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Multi-Leader-Common-Follower Games with Pessimistic Leaders: Approximate and Viscosity Solutions

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We consider a two-stage game with k leaders having pessimistic attitude and one follower common to all leaders. Such a game, called CF game, may fail to have *pessimistic solutions*, even if the leader payoffs are linear and the optimal reaction of the follower to the leaders strategies is unique. So, we introduce two classes of games, called weighted value-potential and weighted potential CF games, and we illustrate their inherent difficulties and properties. For the more tractable class of weighted potential CF games, suitable *approximate and viscosity solutions* are introduced and are proven to exist under appropriate conditions, in line with what done for one-leader-one-follower games by the authors.

Keywords: Two-stage game, weighted potential game, pessimistic behavior, viscosity solution, lower semicontinuous set-valued map.

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1. Introduction

In this note, we consider Multi-Leader-Common-Follower games (see, for example, [19]), CF games for short, that is two-stage games [7] with a k-players non-cooperative game at the first stage and a parametric one-player game at the second stage, a particular case of the so-called Multi-Leader-Multi-Follower games [25], [11]. Their hierarchical nature leads to introduce different concepts of solution depending on the behavior of the leaders, in particular on their extreme attitudes, called optimistic or pessimistic [18], in line with what proposed for One-Leader-One-Follower two-stage games (see [10] for an overview). However, the existence of such solutions is complicated by the presence of more leaders, even if the set of the optimal solutions in the second stage is a singleton and the leaders payoffs are linear. Therefore, to our knowledge, only a few number of papers is concerned by existence results for CF games and, except the review [5], they can be divided into three groups:

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- ▷ papers regarding a specific situation, derived from the real-world, which is solved by explicitly computing the solutions set [2], [3], [4];
- ▷ papers considering the mathematical problems associated to these concepts as special cases of equilibrium problems with equilibrium constraints [14], [15], [19];
- ▷ papers considering classes of CF games satisfying specific conditions [16], [17].

We aim to investigate the case of non-unique solution at the second stage, also considered in [19], [16], [17], [5], assuming a pessimistic behavior of the leaders when they cannot influence the common follower that can choose any optimal response to the choice of the leaders. The leaders cannot predict the choice of the follower but they choose to prevent the worst for them, in the same spirit than for One-leader-One-Follower two-stage games (see, for example, [10]). Therefore, in Section 2, a pessimistic solution concept for CF games is presented and, in Section 3, existence of such solutions is established for the class of *Weighted Value-Potential CF games.* The limit of considering this class being that the relative results cannot be formulated in terms of explicit conditions on the data, we also consider the less general class of *Weighted Potential CF games*, which anyway enlarges the ones considered in [16] and [17] in the case of one common follower. Then, we investigate the results which can be obtained in these new classes together with their difficulties, which are in line with those arisen when investigating One-Leader-One-Follower two-stage games with a pessimistic leader (also called weak Stackelberg games [20],[10]). Indeed, pessimistic solutions to weighted potential games may fail to exist even for nice data, so, suitable regularizations and related approximate solutions for CF games with pessimistic leaders are introduced and investigated. This leads to the introduction of suitable *approximate and viscosity* solutions, in line with what we did for One-Leader-Multi-Follower games in [21] and for One-Leader-One-Follower games in [22].

In this paper, in order to get our results easy to use in concrete applications, only finite-dimensional spaces are considered; nevertheless, our results could be naturally extended to infinite dimensional Banach spaces by appropriately balancing the use of strong and weak convergence in the hypotheses, as already done in [22].

2. Setting, preliminaries and basic results

Let Y be a nonempty subset of $\mathbb{R}^h$ and let $k > 1$. If H_i is a nonempty closed subset of $\mathbb{R}^{m_i}$, for $i = 1, ..., k$, we consider $H = \prod_{i=1,..,k} H_i$ and, for any given $\bar{\boldsymbol{x}} = (\bar{x}_1, ..., \bar{x}_k) \in H$, we denote by

$\bar{\boldsymbol{x}}_{-i}$ the point $(\bar{x}_1, ..., \bar{x}_{i-1}, \bar{x}_{i+1}, ..., \bar{x}_k) \in H_{-i} = \prod_{j\neq i} H_j$, and by

$(x_i, \bar{\boldsymbol{x}}_{-i})$ the point $(\bar{x}_1, , .., \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, ..., \bar{x}_k) \in H$.

Let $L_1, ..., L_k$ be real-valued functions defined on $H \times Y$. Given $\boldsymbol{x} = (x_1, ..., x_k) \in H$, we assume that $\boldsymbol{x}$ corresponds to a strategy profile of k leaders playing first non-cooperatively, and then the follower will choice $y \in Y$ knowing the strategy profile $\boldsymbol{x}$ chosen by the leaders. Then, we consider a model in which the follower aims to minimize with respect to y his objective function

$$F : (\boldsymbol{x}, y) \in H \times Y \to F(\boldsymbol{x}, y) \in \mathbb{R} \cup \{+\infty\},$$

knowing that y is constrained in $K(\boldsymbol{x})$, where $K : \boldsymbol{x} \in H \rightrightarrows K(\boldsymbol{x}) \subseteq Y$ is a set-valued map with nonempty values, so that he solves, for any $\boldsymbol{x} \in H$, the problem

$$P(\boldsymbol{x}) \qquad \text{find } \ \bar{y} \in K(\boldsymbol{x}) \ \text{ such that } \ F(\boldsymbol{x}, \bar{y}) \leq F(\boldsymbol{x}, y) \ \ \forall\, y \in K(\boldsymbol{x}).$$

The argmin map

$$\mathcal{M} : \boldsymbol{x} \in H \rightrightarrows \mathcal{M}(\boldsymbol{x}) = \{y \in K(\boldsymbol{x}) : F(\boldsymbol{x}, y) \leq F(\boldsymbol{x}, z) \ \forall\, z \in K(\boldsymbol{x})\}$$

comes to be defined and could be set-valued; we also assume that $\mathcal{M}$ is nonempty-valued. The case where the follower aims to maximize with respect to y his objective function F can be easily analized observing that

$$\max_{y \in K(\boldsymbol{x})} F(\boldsymbol{x}, y) \ = \ - \min_{y \in K(\boldsymbol{x})} -F(\boldsymbol{x}, y).$$

All leaders, when prepared for the worst and having in mind to minimize their payoffs, consider the functions

$$\mathscr{P}_i : \boldsymbol{x} \in H \to \mathscr{P}_i(\boldsymbol{x}) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} L_i(\boldsymbol{x}, y), \ \text{for } i = 1, ..., k \tag{1}$$

and modelize the following classical Nash Equilibrium Problem [24]

$$\text{find } \bar{\boldsymbol{x}} \in H \ \text{ such that } \ \mathscr{P}_i(\bar{\boldsymbol{x}}) \ = \ \inf_{x_i \in H_i} \mathscr{P}_i(x_i, \bar{\boldsymbol{x}}_{-i}) \ \text{ for } i = 1, .., k.$$

This leads to formulate the *Pessimistic Multi-Leader-Common-Follower problem*, pessimistic CF problem for short:

$$(PCF) \qquad \text{find } \ \bar{\boldsymbol{x}} \in H \ \text{ such that}$$
$$\sup_{y \in \mathcal{M}(\bar{\boldsymbol{x}})} L_i(\bar{\boldsymbol{x}}, y) = \inf_{x_i \in H_i} \sup_{y \in \mathcal{M}(x_i, \bar{\boldsymbol{x}}_{-i})} L_i(x_i, \bar{\boldsymbol{x}}_{-i}, y) \ \ \forall\, i = 1, .., k \tag{2}$$

A solution to (PCF) is called a *pessimistic solution to the CF game.*

The next example, considered by Pang and Fukushima in [25], shows that CF games may fail to have pessimistic solutions even if the map $\mathcal{M}$ is single-valued and the leaders' payoffs are linear.

Example 2.1. [25, Example 4] Consider the CF game defined by:

$$H_1 = H_2 = [0, 1], Y = [0, +\infty[, F(x_1, x_2, y) = y(x_1 + x_2 - 1) + \frac{1}{2} y^2,$$
$$L_1(x_1, x_2, y) = \frac{1}{2}\, x_1 + y, \qquad L_2(x_1, x_2, y) = -\left(\frac{1}{2}\, x_2 + y\right).$$

The argmin map $\mathcal{M}$ of the follower is single-valued:

$$\mathcal{M} : \boldsymbol{x} = (x_1, x_2) \in H \ \rightrightarrows \ \mathcal{M}(\boldsymbol{x}) \ = \{\max(0, 1 - x_1 - x_2)\} \subseteq Y,$$

and one can see that the normal form game $(\mathscr{P}_1, \mathscr{P}_2, H_1, H_2)$, where

$$\mathscr{P}_1(x_1, x_2) = \max\left(\frac{1}{2}\, x_1, 1 - \frac{1}{2}\, x_1 - x_2\right), \quad \mathscr{P}_2(x_1, x_2) = \min\left(-\frac{1}{2}\, x_2, -1 + x_1 + \frac{1}{2}\, x_2\right),$$

does not have any Nash equilibrium and the CF game does not have any pessimistic solution. □

Now, we review some basic definitions of set-valued analysis that we will use in the next section. If $(C_n)_n$ is a sequence of nonempty subsets of $\mathbb{R}^p$, the *Painlevé-Kuratowski upper and lower limits* [1] of the sequence $(C_n)_n$ are defined by:

- $z \in \limsup_n C_n$ if there exists $(z_k)_k$ converging to z such that, for a subsequence $(C_{n_k})_k$ of $(C_n)_n$, $z_k \in C_{n_k}$ for any $k \in \mathbb{N}$;
- $z \in \liminf_n C_n$ if there exists $(z_n)_n$ converging to z such that $z_n \in C_n$ for n sufficiently large.

A set-valued map T from $H \subseteq \mathbb{R}^m$ to $Y \subseteq \mathbb{R}^h$ is:

- *lower semicontinuous* over H if for every $\boldsymbol{x} \in H$, every sequence $(\boldsymbol{x}_n)_n$ converging to $\boldsymbol{x}$ in H and every $y \in T(\boldsymbol{x})$ there exists a sequence $(y_n)_n$ converging to y such that $y_n \in T(\boldsymbol{x}_n)$ for n sufficiently large, i.e.
$$T(\boldsymbol{x}) \ \subseteq \ \liminf_n \ T(\boldsymbol{x}_n);$$
- *closed* over H if for any $\boldsymbol{x} \in H$ and any $(\boldsymbol{x}_n)_n$ converging to $\boldsymbol{x}$ in H, if $(y_k)_k$ converges to y in Y and $y_k \in T(\boldsymbol{x}_{n_k})$ for any $k \in \mathbb{N}$, then we have that $y \in T(\boldsymbol{x})$, i.e.:
$$\limsup_n \ T(\boldsymbol{x}_n) \ \subseteq \ T(\boldsymbol{x});$$
- *concave* over H [13] if the set H is convex and for any $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in H and any $\lambda \in [0,1]$ one has:
$$T(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \subseteq \lambda T(\boldsymbol{x}_1) + (1-\lambda) T(\boldsymbol{x}_2).$$

The formulation of problem (PCF) in (2) and the existence theorem of Nash equilibria for normal form games [24] lead to consider semicontinuity and quasi-convexity properties of the so called *marginal functions* of the sup-type, so, classical conditions for achieving such properties are briefly recalled below.

Proposition 2.2. *Let g be a real-valued function defined in $U \times W \subseteq \mathbb{R}^m \times \mathbb{R}^h$, where U and W are nonempty closed sets, and let T be a set-valued map from U to W.*

(1) *If we assume that:*

(S_1) *the set-valued map T is lower semicontinuous over U;*

(S_2) *the function g is lower semicontinuous over $U \times W$;*

then, the marginal function
$$s : u \in U \to s(u) = \sup_{w \in T(u)} g(u, w) \tag{3}$$
is lower semicontinuous over U.

(2) *If we assume that:*

(S_3) *the set W is compact and the set-valued map T is closed over U;*

(S_4) *the function g is upper semicontinuous over $U \times W$;*

then, the marginal function s is upper semicontinuous over U.

(3) *If we assume that:*

(S_5) *the sets U and W are convex and the set-valued map T is concave over U;*

(S_6) *the function g is quasi-convex over $U \times W$;*

then, the marginal function s is quasi-convex over U.

Proof. The proof of points (1) and (2) can be found in [8] or in [1], the proof of (3) can be found in [13]. □

It is well known that, although condition (S_3) is satisfied by the argmin map $\mathcal{M}$ under reasonable assumptions [8], conditions (S_1) and (S_5) may fail to be satisfied by $\mathcal{M}$.

Thus, in Section 3, we consider particular classes of CF games, having in mind that the search of a Nash equilibrium for a normal form game $(G_1, ..., G_k, H_1, ..., H_k)$ can be reduced to the search of a minimum point whenever we deal with a *weighted potential game* in the sense of Monderer and Shapley [23].

We recall that the game $(G_1, ..., G_k, H_1, ..., H_k)$ is a weighted potential game whenever there exist a real-valued function Φ defined on H and a vector

$$\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_k) \in \mathbb{R}^k_{++} = \{(\alpha_1, ..., \alpha_k) \in \mathbb{R}^k : \alpha_i > 0 \ \forall\ i \ = \ 1, ..., k\}$$

such that, for any $i \ = \ 1, ..., k$, any $\boldsymbol{x} \in H$ and any $x'_i \in H_i$, one has

$$G_i(x_i, \boldsymbol{x}_{-i}) - G_i(x'_i, \boldsymbol{x}_{-i}) \ = \ \alpha_i(\Phi(x_i, \boldsymbol{x}_{-i}) \ - \ \Phi(x'_i, \boldsymbol{x}_{-i})).$$

In [12, Thm. 2.1] it was proved that the game $(G_1, ..., G_k, H_1, ..., H_k)$ is a weighted potential game if and only if there exist a real-valued function P, defined on H and called weighted potential of the normal form game, a vector $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_k) \in \mathbb{R}^k_{++}$ and k real-valued functions p_i, defined on H_{-i}, such that

$$G_i(\boldsymbol{x}) = \alpha_i P(\boldsymbol{x}) \ + \ p_i(\boldsymbol{x}_{-i}), \quad \forall\ i \ = \ 1, ..., k.$$

Weighted potential normal form games are tractable since the set of their Nash equilibria contains the set of the minimum points to the weighted potential function, so that existence of Nash equilibria is guaranteed by the lower semicontinuity of the function P and no convexity condition on the payoffs is needed; note that new existence and uniqueness results of Nash equilibria for such games can be found in [9]. Then, exploiting this feature of weighted potential normal form games, one is induced to consider suitable classes of CF games in order to overcome the lack of pessimistic solutions, which may fail to exist even for linear payoffs (see Example 2.1). First investigations in this direction can be found in [16], [17]. In the next section, two classes of CF games satisfying a potentiality-like property are considered and their peculiarities are investigated.

3. Weighted value-potentials and weighted potentials for a CF game

With the same notations as in Section 2, we start this section by introducing the concept of weighted value-potential for a CF game.

Definition 3.1. A CF game is said to be a *weighted value-potential game* if there exist a real-valued function φ, defined on H and called a weighted value-potential of the game CF, and a vector $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_k) \in \mathbb{R}^k_{++}$ such that, for any $i \ = \ 1, ..., k$, any $\boldsymbol{x} \in H$ and any $x'_i \in H_i$, one has

$$\mathscr{P}_i(x_i, \boldsymbol{x}_{-i}) - \mathscr{P}_i(x'_i, \boldsymbol{x}_{-i}) = \alpha_i \left[\varphi(x_i, \boldsymbol{x}_{-i}) - \varphi(x'_i, \boldsymbol{x}_{-i})\right]. \tag{4}$$

i.e. the game $(\mathscr{P}_1, .., \mathscr{P}_k, H_1, .., H_k)$ is a weighted potential normal form game in the sense of Monderer and Shapley.

Then, the next proposition follows from [12, Theorem 2.1].

Proposition 3.2. *A CF game is a weighted value-potential game if and only if there exists a real-valued function $\mathcal{P}$ defined on H and, for any $i = 1, ..., k$, there exist $\alpha_i \in \mathbb{R}_{++}$ and a real-valued function h_i defined on H_{-i}, such that, for all $\boldsymbol{x} \in H$, one has:*

$$\mathscr{P}_i(\boldsymbol{x}) \;=\; \sup_{y \in \mathcal{M}(\boldsymbol{x})} L_i(\boldsymbol{x}, y) \;=\; \alpha_i \mathcal{P}(\boldsymbol{x}) \,+\, h_i(\boldsymbol{x}_{-i}) \,. \tag{5}$$

Moreover, the function $\mathcal{P}$ is a weighted value-potential of the CF game.

A simple, illustrative example of a nonlinear weighted value-potential CF game is the next one.

Example 3.3. Consider the CF game defined by:

$$E = V_1 = V_2 = \mathbb{R}, \quad H_1 = H_2 = [0,1], \quad Y = [0,1], \quad F(x_1, x_2, y) = 0,$$

$$K(x_1, x_2) = [0,1], \; L_1(x_1, x_2, y) = -x_1 + y - yx_2, \; L_2(x_1, x_2, y) = -x_1 y + y - x_1.$$

It is a weighted value-potential game since

$$\mathscr{P}_1(x_1, x_2) = 1 - x_1 - x_2, \quad \mathscr{P}_2(x_1, x_2) = 1 - 2\,x_1,$$

and we can take $\alpha_1 = 1$, $\alpha_2 = 1$, $\mathcal{P}(\boldsymbol{x}) = 1 - x_1$, $h_1(x_2) = -x_2$, $h_2(x_1) = -x_1$. □

Concerning the pessimistic solutions to weighted value-potential CF games, the next existence result holds.

Theorem 3.4. *Assume that a CF game with pessimistic leaders is a weighted value-potential game with $\mathcal{P}$ as a weighted value-potential and that*

(P_1) *the set H_i is compact for any $i = 1, ..., k$;*

(P_2) *the weighted value-potential $\mathcal{P}$ is lower semicontinuous over H.*

Under these conditions there exists a Nash equilibrium for the normal form game $(\mathscr{P}_1, ..., \mathscr{P}_k, H_1, ..., H_k)$, i.e. there exists a pessimistic solution to the CF game.

Proof. Assumptions (P_1) and (P_2) imply that $\mathcal{P}$ has a minimum point in H, which turns out to be a Nash equilibrium for the game $(\mathscr{P}_1, ..., \mathscr{P}_k, H_1, ..., H_k)$. □

Despite Example 3.3, we are aware that the property of being a weighted value-potential game can present inherent difficulties:

- the corresponding class of games is not always easy to be described since a constrained marginal function has to be explicitly computed;
- Theorem 3.4 does not contain direct assumptions on the payoffs of the leaders as well on the constraints and the payoff of the follower.

Thus, we consider a smaller but more tractable class of CF games.

Definition 3.5. A CF game is said to be a *weighted potential game* if there exists a real-valued function π defined on $H \times Y$, called a weighted potential of the CF game, and, for any $i = 1, ..., k$, there exists $\beta_i \in \mathbb{R}_{++}$ such that one has for all $(\boldsymbol{x}, y) \in H \times Y$ and for all $(x'_i, y') \in H_i \times Y$:

$$L_i(x_i, \boldsymbol{x}_{-i}, y) - L_i(x'_i, \boldsymbol{x}_{-i}, y') = \beta_i \left[\pi(x_i, \boldsymbol{x}_{-i}, y) - \pi(x'_i, \boldsymbol{x}_{-i}, y')\right]. \tag{6}$$

The following proposition says that the weighted potential property for CF games can be equivalently formulated in a more handy way.

Proposition 3.6. *A CF game is a weighted potential game if and only if there exists a real-valued function* Π *defined on* $H \times Y$*, and, for any* $i = 1, ..., k$*, there exist* $\beta_i \in \mathbb{R}_{++}$ *and a real-valued function* Ψ_i*, defined on* H_{-i}*, such that, for all* $(\boldsymbol{x}, y) \in H \times Y$*, one has:*

$$L_i(\boldsymbol{x}, y) \ = \ \beta_i \, \Pi(\boldsymbol{x}, y) \ + \ \Psi_i(\boldsymbol{x}_{-i}). \tag{7}$$

Moreover, the function Π *is a weighted potential of the game CF.*

Proof. If the game is a weighted potential game, then, from (7) we get (6) by considering $\pi = \Pi$. Conversely, for any $i = 1, ..., k$, equality (6) implies that the difference function $L_i - \beta_i \pi$ depends only on $\boldsymbol{x}_{-i}$, so that condition (7) is satisfied by taking $\pi = \Pi$ and $\Psi_i = L_i - \beta_i \Pi$. □

We now show that the class of weighted potential CF games is strictly included in the class of weighted value-potential CF games.

Proposition 3.7. *If* Y *is a compact set, then any weighted potential CF game is also a weighted value-potential game. If* Π *is a weighted potential for CF then* $\mathscr{P}(\boldsymbol{x}) = \sup\limits_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$ *is a weighted value-potential for CF.*

Proof. If a CF game satisfies condition (7), then it satisfies condition (5) in Proposition 3.2 by setting, for any $i = 1, ..., k$, $\alpha_i = \beta_i$, $\mathcal{P}(\boldsymbol{x}) = \mathscr{P}(\boldsymbol{x}) = \sup\limits_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$ and $h_i = \Psi_i$. □

Example 3.3 shows that the converse of the above result does not hold in general, since the game therein is clearly not a weighted potential game.

Remark 3.8. In [16, Definition 2.2] and in [17, Definition 3.2], potential and quasi-potential Multi-Leader-Multi-Follower games have been respectively defined and we remark that:

- the concept of weighted potential game of Definition 3.2 generalizes both of them, which coincide when a single follower is common to all leaders;
- weighted value-potential games may be not quasi-potential in the sense of [17], in general. Indeed, the game presented in Example 3.3 is not a quasi-potential game but it is a weighted value-potential game. □

For what concerning weighted potential CF games, the next existence result holds.

Theorem 3.9. *Assume that a CF game is a weighted potential game with* Π *as a weighted potential and that*

(P_1) *the set* H_i *is compact for any* $i = 1, ..., k$*;*

(P_3) *the function* $\mathscr{P}(x) = \sup\limits_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$ *is lower semicontinuous over* H*.*

Under these conditions there exists a Nash equilibrium point for the normal form game $(\mathscr{P}_1, ..., \mathscr{P}_k, H_1, ..., H_k)$*, i.e. there exists a pessimistic solution to the CF game.*

The proof of this theorem is straightforward and it is omitted.

However, even if the leader's payoffs are linear, assumption (P_3) may fail to be satisfied and weighted potential games may fail to have equilibria, as in the next example.

Example 3.10. Consider the CF game defined by:

$$E = V_1 = V_2 = \mathbb{R}, \quad H_1 = H_2 = [0,1], \quad Y = [0,1], \quad F(x_1, x_2, y) = x_1 y,$$
$$K(x_1, x_2) = [0,1], \quad L_1(x_1, x_2, y) = x_1 + y - x_2, \quad L_2(x_1, x_2, y) = \frac{1}{2}(x_1 + y).$$

Then, the CF game is a weighted potential game (so, it is a weighted value-potential game too) and a weighted potential is $\Pi(\boldsymbol{x}, y) = x_1 + y$ with $\beta_1 = 1$ and $\beta_2 = \frac{1}{2}$. However, the problem (PCF) does not have pessimistic solutions. Indeed, the second stage argmin map $\mathcal{M}$ is defined by:

$$\mathcal{M}(x_1, x_2) = [0,1] \text{ when } x_1 = 0 \text{ and } \mathcal{M}(x_1, x_2) = \{0\} \text{ when } x_1 > 0,$$

so we have:
$$\mathscr{P}_1(x_1, x_2) = 1 - x_2, \qquad \mathscr{P}_2(x_1, x_2) = \frac{1}{2} \qquad \text{when } x_1 = 0,$$
$$\mathscr{P}_1(x_1, x_2) = x_1 - x_2, \qquad \mathscr{P}_2(x_1, x_2) = \frac{x_1}{2} \qquad \text{when } x_1 > 0.$$

The normal form game $(\mathscr{P}_1, \mathscr{P}_2, H_1, H_2)$ does not have Nash equilibria since, for any $(x_1, x_2) \in H$, $\mathscr{P}_1(x_1, x_2) > -x_2 = \inf_{x_1' \in H_1} \mathscr{P}_1(x_1', x_2)$ and the best reply of leader 1 is empty-valued.

Therefore, there is no pessimistic solution to the CF game.

Moreover, by taking $\alpha_1 = 1$, $\alpha_2 = \frac{1}{2}$, one can check that this game is a weighted value-potential game with weighted value-potential $\mathscr{P}(\boldsymbol{x}) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$ and that $\mathscr{P}$ is not lower semicontinuous and does not have any minimum point in H since $\mathscr{P}(\boldsymbol{x}) = 1$ if $x_1 = 0$ and $\mathscr{P}(\boldsymbol{x}) = x_1$ if $x_1 \in]0,1]$. □

In spite of the previous example, the following one shows that a weighted value-potential game may have pessimistic solutions even if the weighted value-potential $\mathcal{P}$ in Definition 3.1 is not lower semicontinuous and does not have any optimal point.

Example 3.11. Consider the CF game defined by:

$$E = V_1 = V_2 = \mathbb{R}, \quad H_1 = H_2 = [0,1], \quad Y = [0,1], \quad F(x_1, x_2, y) = x_1 x_2 \, y,$$
$$K(x_1, x_2) = [0,1], \quad L_1(x_1, x_2, y) = x_1 x_2 + y + x_2, \quad L_2(x_1, x_2, y) = x_1 x_2 + y + x_1.$$

In consequence the CF game is a weighted potential game with weighted potential $\Pi(x_1, x_2, y) = x_1 x_2 + y$, so that, in the light of Proposition 3.6, it is also a weighted value-potential game and a weighted value-potential is $\mathscr{P}(\boldsymbol{x}) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$.

The argmin map $\mathcal{M}$ is defined by $\mathcal{M}(x_1, x_2) = [0,1]$ when $x_1 x_2 = 0$ and $\mathcal{M}(x_1, x_2) = \{0\}$ when $x_1 x_2 \neq 0$, so we get that $\mathscr{P}(x_1, x_2) = 1$, when $x_1 x_2 = 0$, $\mathscr{P}(x_1, x_2) = x_1 x_2$, when $x_1 x_2 \neq 0$, so $\mathscr{P}$ is not lower semicontinuous and does not have any minimum point in H. However, since we have:

$$\mathscr{P}_1(x_1, x_2) = 1 + x_2 \text{ and } \mathscr{P}_2(x_1, x_2) = 1 + x_1 \text{ when } x_1 x_2 = 0,$$
$$\mathscr{P}_1(x_1, x_2) = x_1 x_2 + x_2, \text{ and } \mathscr{P}_2(x_1, x_2) = x_1 x_2 + x_1 \text{ when } x_1 x_2 \neq 0,$$

the point $(0,0)$ is a Nash equilibrium for the normal form game $(\mathscr{P}_1, \mathscr{P}_2, H_1, H_2)$, i.e. the CF game has $(0,0)$ as a pessimistic solution. □

Therefore, we can conclude that, for both classes of CF games previously considered, the existence of solutions is not necessarily guaranteed. Then, in line with what has been done for one-leader-multi-followers games [21] and for one-leader-one-follower games with pessimistic behavior of the leaders [22], we will face a weighted potential CF game with pessimistic behavior of the leaders through appropriate regularizations of the optimal response map of the common follower. We start by illustrating this regularization method considering the CF game in Example 3.10.

Let ε be a positive number smaller than 1/4 and consider the ε-minimum map

$$\mathcal{M}^\varepsilon : \boldsymbol{x} \in H \rightrightarrows \mathcal{M}^\varepsilon(\boldsymbol{x}) = \left\{ y \in K(\boldsymbol{x}) : F(\boldsymbol{x}, y) \leq \inf_{z \in K(\boldsymbol{x})} F(\boldsymbol{x}, z) + \varepsilon \right\}.$$

With data of Example 3.10, we get:

$$\mathcal{M}^\varepsilon(x_1, x_2) = [0,1] \text{ if } x_1 \in [0, \varepsilon], \quad \mathcal{M}^\varepsilon(x_1, x_2) = \left[0, \frac{\varepsilon}{x_1}\right] \text{ if } x_1 \in [\varepsilon, 1].$$

Then, the marginal function of Π over this regularized map, denoted by $\mathscr{P}^\varepsilon$, is continuous on H since

$$\mathscr{P}^\varepsilon(\boldsymbol{x}) = \sup_{y \in \mathcal{M}^\varepsilon(\boldsymbol{x})} \Pi(\boldsymbol{x}, y) = \begin{cases} x_1 + 1 & \text{if } x_1 \in [0, \varepsilon], \\ x_1 + \dfrac{\varepsilon}{x_1} & \text{if } x_1 \in [\varepsilon, 1]. \end{cases}$$

One can check that, for any $\varepsilon \in]0, 1/4[$, the minimum points of $\mathscr{P}^\varepsilon$ are $(\sqrt{\varepsilon}, x_2)$ for any $x_2 \in [0,1]$; so, the minimum value of $\mathscr{P}^\varepsilon$ is $2\sqrt{\varepsilon}$.

Therefore, even if the considered CF game does not have pessimistic solutions, we can bypass this lack by considering the limits when ε tends to zero of the above approximate pessimistic solutions, i.e. points $(0, x_2)$, as reasonable solutions to the problem (PCF), since we have that:

$$\lim_{\varepsilon \to 0} \mathscr{P}^\varepsilon(\sqrt{\varepsilon}, x_2) = \lim_{\varepsilon \to 0} \inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}^\varepsilon(\boldsymbol{x})} \Pi(\boldsymbol{x}, y) = 0 = \inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y),$$

and points $(0, x_2)$ allow both the leaders to realize the security value of a new two-stage game, with only one leader, having H as the set of strategies, namely:

$$\text{find} \quad \bar{\boldsymbol{x}} \in H \ \text{ such that } \sup_{y \in \mathcal{M}(\bar{\boldsymbol{x}})} \Pi(\bar{\boldsymbol{x}}, y) = \inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y).$$

Thus, in line with the terminology we adopted in One-Leader-Multi-Follower games [21] or in Bilevel Optimization problems [22], we can introduce the concept of *pessimistic viscosity solution* to (PCF) in the class of weighted CF potential games.

Definition 3.12. Let the CF game be a weighted potential game with Π as a weighted potential. A point $\bar{\boldsymbol{x}} \in H$ is a *pessimistic viscosity solution* to the problem (PCF) if for every sequence of positive numbers $(\varepsilon_n)_n$ decreasing to zero there exists a sequence $(\bar{\boldsymbol{x}}_n)_n$, $\bar{\boldsymbol{x}}_n \in H$ for any $n \in \mathbb{N}$, such that:

(V_1) a subsequence $(\bar{\boldsymbol{x}}_{n_k})_k$ converges towards $\bar{\boldsymbol{x}}$;

(V_2) for any $n \in \mathbb{N}$, $\bar{\boldsymbol{x}}_n$ is a minimum point for the function

$$\mathscr{P}^{\varepsilon_n} : \boldsymbol{x} \in H \to \mathscr{P}^{\varepsilon_n}(\boldsymbol{x}) = \sup_{y \in \mathcal{M}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$$

i.e. $$\mathscr{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \leq \mathscr{P}^{\varepsilon_n}(\boldsymbol{x}) \quad \forall \ \boldsymbol{x} \in H;$$

(V_3) $$\lim_n \mathscr{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) = \inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y).$$

Roughly speaking, a pessimistic viscosity solution to a weighted potential CF game is a cluster point of a sequence of minimum points of suitable regularizations $\mathscr{P}^{\varepsilon_n}$ of the function $\mathscr{P}$, as defined in Proposition 3.7, whose values approach the security value of a one-leader-one-follower two-stage game with an hypothetic pessimistic leader having the weighted potential Π as payoff.

Therefore, in order to investigate such a concept, it is primarily interesting to determine conditions which guarantee that for any fixed positive number ε, the function $\mathscr{P}^\varepsilon$ defined by

$$\mathscr{P}^\varepsilon(\boldsymbol{x}) = \sup_{y \in \mathcal{M}^\varepsilon(\boldsymbol{x})} \Pi(\boldsymbol{x}, y),$$

and called *approximate weighted value-potential*, has a minimum point in H, called *ε-pessimistic solution* to the CF game.

Theorem 3.13. *Assume that a CF game is a weighted potential game with* Π *as a weighted potential. If the following assumptions hold:*

(i) *the sets Y and H_i are compact for any $i = 1, .., k$;*

(ii) *the set-valued map K is closed, lower semicontinuous and convex-valued over H;*

(iii) *the function F is continuous over $H \times Y$;*

(iv) *the function $F(\boldsymbol{x}, \cdot)$ is strictly quasi-convex* [6] *over $K(\boldsymbol{x})$ for every $\boldsymbol{x} \in H$;*

(v) *the weighted potential* Π *is lower semicontinuous over $H \times Y$;*

then, for every $\varepsilon > 0$, there exists a minimum point for the approximate weighted value-potential $\mathscr{P}^\varepsilon$.

Proof. Due to the compactness of the set H, we have only to prove that the function $\mathscr{P}^\varepsilon$ is lower semicontinuous over H. Since also the set Y is assumed to be compact, assumptions (ii)–(iv) imply that the map $\mathcal{M}^\varepsilon$ is lower semicontinuous over H (see, for example, [22, Proposition 2.5]) and, by point (1) in Proposition 2.2, assumption (v) guarantees that $\mathscr{P}^\varepsilon$ is lower semicontinuous on H. □

Finally, we present an existence result for pessimistic viscosity solutions of weighted potential CF games.

Theorem 3.14. *Assume that the CF game is a weighted potential game with* Π *as a weighted potential. If the assumptions in Theorem* 3.13 *and the following hold:*

(vi) *for every $\boldsymbol{x} \in H$ there exists a sequence $(\boldsymbol{x}_n)_n$ converging to $\boldsymbol{x}$ in H such that for every $y \in Y$ and every sequence $(y_n)_n$ converging to y in Y one has*

$$\limsup_n \Pi(\boldsymbol{x}_n, y_n) \leq \Pi(\boldsymbol{x}, y);$$

then, there exists a pessimistic viscosity solution for the CF game.

Proof. Let $(\varepsilon_n)_n$ be a sequence of positive numbers decreasing to zero and let $(\bar{\boldsymbol{x}}_n)_n$ be a sequence of points of H, existing by Theorem 3.13, such that $\bar{\boldsymbol{x}}_n$ is a minimum point of $\mathscr{P}^{\varepsilon_n}$ for any $n \in \mathbb{N}$. Such a sequence has a subsequence converging towards a point $\bar{\boldsymbol{x}} \in H$. In order to prove that $\bar{\boldsymbol{x}}$ is a pessimistic viscosity solution to the CF game, we need only to prove condition (V_3), that is

$$\lim_n \mathscr{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) = \lim_n \inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y) = \inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y).$$

We note that, for any $n \in \mathbb{N}$ and any $\boldsymbol{x} \in H$,

$$\sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y) \leq \sup_{y \in \mathcal{M}^{\varepsilon_n}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y),$$

so that
$$\inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y) \leq \mathscr{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \quad \text{for any } n \in \mathbb{N}$$

and
$$\inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y) \leq \liminf_n \mathscr{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n).$$

If a is a real number such that

$$\inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y) < a < \limsup_n \mathscr{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n),$$

there exist a subsequence $(\bar{\boldsymbol{x}}_{n_k})$ of $(\bar{\boldsymbol{x}}_n)$ such that $a < \mathscr{P}^{\varepsilon_{n_k}}(\bar{\boldsymbol{x}}_{n_k})$ for any $k \in \mathbb{N}$, and a point $\widetilde{\boldsymbol{x}} \in H$ such that

$$\Pi(\widetilde{\boldsymbol{x}}, y) < a \quad \text{for any } y \in \mathcal{M}(\widetilde{\boldsymbol{x}}). \tag{8}$$

Due to condition (vi), there exists a sequence $(\widetilde{\boldsymbol{x}}_n)_n$ converging to $\widetilde{\boldsymbol{x}}$ in H such that for every $y \in Y$ and every sequence $(y_n)_n$ converging to y in Y one has $\limsup_n \Pi(\widetilde{\boldsymbol{x}}_n, y_n) \leq \Pi(\widetilde{\boldsymbol{x}}, y)$.

But condition (V_2) implies that $\mathscr{P}^{\varepsilon_{n_k}}(\bar{\boldsymbol{x}}_{n_k}) \leq \mathscr{P}^{\varepsilon_{n_k}}(\widetilde{\boldsymbol{x}}_{n_k})$ for any k, so that, since

$$a < \limsup_k \mathscr{P}^{\varepsilon_{n_k}}(\bar{\boldsymbol{x}}_{n_k}) \leq \limsup_k \mathscr{P}^{\varepsilon_{n_k}}(\widetilde{\boldsymbol{x}}_{n_k}),$$

there exists a subsequence of $(\widetilde{\boldsymbol{x}}_{n_k})$, still denoted by $(\widetilde{\boldsymbol{x}}_{n_k})$ for simplicity, such that $a < \mathscr{P}^{\varepsilon_{n_k}}(\widetilde{\boldsymbol{x}}_{n_k})$ for any $k \in \mathbb{N}$. Therefore, there exists a sequence $(y_k)_k$ in Y such that $y_k \in \mathcal{M}^{\varepsilon_{n_k}}(\widetilde{\boldsymbol{x}}_{n_k})$ and $a < \Pi(\widetilde{\boldsymbol{x}}_{n_k}, y_k)$ for any $k \in \mathbb{N}$.

A subsequence of $(y_k)_k$ has to converge to $\widetilde{y} \in \mathcal{M}(\widetilde{\boldsymbol{x}})$ (see, for example, [22, Proposition 2.2]) so, by assumption vi), we get $a \leq \Pi(\widetilde{\boldsymbol{x}}, \widetilde{y})$, which contradicts (8). We conclude that $\limsup_n \mathscr{P}^{\varepsilon_n}(\bar{\boldsymbol{x}}_n) \leq \inf_{\boldsymbol{x} \in H} \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$ and the proof is completed. □

It is worth noting that condition (vi) is satisfied whenever the function $\Pi(\boldsymbol{x}, \cdot)$ is upper semicontinuous over Y for every $\boldsymbol{x} \in H$.

Finally, in the same spirit of [21, Proposition 3.2] and [22, Proposition 4.2], we show that a viscosity solution for a weighted potential CF game, when existing, turns out to be a minimum point of the lower semicontinuous regularization of a weighted value-potential of CF.

Proposition 3.15. *Let CF be a weighted potential game. If the assumptions of Theorem* 3.13 *hold and the weighted potential* Π *is upper semicontinuous over* $H \times Y$*, then, any viscosity solution for CF is a minimum point for cl* $\mathscr{P}$*, the lower semicontinuous regularization of the weighted value-potential* $\mathscr{P}$*, defined by* $\mathscr{P}(\boldsymbol{x}) = \sup_{y \in \mathcal{M}(\boldsymbol{x})} \Pi(\boldsymbol{x}, y)$.

Proof. The proof, which consists in proving that, for any sequence $(\varepsilon_n)_n$ of positive numbers decreasing to zero, the sequence $(\mathscr{P}^{\varepsilon_n})_n$ epiconverges [22] to *cl* $\mathscr{P}$, is omitted since it is similar to the proof of [22, Proposition 4.2]. We only stress that assumption (ii) in [22, Proposition 4.2] is satisfied by the value-potential $\mathscr{P}$ due to point (2) in Proposition 2.2: indeed, under the made assumptions, the argmin map $\mathcal{M}$ is closed, so the upper semicontinuity of Π over $H \times Y$ guarantees the upper semicontinuity of $\mathscr{P}$ over H. Then, by a well known property of epiconvergence, the limit of a sequence of minimum points of $(\mathscr{P}^{\varepsilon_n})_n$, that is a viscosity solution for CF, is a minimum point for *cl* $\mathscr{P}$. $\square$

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