharpoonup y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega),\ \ y \ge 0\,. \tag{36}$$

From (32) we have

$$\|u_n\|^{p-2} A_p(y_n) + A(y_n) = \frac{N_f(u_n)}{\|u_n\|} \text{ for all } n \ge 1. \tag{37}$$

Hypotheses H_2(i), (iii) imply that

$$|f(z,x)| \le c_4(|x| + |x|^{p-1}) \text{ for a.a. } z \in \Omega,\ \text{ all } x \in \mathbb{R},\ \text{ for some } c_4 > 0,$$

so that

$$\left\{ \frac{N_f(u_n)}{\|u_n\|} \right\}_{n\ge1} \subseteq L^2(\Omega) \tag{38}$$

is bounded. From (38) and hypothesis H_2(ii), we have, at least for a subsequence, that

$$\frac{N_f(u_n)}{\|u_n\|} \rightharpoonup \eta_0 y \text{ in } L^2(\Omega), \text{with } \eta(z) \le \eta_0(z) \le \hat{\eta}(z) \text{ for a.a. } z \in \Omega\,. \tag{39}$$

From (36) we have that $\{A_p(y_n)\}_{n\ge1} \subseteq W^{-1,p'}(\Omega)$ is bounded. Hence, from (35) and since $2 < p$

$$\|u_n\|^{p-2} A_p(y_n) \to 0 \text{ in } W^{-1,p'}(\Omega)\,. \tag{40}$$

So, if in (37) we pass to the limit as $n \to \infty$ and use (36), (39) and (40), then $A(y) = \eta_0 y$, that is

$$-\Delta y(z) = \eta_0(z)y(z) \text{ a.e. in } \Omega,\ y_{|\partial\Omega} = 0,\ y \geq 0\,, y \neq 0. \tag{41}$$

From (7) and the hypothesis on η, we have $\hat{\lambda}_1(2, \eta_0) < \hat{\lambda}_1(2, \hat{\lambda}_1(2)) = 1$. Then from (41) it follows that $y = 0$. On the other hand from (37), we have

$$\|u_n\|^{p-2}(-\Delta_p y_n(z)) - \Delta y_n(z) = \frac{f(z, u_n(z))}{\|u_n\|} \text{ a.e. in } \Omega,\ y_{n|\partial\Omega} = 0. \tag{42}$$

From (35) we have $\|u_n\|^{p-2} \to 0$, so from [19] we know that we can find $M_4 > 0$ s.t. $\|y_n\|_\infty \leq M_4$ for all $n \geq 1$. Then we can use Theorem 1 of [20] and produce $\gamma \in (0, 1)$ and $M_5 > 0$ s.t.

$$y_n \in C_0^{1,\gamma}(\bar{\Omega}) \text{ and } \|y_n\|_{C_0^{1,\gamma}(\bar{\Omega})} \leq M_5 \text{ for all } n \geq 1\,.$$

Exploiting the compact embedding of $C_0^{1,\gamma}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$ and using (36), we have $y_n \to y = 0$ in $C_0^{1,\gamma}(\bar{\Omega})$, so

$$y_n \to 0 \text{ in } W_0^{1,p}(\Omega)\,,$$

which contradicts the fact that $\|y_n\| = 1$ for all $n \geq 1$. Therefore $u_* \neq 0$ and so

$$u_* \in S_+ \subseteq \operatorname{int} C_+, u_* = \inf S_+\,.$$

In a similar fashion, using the set S_-, we produce $v_* \in -\operatorname{int} C_+$ the biggest negative solution of (1). $\square$

4. Nodal Solutions

In this section, using the extremal constant sign solutions of (1) produced in Section 3, we will generate nodal solutions. Some of these techniques have also been employed in proving the existence of three nontrivial solutions in the works of Carl-Motreanu (see, for instance, [7, 8]).

We need to strengthen further the conditions on $f(z, \cdot)$ near zero, the new hypotheses on $f(z, x)$ are the following:

H_3: The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function s.t. $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses H_3(i), (ii), (iv) are the same as the corresponding hypotheses H_1(i), (ii), (iv) and

(iii) there exist functions η, $\hat{\eta} \in L^\infty(\Omega)_+$, s.t.

$$\hat{\lambda}_2(2) < \eta(z) \leq \hat{\eta}(z) \text{ for a.a. } z \in \Omega\,,$$

$$\eta(z) \leq \liminf_{x\to 0} \frac{f(z, x)}{x} \leq \limsup_{x\to 0} \frac{f(z, x)}{x} \leq \hat{\eta}(z) \text{ uniformly for a.a. } z \in \Omega.$$

In what follows $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are the two extremal constant sign solutions produced in Proposition 3.6.

Proposition 4.1. *If hypotheses* H_3 *hold, then problem* (1) *admits a nodal solution* $y_0 \in [v_*, u_*] \bigcap C_0^1(\bar{\Omega})$.

Proof. We introduce the following Carathéodory truncation of the reaction $f(z, \cdot)$:

$$\hat{f}(z,x) = \begin{cases} f(z, v_*(z)) & \text{if}\, x < v_*(z) \\ f(z,x) & \text{if}\, v_*(z) \leq x \leq u_*(z), \\ f(z, u_*(z)) & \text{if}\, u_*(z) < x. \end{cases} \tag{43}$$

Let $\hat{F}(z,x) = \int_0^x \hat{f}(z,s)\, ds$ and consider the C^1-functional $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \hat{F}(z, u(z))dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Also, we consider the positive and negative truncations of $\hat{f}(z, \cdot)$, namely the Carathéodory functions

$$\hat{f}_\pm(t,x) = \hat{f}(z, \pm x^\pm)\,.$$

We set $\hat{F}_\pm(t,x) = \int_0^x \hat{f}_\pm(z,s)ds$ and consider the C^1-functionals $\psi^\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi^\pm(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \hat{F}_\pm(z, u(z))dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Claim 1: $K_\psi \subseteq [v_*, u_*] = \{u \in W_0^{1,p}(\Omega) : v_*(z) \leq u(z) \leq u_*(z) \text{ a.e. in } \Omega\}$ and $K_{\psi^+} = \{0, u_*\}$, $K_{\psi^-} = \{v_*, 0\}$. Let $u \in K_\psi$. Then we have

$$A_p(u) + A(u) = N_{\hat{f}}(u)\,. \tag{44}$$

On (44) first we act with $(u - u_*)^+ \in W_0^{1,p}(\Omega)$. We use (43) and the fact that $u^* \in S_+$ and we obtain

$$\begin{aligned} \langle A_p(u), (u-u_*)^+\rangle + \langle A(u), (u-u_*)^+\rangle = & \int_\Omega f(z,u_*)(u-u_*)^+dz \\ & + \langle A_p(u_*), (u-u_*)^+\rangle + \langle A(u), (u-u_*)^+\rangle\,, \end{aligned}$$

so $$\langle A_p(u) - A_p(u_*), (u-u_*)^+\rangle + \|D(u-u_*)^+\|_2^2 = 0\,,$$

and $|\{u > u_*\}|_N = 0$, hence $u \leq u_*$.

Similarly, acting on (44) with $(v_* - u)^+ \in W_0^{1,p}(\Omega)$, we show that $v_* \leq u$. Hence $u \in [v_*, u_*]$ and so $K_\psi \subseteq [v_*, u_*]$. In a similar fashion we show that

$$K_{\psi^+} \subseteq [0, u_*] \text{ and } K_{\psi^-} \subseteq [v_*, 0]\,.$$

The extremality of the solutions u_* and v_* implies that

$$K_{\psi^+} = \{0, u_*\} \text{ and } K_{\psi^-} = \{v_*, 0\}\,.$$

This proves Claim 1.

Claim 2: $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are both local minimizers of ψ.

From (43) it is clear from that ψ^+ is coercive and sequentially weakly lower semicontinuous. So, we can find $\hat{u}_* \in W_0^{1,p}(\Omega)$ s.t.

$$\psi^+(\hat{u}_*) = \inf[\psi^+(u) : \ u \in W_0^{1,p}(\Omega)]. \tag{45}$$

As in the proof of Proposition 3.4, using hypothesis H_3(iii), we show that for $t \in (0,1)$ small, at least such that $t\hat{u}_1(2) \leq u_*$ (recall that $u_* \in \operatorname{int} C_+$ and use Lemma 3.3 of [14]), we have $\psi^+(t\hat{u}_1(2)) < 0$, hence $\psi^+(\hat{u}_*) < 0 = \psi^+(0)$, and $\hat{u}_* \neq 0$. From (45) we have $\hat{u}_* \in K_{\psi^+}$ and so by virtue of Claim 1, $\hat{u}_* = u_* \in \operatorname{int} C_+$. Since $\psi^+_{|C_+} = \psi_{|C_+}$, we see that u_* is a local $C_0^1(\bar{\Omega})$-minimizer of ψ and so by Proposition 2.2 it is also a $W_0^{1,p}(\Omega)$-minimizer of ψ.

Similarly for $v_* \in -\operatorname{int} C_+$, using this time the functional ψ^-, proving Claim 2.

Without loss of generality we may assume that $\psi(v_*) \leq \psi(u_*)$ (the analysis is similar if the opposite inequality is true). Also, we assume that K_ψ is finite or otherwise, we already have an infinity of nodal solutions. By virtue of Claim 2, we can find $\rho \in (0,1)$ small s.t.

$$\psi(v_*) \leq \psi(u_*) < \inf[\, \psi(u) \ : \ \|u - u_*\| = \rho\,] = m_\rho, \quad \|v_* - u_*\| > \rho. \tag{46}$$

Recall that ψ is coercive (see (43)), therefore it satisfies the $PS-$condition. This fact and (46) permit the use of Theorem 2.1. So, we can find $y_0 \in W_0^{1,p}(\Omega)$ s.t.

$$y_0 \in K_\psi \text{ and } m_\rho \leq \psi(y_0)\,. \tag{47}$$

From (47) and Claim 1, we have that y_0 is a solution of problem (1) and the nonlinear regularity theory implies that $y_0 \in [v_*, u_*] \cap C_0^1(\bar{\Omega})$. Also from (46) and (47), we have $y_0 \notin \{u_*, v_*\}$. It remains to show that $y_0 \neq 0$. Indeed since $y_0 \in [v_*, u_*]$, by virtue of the extremality of $u_* \in \operatorname{int} C_+$ and of $v_* \in -\operatorname{int} C_+$, we can conclude that y_0 is nodal. From the mountain pass theorem (see Theorem 2.1), we have

$$\psi(y_0) = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \psi(\gamma(t)), \tag{48}$$

where

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) \ : \ \gamma(0) = v_*, \, \gamma(1) = u_*\}.$$

According to (48), in order to establish the nontriviality of y_0, it suffices to produce a path $\gamma_* \in \Gamma$ s.t. $\psi_{|\gamma_*} < 0$. To this end, we define $M = H_0^1(\Omega) \cap \partial B_1^{L^2}$ and $M_c = M \cap C_0^1(\bar{\Omega})$ (here $\partial B_1^{L^2} = \{u \in L^2(\Omega) : \ \|u\|_2 = 1\}$). Using these manifolds, we introduce the following sets of paths:

$$\hat{\Gamma} = \{\hat{\gamma} \in C([-1,1], M) : \ \hat{\gamma}(-1) = -\hat{u}_1(2), \ \hat{\gamma}(1) = \hat{u}_1(2)\} \text{ and}$$

$$\hat{\Gamma}_c = \{\hat{\gamma} \in C([-1,1], M_c) : \ \hat{\gamma}(-1) = -\hat{u}_1(2), \ \hat{\gamma}(1) = \hat{u}_1(2)\}\,.$$

Claim 3: $\hat{\Gamma}_c$ is dense in $\hat{\Gamma}$ (see also [6]).

Let $\hat{\gamma} \in \hat{\Gamma}$. For every $n \geq 1$, we introduce the multifunction $T_n : [-1,1] \to 2^{\hat{C}_0^1(\bar{\Omega})}$ defined by

$$T_n(t) = \begin{cases} \{-\hat{u}_1(2)\} & \text{if } t = -1 \\ \{u \in C_0^1(\bar{\Omega}) : \; \|u(t) - \hat{\gamma}(t)\| < \frac{1}{n}\} & \text{if } t \in (-1,1) \\ \{\hat{u}_1(2)\} & \text{if } t = 1\ . \end{cases}$$

Evidently T_n has nonempty and convex values and as a multifunction is lower semicontinuous (see Papageorgiou-Kyritsi [27], (p.457)). Also, the values of $T_n(\cdot)$ are either open sets (for $t \in (-1,1)$) or singletons (for $t = \pm 1$). So, we can apply Theorem 3.1‴ of Michael [23] and obtain a continuous map $\hat{\sigma}_n$ s.t. $\hat{\sigma}_n(t) \in T_n(t)$ for all $t \in [-1,1]$. We have

$$\|\hat{\sigma}_n(t) - \hat{\gamma}(t)\| < \frac{1}{n}, \text{ for all } t \in [-1,1], \; \hat{\sigma}_n(\pm 1) = \pm \hat{u}_1(2)\,. \tag{49}$$

It follows that

$$\|\hat{\sigma}_n(t)\|_p \to \|\hat{\gamma}(t)\|_p = 1, \text{ uniformly in } t \in [-1,1]\,. \tag{50}$$

Therefore, for $n \geq n_0 \geq 1$ we can define

$$\sigma_n(t) = \frac{\hat{\sigma}_n(t)}{\|\hat{\sigma}_n(t)\|_p} \quad \text{for all } t \in [-1,1]\,,$$

and we have $\sigma_n \in \hat{\Gamma}_c$ for $n \geq n_0$. Also, thanks to (49) and (50) we can find $c_5 > 0$ such that:

$$\begin{aligned} \|\sigma_n(t) - \hat{\gamma}(t)\| &\leq \|\sigma_n(t) - \hat{\sigma}_n(t)\| + \|\hat{\sigma}_n(t) - \hat{\gamma}(t)\| \\ &\leq \frac{|1 - \|\hat{\sigma}_n(t)\|_p|}{\|\hat{\sigma}_n(t)\|_p}\|\hat{\sigma}_n(t)\| + \frac{1}{n} \\ &\leq c_5 |1 - \|\hat{\sigma}_n(t)\|_p| + \frac{1}{n} \quad \text{for all } t \in [-1,1]\,. \end{aligned} \tag{51}$$

Note that $\|\hat{\sigma}_n(t)\|_p = 1$ so, using (49), we can find $c_6 > 0$ s.t. for all $n \geq 1$ one has

$$\begin{aligned} &\max_{-1 \leq t \leq 1} |1 - \|\hat{\sigma}_n(t)\|_p| = \max_{-1 \leq t \leq 1} |\|\hat{\gamma}(t)\|_p - \|\hat{\sigma}_n(t)\|_p| \\ &\leq \max_{-1 \leq t \leq 1} \|\hat{\gamma}(t) - \hat{\sigma}_n(t)\|_p \leq c_6 \max_{-1 \leq t \leq 1} \|\hat{\gamma}(t) - \hat{\sigma}_n(t)\| < \frac{c_6}{n}\,. \end{aligned} \tag{52}$$

Then (51) and (52) imply that

$$\lim_{n \to +\infty} \max_{-1 \leq t \leq 1} \|\tilde{\sigma}_n(t) - \hat{\gamma}(t)\| = 0\,.$$

Since $\sigma_n \in \hat{\Gamma}_c$ for all $n \geq n_0$, we conclude that $\hat{\Gamma}_c$ is dense in $\hat{\Gamma}$.

This proves Claim 3. Using Claim 3 and Proposition 2.4 we see that given $\varepsilon > 0$, we can find $\hat{\gamma}_0 \in \hat{\Gamma}_c$ such that

$$\|D\hat{\gamma}_0(t)\|_2^2 \leq \hat{\lambda}_2(2) + \varepsilon \text{ for all } t \in [-1, 1]\,. \tag{53}$$

Hypotheses H_3(iii) implies that we can find $\delta = \delta(\varepsilon) > 0$ s.t.

$$F(z, x) \geq \frac{\eta(z) - \varepsilon}{2}|x|^2 \ \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta\,. \tag{54}$$

Since $\hat{\gamma}_0 \in \hat{\Gamma}_c$, we have that $\gamma_0([-1, 1]) \subseteq C_0^1(\bar{\Omega})$ is compact. Also, recall that $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$. So using Lemma 3.3 of Filippakis-Kristàly-Papageorgiou [14], we can find $s \in (0, 1)$ small s.t.

$$s\hat{\gamma}_0(t) \in [v_*, u_*], \ |s\hat{\gamma}_0(t)(z)| \leq \delta \text{ for all } z \in \bar{\Omega}, \text{ all } t \in [-1, 1]\,. \tag{55}$$

Also, we can find $M_6 > 0$ s.t.

$$\|D\hat{\gamma}_0(t)\|_p^p \leq M_6 \ \text{ for all } t \in [-1, 1]\,. \tag{56}$$

Then for all $t \in [-1, 1]$, if we use (53), (54), (55), (56) and recall that $\|\tilde{\gamma}(t)\|_2 = 1$, then we have

$$\begin{aligned}
\psi(s\hat{\gamma}_0(t)) &= \frac{s^p}{p}\|D\hat{\gamma}_0(t)\|_p^p + \frac{s^2}{2}\|D\hat{\gamma}_0(t)\|_2^2 - \int_\Omega \hat{F}(z, s\hat{\gamma}_0(t))dz \\
&\leq \frac{s^p}{p}M_6 + \frac{s^2}{2}\left[\hat{\lambda}_2(2) + 2\varepsilon - \int_\Omega \eta(z)\hat{\gamma}_0(t)^2 dz\right].
\end{aligned} \tag{57}$$

Let $\xi(t) = \int_\Omega(\eta(z) - \hat{\lambda}_2(2))\hat{\gamma}_0(t)^2 dz - 2\varepsilon$. Evidently $\xi(\cdot)$ is continuous. So, we can find $t_* \in [-1, 1]$ s.t. $\xi(t_*) = \min_{-1 \leq t \leq 1} \xi(t)$. From (57), we have

$$\psi(s\hat{\gamma}_0(t)) \leq \frac{s^p}{p}M_6 - \frac{s^2}{2}\xi(t_*) \text{ for all } t \in [-1, 1]\,.$$

Since $p > 2$, choosing $s \in (0, 1)$ even smaller if necessary, we have

$$\psi(s\hat{\gamma}_0(t)) < 0 \ \text{ for all } t \in [-1, 1]\,.$$

Let $\gamma_0 = s\hat{\gamma}_0$. Then γ_0 is a continuous path in $C_0^1(\bar{\Omega})$ hence also in $W_0^{1,p}(\Omega)$. It connects $-s\hat{u}_1(2)$ and $s\hat{u}_1(2)$ and we have

$$\psi_{|\gamma_0} < 0\,. \tag{58}$$

Next we will produce a continuous path in $W_0^{1,p}(\Omega)$ connecting $s\hat{u}_1(2)$ and u_* and along which the functional ψ is strictly negative.

Let $a = \inf \psi^+ < 0 = \psi^+(0)$. Claim 1 implies that $(\psi^+)^a = \{u_*\}$. Invoking the second deformation theorem (see, for example, Gasinski-Papageorgiou [15] (p.628)), we can find a deformation $h_+ : [0, 1] \times ((\psi^+)^0 \setminus K_{\psi^+}^0) \to (\psi^+)^0$ s.t.

$$h_+(t, u_*) = u_* \ \text{ for all } t \in [0, 1], \ h_+(1, (\psi^+)^0 \setminus K_{\psi^+}^0) = \{u_*\}\,, \tag{59}$$

$$\psi^+(h_+(t,u)) \le \psi^+(h(\tau,u)) \text{ for all } t,\tau \in [0,1], \ , t \le \tau, \text{ all } u \in (\psi^+)^0 \setminus K^0_{\psi^+} . \quad (60)$$

Let $\gamma_+(t) = h_+(t, s\hat{u}_1(2))^+$ be the continuous path in $W^{1,p}_0(\Omega)$ with values in $W_+ = \{u \in W^{1,p}_0(\Omega) : u(z) \ge 0 \text{ a.e. in } \Omega\}$. We have

$$\gamma_+(0) = h_+(0, s\hat{u}_1(2))^+ = s\hat{u}_1(2),\ \gamma_+(1) = h_+(1, s\hat{u}_1(2))^+ = u_* .$$

Therefore γ_+ connects $s\hat{u}_1(2)$ and u_*. Using (60), (58) and the fact that $\psi^+_{|W_+} = \psi_{|W_+}$, we have for all $t \in [0,1]$

$$\psi(\gamma_+(t)) = \psi^+(\gamma_+(t)) \le \psi^+(s\hat{u}_1(2)) = \psi(s\hat{u}_1(2)) < 0 , \quad \text{so}$$

$$\psi_{|\gamma_+} < 0 . \quad (61)$$

In a similar fashion we produce γ_-, another continuous path in $W^{1,p}_0(\Omega)$, which connects $-s\hat{u}_1(2)$ and v_* and such that

$$\psi_{|\gamma_-} < 0 . \quad (62)$$

We concatenate paths γ_-, γ_0, γ_+ and produce $\gamma_* \in \Gamma$ s.t. $\psi_{|\gamma_*} < 0$, so $y_0 \ne 0$ and $y_0 \in [v_*, u_*] \cap C^1_0(\bar{\Omega})$ is a nodal solution of (1). □

We can improve the conclusion of this proposition, provided we strengthen hypothesis H_3(iv). So, the new conditions on the reaction $f(z,x)$ are the following:

H_4: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function s.t. $f(z,0) = 0$ for a.a. $z \in \Omega$, hypotheses H_4(i), (ii), (iii) are the same as the corresponding hypotheses H_3(i), (ii), (iii) and

(iv) for every $\rho > 0$ there exists $\hat{\xi}_\rho > 0$ s.t. for a.a. $z \in \Omega$,

$$x \to f(z,x) + \hat{\xi}_\rho |x|^{p-2}x$$

is non decreasing on $[-\rho, \rho]$.

In what follows $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are the two extremal constant sign solutions produced in Proposition 4.1.

Remark 4.2. Recall the elementary inequality which says that for $p \ge 2$, we have

$$||x|^{p-2}x - |u|^{p-2}u| \le c_\rho |x-u|(|x|+|u|)^{p-2} \text{ for all } x, u, \in \mathbb{R}, \text{ for some } c_\rho > 0 .$$

That is $x \to |x|^{p-2}x$ is locally Lipschitz and so hypotheses H_4(iv) implies hypotheses H_3(iv).

Proposition 4.3. *If hypotheses H_4 hold, then problem* (1) *admits a nodal solution $y_0 \in \operatorname{int}_{C^1_0(\bar{\Omega})}[v_*, u_*]$.*

Proof. From Proposition 4.1, we already have a nodal solution

$$y_0 \in [v_*, u_*] \cap C^1_0(\bar{\Omega}) .$$

As in the proof of Proposition 3.4, using the tangency principle of Pucci-Serrin [30] (p.35), we have

$$v_*(z) < y_0(z) < u_*(z) \text{ for all } z \in \Omega\,. \tag{63}$$

Let $\rho = \max\{\|u_*\|_\infty, \|v_*\|_\infty\}$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis H_4(iv) and let $\tilde{\xi}_\rho > \hat{\xi}_\rho > 0$. Using hypothesis H_4(iv) and the inequality $y_0 \le u_*$, we have

$$\begin{aligned} -\Delta_p y_0(z) - \Delta y_0(z) + \hat{\xi}_\rho |y_0(z)|^{p-2} y_0(z) &= f(z, y_0(z)) + \hat{\xi}_\rho |y_0(z)|^{p-2} y_0(z) \\ \le f(z, u_*(z)) + \hat{\xi}_\rho u_*(z)^{p-1} &= -\Delta_p u_*(z) - \Delta u_*(z) + \hat{\xi}_\rho u_*(z)^{p-1} \text{ a.a. in } \Omega\,. \end{aligned} \tag{64}$$

Let $h_1(z) = f(z, y_0(z)) + \hat{\xi}_\rho |y_0(z)|^{p-2} y_0(z)$, $\hat{h}_1(z) = h_1(z) + (\tilde{\xi}_\rho - \hat{\xi}_\rho)|y_0(z)|^{p-2} y_0(z)$ and $h_2(z) = f(z, u_*(z)) + \hat{\xi}_\rho |u_*(z)|^{p-1}$, $\hat{h}_2(z) = h_2(z) + (\tilde{\xi}_\rho - \hat{\xi}_\rho)|u_*(z)|^{p-1}$. From hypothesis H_4(iv) and (63), we see that $\hat{h}_1 \prec \hat{h}_2$. Then (64) and Proposition 2.3, imply

$$u_* - y_0 \in \operatorname{int} C_+\,.$$

In a similar fashion, we show that

$$y_0 - v_* \in \operatorname{int} C_+\,.$$

Therefore we conclude that $y_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[v_*, u_*]$. ☐

Theorem 4.4. *If hypotheses H_4 hold, then problem* (1) *has at least three nontrivial solutions $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ and $y_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[v_0, u_0]$ nodal.*

Next we will try to improve this theorem and produce additional solutions for problem (1). It is worth noticing that the method to find an additional solution based on Morse theory is contained in [24, Cap.12] and references therein.

Here, we need to strengthen the conditions on the reaction $f(z,\cdot)$. The new hypotheses on $f(z,x)$ are the following:

H_5: The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function s.t. $f(z,0) = 0$ for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z,x)| \le a(z)(1 + |x|^{p-2})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$;

(ii) $\limsup_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \le \hat{\lambda}_1(p)$ uniformly for a.a. $z \in \Omega$ and there exists $\xi_0 > 0$ s.t.;

$$f(z,x)x - pF(z,x) \ge -\xi_0 \text{ for a.a } z \in \Omega, \text{ all } x \in \mathbb{R};$$

(iii) $f'_x(z,0) = \lim_{x \to 0} \frac{f(z,x)}{x}$ uniformly for a.a. $z \in \Omega$ and there exists an integer $m \ge 2$ s.t.

$$f'_x(z,0) \in [\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)] \text{ for a.a. } z \in \Omega\,,$$
$$f'_x(\cdot,0) \not\equiv \hat{\lambda}_m(2), \quad f'_x(\cdot,0) \not\equiv \hat{\lambda}_{m+1}(2)\,.$$

Remark 4.5. If $m = 2$, then in hypothesis H_5(iii) we allow partial interaction (nonuniform nonresonance) with the eigenvalue $\hat{\lambda}_2(2)$, in contrast to hypothesis H_4(iii) where the quotient $\frac{f(z,x)}{x}$ asymptotically as $x \to 0$ avoids completely

$\hat{\lambda}_2(2)$. On the other hand, in hypothesis H_5(iii) above, $\limsup_{x\to 0} \frac{f(z,x)}{x}$ is bounded by $\hat{\lambda}_{m+1}(2)$ (that is asymptotically as $x \to 0$ the quotient $\frac{f(z,x)}{x}$ stays within the spectral interval $[\hat{\lambda}_m(2), \hat{\lambda}_{m+1}(2)]$, while in hypothesis H_4(iii), the bound for $\limsup_{x\to 0} \frac{f(z,x)}{x}$ is any $L^\infty(\Omega)$-function $\hat{\eta}$. Finally note that the differentiability hypothesis on $f(z, \cdot)$ and H_5(iii) implies that given any $\rho > 0$, we can find $\xi_\rho > 0$ such that for a.a. $z \in \Omega$ $x \to f(z, x) + \hat{\xi}_\rho |x|^{p-2}x$ is non decreasing on $[-\rho, \rho]$.

Proposition 4.6. *If hypotheses H_5 hold, then $C_k(\varphi, 0) = \delta_{k,d_m} Z$ for all $k \geq 0$, with $d_m = dim \oplus_{i=1}^m E(\hat{\lambda}_i(2))$.*

Proof. Let $h(z) = f'_x(z, 0)$. Evidently $h \in L^\infty(\Omega)$ (see H_5(i)). We consider the C^2-functional $\mu : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\mu(u) = \frac{1}{2}\|Du\|_2^2 - \frac{1}{2}\int_\Omega h(z)u(z)^2 dz \text{ for all } u \in H_0^1(\Omega)\,.$$

Let $\mu_p = \mu_{|W_0^{1,p}(\Omega)}$ and consider the homotopy $h(t, u)$ defined by

$$h(t, u) = (1-t)\varphi(u) + t\mu_p(u) \text{ for all } (t, u) \in [0,1] \times W_0^{1,p}(\Omega)\,.$$

Suppose we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ s.t.

$$t_n \to t \text{ in } [0,1],\ u_n \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ and } h'_u(t_n, u_n) = 0 \text{ for all } n \geq 1. \tag{65}$$

We have

$$(1-t_n)A_p(u_n) + A(u_n) = (1-t_n)N_f(u_n) + t_n h u_n \text{ for all } n \geq 1. \tag{66}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and from (66) we have

$$(1-t_n)\|u_n\|^{p-2}A_p(y_n) + A(y_n) = (1-t_n)\frac{N_f(u_n)}{\|u_n\|} + t_n h y_n \text{ for all } n \geq 1.$$

Then reasoning as in the proof of Proposition 3.6 and using (65), we obtain $y_n \to 0$ in $W_0^{1,p}(\Omega)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geq 1$. Therefore (65) can not occurr and so, from the homotopy invariance of critical groups (see, for example, Chang [10] (p.334)) we have

$$C_k(\varphi, 0) = C_k(\mu_p, 0) \text{ for all } k \geq 0\,. \tag{67}$$

Since $W_0^{1,p}(\Omega)$ is dense in $H_0^1(\Omega)$, from Palais [26], we have

$$C_k(\mu_p, 0) = C_k(\mu, 0) \text{ for all } k \geq 0\,. \tag{68}$$

But, due to nonresonance, $u = 0$ is a nondegenerate critical point of μ, with Morse index $d_m = dim \bigoplus_{i=1}^m E(\hat{\lambda}_i(2))$. Therefore (see, for example, Motreanu-Motreanu-Papageorgiou [24], p.155)

$$C_k(\mu, 0) = \delta_{k,d_m} Z \text{ for all } k \geq 0\,,$$

so from (67) and (68) we deduce

$$C_k(\varphi, 0) = \delta_{k,d_m} Z \quad \text{for all } k \geq 0 . \qquad \square$$

We can use Proposition 4.6 and Morse theoretic arguments, to produce a second nodal solution for problem (1). As before $u_* \in \operatorname{int} C_+$, $v_* \in -\operatorname{int} C_+$ are the extremal constant sign solutions of (1) (see Proposition 3.6).

Proposition 4.7. *If hypotheses H_5 hold, then problem* (1) *has at least two nodal solutions* $y_0,\ \hat{y} \in \operatorname{int}_{C_0^1(\bar{\Omega})}[v_*, u_*]$.

Proof. Note that we can not invoke Proposition 4.3 in order to claim the first nodal solution $y_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[v_*, u_*]$, because of the situation when $m = 2$, described in the remark above. Nevertheless, we can proceed as follows. Using the extremal constant sign solutions of (1), we can introduce the truncated C^1-functional $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ used in the Proposition 4.1. We assume that K_ψ is finite, or otherwise we already have an infinity of nodal solutions for problem (1) and we are done (recall that $K_\psi \subseteq [v_*, u_*]$, see Claim 1 in the proof of Proposition 4.1). Since $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ and, by (43), $\varphi_{|[v_*,u_*]} = \psi_{|[v_*,u_*]}$, we have

$$C_k(\varphi_{|C_0^1(\bar{\Omega})}, 0) = C_k(\psi_{|C_0^1(\bar{\Omega})}, 0) \text{ for all } k \geq 0 . \tag{69}$$

As in the proof of Proposition 4.6,using the homotopy invariance of critical groups, we can show that

$$C_k(\varphi_{|C_0^1(\bar{\Omega})}, 0) = C_k(\mu_{|C_0^1(\bar{\Omega})}, 0) \text{ for all } k \geq 0 . \tag{70}$$

From Palais [26] and the proof of Proposition 4.6, we have

$$C_k(\mu_{|C_0^1(\bar{\Omega})}, 0) = C_k(\mu, 0) = \delta_{k,d_m} Z \text{ for all } k \geq 0 ,$$

so, from (69) and (70), we obtain

$$C_k(\psi, 0) = \delta_{k,d_m} Z \text{ for all } k \geq 0 . \tag{71}$$

On the other hand, reasoning as in the proof of Proposition 4.3, via Theorem 2.1 applied to the functional ψ, we obtain $y_0 \in [v_*, u_*]$ a solution to problem (1). From Motreanu-Motreanu-Papageorgiou [24], (p.168), we have

$$C_1(\psi, y_0) \neq 0 , \text{ so, using (71), we obtain } y_0 \neq 0 . \tag{72}$$

As in the proof of Proposition 4.3, using Proposition 2.3, we show that

$$y_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})}[v_*, u_*] . \tag{73}$$

Consider the homotopy $\hat{h}(t, u)$ defined by

$$\hat{h}(t, u) = (1 - t)\varphi(u) + t\psi(u) \text{ for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega) .$$

Suppose, we can find $\{t_n\}_{n\geq 1} \subseteq [0, 1]$ and $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ s.t.

$$t_n \to t \in [0, 1],\ u_n \to y_0 \text{ in } W_0^{1,p}(\Omega) \text{ and } \hat{h}'_u(t_n, u_n) = 0 \text{ for all } n \geq 1. \tag{74}$$

From (43) we have

$$A_p(u_n) + A(u_n) = (1 - t_n)N_f(u_n) + t_n N_{\hat{f}}(u_n) \text{ for all } n \geq 1,$$

that is

$$-\Delta_p(u_n) - \Delta(u_n) = (1 - t_n)f(z, u_n(z)) + t_n \hat{f}(z, u_n(z)) \text{ a.e. in } \Omega,$$

$$u_{n|\partial\Omega} = 0 \text{ for all } n \geq 1.$$

From Ladyzhenskaya-Uraltseva [19] (p. 286), we know that we can find $M_7 > 0$ s.t.

$$\|u_n\|_\infty \leq M_7 \text{ for all } n \geq 1.$$

Applying Theorem 1 of Lieberman [20], we can find $\alpha \in (0, 1)$ and $M_8 > 0$ s.t.

$$u_n \in C_0^{1,\alpha}(\bar{\Omega}) \text{ and } \|u_n\|_{C_0^{1,\alpha}(\bar{\Omega})} \leq M_8 \text{ for all } n \geq 1.$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$ and using (74) we obtain

$$u_n \to y_0 \text{ in } C_0^1(\bar{\Omega}),$$

so from (73) we deduce

$$u_n \in [v_*, u_*] \text{ for all } n \geq n_0 \geq 1,$$

and Claim 1 of Proposition 4.1 gives

$$\{u_n\}_{n \geq n_0} \subseteq K_\psi.$$

This contradicts our assumption that K_ψ is finite. Therefore (74) can not happen and the homotopy invariance of critical groups implies that

$$C_k(\varphi, y_0) = C_k(\psi, y_0) \text{ for all } k \geq 0, \tag{75}$$

so, from (72) and Papageorgiou-Smyrlis [28] we have respectively

$$C_1(\varphi, y_0) \neq 0, \quad \text{and so} \quad C_k(\varphi, y_0) = \delta_{k,1}Z \quad \text{for all } k \geq 0.$$

Hence, using (75), we conclude

$$C_k(\psi, y_0) = \delta_{k,1}Z \quad \text{for all } k \geq 0. \tag{76}$$

From Claim 2 in the proof of Proposition 4.1 we know that $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are local minimizers of ψ. Therefore

$$C_k(\psi, u_*) = C_k(\psi, v_*) = \delta_{k,0}Z \quad \text{for all } k \geq 0. \tag{77}$$

Finally, recall that ψ is coercive (see (43)). So, we have

$$C_k(\psi, \infty) = \delta_{k,0}Z \quad \text{for all } k \geq 0. \tag{78}$$

Suppose $K_\psi = \{0, u_*, v_*, y_0\}$. Then from (71), (76), (77), (78) and the Morse relation (see (8)) with $t=-1$, we have

$$(-1)^{d_m} + 2(-1)^0 + (-1)^1 = (-1)^0, \text{ that is } (-1)^{d_m} = 0, \text{a contradiction.}$$

Therefore, we can find $\hat{y} \in K_\psi$, $\hat{y} \notin \{0,\, u_*,\, v_*,\, y_0\}$. From Claim 1 of Proposition 4.1 and (43) we know that $\hat{y}$ is a nodal solution of (1). Moreover, as in the proof of Proposition 4.3, using the nonlinear regularity theory and Proposition 2.3, we show that $\hat{y} \in \operatorname{int}_{\hat{C}_0^1(\bar{\Omega})}[v_*, u_*]$. $\square$

We can state the second multiplicity theorem for problem (1).

Theorem 4.8. *If hypotheses H_5 hold, then problem* (1) *admits at least four nontrivial solutions*

$$u_* \in \operatorname{int} C_+,\ v_* \in -\operatorname{int} C_+ \text{ and } y_0,\ \hat{y} \in \operatorname{int}_{C_0^1(\bar{\Omega})}[v_*, u_*] \text{ nodal.}$$

In the above multiplicity result, we do not allow resonance at zero (see hypothesis H_5(iii)). It is natural to ask what can be said when resonance occurs also at zero (double resonance). We can still have a similar multiplicity theorem producing four nontrivial solutions, but in this case we can not say that the fourth solution is nodal.

To this end, we impose the following conditions on the reaction $f(z,x)$:

H_6: The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $f(z,0) = 0$ for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z,x)| \le a(z)(1 + |x|^{p-2})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$;

(ii) $\limsup_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \le \hat{\lambda}_1(p)$ uniformly for a.a. $z \in \Omega$ and there exists $\xi_0 > 0$ s.t.

$$f(z,x)x - pF(z,x) \ge -\xi_0 \text{ for a.a } z \in \Omega, \quad \text{all } x \in \mathbb{R};$$

(iii) there exists an integer $m \ge 3$, a function $\eta \in L^\infty(\Omega)_+$ and $\delta > 0$ s.t.

$$\eta(z)x^2 \le f(z,x)x \text{ for a.a } z \in \Omega, \quad \text{all } |x| \le \delta$$

with $\hat{\lambda}_m(2) \le \eta(z)$ for a.a. $z \in \Omega$, $\hat{\lambda}_m(2) \ne \eta$ and

$$F(z,x) \le \frac{\hat{\lambda}_{m+1}(2)}{2}x^2 + \frac{\hat{\lambda}_1(p)}{p}|x|^p \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Remark 4.9. Evidently the second part of hypothesis H_6(iii) incorporates in our framework also equations which at zero are resonant with respect to $\hat{\lambda}_{m+1}(2)$ from the left, that is the reaction $f(z,x)$ satisfies

$$\lim_{x\to 0} \frac{f(z,x)}{x} = \hat{\lambda}_1(2) \text{ uniformly for a.a. } z \in \Omega\,,$$

$$f(z,x)x \le \hat{\lambda}_1(2)x^2 \text{ for a.a } z \in \Omega, \text{ all } |x| \le \delta\,,$$

and

$$f(z,x)x \le \hat{\lambda}_1(2)x^2 + \hat{\lambda}_1(p)|x|^p \text{ for a.a } z \in \Omega, \text{ all } x \in \mathbb{R}\,.$$

Theorem 4.10. *If hypotheses H_6 hold, then problem* (1) *admits at least four nontrivial solutions*

$$u_0 \in \operatorname{int} C_+,\ v_0 \in -\operatorname{int} C_+,\ y_0, \in \operatorname{int}_{C_0^1(\bar\Omega)}[v_0,u_0] \text{ nodal and } \hat y \in C_0^1(\bar\Omega).$$

Proof. From Theorem 4.4, we already have at least three nontrivial solutions $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ and $y_0 \in \operatorname{int}_{C_0^1(\bar\Omega)}[v_*,u_*]$ nodal. From Proposition 3.4, we know that u_0 and v_0 are both local minimizers of the energy functional φ. Hence

$$C_k(\varphi,u_0) = C_k(\varphi,v_0) = \delta_{k,0}Z \text{ for all } k \geq 0\,. \tag{79}$$

Also from (75) and (76), we have

$$C_k(\varphi,y_0) = \delta_{k,1}Z \text{ for all } k \geq 0\,. \tag{80}$$

Let $V_m = \oplus_{i=1}^m E(\hat\lambda_i(2))$ and $W_m = W_0^{1,p}(\Omega) \cap V_m^\perp$. We have

$$W_0^{1,p}(\Omega) = V_m \oplus W_m\,.$$

Let $u \in V_m$. Since V_m is finite dimensional, all norms are equivalent. So, corresponding to $\delta > 0$ as in hypothesis H_6(iii), we can find $\rho > 0$ s.t.

$$\|u\| \leq \rho \ \Rightarrow\ \|u\|_\infty \leq \delta \text{ for all } u \in V_m\,. \tag{81}$$

Then for $u \in V_m$ with $\|u\| \leq \rho$, taking into account H_6(iii), [25] and (81), we can find $c_7 > 0$ s.t.

$$\begin{aligned}\varphi(u) &= \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u(z))dz \\ &\leq \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\left[\|Du\|_2^2 - \int_\Omega \eta(z)u^2dz\right] \\ &\leq \frac{1}{p}\|u\|^p - c_7\|u\|^2\,.\end{aligned} \tag{82}$$

Since $2 < p$, by choosing $\rho \in (0,1)$ even smaller if necessary, from (82) we have

$$\varphi(u) < 0 \text{ for all } u \in V_m \text{ with } \|u\| = \rho\,. \tag{83}$$

Also, if $u \in W_m$, then using the second part of hypothesis H_6(iii), (3) and (5), we have

$$\begin{aligned}\varphi(u) &= \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u(z))dz \\ &\geq \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{\hat\lambda_{m+1}(2)}{2}\|u\|_2^2 - \frac{\hat\lambda_1(p)}{p}\|u\|_p^p \geq 0\,.\end{aligned}$$

Finally recall that φ is coercive (see Proposition 3.2), hence it is bounded below. Therefore, we can apply Theorem 3.1 of Perera [29] and obtain $\hat y \in W_0^{1,p}$ s.t.

$$\hat y \in K_\varphi,\quad \varphi(\hat y) < 0 = \varphi(0),\quad C_(\varphi,\hat y) \neq 0 \quad (d_m = \dim \oplus_{i=1}^m E(\hat\lambda_i(2)))\,. \tag{84}$$

From (84) we see that $\hat y \neq 0$, solves problem (1) and, by the nonlinear regularity theory, $\hat y \in C_0^1(\bar\Omega)$. Since $m \geq 3$, also $d_m \geq 3$, so from (79), (80) and (84), we infer that $\hat y \notin \{0,u_0,v_0,y_0\}$. So $\hat y \in C_0^1(\bar\Omega)$ is the fourth nontrivial solution of problem (1). ∎

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