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Optimal Control of Stochastic Variational Inequalities

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This work focuses on the optimal control problem for elliptic stochastic variational inequalities where the diffusivity coefficient and the source term are random fields. Besides recalling the existence theorem for the stochastic elliptic variational inequalities, we also give an existence result for the optimal control problem, which is posed as a stochastic optimization problem. We conduct two preliminary computational experiments by coupling the penalty method with the stochastic approximation approach.

Keywords: Stochastic optimal control, partial differential equations with random data, stochastic approximation, regularization, penalization.

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1. Introduction

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, where Ω is a nonempty set of elementary events, $\mathbb{F}$ is a σ-algebra of subsets of Ω, and $\mathbb{P} : \mathbb{F} \to [0, 1]$ is a probability measure. Given a real Banach space X, the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, and an integer $p \in [1, \infty)$, the Bochner space $L^p(\Omega; X)$ consists of equivalence classes of Bochner integrable functions $u : \Omega \to X$ with finite p-th moment, that is,

$$\|u\|_{L^p(\Omega;X)} := \left(\int_\Omega \|u(\omega)\|_X^p d\mathbb{P} \right)^{1/p} = \mathbb{E}\left[\|u(\omega)\|_X^p\right]^{1/p} < \infty.$$

If $p = \infty$, then $L^\infty(\Omega; X)$ is the space of equivalence classes of Bochner measurable functions $u : \Omega \to X$ such that

$$\operatorname{ess\,sup}_{\omega\in\Omega} \|u(\omega)\|_X < \infty.$$

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Before proceeding further, we also need to recall some function spaces. Given the domain D, for $1 \le p < \infty$, by $L^p(D)$, we denote the space of equivalence classes of pth Lebesgue integrable functions. The space $L^\infty(D)$ consists of equivalence classes of measurable functions that are bounded almost everywhere (a.e.) on D. We also recall that the Sobolev spaces are given by

$$H^1(D) = \left\{y \in L^2(D), \quad \partial_{x_i} y \in L^2(D),\ i = 1, \dots, n\right\},$$

$$H_0^1(D) = \left\{y \in H^1(D), \quad y|_{\partial D} = 0\right\},$$

and $H^{-1}(D) = (H_0^1(D))^*$ is the topological dual of $H_0^1(D)$.

Given the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, let $D \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary ∂D, let $a : \Omega \times D \to \mathbb{R}$, $f : \Omega \times D \to \mathbb{R}$, and $v : \Omega \times D \to \mathbb{R}$ be second-order random fields, and let K be a nonempty, closed, and convex subset of $L^2(\Omega; H_0^1(D))$ with $0 \in K$. We consider the stochastic elliptic variational inequality that seeks $y_v(\omega, x) \in K$ such that

$$\begin{aligned}\mathbb{E}&\left[\int_D a(\omega,x)\nabla y_v(\omega,x)\cdot\nabla(z(\omega,x)-y_v(\omega,x))dx\right] \qquad (1)\\ &\ge \mathbb{E}\left[\int_D (f(\omega,x)+v(\omega,x))(z(\omega,x)-y_v(\omega,x))\,dx\right], \quad \text{for every } z \in K.\end{aligned}$$

Let H be a separable Hilbert space compactly embedded into $L^2(\Omega; L^2(D))$, and let $U \subset H$ be a nonempty, closed, and convex set of the admissible controls.

This work aims to study the optimal control problem of finding $u \in U$ such that

$$J_\kappa(u) = \min_{v\in U} J_\kappa(v) := \mathbb{E}\left[\frac{1}{2}\int_D |y_v(\omega,x) - \tilde{u}(\omega,x)|^2 dx\right] + \frac{\kappa}{2}\|v(\omega,x)\|_H^2. \qquad (2)$$

Here $\tilde{u}(\omega, x) \in L^2(\Omega; L^2(D))$ is the target, $\kappa > 0$ is the regularization parameter, and y_v solves the stochastic variational inequality (1).

Besides the formulation (1), which is defined over $\Omega \times D$, we could also consider a pathwise formulation of a variational inequality that seeks, for a fixed realization $\omega \in \Omega$, $y(\omega, \cdot) \in \widetilde{K}$, where $\widetilde{K}$ is a closed and convex subset of $H_0^1(D)$, such that

$$\begin{aligned}&\int_D a(\omega,x)\nabla y(\omega,x)\cdot\nabla(z(\omega,x)-y(\omega,x))dx\\ &\qquad \ge \int_D (f(\omega,x)+v(\omega,x))(z(\omega,x)-y(\omega,x))\,dx, \quad \text{for every } z \in \widetilde{K}. \qquad (3)\end{aligned}$$

A prototypical source of stochastic elliptic variational inequalities is the obstacle problem that seeks $y(\omega, x)$ such that for almost sure (a.s.) $\omega \in \Omega$ for almost every (a.e.) $x \in D$, we have

$$-\mathrm{div}(a(\omega,x)\nabla y) \ge f(\omega,x), \text{ in } D, \qquad (4a)$$

$$y \ge \chi, \text{ in } D, \qquad (4b)$$

$$(\mathrm{div}(a(\omega,x)\nabla y) + f(\omega,x))(y-\chi) = 0, \text{ in } D, \qquad (4c)$$

$$y = 0, \text{ on } \partial D, \qquad (4d)$$

where χ is a deterministic obstacle.

An important special case of (4) is the stochastic boundary value problem that seeks a random field $y : \Omega \times D \to \mathbb{R}$ that almost surely satisfies:

$$-\nabla \cdot (a(\omega, x)\nabla y(\omega, x)) = f(\omega, x), \text{ in } D, \tag{5a}$$

$$y(\omega, x) = 0, \text{ on } \partial D. \tag{5b}$$

During the last decade, extensive research has been carried out on developing theoretical tools and reliable numerical methods for stochastic PDEs, see [19]. Recent research efforts have also focused on optimal control and inverse problems for stochastic PDEs. To mention a few relevant contributions, we note that Badri Narayanan and Zabaras [2] investigated the inverse problem in the presence of uncertainties in the material data and developed an adjoint-approach based identification process by employing the spectral stochastic finite element method. In [30], the authors developed a scalable methodology for the stochastic inverse problem using a sparse grid collocation approach. In [27], the authors developed a robust and efficient approach by employing generalized polynomial chaos expansion to identify uncertain elastic parameters from experimental modal data. Some of the related developments are available in [4, 5, 18, 21, 26, 28, 29] and the cited references therein.

The research directions outlined in some of the above contributions have even been extended to solving stochastic variational inequalities, see [1, 7, 9, 16, 17]. However, the questions related to the optimal control of stochastic variational inequalities have not been addressed yet, even though the optimal control of deterministic variational inequalities has been extensively explored over the last five decades, [22, 24]. This paper attempts to fill this gap by giving initial results for the optimal control of stochastic variational inequalities.

2. Optimal control for stochastic variational inequalities

In the following, we assume that there exist constants k_0 and k_1 such that

$$0 < k_0 \leq a(\omega, x) \leq k_1, \quad \text{a.s. in } \Omega \times D. \tag{6}$$

In particular, $a \in L^\infty(\Omega \times D)$.

Assumption (6) easily extends some known results from deterministic to stochastic variational inequalities. However, a disadvantage is that log-normal random fields, due to the unboundedness of the Gaussian random variables, violate (6).

We define a bilinear form $b\colon L^2(\Omega; H_0^1(D)) \times L^2(\Omega; H_0^1(D)) \to \mathbb{R}$ and a linear functional $\ell : L^2(\Omega; H_0^1(D)) \to \mathbb{R}$ by

$$b(y, z) = \mathbb{E}\left[\int_D a(\omega, x)\nabla y(\omega, x) \cdot \nabla z(\omega, x) dx\right], \tag{7}$$

$$\ell(z) = \mathbb{E}\left[\int_D (f(\omega, x) + v(\omega, x))z(\omega, x)\, dx\right]. \tag{8}$$

Then, the variational inequality (1) can be written as follows:

$$\text{Find } y \in K \text{ such that} \qquad b(y, z - y) \geq \ell(z - y), \quad \text{for every } z \in K. \tag{9}$$

Since,
$$\begin{aligned}|b(y,z)| &= \left|\mathbb{E}\left[\int_D a(\omega,x)\nabla y(\omega,x)\cdot\nabla z(\omega,x)dx\right]\right|\\ &\le \int_{\Omega\times D} |a(\omega,x)\nabla y(\omega,x)\cdot\nabla z(\omega,x)\,|\,dx\,d\mathbb{P}\\ &\le \|a\|_{L^\infty(\Omega\times D)}\int_{\Omega\times D}|\nabla y(\omega,x)\cdot\nabla z(\omega,x)|dx\,d\mathbb{P}\\ &\le \|a\|_{L^\infty(\Omega\times D)}\|y\|_{L^2(\Omega;H_0^1(D))}\|z\|_{L^2(\Omega;H_0^1(D))},\end{aligned}$$

the bilinear form b is continuous. Furthermore, we also have

$$\begin{aligned}b(z,z) &= \mathbb{E}\left[\int_D a(\omega,x)\nabla z(\omega,x)\cdot\nabla z(\omega,x)dx\right]\\ &\ge k_0\int_{\Omega\times D}\nabla z(\omega,x)\cdot\nabla z(\omega,x)dx\,d\mathbb{P} = \alpha\|z\|^2_{L^2(\Omega;H_0^1(D))},\end{aligned}$$

where α is a positive constant involving Poincare's coefficient. Hence b is coercive. Finally, for $v, f\in L^2(\Omega;H^{-1}(D))$ and for any $z\in K$, for the functional $\ell(\cdot)$, we have

$$|\ell(z)| = \left|\int_\Omega\int_D (v(\omega,x)+f(\omega,x))z(\omega,x)\,dx\,d\mathbb{P}\right| \le \|v+f\|_{L^2(\Omega;H^{-1}(D))}\|z\|_{L^2(\Omega;H_0^1(D))},$$

which proves the continuity of ℓ. Thus, the variational inequality (1) is uniquely solvable by standard arguments, see [10].

Summarizing, we have the following simple but important result:

Theorem 2.1. *Assume that* (6) *holds and* $f, v\in L^2(\Omega;H^{-1}(D))$. *Then, the variational inequality* (1) *is uniquely solvable. Moreover, there exists a positive constant* $c>0$ *such that*
$$\|y\|_{L^2(\Omega;H_0^1(D))} \le c\|f+v\|_{L^2(\Omega;H^{-1}(D))}. \tag{10}$$

The following result gives the Lipschitz continuity of the control-to-solution map.

Theorem 2.2. *Assume that* (6) *holds and that* $f, v\in L^2(\Omega;H^{-1}(D))$. *Then the map* $L^2(\Omega;H^{-1}(D))\ni u(\omega,x)\mapsto y_u(\omega,x)\in L^2(\Omega;H_0^1(\Omega))$ *is Lipschitz continuous.*

Proof. Let $y_{u_1}\in K$ be the solution of (1) corresponding to u_1 and let $y_{u_2}\in K$ be the solution of (1) corresponding to u_2. By the definitions of y_{u_1} and y_{u_2}, for every $z\in K$, we have

$$\begin{aligned}&\mathbb{E}\left[\int_D a(\omega,x)\nabla y_{u_1}(\omega,x)\cdot\nabla(z(\omega,x)-y_{u_1}(\omega,x))dx\right]\\ &\qquad\ge \mathbb{E}\left[\int_D (f(\omega,x)+u_1(\omega,x))(z(\omega,x)-y_{u_1}(\omega,x))\,dx\right],\\ &\mathbb{E}\left[\int_D a(\omega,x)\nabla y_{u_2}(\omega,x)\cdot\nabla(z(\omega,x)-y_{u_2}(\omega,x))dx\right]\\ &\qquad\ge \mathbb{E}\left[\int_D (f(\omega,x)+u_2(\omega,x))(z(\omega,x)-y_{u_2}(\omega,x))\,dx\right].\end{aligned}$$

We set $z = y_{u_2}$ in the variational inequality defining y_{u_1} and $v = y_{u_1}$ in the variational inequality defining y_{u_2}, and combine the resulting inequalities to obtain

$$\mathbb{E}\left[\int_D (u_1(\omega,x) - u_2(\omega,x))(y_{u_1}(\omega,x) - y_{u_2}(\omega,x))dx\right]$$
$$\geq \mathbb{E}\left[\int_D a(\omega,x)\nabla(y_{u_1}(\omega,x) - y_{u_2}(\omega,x)) \cdot \nabla(y_{u_1}(\omega,x) - y_{u_2}(\omega,x))dx\right],$$

which implies that

$$\begin{aligned}
&k_0\mathbb{E}\left[\|\nabla(y_{u_1}(\omega,\cdot) - y_{u_2}(\omega,\cdot))\|^2_{L^2(D)}\right] \\
&\leq \mathbb{E}\left[\int_D a(\omega,x)\nabla(y_{u_1}(\omega,x) - y_{u_2}(\omega,x)) \cdot \nabla(y_{u_1}(\omega,x) - y_{u_2}(\omega,x))dx\right] \\
&\leq \mathbb{E}\left[\int_D (u_1(\omega,x) - u_2(\omega,x))(y_{u_1}(\omega,x) - y_{u_2}(\omega,x))dx\right] \\
&\leq \|u_1 - u_2\|_{L^2(\Omega\times D)}\|y_{u_1} - y_{u_2}\|_{L^2(\Omega;H_0^1(D))}.
\end{aligned}$$

Consequently, for a constant $c > 0$ which involves the Poincare's constant, we have

$$\|y_{u_1} - y_{u_2}\|_{L^2(\Omega;H_0^1(D))} \leq c\|u_1 - u_2\|_{L^2(\Omega\times D)},$$

and the proof is complete. □

Although Lipschitz continuous due to Theorem 2.2, the control-to-solution map for variational inequalities is generally non-differentiable, making optimal control a challenging subject.

The following result proves the existence of an optimal control.

Theorem 2.3. *Assume that* (6) *holds and $f, v \in L^2(\Omega; H^{-1}(D))$. Then the optimal control problem* (2) *has a nonempty solution set.*

Proof. Since $J_\kappa(u) \geq 0$, for every $u \in U$, there is a minimizing sequence $\{u_n\}$ in U such that

$$\lim_{n\to\infty} J_\kappa(u_n) = \inf\{J_\kappa(u)|\, u \in U\}.$$

Therefore, $\{J_\kappa(u_n)\}$ is bounded, and hence $\{u_n\}$ is also bounded in H. Then, there is a subsequence, still denoted by $\{u_n\}$, which converges to some $\bar{u}(\omega,x) \in U$ in $L^2(\Omega; L^2(D))$. Let y_n be the solution of the variational inequality that corresponds to the control u_n. That is, for all $z \in K$ we have

$$\mathbb{E}\left[\int_D a(\omega,x)\nabla y_n \cdot (\nabla z(\omega,x) - \nabla u_n)\,dx\right] \geq \mathbb{E}\left[\int_D (f(\omega,x) + u_n)(z(\omega,x) - y_n)\,dx\right].$$

Setting $z = 0 \in K$, for a constant $\alpha > 0$, we obtain

$$\alpha\,\|y_n\|^2_{L^2(\Omega;H_0^1(D))} \leq \mathbb{E}\left[\int_D a(\omega,x)\nabla y_n \cdot \nabla y_n dx\right] \leq \mathbb{E}\left[\int_D (f(\omega,x) + u_n)y_n\,dx\right],$$

which confirms the boundedness of $y_n = y(u_n)$ in $L^2(\Omega; H_0^1(D))$.

Therefore, any subsequence of $\{y_n\}$ has a subsequence that converges weakly to some $\bar{y} \in K$. We will show that $\bar{y} = y(\bar{u})$. By the definition of y_n, for every $z \in L^2(\Omega; H_0^1(D))$, we have

$$\mathbb{E}\left[\int_D a(\omega,x)\nabla y_n \cdot (\nabla z(\omega,x) - \nabla y_n) dx\right] \geq \mathbb{E}\left[\int_D (f(\omega,x) + u_n)(z(\omega,x) - y_n)\, dx\right],$$

which can be written as

$$\mathbb{E}\left[\int_D a(\omega,x)\nabla z(\omega,x) \cdot (\nabla z(\omega,x) - \nabla y_n) dx\right] \geq \mathbb{E}\left[\int_D (f(\omega,x) + u_n)(z(\omega,x) - y_n)\, dx\right],$$

which under the limit $n \to \infty$ implies that for every $z \in L^2(\Omega; H_0^1(D))$, we have

$$\mathbb{E}\left[\int_D \left[a(\omega,x)\nabla z(\omega,x) \cdot (\nabla z(\omega,x) - \nabla \bar{y}(\omega,x))\right]\, dx\right] \geq \mathbb{E}\left[\int_D (f(\omega,x) + \bar{u}(\omega,x)(z(\omega,x) - \bar{y}(\omega,x))\, dx\right].$$

For $t \in (0,1)$, setting $z = \bar{y} + t(w - \bar{y})$, where $w \in K$ is arbitrary, we get

$$\mathbb{E}\left[\int_D \left[a(\omega,x)\nabla(\bar{y}(\omega,x) + t(w(\omega,x) - \bar{y}(\omega,x)) \cdot (\nabla w(\omega,x) - \nabla \bar{u}(\omega,x))\right]\, dx\right] \geq \mathbb{E}\left[\int_D (f(\omega,x) + \bar{u}(\omega,x))(w(\omega,x) - \bar{y}(\omega,x))\, dx\right].$$

By passing $t \to 0$, we obtain

$$\mathbb{E}\left[\int_D \left[a(\omega,x)\nabla \bar{y}(\omega,x) \cdot (\nabla w(\omega,x) - \nabla \bar{y}(\omega,x))\right]\, dx\right] \geq \mathbb{E}\left[\int_D (f(\omega,x) + \bar{u}(\omega,x))(w(\omega,x) - \bar{y}(\omega,x))\, dx\right].$$

Since the variational inequality (1) is uniquely solvable because of Theorem 2.1, we deduce that $\bar{y} = y(\bar{u})$. The optimality concerning (2) follows from the weak lower-semicontinuity of any norm. □

3. Stochastic approximation for optimal control

We provide some heuristic numerical results using a coupling of the well-known penalty method and the stochastic approximation approach. We note that the vital field of stochastic approximation, initiated by Robbins and Monro [25], has been recently used for optimal control in stochastic PDEs by Martin, Krumscheid, and Nobile [20] and Geiersbach and Pflug [8]. The stochastic approximation approach was recently used in [13, 14, 15] for the nonlinear inverse problem of parameter identification. For a general overview of the stochastic approximation approach, see the recent monograph [10].

In the following, we consider two numerical examples; in both cases, we consider the optimal control associated with obstacle problem (4). For convenience, we recall that we consider the optimal control problem of finding $u \in U$ such that

$$J(u) = \min_{v \in U} J(v) := \frac{1}{2}\mathbb{E}\left[\int_D |y_v(\omega,x) - \tilde{u}(\omega,x)|^2 dx\right] + \frac{\kappa}{2}\mathbb{E}\left[\int_D |v(\omega,x)|^2\, dx\right], \quad (11)$$

where $\kappa > 0$ and $y_v \in L^2(\Omega; H_0^1(D))$ solves the obstacle problem:

$$-\mathrm{div}(a(\omega,x)\nabla y_v(\omega,x)) \geq f(\omega,x) + v(\omega,x), \text{ in } D, \quad (12a)$$

$$y(\omega,x) \geq \chi, \text{ in } D, \quad (12b)$$

$$(\mathrm{div}(a(\omega,x)\nabla y_v(\omega,x)) + f(\omega,x) + v(\omega,x))(y_v - \chi) = 0, \text{ in } D, \quad (12c)$$

$$y_v(\omega,x) = 0, \text{ on } \partial D. \quad (12d)$$

For a numerical solution, we will reduce (12) into a system of equations by employing a standard penalization approach (see for example [12, 11]). Given a penalization parameter $\varepsilon > 0$, we consider the following penalized optimal control problem for $\kappa > 0$:

$$\begin{aligned} J_\varepsilon(u) &= \min_{v \in U} \frac{1}{2}\mathbb{E}[J_{\varepsilon,\kappa}(\omega,x)] \quad (13)\\ &:= \mathbb{E}\left[\frac{1}{2}\int_D |y_{\varepsilon,v}(\omega,x) - \tilde{u}(\omega,x)|^2 dx + \frac{\kappa}{2}\int_D |v(\omega,x)|^2\right], \end{aligned}$$

where $y_{\varepsilon,v}(\omega,x)$ solves the following penalized variational equation:

$$-\mathrm{div}(a(\omega,x)\nabla y_{\varepsilon,v}) + \varphi_\varepsilon(y_{\varepsilon,v}(\omega,x) - \chi) = f(\omega,x) + v(\omega,x), \text{ in } D, \quad (14a)$$

$$y_{\varepsilon,v}(\omega,x) = 0, \text{ on } \partial D. \quad (14b)$$

In the numerical experiments, we use the following penalization function:

$$\varphi_\varepsilon(x) := \begin{cases} x & \text{if } \quad x > \frac{\varepsilon}{2}, \\ \frac{1}{2\varepsilon}\left(x + \frac{\varepsilon}{2}\right)^2 & \text{if } \quad |x| \leq \frac{\varepsilon}{2}, \\ 0 & \text{if } \quad x \leq -\frac{\varepsilon}{2}. \end{cases}$$

Algorithm 1: Stochastic approximation for the control problem

1: Choose $u_0 \in U$, $\alpha_0 > 0$, $\varepsilon > 0$, $\kappa > 0$, and tol> 0.
2: **while** $\|u_n - u_{n-1}\| >$tol **do**
3: Generate sample ω_n and step $\alpha_n = \alpha_0/n$.
4: Compute state solution $y_{u_n} = y(\omega_n,\cdot)$ to (14).
5: Compute adjoint solution $p_n = p(\omega_n,\cdot)$ to (15).
6: Compute gradient $G(\omega_n,\cdot) = p_n + \kappa u_n$.
7: $u_{n+1} := u_n - \alpha_n G(\omega_n,\cdot)$.
8: **end while**

The stochastic approximation approach is described in Algorithm 1, which requires computing the gradient $\nabla J_\varepsilon(v) = \mathbb{E}[\nabla J_{\varepsilon,\kappa}(\omega,x)]$, where $\nabla J_{\varepsilon,\kappa}(\omega,x)$ is given by

$$\nabla J_{\varepsilon,\kappa}(\omega,x) = p_\varepsilon(\omega,x) + v(\omega,x),$$

with $p_\varepsilon(\omega,x)$ being the solution of the corresponding adjoint problem:

$$-\mathrm{div}(a(\omega,x)\nabla p_\varepsilon(\omega,x)) + \varphi_\varepsilon'^*(p_\varepsilon(\omega,x)-\chi) = y_v(\omega,x) - \tilde{u}(\omega,x), \text{ in } D, \tag{15a}$$

$$p_\varepsilon(\omega,x) = 0, \text{ on } \partial D. \tag{15b}$$

Algorithm 1 is a stochastic analogue of the adjoint approach used heavily in deterministic optimal control problems for variational inequalities, see [6].

Example 1. We consider a one dimensional example with two degrees of stochasticity. It is based on the deterministic flat obstacle problem studied in [23, Example 1]. Here $D = (0,1)$ and we consider the unconstrained problem $U = L^2(\Omega; L^2(D))$. The parameter a and the obstacle χ are taken to be constant $a(\omega,x) = 1$, $\chi = -1$.

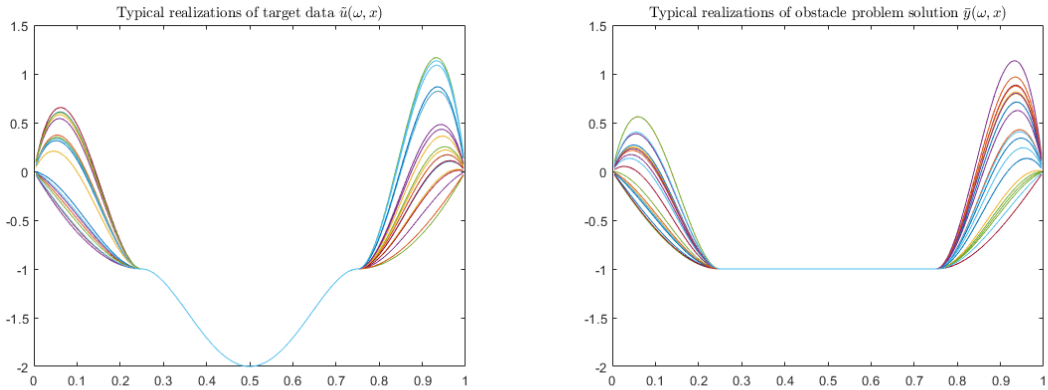

Figure 3.1: Example 1

Furthermore, we define the following second order random field as solution (see Figure 3.1):

$$\bar{y}(\omega,x) := \begin{cases} 16x^2 - 8x + 500Y_1(\omega)\left(x-\frac{1}{4}\right)^2 \sin(x) & \text{if } \; x \le \frac{1}{4}, \\ -1 & \text{if } \; \frac{1}{4} \le x \le \frac{3}{4}, \\ 16x^2 - 24x + 8 + 250Y_2(\omega)\left(x-\frac{3}{4}\right)^2 \sin(\pi x) & \text{if } \; \frac{3}{4} \le x \le 1, \end{cases}$$

where $Y_1(\omega)$, $Y_2(\omega) \sim U[0,1]$ are independent and uniformly distributed over $[0,1]$. We compute the corresponding $f(\omega,x)$ using (12). Taking target data

$$\tilde{u} := \begin{cases} 16x^2 - 8x + 500Y_1(\omega)\left(x-\frac{1}{4}\right)^2 \sin(x) & \text{if } \; x \le \frac{1}{4}, \\ -10 + 96x - 352x^2 + 512x^3 - 256x^4 & \text{if } \; \frac{1}{4} \le x \le \frac{3}{4}, \\ 16x^2 - 24x + 8 + 250Y_2(\omega)\left(x-\frac{3}{4}\right)^2 \sin(\pi x) & \text{if } \; \frac{3}{4} \le x \le 1, \end{cases}$$

in (11), we observe that $\bar{u} = 0$ solves problem (11) such that $\bar{y}(\omega,\cdot)$ corresponds to the solution to the random obstacle problem (12) and the optimal value is given by

$$J(\bar{u}) = \frac{1}{2}\int_{\frac{1}{4}}^{\frac{3}{4}} (-10 + 96x - 352x^2 + 512x^3 - 256x^4 + 1)^2 dx \approx 1.0159e-01.$$

In the numerical computations, we take $\kappa = 1$ and $\varepsilon = 10^{-3}$. Numerical results, presented in Table 3.1, correspond to the average of 20 runs of the algorithm, where we take $u_0(x) = 2+x^2$ as the initial guess, $\alpha_0 = 1$ as initial step-length, and tolerance $\mathrm{tol} = 1e-03$.

n (number of nodes)	$\left(\sum_{i=1}^{20} \lVert u_i - \bar{u} \rVert_{L^2(\Omega)}\right)/20$	$\left(\sum_{i=1}^{20} J(u_i)\right)/20$	avg SG iterations
25	5.7135e-05	1.0123e-01	2.85
50	2.8022e-05	1.0111e-01	3
75	2.1471e-05	1.0110e-01	3
100	1.8091e-05	1.0110e-01	3.6
125	1.7089e-05	1.0111e-01	4
125	1.7097e-05	1.0110e-01	4
175	1.7097e-05	1.0110e-01	4
200	1.6943e-05	1.0110e-01	4

Table 3.1: Numerical results for Example 1.

Example 2. We now consider a two dimensional framework with two degrees of stochasticity. We consider a random version of the deterministic example presented in [3]. We take $D = (0,1) \times (0,1)$, and $U = L^2(\Omega; L^2(D))$. For this case, we set $a(\omega, x_1, x_2) = 1$ and $\chi(\omega, x_1, x_2) = 0$. The solution to variational inequality and the targent data are given by the following second order random fields (see Figures 3.2 and 3.3)

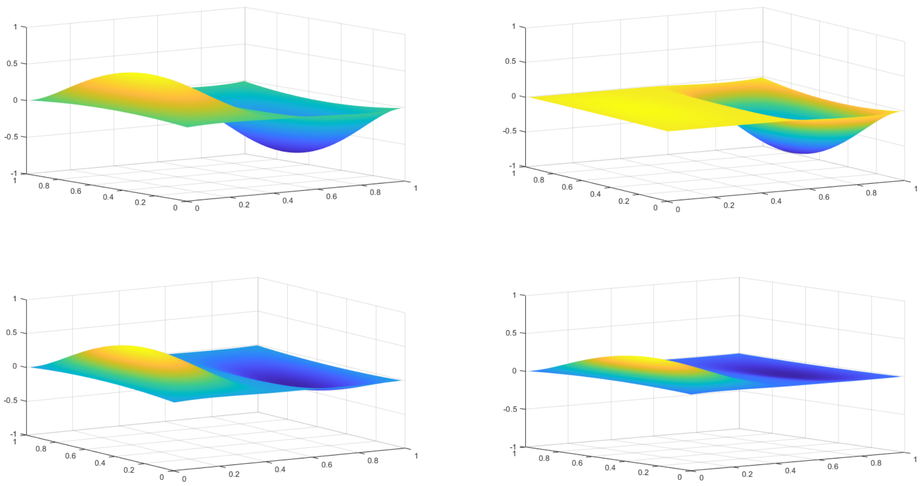

Figure 3.2: Example 2. Four realizations of target data $\tilde{u}(\omega, x)$

$$\bar{y}(\omega, x_1, x_2) := \begin{cases} 200 x_1 x_2 (x_1 - 0.5)^2 (1 - x_2) Y_1(\omega) & \text{if} \quad x_1 \leq 0.5, \\ 0 & \text{if} \quad x_1 \geq 0.5, \end{cases}$$

$$\tilde{u}(\omega, x_1, x_2) := \begin{cases} 200 x_1 x_2 (x_1 - 0.5)^2 (1 - x_2) Y_1(\omega) & \text{if} \quad x_1 \leq 0.5, \\ 200 x_2 (x_1 - 1)(x_1 - 0.5)^2 (1 - x_2) Y_2(\omega) & \text{if} \quad x_1 \geq 0.5, \end{cases}$$

where $Y_1(\omega)$, $Y_2(\omega) \sim U[0,1]$ are independent and uniformly distributed over $[0,1]$. We compute $f(\omega, x)$ using (12). Again, $\bar{u} = 0$ solves problem (11) such that $\bar{y}(\omega, x)$ corresponds to the solution to the random obstacle problem (12) and the optimal value is given by

$$J(\bar{u}) = \frac{1}{2}\int_0^1\int_0^1\int_{\frac{1}{2}}^1 200^2 x_2^2(x_1-1)^2(x_1-0.5)^2(1-x_2)y_2^2 dx_1 dx_2 dy_2 \approx 1.6534\text{e-}02.$$

In the numerical computations, we take $\kappa = 1$, $\varepsilon = 10^{-3}$, $u_0(x) = 2$, $\alpha_0 = 1$, and tol $= 1e - 03$. Numerical results, corresponding again to the average of of 20 runs of algorithm, are given in Table 3.2. In both cases, we can check the effectiveness of the method.

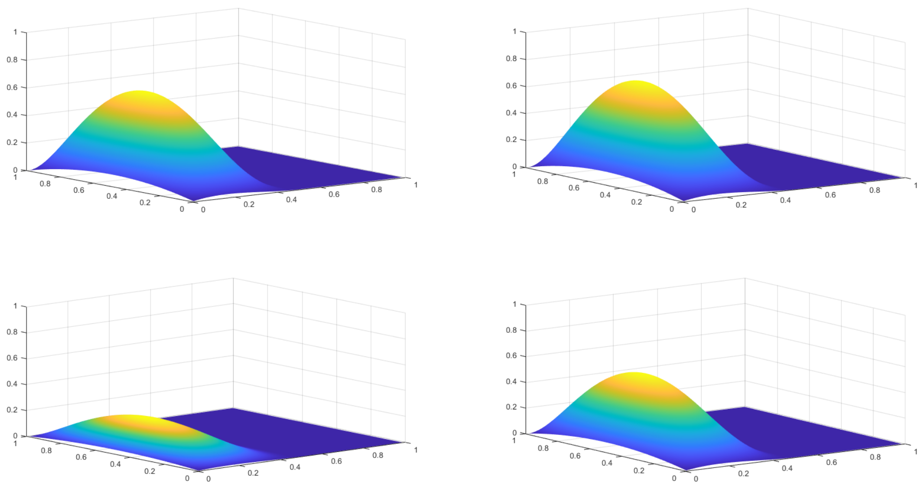

Figure 3.1: Example 2. Four realizations of obstacle problem solution $\bar{y}(\omega, x)$

n^2 (number of nodes)	$\left(\sum_{i=1}^{20} \|\|u_i - \bar{u}\|\|_{L^2(\Omega)}\right)/20$	$\left(\sum_{i=1}^{20} J(u_i)\right)/20$	avg SG iterations
10^2	7.0633e-06	1.6104e-02	2
20^2	7.6343e-06	1.6431e-02	2.2
30^2	6.8373e-06	1.6261e-02	2.5
40^2	8.3325e-06	1.6563e-02	2.65
50^2	7.4246e-06	1.6377e-02	2.85
75^2	6.9939e-06	1.6341e-02	3.65
100^2	8.3325e-06	1.6563e-02	2.65

Table 3.2: Numerical results for Example 2.

4. Concluding remarks

We studied optimal control for stochastic variational inequalities. The preliminary numerical experiments conducted using the stochastic approximation approach seem quite promising. However, a rigorous treatment of the penality approach and convergence analysis for the stochastic approximation scheme needs to be carried out. Using the stochastic Galerkin and the stochastic collocation methods for solving the optimality systems is also an important topic.

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