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Remarks on Fixed Point and Equilibrium Problems

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We consider the fixed point problem as two cases of the equilibrium problem in the sense of Blum-Oettli. The first case explores the properties of maximal σ monotone operators, to propose necessary and sufficient conditions to guarantee the existence of fixed points. The second case transforms the fixed point problem, defined by an operator that has closed images not necessarily σ monotone, into an equilibrium problem, with a different approach than the first case. Also, we establish a methodology to extend any σ (with function σ bounded) monotone operator into a maximal σ monotone operator.

Keywords: Fixed point problem, equilibrium problem, σ monotone operator.

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1. Introduction

The history of the Fixed Point Problems (FPP, in short), according to the literature, begins in 1910 for the differentiable case and was proved by Hadamard. After a year, the continuous case was proved by Brouwer in 1911. Here we are interested in the FPP for point-to-set operators $T \subset \mathbb{R}^n \times \mathbb{R}^n$, where $T(x) = \{u \in \mathbb{R}^n : (x, u) \in T\}$. Here, the problem is to find a point x such that $x \in T(x)$. In [3], the authors transform the FPP into an equilibrium problem (EP, in short), as an invitation to analyze the FPP defined by a point-to-set operator. Of course there are many ways to transform the FPP into an EP, here we consider two approaches different from the one suggested by Blum-Oettli in [3].

The σ monotone operators were first introduced in 2009 (for more details see [6]), for the case where the effective domain of the operator is the entire space. Later in 2020 (for more details see [9]) the authors deal with the case of convex domains, non necessarily the entire space. In general, the effective domain of a σ monotone operator is not a convex set. In this last case the problem is how to construct maximal σ monotone operators from σ monotone operators with non-convex effective domain. To answer this question, we propose a methodology, following the case of monotone operators, i.e. construct a major operator containing the σ monotone operator and for any point of the major operator added to the monotone σ operator, the new operator remains σ monotone. This idea, for the monotone case, was introduced by Crouzeix-Ocaña-Sosa in [4].

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The structure of this manuscript is: After this introduction, we consider some notations used throughout this manuscript. In Section 2 we deal with σ monotone operators. In Section 3 we deal with the FPP for maximal σ monotone operators. Finally, in section 4 we transform the FPP defined by an operator with closed images into an EP.

Notations

Let A be a nonempty set in $\mathbb{R}^n$, we use the following notations:

1. $\overline{A}$ denotes the *closure* of the set A.
2. $int(A)$ denotes the *interior* of the set A.
3. $\partial(A)$ denotes the *boundary* of the set A.
4. $co(A)$ denotes the *convex hull* of the set A.
5. A^∞ denote the *recession cone* of A.
6. $\langle A, y\rangle := \{\langle u, y\rangle : u \in A\}$
7. $B(0,r) := \{y \in \mathbb{R}^n : \|y\| < r\}$, resp. $CB(0,r) := \{y \in \mathbb{R}^n : \|y\| \leq r\}$ denote the *open ball*, resp. the *closed ball*, with center in the origin and radius r.
8. S denotes the *boundary* of $B(0,1)$.
9. $T \subset \mathbb{R}^n \times \mathbb{R}^n$ denotes an operator, here $T(x) = \{u \in \mathbb{R}^n : (x,u) \in T\}$.
10. $D(T) = \{x \in \mathbb{R}^n : T(x) \neq \emptyset\}$ denotes the *effective domain* of the operator $T \subset \mathbb{R}^n \times \mathbb{R}^n$.
11. $g : D(T) \times D(T) \to \mathbb{R} \cup \{+\infty\}$ is defined by $g(x,y) = \sup_{u\in T(x)}\langle u, \frac{y-x}{\|y-x\|}\rangle$ with $x \neq y$.
12. A set A is called *Motzkin decomposable* if $A = B + C$, where C is a nonempty convex closed cone and B is a nonempty convex and compact set.

2. Maximal σ-monotone operators

The notion of σ monotonicity was introduced by Iusem-Kassay-Sosa in 2009 as pre-monotonicity (for more details see [6], of course there are other contributions about that notion, for instance in [2], [1] and [9]). In order to understand the maximality of a σ monotone operator, we develop here some ideas about σ monotonicity. So, in [9], the authors introduce the function $\sigma_T : D(T) \to \mathbb{R} \cup \{+\infty\}$ defined by:

$$\sigma_T(x) = \max\{0, \sup_{y\neq x,\ y\in D(T)} [g(x,y) + g(y,x)]\}. \tag{1}$$

The following result holds for any operator $T \subset \mathbb{R}^n \times \mathbb{R}^n$.

Lemma 2.1. *Let $T \subset \mathbb{R}^n \times \mathbb{R}^n$ be an operator.*

(1) *If $\{x,y\} \subset D(T)$, then*

$$\langle T(x), y - x\rangle + \langle T(y), x - y\rangle \leq \min\{\sigma_T(x), \sigma_T(y)\}\|x - y\| \tag{2}$$

(2) *If $x \in D(T)$, then $\sigma_T(x) \leq \sup_{y\in D(T), y\neq x} \sigma_T(y)$.*

Proof. (1) follows from the fact that for each $(w,z) \in D(T) \times D(T)$ with $w \neq z$ we have $\sigma_T(w) \geq g(w,z) + g(z,w)$. First take $x = w$ and $y = z$. Finally, take $y = w$ and $x = z$. (2) follows directly from (1). □

Here, we define the σ monotonicity as follows:

Definition 2.2. Let $T \subset \mathbb{R}^n \times \mathbb{R}^n$ be an operator. T is called σ monotone, if $\sigma : D(T) \to [0, +\infty)$ and $\sigma_T \leq \sigma$ on $D(T)$. □

When σ_T is the null function, T is called monotone.

Proposition 2.3. *Let $T \subset \mathbb{R}^n \times \mathbb{R}^n$ be an operator. T is σ_T monotone, if and only if $\sigma_T : D(T) \to [0, +\infty)$. Moreover, for $\{x, y\} \subset D(T)$, we have the following inequality:*

$$\langle T(x), y - x\rangle + \langle T(y), x - y\rangle \leq \min\{\sigma_T(x), \sigma_T(y)\}\|x - y\| < +\infty \tag{3}$$

Proof. Follows from its definition 2.2 and Lemma 2.1 . □

In the case of monotone operators, the authors in [4] construct an operator $\tilde{T}$, greater than T, with the following property: for every $(x, u) \in \tilde{T}$, $T \cup \{(x, u)\}$ remains monotone. Here, we consider the following operator $T^c \subset \mathbb{R}^n \times \mathbb{R}^n$, defined as follows:

$$T^c = \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n : \langle u, y - x\rangle + \langle v, x - y\rangle \leq \sigma_T(y)\|x - y\| \ \forall (y, v) \in T\}.$$

Lemma 2.4. *For each $x \in \mathbb{R}^n$: $T^c(x)$ is closed and convex.*

Proof. The statement follows because

$$T^c(x) = \{u \in \mathbb{R}^n : \langle u, y - x\rangle \leq \sigma_T(y)\|y - x\| + \langle v, y - x\rangle \ \ \forall (y, v) \in T\},$$

so, it is the intersection of half spaces. □

The following result, when $x \in \partial(\overline{co(D(T))}) \cap D(T^c)$, gives us information about $T^c(x)$, in addition to the fact of being convex and closed.

Proposition 2.5. *Let $T \subset \mathbb{R}^n \times \mathbb{R}^n$ be a σ_T monotone operator. If we have $x \in \partial(\overline{co(D(T))}) \cap D(T^c)$, then $(T^c(x))^\infty = N_{D(T)}(x)$.*

Proof. Take $x \in \partial(\overline{co(D(T))}) \cap D(T^c)$ and $d \in N_{\overline{co(D(T))}}(x) = N_{D(T)}(x)$. Then for $u \in T^c(x)$ we obtain

$$\langle u, y - x\rangle + \langle v, x - y\rangle \leq \sigma_T(y)\|x - y\| \ \ \forall (y, v) \in T \tag{4}$$

and

$$\langle d, y - x\rangle \leq 0 \ \ \forall y \in D(T) \tag{5}$$

Multiplying (5) by $t \geq 0$ and adding (4), we obtain

$$\langle u + td, y - x\rangle + \langle v, x - y\rangle \leq \sigma(y)\|x - y\| \ \ \forall (y, v) \in T, \ \forall t \geq 0.$$

It follows that $u + td \in T^c(x)$ for all $t \geq 0$, and hence $d \in (T^c(x))^\infty$.

Conversely, take $d \in (T^c(x))^\infty$, $x \in \partial(\overline{co(D(T))}) \cap D(T^c)$ and $u \in T^c(x)$. Then,

$$\langle u + td, y - x\rangle + \langle v, x - y\rangle \leq \sigma_T(y)\|x - y\| \ \ \forall (y, v) \in T, \ \forall t > 0.$$

So,

$$\frac{\langle u, y - x\rangle + \langle v, x - y\rangle}{t} + \langle d, y - x\rangle \leq \frac{\sigma_T(y)\|y - x\|}{t} \tag{6}$$

for all $(y, v) \in T$ and all $t > 0$.

Taking limits with $t \to \infty$ in (6), we obtain $\langle d, y - x\rangle \le 0$ for all $y \in D(T)$. Thus $d \in N_{D(T)}(x)$. □

The next result appears in [4]. For the sake of completeness, we consider it here.

Lemma 2.6. *Let $D = co(\{x^0, x^1, \cdots, x^n\})$ be an n-dimensional simplex of $\mathbb{R}^n$. Take a closed and convex set $V \subset int(D)$. Then, for all $x \in V$ and all $c \in \mathbb{R}^n$, the linear programming problem*

$$\min_u \sum_{i=0}^{n} u_i \ \text{ s.t.} \sum_{i=0}^{n} u_i(x^i - x) = c, \ \ u \ge 0, \tag{7}$$

is feasible and has a unique optimal solution, say $u(c, x)$, which is continuous as a function of $(c, x) \in \mathbb{R}^n \times V$.

Proof. See Lemma 2.2 in [4] □

Now, we prove the local boundedness of T^c on $DT = int(\overline{co(D(T))})$, when $DT \ne \emptyset$.

Theorem 2.7. *Let $T \subset \mathbb{R}^n \times \mathbb{R}^n$ be a σ_T monotone operator. If $DT \ne \emptyset$ and $\bar{x} \in DT$, then,*

(1) *$T \subset T^c$,*

(2) *there exists a compact set K and a neighborhood V of $\bar{x}$ such that $\emptyset \ne T^c(x) \subset K$ for all $x \in V$.*

Proof. The first item is evident. For the second item, since $\bar{x} \in DT$, there exists $\{\bar{t}_i\}_{i=1}^n \subset (0, 1)$ and $\{(x^i, u^i)\}_{i=0}^n \subset T$ such that the following vectors $x^1 - x^0$, $x^2 - x^0$, $\ldots, x^n - x^0$ are linearly independent, $\bar{x} = \sum_{i=0}^n \bar{t}_i x^i$ and $\sum_{i=0}^n \bar{t}_i = 1$. Taking $\varepsilon > 0$ such that $\varepsilon < \bar{t}_i$ for all $i \in \{0, 1, \cdots, n\}$, we consider

$$V = \left\{ x = \sum_{i=0}^{n} t_i x^i : \sum_{i=0}^{n} t_i = 1, \ \{t_i\}_{i=0}^n \subset [\varepsilon, 1) \right\}.$$

By construction, V is a nonempty, convex and compact neighborhood of $\bar{x}$ contained in DT. Given $c \in \mathbb{R}^n$ and $x \in DT$, we define

$$\alpha(c, x) = \sup_u \{\langle c, u\rangle : u \in T^c(x)\}, \tag{8}$$

with, convention, $\alpha(c, x) = -\infty$ if $T^c(x) = \emptyset$. Then,

$$-\infty \le \alpha(c, x) \le \beta(c, x)$$

where,
$$\beta(c, x) = \sup_u \{\langle c, u\rangle : A^t u \le b\}, \tag{9}$$

$A \in \mathbb{R}^{n\times(n+1)}$ is the matrix with columns $x^i - x$ $(0 \le i \le n)$, and $b \in \mathbb{R}^{n+1}$ is defined as $b_i = \sigma_T(x^i)\|x^i - x\| + \langle u^i, x^i - x\rangle$.

Let P_e be the linear programming problem defined by the right hand side of (9). Its dual is the problem D_e given by

$$\widetilde{\beta}(c, x) = \min_y \{\langle y, b\rangle : y \ge 0, \ Ay = c\}. \tag{10}$$

The restriction set of Problem D_e satisfies the assumptions of Lemma 2.6. Hence, D_e is feasible, there is no duality gap between P_e and D_e, and

$$\beta(c,x) = \widetilde{\beta}(c,x) \leq \rho(c,x) := \langle u(c,x), b\rangle, \tag{11}$$

where $u(c,x)$ is the solution of the linear problem (7) of Lemma 2.6.

Let $M = \sup_{x,c}\{\rho(c,x) : x \in V,\ \|c\| \leq 1\}, K = CB(0,M)$, with ρ defined as in (11).

Since V is compact and the function u is continuous on $\mathbb{R}^n \times V$ by Lemma 2.6 M is finite and K is a nonempty compact convex set.

Then, for all c with $\|c\| \leq 1$ and all $x \in V$, we obtain, using (10) and ρ as in (11),

$$\alpha(c,x) \leq \beta(c,x) \leq \rho(c,x) \leq M. \tag{12}$$

Define the quantities

$$A(x) = \sup_u\{\|u\| : u \in T^c(x)\},$$

$$B(x) = \sup_u\{\|u\| : \langle u, x^i - x\rangle \leq b_i, \quad (0 \leq i \leq n)\}.$$

Then, we obtain

$$A(x) = \sup_{c,u}\{\langle c,u\rangle\ :\ u \in T^c(x),\ \|c\| \leq 1\} = \sup_c\{\alpha(c,x) : \|c\| \leq 1\},$$

$$B(x) = \sup_c\{\beta(c,x) : \|c\| \leq 1\}.$$

In view of (12), we have $A(x) \leq B(x) \leq M$ for all $x \in V$, implying

$$T^c(x) \subset \{u\ :\ \langle u, x^i - x\rangle \leq b_i,\ i = 0,1,\cdots,n\} \subset K \tag{13}$$

Now, suppose there exists $y \in V$ such that $T^c(y) = \emptyset$, here $y \in DT$. So,

$$T^c(y) = \bigcap_{(x,u)\in T}\{v \in \mathbb{R}^n : \langle v, x-y\rangle \leq b(x,u,y)\},$$

where $b(x,u,y) = \sigma_T(x)\|x-y\| + \langle u, x-y\rangle$. Since K is a nonempty convex compact set, there exists q elements of T, $\{(x^i,u^i)\}_{i=n+1}^{n+q} \subset T$, such that

$$\emptyset = \left[\bigcap_{j=n+1}^{n+q}\{v \in \mathbb{R}^n : \langle v, x^j - y\rangle \leq b_j\}\right]\bigcap K$$

where $b_j = b(x^j,u^j,y)$. Next, in view of (13)

$$\emptyset = \bigcap_{j=0}^{n+q}\{v \in \mathbb{R}^n : \langle v, x^j - y\rangle \leq b_j\}. \tag{14}$$

Consider the $n \times (n+q+1)$ matrix Q with with columns $(x^j - y)$, $j = 0,1,\cdots,n+q$ and the vector $b \in \mathbb{R}^{n+q+1}$ as defined before. Next, consider the pair of dual linear programs

$$m_1 = \sup_v\left[\langle 0,v\rangle : Q^T v \leq b\right] \quad \text{and} \quad m_2 = \inf_u\left[\langle b,u\rangle : u \geq 0,\ Qu = 0\right].$$

The second problem is feasible and therefore $m_1 = m_2$. Next (14) is equivalent to $m_1 = -\infty$ and therefore to

$$\exists u \in \mathbb{R}^{n+q+1} \text{ such that } u \geq 0,\ Qu = 0,\ \textit{and } \langle b, u\rangle < 0.$$

For each u feasible for the dual problem, we have that $Qu = 0$ implies that

$$y = \sum_{i=0}^{n+q} \frac{u_i x^i}{\sum_{j=0}^{n+q} u_j}.$$

Without loss of generality, consider $\sum_{j=0}^{n+q} u_i = 1$. So there exists a feasible u with $\sum_{j=0}^{n+q} u_i = 1$, such that $\langle u, b\rangle < 0$. On the other hand,

$$\langle u^i, x^j - x^i\rangle + \langle u^j, x^i - x^j\rangle \leq \min\{\sigma_T(x^i), \sigma_T(x^j)\}\|x^i - x^j\|,\ \forall i, j \tag{15}$$

For the term after the inequality symbol, we have:

$$\begin{array}{rl} & \sum_{i,j=0}^{n+q} u_i u_j \min\{\sigma_T(x^i), \sigma_T(x^j)\}\|x^i - x^j\| \\ \leq & \sum_{i,j=0}^{n+q} u_i u_j \min\{\sigma_T(x^i), \sigma_T(x^j)\}\|x^i - y\| \\ + & \sum_{i,j=0}^{n+q} u_i u_j \min\{\sigma_T(x^i), \sigma_T(x^j)\}\|x^j - y\| \\ \leq & \sum_{i=0}^{n+q} u_i \sigma_T(x^i)\|x^i - y\| + \sum_{j=0}^{n+q} \sigma_T(x^j)\|x^j - y\| \\ = & 2\sum_{i=0}^{n+q} u_i \sigma_T(x^i)\|x^i - y\|. \end{array} \tag{16}$$

For the term before the inequality symbol, we have:

$$\begin{array}{rl} & \sum_{i,j=0}^{n+q} u_i u_j [\langle u^i, x^j - x^i\rangle + \langle u^j, x^i - x^j\rangle \\ = & \sum_{i,j=0}^{n+q} u_i u_j \langle u^i, x^j - x^i\rangle + \sum_{i,j=0}^{n+q} u_i u_j \langle u^j, x^i - x^j\rangle \\ = & \sum_{i=0}^{n+q} u_i \langle u^i, y - x^i\rangle + \sum_{j=0}^{n+q} u_j \langle u^j, y - x^j\rangle \\ = & 2\sum_{i=0}^{n+q} u_i \langle u^i, y - x^i\rangle. \end{array} \tag{17}$$

Finally, from (15), (16) and (17), it follows that

$$0 \leq 2\sum_{i=0}^{n+q} u_i [\sigma_T(x^i)\|x^i - y\| + \langle u^i, x^i - y\rangle = 2\langle u, b\rangle] < 0, \tag{18}$$

which is impossible. □

As a consequence of Theorem 2.7, we have the following results.

Corollary 2.8. *Let $\{u^i\} \subset \mathbb{R}^n$ be such that $u^i \in T^c(x^i)$ and $\{x^i\} \subset DT$ converges to $x \in \partial DT$.*

(1) *If $T^c(x) \neq \emptyset$, then every cluster point of $\{u^i\}$ belong to $T^c(x)$.*

(2) *If $T^c(x) = \emptyset$, then $\{u^i\}$ is unbounded.*

Proof. For the first item, take a cluster point u of $\{u^i\}$. Without loss of generality, we can consider that $\{u^i\}$ converges to u. The statement follows because

$$\langle u^i, y - x^i\rangle + \langle v, x^i - y\rangle \leq \sigma_T(y)\|x^i - y\|\ \forall (y, v) \in T.$$

For the second item, suppose that $\{u^i\}$ is bounded, taking u, a cluster point of $\{u^i\}$ and using the same argument as in the previous item we have a contradiction. □

Now, we are able to prove that T^c has the same property as $\tilde{T}$ for the monotone case, when σ_T is bounded.

Theorem 2.9. *Let $T \subset \mathbb{R}^n \times \mathbb{R}^n$ be a σ_T monotone operator, σ_T is a bounded function and $(\bar{x}, \bar{u}) \in (D(T^c) \setminus D(T)) \times \mathbb{R}^n$.*

(1) *$T' = T \cup \{(\bar{x}, \bar{u})\}$ is $\sigma_{T'}$ monotone if and only if $\bar{u} \in T^c(\bar{x})$.*
Moreover $\sigma_{T'}(x) = \sigma_T(x)$ $\forall x \in D(T)$.

(2) *$\hat{T} = T \cup (\{\bar{x}\} \times T^c(\bar{x}))$ is $\sigma_{\hat{T}}$ monotone. Moreover $\sigma_{\hat{T}}(x) = \sigma_T(x)$ $\forall x \in D(T)$.*

(3) *$T \subset T' \subset \hat{T} \subset (\hat{T})^c \subset (T')^c \subset T^c$.*

Proof. Note that, from the boundedness of σ_T and definition of T' and $\hat{T}$, we have:

(a) There exists $M > 0$ such that $\sigma_T(x) \in [0, M]$ $\forall x \in D(T)$.

(b) $T(x) = T'(x) = \hat{T}(x)$ and $\sigma_T(x) \leq \sigma_{T'}(x) \leq \sigma_{\hat{T}}(x)$ $\forall x \in D(T)$

If we prove $\sigma_{\hat{T}}(x) \leq \sigma_T(x)$ $\forall x \in D(T)$, then $\sigma_T(x) = \sigma_{T'}(x) = \sigma_{\hat{T}}(x)$ $\forall x \in D(T)$. Indeed, taking $x \in D(T)$, if $y \in D(\hat{T})\setminus\{x\} = (D(T)\setminus\{x\}) \cup \{\bar{x}\}$, then $y \in D(T)\setminus\{x\}$ or $y = \bar{x}$. So,

$$\sup_{u \in \hat{T}(x)} \langle u, \frac{y-x}{\|y-x\|}\rangle + \sup_{v \in \hat{T}(y)} \langle v, \frac{x-y}{\|x-y\|}\rangle \leq \sigma_T(x) \quad \forall y \in D(T) \setminus \{x\}.$$

If $y = \bar{x}$,
$$\sup_{u \in \hat{T}(x)} \langle u, \frac{\bar{x}-x}{\|\bar{x}-x\|}\rangle + \sup_{v \in \hat{T}(\bar{x})} \langle v, \frac{x-\bar{x}}{\|x-\bar{x}\|}\rangle \leq \sigma_T(x)$$

(this follows from the definition of $T^c(\bar{x}) = \hat{T}(\bar{x})$ and $\hat{T}(x) = T(x)$ $\forall x \in D(T)$). It implies that $\sigma_{\hat{T}}(x) \leq \sigma_T(x)$. So, $\sigma_T(x) = \sigma_{T'}(x) = \sigma_{\hat{T}}(x)$ $\forall x \in D(T)$.

(1) If T' is $\sigma_{T'}$ monotone, then

$$\langle \bar{u}, y - \bar{x}\rangle + \langle v, \bar{x} - y\rangle \leq \min\{\sigma_{T'}(\bar{x}), \sigma_{T'}(y)\}\|y - \bar{x}\| \leq \sigma_T(y)\|y - \bar{x}\|$$

for all $(y, v) \in T$. This implies that $\bar{u} \in T^c(\bar{x})$.
If $\bar{u} \in T^c(\bar{x})$, we only need to prove that $\sigma_{T'}(\bar{x}) < +\infty$. Indeed,

$$\langle \bar{u}, \frac{y-\bar{x}}{\|y-\bar{x}\|}\rangle + \sup_{v \in T(y)} \langle v, \frac{\bar{x}-y}{\|\bar{x}-y\|}\rangle \leq \sigma_T(y) \leq M \quad \forall y \in D(T).$$

This implies that $\sigma_{T'}(\bar{x}) \leq M < +\infty$.

(2) If $y \in D(T)$, $\sup_{u \in \hat{T}(\bar{x})} \langle u, \frac{y-\bar{x}}{\|y-\bar{x}\|}\rangle + \sup_{v \in \hat{T}(y)} \langle v, \frac{\bar{x}-y}{\|\bar{x}-y\|}\rangle \leq \sigma_T(y) \leq M$. This implies that $\sigma_{\hat{T}}(\bar{x}) \leq M < +\infty$, because $\hat{T}(\bar{x}) = T^c(\bar{x})$ and $\hat{T}(y) = T(y)$.

(3) Follows from definitions of T, T', $\hat{T}$ and F^c for $F \in \{T, T', \hat{T}\}$. □

When $T \subset \mathbb{R}^n \times \mathbb{R}^n$ is a σ_T monotone, $D(T) = D(T^c) \cap \overline{DT}$, we use the following operator (introduced in [9]):

$$T^h(x) = \{u \in \mathbb{R}^n : \langle u - v, y - x\rangle \leq \min\{\sigma_T(x), \sigma_T(y)\}\|y - x\| \ \ \forall (y, v) \in T\} \quad (19)$$

Corollary 2.10. *Take $(\bar{x}, \bar{u}) \in D(T) \times \mathbb{R}^n$ and $T' = T \cup \{(\bar{x}, \bar{u})\}$. T' is $\sigma_{T'}$ monotone if and only if $\bar{u} \in T^h(\bar{x})$. Moreover, $\hat{T} = T \cup (\{\bar{x}\} \times T^h(\bar{x}))$ is $\sigma_{\hat{T}}$ monotone.*

Proof. The statement follows because $T(x) \subset T^h(x) \subset T^c(x)\ \forall x \in D(T)$. Use Definition of T^h and Theorem 2.9. □

Definition 2.11. An operator $T \subset \mathbb{R}^n \times \mathbb{R}^n$ is called *maximal σ_T monotone*, if $\emptyset \neq DT \subset D(T) \subset \overline{DT}$ and if there exist other σ_T monotone F such that $T \subset F$, then $T = F$.

Proposition 2.12. *Let $T \subset \mathbb{R}^n \times \mathbb{R}^n$ be an operator satisfying the inclusion $DT \subset D(T) = D(T^c) \cap \overline{DT}$. T is maximal σ_T monotone if and only if $T = T^h$.*

Proof. It was proved in Proposition 3.2 in [9]. □

Theorem 2.13. *Let T be a σ_T monotone operator. If $DT \neq \emptyset$ and if σ_T is bounded, then T has at least one maximal σ_{T_e} extension T_e of T, which satisfies $DT \subset D(T_e) = D(T^c) \cap \overline{DT}$ and $\sigma_{T_e}(x) = \sigma_T(x)\ \forall x \in D(T)$.*

Proof. The proof requires Zorn's Lemma. Consider

$$P = \left\{ F \subset \mathbb{R}^n \times \mathbb{R}^n : \begin{array}{l} T \subset F \subset T^c,\ F \textit{ is } \sigma_F \textit{ monotone}, \\ D(F) \subset D(T^c) \cap \overline{DT},\ \sigma_F = \sigma_T \textit{ on } D(T) \end{array} \right\}.$$

P is endowed with the order defined by inclusion, so P is partially ordered and not empty (since $T \in P$). Let $Q = \{F_i\}_{i \in I} \subset P$ be totally ordered (a chain), it is easy to see that $F = \cup_{i \in I} F_i$ is an upper bound of Q. This implies that P is inductive. By Zorn's Lemma, P admits a maximal element. Denote by T_e a maximal element of P. We prove that $D(T_e) = D(T^c) \cap \overline{DT}$, by contradiction. We suppose that there exists $x \in D(T^c) \cap \overline{DT}$ such that $x \notin D(T_e)$, then for any $u \in T^c(x)$, from Theorem 2.9, we have that $(T_e)' = T_e \cup \{(x, u)\} \in P$. This contradiction implies (because T_e is a maximal element) $D(T_e) = D(T^c) \cap \overline{DT}$.

Now, we prove that $(T_e)^h$ is σ_{T_e} monotone. Since $T(x) \subset T^h(x)$ and $T^h(x)$ is convex, then $(T_e)^h(x) = \cup_{t \in [0,1]}(tT_e(x) + (1-t)(T_e)^h(x))$. From definition of T^h, we obtain

$$\langle (T_e)^h(x), \frac{y-x}{\|y-x\|} \rangle + \langle T_e(y), \frac{x-y}{\|x-y\|} \rangle \leq \min\{\sigma_{T_e}(x), \sigma_{T_e}(y)\} \tag{20}$$

$$\langle T_e(x), \frac{y-x}{\|y-x\|} \rangle + \langle (T_e)^h(y), \frac{x-y}{\|x-y\|} \rangle \leq \min\{\sigma_{T_e}(x), \sigma_{T_e}(y)\} \tag{21}$$

Multiplying 20 and 21 by t and $(1-t)$ respectively and add them, and the fact that $(T_e)^h(w) = \cup_{t \in [0,1]}(tT_e(w) + (1-t)(T_e)^h(w)$, we have that

$$\langle (T_e)^h(x), \frac{y-x}{\|y-x\|} \rangle + \langle (T_e)^h(y), \frac{x-y}{\|x-y\|} \rangle \leq \min\{\sigma_{T_e}(x), \sigma_{T_e}(y)\} \tag{22}$$

But, equation 22 implies that $(T_e)^h$ is σ_{T_e} monotone. Note that $T_e \subset (T_e)^h$, it implies that $(T_e)^{hh} = ((T_e)^h)^h \subset (T_e)^h$. Since $(T_e)^h$ is σ_{T_e} monotone, we have that $(T_e)^h \subset (T_e)^{hh}$ and so, $(T_e)^h$ is maximal σ_{T_e} monotone, because $(T_e)^{hh} = (T_e)^h$. □

3. The problem

Let $T \subset \mathbb{R}^n \times \mathbb{R}^n$ be an operator. We are interested in the following two problems:

$$(FPP(T)) \qquad \text{Find } \; x \in \mathbb{R}^n : x \in T(x), \tag{23}$$

$$(ZIP(T)) \qquad \text{Find } \; x \in \mathbb{R}^n : 0 \in T(x) \tag{24}$$

The following result is well known (it is given without proof) and establishes the relation between $FPP(T)$ and $ZIP(T)$ problems.

Lemma 3.1. $0 \in T(x)$ *if and only if* $x \in (I - \lambda T)(x)$ $\forall \lambda \neq 0$.

From now on, we consider only the Fixed Point Problem and the following assumptions:

[A1] The operator $T \subset \mathbb{R}^n \times \mathbb{R}^n$ is maximal σ_T monotone.

[A2] $D(T)$ is a closed and convex set and $\emptyset \neq int(D(T))$.

[A3] $\sigma_T : D(T) \to [0, +\infty)$ is upper semicontinuous function.

[A4] For each $x \in \partial(D(T))$, $T(x)$ is Motzkin decomposable.

The assumption A4, is suggested by Corollary 2.8 and may be, in this context is superfluous (for example when T is the sub differential of a convex function and $D(T) \subset int(dom(f))$, apparently the set of all cluster point, named C, of sequences $\{u^i\}$ generated by images of convergent sequences $\{x^i\} \subset int(D(T))$ $(u^i \in T(x^i))$ to $x \in \partial(D(T))$ is compact and $T(x) = C + N_{D(T)}(x)$). Maybe it will be material for a new research.

Lemma 3.2. *Assume that T satisfies assumptions* A1, A2, A3 *and* A4. *The following statements hold.*

(1) *T has a closed graph.*

(2) *T is locally compact on $int(D(T))$.*

(3) $\sup_{u \in B(x)} \langle u - x, y - x \rangle < +\infty$ *for each $(x, y) \in D(T) \times D(T)$, where $B(x) = T(x)$ if $x \in int(D(T))$ and $B(x)$ is the compact part of the decomposition if $x \in \partial D(T)$. Moreover*

$$\sup_{u \in B(x)} \langle u - x, y - x \rangle = \sup_{u \in T(x)} \langle u - x, y - x \rangle \quad \forall (x, y) \in D(T) \times D(T).$$

Proof. (1) follows from the assumption A1, A2 and A3. (2) follows from A1, A2 and Theorem 2.7. And the last item follows from A1, A2 and A4. □

Now, we define $f : D(T) \times D(T) \to \mathbb{R}$ as follows:

$$f(x, y) := \sup_{u \in T(x)} \langle u - x, y - x \rangle \tag{25}$$

This function is important in order to transform the $FPP(T)$ in an $EP(f, D(T))$ (more information about the EP, can be find in [3, 6, 7, 8, 9]), which is defined as follows:

$$\text{EP(f,D(T))} \qquad \text{Find } \; \bar{x} \in D(T) : f(\bar{x}, y) \geq 0 \;\; \forall y \in D(T). \tag{26}$$

We denote by $S(EP(f, D(T)))$ the solution set of $EP(f, D(T))$.

Lemma 3.3. *Assume that* T *satisfies assumptions* A1, A2, A3 *and* A4. *If* $\bar{x} \in S(EP(f, D(T)))$*, then there exists* $\bar{u} \in T(\bar{x})$*, such that*

$$\langle \bar{u} - \bar{x}, y - \bar{x} \rangle \geq 0 \quad \forall y \in D(T).$$

Proof. From item 3 of Lemma 3.2, we can consider $B(x)$ instead of $T(x)$. Assume by contradiction, that the conclusion is not true. For every $u \in B(\bar{x})$ there exists $y_u \in D(T)$ and $\epsilon_u > 0$ such that

$$\langle u - \bar{x}, y_u - \bar{x} \rangle < -\epsilon_u.$$

For each $u \in B(\bar{x})$, consider $S(y_u, \epsilon_u) = \{v \in \mathbb{R}^n : \langle v - \bar{x}, y_u - \bar{x} \rangle < -\epsilon_u\}$. Since $B(\bar{x})$ is nonempty and compact, and $B(\bar{x}) \subset \bigcup_{u \in T} S(y_u, \epsilon_u)$ (it is $\{S(y_u, \epsilon_u)\}_{u \in T}$ is an open cover of $B(\bar{x})$), then there exists a finite subcover

$$B(\bar{x}) \subset \bigcup_{i=1}^{q} S(y_{u_i}, \epsilon_{u_i}).$$

Take $\epsilon = \min_{i \in \{1, \cdots, q\}} \epsilon_{u_i}$, then $\min_{i \in \{1, \cdots, q\}} \langle u - \bar{x}, y_{u_i} - \bar{x} \rangle < -\epsilon \ \forall u \in B(\bar{x})$. From standard result of convex analysis [[10], Theorem 21.1] that there exist positive real numbers $\{\alpha_i\}_{i=1}^{q}$ (without loss of generality consider $\sum_{i=1}^{q} \alpha_i = 1$) such that $\sum_{i=1}^{q} \alpha_i \langle u - \bar{x}, y_{u_i} - \bar{x} \rangle \leq -\epsilon$. Taking $\bar{y} = \sum_{i=1}^{q} \alpha_i y_{u_i}$, we have that $\langle u - \bar{x}, \bar{y} - \bar{x} \rangle \leq -\epsilon$ $\forall u \in B(\bar{x})$. This implies that $f(\bar{x}, \bar{y}) \leq -\epsilon$, which is impossible, and so the proof is complete. □

Theorem 3.4. *Assume that* T *satisfies assumptions* A1, A2, A3 *and* A4. *Let* x *and* y *be two points in* $D(T)$.

(1) $x \in T(x)$ *if and only if* $x \in S(EP(f, D(T)))$.

(2) *If* $f(x, z) \leq 0$ *for all* $x \in D(T)$*, then* $z \in T(z)$.

Proof. (1) If $x \in T(x)$, it is easy to verify that $x \in S(EP(f, D(T)))$. Conversely, if $x \in S(EP(f, D(T)))$, from Lemma 3.3, there exists $u \in T(x)$ such that we have $\langle u - x, y - x \rangle \geq 0$ for all $y \in D(T)$. The statement follows now dirctly because,

$$\langle x, y - x \rangle + \langle v, x - y \rangle \leq \langle u, y - x \rangle + \langle v, x - y \rangle \leq \min\{\sigma_T(x), \sigma_T(y)\} \|x - y\|$$

for all $(y, v) \in T$.

(2) If $f(x, z) \leq 0$ for all $x \in D(T)$, then $\langle u - x, z - x \rangle \leq 0$ for all $(x, u) \in T$. The statement follows because, $\langle u - z, z - x \rangle \leq \langle u - z, z - x \rangle + \langle z - x, z - x \rangle = \langle u - x, z - x \rangle \leq 0 \leq \min\{\sigma_T(x), \sigma_T(z)\} \|x - z\|$ for all $(x, u) \in T$. □

The following results appear as hypotheses to guarantee that $EP(f, D(T))$ has a solution (for more details see, for instance [3], [6], [7], [8], [9] and references therein).

Lemma 3.5. *Assume that* T *satisfies the assumptions* A1, A2, A3 *and* A4. *The following statements hold.*

(1) $f(x, x) = 0 \ \forall x \in D(T)$.

(2) $f(x, \cdot) : \mathbb{R}^n \to \mathbb{R}$ *is convex and continuous function for each* $x \in D(T)$.

(3) *For each* $y \in D(T)$*:* $U_f(y) = \{x \in D(T) : f(x, y) \geq 0\}$ *is closed.*

(4) *For each finite set* $\{x^i\}_{i=1}^{m} \subset D(T)$*:* $co\{x^i\}_{i=1}^{m} \subset \cup_{i=1}^{m} U_f(x^i)$

Proof. (1) and (2), follow from the definition of f. (3) follows from (1) of Lemma 3.2 and the last item follows from (2) of this Lemma. $\square$

The next result is the key for the existence of solution of our equilibrium problem.

Lemma 3.6. ([5], Lemma 1) *Let* Y *be an arbitrary subset of* $\mathbb{R}^n$. *For each* $y \in Y$, *consider a closed subset* $C(y)$ *of* $\mathbb{R}^n$. *If the following two conditions hold:*

(C1) *the convex hull of any finite subset* $\{x_1, .., x_n\}$ *of* Y *(*$co\{x_1, ..., x_n\}$*), is contained in* $\bigcup_{i=1}^{n} C(x_i)$,

(C2) $C(x)$ *is compact for at least some* $x \in Y$,

then $\bigcap_{y \in Y} C(y) \neq \emptyset$

Now, take some $x \in D(T)$ and for each $n \in \mathbb{N}$ we consider $r_n = \|x\| + n$ and $D_n = CB(0, r_n) \cap D(T)$.

Lemma 3.7. *Assume that* T *satisfies assumption A1, A2, A3 and A4. If* $D(T)$ *is bounded, then* T *has at least one fixed point..*

Proof. It follows by applying Lemma 3.6 with $C(y) = U_f(y)$ for all $y \in D(T)$. $\square$

Now, we consider another assumption, called separation property, because when $x \notin T(x)$ and $T(x)$ is nonempty closed and convex, we can separate x and $T(x)$.

[A5] For each sequence $\{x_n\} \subset D(T)$ such that $x_n \notin T(x_n)$ and $\|x_n\| \to +\infty$, there exists $m \in \mathbb{N}$ and $\bar{y} \in D(T) \cap B(0, \|x_m\|)$ such that $\{x_m\}$ and $T(x_m)$ are separated by the hyperplane containing x_m and the normal vector $\bar{y} - x_m$. This means $\langle u, \bar{y} - x_m \rangle \leq \langle x_m, \bar{y} - x_m \rangle \ \forall u \in T(x_m)$.

Theorem 3.8. *Let* T *be satisfying assumption* A1, A2, A3, A4 *and* A5. *Then the mapping* T *has at least one fixed point.*

Proof. Take $\hat{x} \in D(T)$. Consider $r_n = \|\hat{x}\| + n$, $K_n = D(T) \cap CB(0, r_n)$ for each $n \in \mathbb{N}$. Applying Lemma 3.6 for each $n \in \mathbb{N}$ (considering $EP(f, K_n)$), we have that $S(EP(f, K_n)) \neq \emptyset$. Now, take $x^n \in \operatorname{argmin}\{\|x\| : x \in S(EP(f, K_n))\}$. The statement follows if there exists $m \in \mathbb{N}$ such that $x^m \in T(x^m)$. Otherwise (when $x^n \notin T(x^n) \ \forall n \in \mathbb{N}$), we claimed that $\{x^n\}$ is bounded. By contradiction, suppose that $\{x^n\}$ is unbounded. From assumption A5, there exists $m \in \mathbb{N}$ and $\bar{y} \in D(T) \cap B(0, \|x^m\|) \subset K_m$ such that $\langle u - x^m, \bar{y} - x^m \rangle \leq 0 \ \forall u \in T(x^m)$. This implies that $f(x^m, \bar{y}) \leq 0$, here $\bar{y}$ is a local minimizer of $f(x^m, \cdot)$, because we have $x^m \in S(EP(f, K_n))$. So, $\bar{y}$ is a global minimizer, because $f(x^m, \cdot)$ is convex. This is a contradiction, because from Theorem 2.13 we have that $x^m \in T(x^m)$. Finally, taking $y \in D(T)$, for each $n \in \mathbb{N}$ such that $r_n > \|y\|$, we have that $x^n \in U_f(y)$. Any cluster point of $\{x^n\}$ belong to $U_f(y)$. This implies that $S(EP(f, D(T))) \neq \emptyset$. $\square$

4. The general case

In this section, consider $T \subset \mathbb{R}^n \times \mathbb{R}^n$, associated to T, we consider the following operator $T^o : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, defined by

$$T^o(x) := co(\overline{T(x) \cap CB(0, \|x\|)}).$$

Lemma 4.1. *Let T be an operator and T^o its associated operator. The following statement hold.*

(1) $D(T^o) \subset D(T)$.

(2) *The operator T^o has convex and compact values.*

Proof. Follows directly from the definition of T^o □

Corollary 4.2. *For each $x \in D(T^o)$ we have that $T^o(x)$ is nonempty convex and compact.*

Lemma 4.3. *Let A be a subset of $\mathbb{R}^n$ and $x \in co(\overline{A \cap CB(0,r)})$. If $\|x\| = r$, then $x \in \overline{A}$.*

Proof. Note that $co(\overline{A \cap CB(0,r)}) \subset CB(0,r)$. If $\|x\| = r$, then x is a an extremal point of $\overline{A \cap CB(0,r)}$. And so the statement follows. □

The following result is the key in our analysis.

Theorem 4.4. *Let T be an operator with closed images and T^o its associated operator. The following statements are equivalent.*

(1) *x is a fixed point of T.*

(2) *x is a fixed point of T^o.*

Proof. Evidently (1) implies (2). The converse implication follows from the previous Lemma. □

Now, we are able to establish a necessary condition for the existence of fixed point of mapping T.

Theorem 4.5. (Necessary condition) *Let T be an operator with closed images and T^o its associated operator. If T has at least one fixed point, then $D(T^o) \neq \emptyset$.*

Proof. The statement follows because $x \in CB(0, \|x\|)$ for all $x \in \mathbb{R}^n$. □

Lemma 4.6. *For each $x \in D(T^o)$ and each $y \in \mathbb{R}^n$, the following equality holds*

$$-\infty < \sup_{u \in \overline{T(x) \cap CB(0,\|x\|)}} \langle x-u, y-x \rangle = \sup_{u \in T^o(x)} \langle x-u, y-x \rangle < +\infty. \tag{27}$$

Proof. If $x \in T^o(x)$, then $\emptyset \neq T(x) \cap CB(0,\|x\|) \subset T^o(x)$. Moreover, we derive that $\overline{T(x) \cap CB(0,\|x\|)}$ and $T^o(x)$ are nonempty compact sets. Hence, there exist $v_y \in \overline{T(x) \cap CB(0,\|x\|)}$ and $u_y \in T^o(x)$ such that

$$\sup_{v \in \overline{T(x) \cap CB(0,\|x\|)}} \langle x-v, y-x \rangle = \langle x-v_y, y-x \rangle$$

$$\sup_{u \in T^o(x)} \langle x-u, y-x \rangle = \langle x-u_y, y-x \rangle$$

$$-\infty < \langle x-v_y, y-x \rangle \leq \langle x-u_y, y-x \rangle < +\infty.$$

Here, $u_y = \sum_{i=1}^q \alpha_i v_y^i$, $v_y^i \in \overline{T(x) \cap CB(0, \|x\|)}$, $\alpha_i \geq 0\ \forall i$ and $\sum_{i=1}^q \alpha_i = 1$. So,

$$\begin{array}{rcl} \langle x - v_y^i, y - x\rangle & \leq & \langle x - v_y, y - x\rangle\ \forall i \\ \langle \alpha_i x - \alpha_i v_y^i, y - x\rangle & \leq & \langle \alpha_i x - \alpha_i v_y, y - x\rangle\ \forall i \\ \langle \sum_{i=1}^q \alpha_i x - \sum_{i=1}^q \alpha_i v_y^i, y - x\rangle & \leq & \langle \sum_{i=1}^q \alpha_i x - \sum_{i=1}^q \alpha_i v_y, y - x\rangle \\ \langle x - u_y, y - x\rangle & \leq & \langle x - v_y, y - x\rangle \\ \langle x - v_y, y - x\rangle & = & \langle x - u_y, y - x\rangle. \end{array}$$

So, the statement follows because

$$\sup_{u \in T(x) \cap CB(0, \|x\|)} \langle x - u, y - x\rangle = \langle x - v_y, y - x\rangle. \qquad \square$$

Define $f : D(T^o) \times \mathbb{R}^n \to \mathbb{R}$ as follows

$$f(x, y) = \sup_{u \in T^o(x)} \langle x - u, y - x\rangle. \tag{28}$$

The following Lemma is a direct consequence of the definition of function f.

Lemma 4.7. (1) $f(x, x) = 0\ \forall x \in D(T^o)$.
(2) $f(x, \cdot) : \mathbb{R}^n \to \mathbb{R}$ *is convex and continuous function for each* $x \in D(T^o)$.

The following result is equivalent to the Lemma 3.3. The proof is included here by completeness.

Lemma 4.8. *Let* $F \subset \mathbb{R}^n$ *be a nonempty convex set. If there exists* $\bar{x} \in D(T^o)$ *such that* $f(\bar{x}.y) \geq 0\ \forall y \in F$, *then there exists* $\bar{u} \in T^o(x)$, *such that*

$$\langle \bar{x} - \bar{u}, y - \bar{x}\rangle \geq 0 \quad \forall y \in F.$$

Proof. Assume by contradiction, that the conclusion is not true. Then for every $u \in T^o(\bar{x})$ there exists $y_u \in F$ and $\epsilon_u > 0$ such that

$$\langle \bar{x} - u, y_u - \bar{x}\rangle < -\epsilon_u.$$

For each $u \in T^o(\bar{x})$, consider $S(y_u, \epsilon_u) = \{v \in \mathbb{R}^n : \langle \bar{x} - v, y_u - \bar{x}\rangle < -\epsilon_u\}$.

Since $T^o(\bar{x})$ is nonempty and compact and $T^o(\bar{x}) \subset \bigcup_{u \in T^o} S(y_u, \epsilon_u)$ (note that $\{S(y_u, \epsilon_u)\}_{u \in T^o}$ is an open cover of $T^o(\bar{x})$), then there exists a finite subcover

$$T^o(\bar{x}) \subset \bigcup_{i=1}^q S(y_{u_i}, \epsilon_{u_i}).$$

Take $\epsilon = \min_{i \in \{1, \cdots, q\}} \epsilon_{u_i}$, then $\min_{i \in \{1, \cdots, q\}} \langle \bar{x} - u, y_{u_i} - \bar{x}\rangle < -\epsilon\ \forall u \in T^o(\bar{x})$. A standard result of convex analysis [10, Theorem 21.1] implies that there exist positive real numbers $\{\alpha_i\}_{i=1}^q$ (without loss of generality consider $\sum_{i=1}^q \alpha_i = 1$) such that

$$\sum_{i=1}^q \alpha_i \langle \bar{x} - u, y_{u_i} - \bar{x}\rangle \leq -\epsilon.$$

Taking $\bar{y} = \sum_{i=1}^q \alpha_i y_{u_i}$, we have that $f(\bar{x}, \bar{y}) \leq -\epsilon$, which is not possible, and so the proof is complete. $\square$

Theorem 4.9. *Let T be an operator with closed images, let F be a convex set and let $x \in \mathbb{R}^n$ be such that $T^o(x) \subset F$. The following statements are equivalent.*

(1) $x \in T(x)$.

(2) $x \in D(T^o)$ *and* $f(x,y) \geq 0 \ \forall y \in F$.

Proof. (1) ⇒ (2): If $x \in T(x)$, then $x \in T^o(x)$. This implies that $x \in D(T^o)$ and $f(x,y) \geq 0 \ \forall y \in F$.

(2) ⇒ (1): If $x \in D(T^o)$ and $f(x,y) \geq 0 \ \forall y \in F$, then from Lemma 4.8 there exists $u \in T^o(x)$ such that $\langle x-u, y-x\rangle \geq 0 \ \forall y \in F$. Since $T^o(x) \subset F$, then $u \in F$. By taking $y = u$, we have that $x = u$, implying that $x \in T^o(x)$. This means that $x \in T(x)$. □

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