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On the Convergence of Efficient Solutions to Semi-Infinite Set-Optimization Problems

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We consider semi-infinite set optimization problems and study the stability of efficient solutions for such problems. We first discuss upper and lower semicontinuity properties of the constraint sets. Next, we introduce a concept of converse property for the original problem and employ it to establish the upper convergence for efficient solutions. Besides, we also propose concepts of sequential domination to consider the lower convergence for efficient solutions.

Keywords: Semi-infinite set optimization problem, stability, efficient solution, converse property, domination.

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1. Introduction

In recent years, set-valued optimization problems have been extensively studied because of their applications in many fields such as mathematical economics, finances, game theories, engineerings, and so forth. Based on extending the concepts of solution regularly defined in the framework of vector optimization, two separate approaches dealing with the class of such problems have been established. The first one, say vector criterion, considers the optimal boundary of the union of all images under the constraint set through the set-valued objective map; see, for instance, [22, 31]. The second one, say set criterion, concerns a comparison among values of the set-valued objective map based on an order set relation, and considers optimality concepts induced by this order. In view of applications, this approach is more suitable than the vector one, and hence it has been studied extensively in the literature; see, e.g., [4, 12, 15, 16, 20, 23].

On the other hand, semi-infinite programming has been applied in many areas of relevance to approximation theory, engineering design, production plan, and so

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forth; see [10, 37, 39]. A number of specific topics related to generalized semi-infinite optimization has already been investigated including optimality conditions, numerical methods, stability analysis. Indeed, besides the monograph [34] presenting a detailed study on generalized semi-infinite optimization, we recall the papers [11, 13, 14, 17, 18, 29, 35, 38].

It is well-known that the stability of a problem, which investigates the behaviour of solutions when input data of the reference problem undergo small variations, is among the important aspects in mathematical programming [2, 8, 19, 26, 27, 28]. In view of applications, input data of problems modeling real-world applications usually contain errors due to measurements or empirical statistics, and hence this study is needed. The stability in the sense of (semi)continuity properties of solution maps studies the behaviour of parametric problems with the parameter perturbed in a parameters space; see [6, 7, 33, 40] and the references therein. Another approach of stability deals with investigating the convergence of a sequence of solutions of perturbed problems to a solution of the given problem. In this work, we focus on the second approach, and hence we would like to recall some main developments in recent years of this topic in vector and set optimization. Anh and Duy [1] studied concepts of well-posedness under perturbations by a sequence of approximating problems for vector equilibrium problems involving a total ordering cone in Euclidean space. Based on the Slater condition, Peng et al. [32] gave sufficient conditions for the lower Painlevé-Kuratowski convergence of weak solutions for convex semi-infinite vector problems under functional perturbations of both objective functions and constraint sets. Kapoor and Lalitha [24] considered unified semi-infinite vector optimization problems induced by general preference sets and established the Painlevé-Kuratowski convergence under the functional perturbations of both objective function and constraint sets. Tung and Duy [36] established the Painlevé-Kuratowski convergence of infimal and minimal point sets for vector optimization problems with abstract constraints and also discussed for the case of semi-infinite vector optimization problems. The convergence in the senses of Panilevé–Kuratowski and Hausdorff of the solution sets in the image space for a set optimization problem perturbed by convergent sequences of constraint sets was first introduced by Gutiérrez et al. [19]. Under certain compactness and domination assumptions, Karuna and Lalitha [26] established internal stability in terms of convergence for a (sub)sequence of (weakly) efficient solutions of perturbed set optimization problems. Moreover, by using the strict quasi-convexity on the objective map which guarantees the coincidence of efficient and weak solution sets, the authors in [26] also obtained the external stability for the reference problems. Anh et al. [5] modified and extended the convergence results concerning the external and internal stability for weak efficient solutions to set-optimization problems considered in [19, 26]. Anh et al. [3] employing a converse property of an objective map to establish the external stability for set optimization problems where the efficient and weak solution sets are distinct. They also discussed the internal stability for such problems via domination properties and a compact convergence condition in the sense of Painlevé-Kuratowski. Very recently, to study the stability of solution sets to perturbed set optimization problems by perturbing both objective maps and constraint sets, Karuna and Lalitha [27] have employed pre-compactness assumptions on the union of all images of perturbing objective maps to establish external stability of weak solution sets, and the strict quasi-convexity to establish the external stability for efficient solution sets. They have also formulated the internal stability by using strong domination properties of the family of perturbing objective maps. To the best of our knowledge, there is no work discuss convergence conditions for semi-infinite set optimization problems, and hence this is a key motivation for us to study this issue.

The layout of this paper is organized as follows. In Section 2, we introduce some notations, and standard facts on the theory of set-valued analysis and vector optimization, as well as present the statement of semi-infinite set optimization problems. Some sufficient conditions for the stability including lower and upper semicontinuity properties of the constraint map are discussed in Section 3. Finally, the main results on external and internal stability for the reference problems are presented in Section 4.

2. Preliminaries

Let X, Y and Z be metric linear spaces. By $B(x, r)$ we denote the open ball centered at $x \in X$ with radius $r > 0$. The *closure* and *interior* of a subset A of X are denoted by $\operatorname{cl} A$ and $\operatorname{int} A$, respectively. Let $K \subset Z$ be a nonempty, convex, closed, pointed and solid cone in Z. We consider the following order relations on Z, as follow:

For $a, b \in Z$, $$a \preceq_K b \ \Leftrightarrow\ b - a \in K,$$

and $$a \prec_K b \ \Leftrightarrow\ b - a \in \operatorname{int} K.$$

Let $C \subset Y$ be a nonempty, convex, closed, and pointed cone in Y. Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y. For $A, B \in \mathcal{P}(Y)$, we denote by $\preceq_C^u$ and $\preceq_C^l$ the following relations on $\mathcal{P}(Y)$, respectively, as follows (see, for instance, [30, 23]):

$$A \preceq_C^u B \Leftrightarrow A \subset B - C,$$

$$A \preceq_C^l B \Leftrightarrow B \subset A + C.$$

We say that $A \sim^u B\, (A \sim^l B,$ respectively) if

$$A \preceq_C^u B \text{ and } B \preceq_C^u A$$

$$(A \preceq_C^l B \text{ and } B \preceq_C^l A, \text{ respectively}).$$

It is easy to check that $\sim^u$ and $\sim^l$ are equivalence relations on $\mathcal{P}(Y)$. Let $F : X \rightrightarrows Y$ be a set-valued mapping. For $x, z \in X$, we say that $x \sim_F^\alpha z$ is valid if and only if $F(x) \sim^\alpha F(z)$, where $\alpha \in \{u, l\}$.

For a given nonempty compact, convex subset A of X, and a nonempty compact subset T of a metric space Ω, let $F : A \rightrightarrows Y$ be a set-valued map and $h : A \times T \to Z$ be a vector-valued map, we consider the following semi-infinite set optimization problem:

$(\mathrm{SIP})^\alpha$: $\quad \operatorname{Min}_{\preceq_C^\alpha} F(x) \quad$ subject to $\quad x \in M(h)$,

where $M(h) := \{x \in A \mid h(x, t) \preceq_K 0, \forall t \in T\}$ and $\alpha \in \{u, l\}$.

Define $\mathcal{K}[A \times T, Z] := \{h \mid h : A \times T \to Z\}$, $\mathcal{L}[A, Y] := \{F \mid F : A \rightrightarrows Y\}$.

Now we consider the problem $(\mathrm{SIP})^{\alpha}$ under perturbations, which are expressed in terms of perturbing both objective maps and constraint sets on the parameter space $\mathbb{P} = \mathcal{K}[A \times T, Z] \times \mathcal{L}[A, Y]$:

$$(\mathrm{SIP})^{\alpha}_{p_n}: \qquad \mathrm{Min}_{\preceq^{\alpha}_{C}} F_n(x) \quad \text{subject to} \quad x \in M(h_n),$$

where $\alpha \in \{u, l\}$ and $p_n = (h_n, F_n) \in \mathbb{P}$. For each $p = (h, F) \in \mathbb{P}$, the *efficient solution maps* $\mathrm{Eff}^{\alpha} : \mathbb{P} \rightrightarrows A$ for the problems $(\mathrm{SIP})^{\alpha}_{p}$ are defined as follows:

$$\mathrm{Eff}^{\alpha}(p) := \{\bar{x} \in M(h) \mid [\forall x \in M(h), F(x) \preceq^{\alpha}_{C} F(\bar{x})] \Rightarrow [F(\bar{x}) \preceq^{\alpha}_{C} F(x)]\},$$

for every $\alpha \in \{u, l\}$.

For each $p = (h, F), q = (g, G) \in \mathbb{P}$, we define the following distance.

$$d(p, q) = \sup_{(x,t)\in A\times T} d(h(x, t), g(x, t)) + \sup_{x\in A} H(F(x), G(x)),$$

where $H(M, N)$ is the Hausdorff distance between M and N. Then, $(\mathbb{P}, d)$ is a pseudometric space.

Definition 2.1. For a given $p = (h, F) \in \mathbb{P}$, we say that p satisfies the *Slater condition* if and only if there exists $\hat{x} \in A$ such that

$$h(\hat{x}, t) \prec 0_Z, \ \forall t \in T.$$

Definition 2.2. For $\alpha \in \{u, l\}$, and $p = (h, F) \in \mathbb{P}$

(a) the problem $(\mathrm{SIP})^{\alpha}_{p}$ is said to have the *α-converse property* on A if for each $x \in A$ and $y \in A \setminus \{x\}$ either $F(y) \not\preceq^{\alpha}_{C} F(x)$ or for all sequences $\{F_n\}$ with $\sup_{x\in A} H(F_n(x), F(x)) \to 0$, $(x_n, y_n) \to (x, y)$, $F_{\hat{n}}(y_{\hat{n}}) \preceq^{\alpha}_{C} F_{\hat{n}}(x_{\hat{n}})$ for some $\hat{n}$.

(b) the problem $(\mathrm{SIP})^{\alpha}_{p}$ satisfies the *α-domination property* if for every sequence $\{p_n\} \subset \mathbb{P}$ converging to p, for each sequence $\{x_n\}$ with $x_n \in M(h_n)$ there exists $\hat{n}$ such that either $x_{\hat{n}} \in \mathrm{Eff}^{\alpha}(p_n)$ or there exists $u_{\hat{n}} \in \mathrm{Eff}^{\alpha}(p_{\hat{n}})$ satisfying $F_{\hat{n}}(u_{\hat{n}}) \preceq^{\alpha}_{C} F_{\hat{n}}(x_{\hat{n}})$.

(c) the problem $(\mathrm{SIP})^{\alpha}_{p}$ satisfies the *strong α-domination property* if for every sequence $\{p_n\} \subset \mathbb{P}$ converging to p, and for each sequence $\{x_n\}$ with $x_n \in M(h_n)$ either $x_n \in \mathrm{Eff}^{\alpha}(p_n)$ or there exist $u_n \in \mathrm{Eff}^{\alpha}(p_n)$ satisfying $F_n(u_n) \preceq^{\alpha}_{C} F_n(x_n)$.

Remark 2.3. When objective maps of the problem $(\mathrm{SIP})^{\alpha}$ are fixed, that is $F_n = F$ for all n, Definition 2.2 (a) collapses to the concept of converse property introduced in [3], while Definitions (b) and (c) are coincident and reduce to the concept of domination property introduced in [21] if $M(h) \equiv A$ for all $h \in \mathcal{K}[A \times T, Z]$.

In the rest of this section we recall semicontinuity concepts for maps and some their important properties used in the sequel.

Definition 2.4. [28, Definition 3.1.1] Let $x_0 \in X$, a set-valued map $Q : X \rightrightarrows Y$ is said to be

(a) *upper semicontinuous* (u.s.c.) at x_0 if for any neighborhood U of $Q(x_0)$, there is a neighborhood N of x_0 such that $Q(N) \subset U$.

(b) *lower semicontinuous* (l.s.c.) at x_0 if for any open set U with $Q(x_0) \cap U \neq \emptyset$, there is a neighborhood N of x_0 such that $Q(x) \cap U \neq \emptyset$ for every $x \in N$.

(c) *continuous* at x_0 if it is both u.s.c. and l.s.c. at x_0.

We say that the map Q is u.s.c. (l.s.c., continuous) on X if it is u.s.c. (l.s.c., continuous) at every $x \in X$.

Lemma 2.5. [28, Propositions 3.1.9 and 3.1.10] *Let $Q : X \rightrightarrows Y$ be a set-valued map and $x_0 \in X$. The following statements are true.*

(a) *Q is l.s.c. at x_0 if and only if for any sequence $\{x_n\}$ converging to x_0 and for any $y_0 \in Q(x_0)$, there exist $y_n \in Q(x_n)$ such that $\{y_n\}$ converges to y_0.*

(b) *Q is l.s.c. at x_0 if and only if for any sequence $\{x_n\}$ converging to x_0, one has $Q(x_0) \subset \liminf Q(x_n)$, where*

$$\liminf Q(x_n) := \{y_0 \in Y \mid \exists y_n \in Q(x_n), y_n \to y_0\}.$$

(c) *If $Q(x_0)$ is compact, then Q is u.s.c. at x_0 if and only if for any sequence $\{x_n\}$ converging to x_0 and for any sequence $\{y_n\}$ with $y_n \in Q(x_n)$, there is a subsequence $\{y_{n_k}\}$ converging to some $y_0 \in Q(x_0)$.*

Definition 2.6. [28, Definition 3.1.28] Let $x_0 \in X$. Then a vector-valued mapping $h : X \to Z$ is said to be

(a) *K-lower semicontinuous* (K-lsc) at x_0 if for all neighborhood V of $h(x_0)$, there exists a neighborhood U of x_0 such that for all $x \in U$, $h(x) \in V + K$.

(b) *K-upper semicontinuous* (K-usc) at x_0 if $(-h)$ is K-lsc at x_0.

(c) *K-continuous* at x_0 if it is both K-usc and K-lsc at x_0.

For a nonempty subset A of X, the map h is K-usc (K-lsc, K-continuous) on A if it is K-usc (K-lsc, K-continuous) at every element $x \in A$.

In the following, for each $z \in Z$ we denote $\operatorname{lev}(h, z, A) = \{x \in A : h(x) \preceq_K z\}$. We have the following statements.

Proposition 2.7. *For a nonempty subset A of X, let $h : A \to Z$ be a vector-valued map.*

(a) *If h is K-lsc on A, then* $\operatorname{lev}(h, z, A)$ *is closed for all $z \in Z$.*

(b) *If h is K-usc on A, then* $\operatorname{lev}(-h, z, A)$ *is closed for all $z \in Z$.*

Proof. For a given $z \in Z$, let $\{x_n\}$ be an arbitrary sequence in $\operatorname{lev}(h, z, A)$ with $x_n \to \bar{x}$. If $h(\bar{x}) \in Z \setminus (z - K)$, we would have, from the K-lower semicontinuity of h at $\bar{x}$ and the convexity of K, that $h(x_n) \in Z \setminus (z - K) + K = Z \setminus (z - K)$ for n sufficiently large. This contradicts $h(x_n) \preceq_K z$. Therefore, $\bar{x} \in \operatorname{lev}(h, z, A)$, i.e., the set $\operatorname{lev}(h, z, A)$ is closed. □

Definition 2.8. [9] Let D be a nonempty convex subset of X. A vector-valued map $h : D \to Z$ is said to be *generalized K-quasiconvex* on D if for each pair of $x, y \in D$, and $t \in (0, 1)$ with $h(x) \in -K$ and $h(y) \in -\operatorname{int} K$, one has $h(tx + (1-t)y) \in -\operatorname{int} K$.

3. Stability of constraint conditions

This section aims to study properties of the constraint map M. These properties will be employed to investigate the upper and lower convergences of solutions to the problem $(\mathrm{SIP})^{\alpha}$ in the next sections. We start with the following lemma.

Lemma 3.1. *Let $p_0 = (h_0, F_0) \in \mathbb{P}$. For each $t \in T$, if $\operatorname{lev}(h_0(\cdot, t), z, A)$ is K-lsc on A for all $z \in Z$, then the constraint map M is u.s.c. and compact-valued at h_0.*

Proof. Suppose to the contrary that M is not u.s.c. at h_0. Then, we can choose an open set $\hat{V} \supset M(h_0)$, and a sequence $\{h_n\}$ converging to h_0, such that for each n, there is $x_n \in M(h_n)\backslash\hat{V} \subset A\backslash\hat{V}$. Since A is compact, one can assume that there is subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some point x_0 of A. Suppose that $x_0 \notin M(h_0)$, then there is $t_0 \in T$ satisfying $h_0(x_0, t_0) \not\preceq 0_Z$, that is,

$$h_0(x_0, t_0) \notin -K. \tag{1}$$

We prove that there exists $e \in \operatorname{int} K$ such that $h_0(x_0, t_0) \notin e - K$. Indeed, suppose to the contrary that for all $e \in \operatorname{int} K$, $h_0(x_0, t_0) \in e - K$. Let $e_n \in \operatorname{int} K, e_n \to 0_Z$. Since $h_0(x_0, t_0) \in e_n - K$, for each $n \in \mathbb{N}$ there is $k_n \in K$ such that $h_0(x_0, t_0) = e_n - k_n$. Consequently, there is a sequence $\{k_n\}$ such that $k_n \in K$ and it converges to $-h_0(x_0, t_0)$. This together with the closedness of K, we get $h_0(x_0, t_0) \in -K$, which is a contradiction. Thus, $h_0(x_0, t_0) \in Z \setminus (e - K)$. Combining this with the closedness of $\operatorname{lev}(h_0(\cdot, t_0), e, A)$, one has

$$h_0(x_n, t_0) \in Z \setminus (e - K).$$

Note further that $-K \subsetneq e - K$ and $\inf_{k \notin (e-K)} d(k, -K) = r > 0$ as $e \in \operatorname{int} K$. On the other hand,

$$0 \leq d(h_n(x_n, t_0), h_0(x_n, t_0)) \leq \sup_{(x,t) \in A \times T} d(h_n(x, t), h_0(x, t)) \to 0 \text{ as } n \to +\infty.$$

Hence, $d(h_n(x_n, t_0), h_0(x_n, t_0))$ converges to 0, which is impossible since

$$d(h_n(x_n, t_0), h_0(x_n, t_0)) \geq d(h_0(x_n, t_0), -K) \geq \inf_{k \notin (e-K)} d(k, -K) = r > 0.$$

Therefore, $x_0 \in M(h_0) \subset \hat{V}$, which again contradicts $x_n \notin \hat{V}$ for all n.

Because A is compact, we prove the compactness of $M(h_0)$ by checking its closedness. Let $\{x_n\} \subset M(h_0)$ be an arbitrary sequence converging to x_0. For each $t \in T$, one has $h_0(x_n, t) \preceq 0_Z$. The closedness of $\operatorname{lev}(h_0(\cdot, t), 0_Z, A)$ at x_0 gives $x_0 \in M(h_0)$. This brings the proof to its end. □

Remark 3.2. By using the cone lower semicontinuity of a vector-valued map, Peng et al. [32] provided a sufficient condition for the closedness of the constraint map M (see Theorem 3.1 in [32]). Because a closed set-valued map defined on a compact set is upper semicontinuous, the conclusion of Theorem 3.1 is similar to that of Lemma 3.1. It follows from Proposition 2.7 that the closedness of the level sets is weaker than the cone lower semicontinuity, and so Lemma 3.1 is a slight generalization of Theorem 3.1 in [32].

Lemma 3.3. *Let* $p_0 = (h_0, F_0) \in \mathbb{P}$*. Suppose that for each* $x \in A$, $t \in T$*, the following conditions are satisfied.*

(i) $h_0(x, \cdot)$ *is* K*-usc on* T*;*

(ii) $h_0(\cdot, t)$ *is generalized* K*-quasiconvex on* A*;*

(iii) p_0 *satisfies the Slater condition.*

Then, M *is l.s.c. at* h_0*.*

Proof. Letting $\hat{M}(h) := \{x \in A \mid h(x,t) \prec 0, \forall t \in T\}$. Because h satisfies the Slater condition, $\hat{M}(h) \neq \emptyset$. Suppose to the contrary that $\hat{M}$ is not l.s.c. at h_0. Then, there are a sequence $\{h_n\}$ converging to h_0 and an element $x_0 \in \hat{M}(h_0)$ such that for all sequences $\{x_n\}$, where x_n belongs to $\hat{M}(h_n)$, $\{x_n\}$ does not converge to x_0.

Without loss of generality, we can assume that $x_0 \notin \hat{M}(h_n)$, thus there exist $t_n \in T$ such that $h_n(x_0, t_n) \notin -\operatorname{int} K$. Because of the compactness of T, there is a subsequence $\{t_{n_m}\}$ of $\{t_n\}$ with $t_{n_m} \to t_0$ for some $t_0 \in T$. From $x_0 \in \hat{M}(h_0)$, we have $h_0(x_0, t_0) = b \prec 0$ and hence, $h_0(x_0, t_0) \in \frac{b}{2} - \operatorname{int} K$. By the K-usc of $h_0(x_0, \cdot)$ at t_0, there is $n_0 \in \mathbb{N}$ such that

$$h_0(x_0, t_{n_m}) \in \frac{b}{2} - \operatorname{int} K - K = \frac{b}{2} - \operatorname{int} K, \ \forall n_m \geq n_0.$$

Note further that $\frac{b}{2} - \operatorname{int} K \subsetneq -\operatorname{int} K$ and $\inf_{k \in Y \setminus (-\operatorname{int} K)} d(k, \frac{b}{2} - \operatorname{int} K) = r > 0$ as $\frac{b}{2} \in -\operatorname{int} K$. On the other hand,

$$d(h_{n_m}(x_0, t_{n_m}), h_0(x_0, t_{n_m})) \leq \sup_{t \in T} d(h_{n_m}(x_0, t), h_0(x_0, t)) \to 0 \quad \text{as } n_m \to \infty.$$

Hence, $d(h_n(x_0, t_{n_m}), h_0(x_0, t_{n_m}))$ converges to 0. It is a contradiction as

$$\begin{aligned} d\left(h_{n_m}(x_0, t_{n_m}), h_0(x_0, t_{n_m})\right) &\geq d\Big(h_{n_m}(x_0, t_{n_m}), \frac{b}{2} - \operatorname{int} K\Big) \\ &\geq \inf_{k \in Y \setminus (\operatorname{int}(-K))} d\Big(k, \frac{b}{2} - \operatorname{int} K\Big) = r > 0, \end{aligned}$$

due to $h_{n_m}(x_0, t_{n_m}) \notin \operatorname{int} K$. Therefore, $\hat{M}$ is lsc at h_0.

Now we prove that $M(h_0) \subseteq \operatorname{cl}\hat{M}(h_0)$. For every $\bar{x} \in M(h_0)$, let $\bar{x}_1 \in \hat{M}(h_0)$ and $\lambda \in (0,1)$. Taking $x_\lambda = (1-\lambda)\bar{x} + \lambda \bar{x}_1$. Taking into account the generalized K-quasiconvexity for the first component of h_0, we also obtain that $x_\lambda \in \hat{M}(h_0)$. On the other hand, $x_\lambda \to \bar{x}$ when $\lambda \to 0$. That is, $\bar{x} \in \operatorname{cl}\hat{M}(h_0)$.

Hence we get $M(h_0) \subset \operatorname{cl}\hat{M}(h_0)$. Thanks to the lower semicontinuity of $\hat{M}$ at h_0 and the closedness of the set $\liminf \hat{M}(h_n)$ we obtain

$$M(h_0) \subset \operatorname{cl}\hat{M}(h_0) \subset \liminf \hat{M}(h_n) \subset \liminf M(h_n).$$

So, M l.s.c. at h_0. □

Finally, Lemmas 3.1 and 3.3 bring us to a position to present the main result of this section.

Theorem 3.4. *Let* $p_0 = (h_0, F_0) \in \mathbb{P}$. *Suppose that for each* $x \in A$, $t \in T$, *the following conditions are satisfied.*

(i) $h_0(x, \cdot)$ *is* K*-usc on* T;

(ii) $h_0(\cdot, t)$ *is* K*-lsc and generalized* K*-quasiconvex on* A;

(iii) p_0 *satisfies the Slater condition.*

Then, M *is continuous and compact-valued at* h_0.

4. Convergence of efficient solutions to the problem (SIP)

In this section, using properties of the constraint map obtained in Section 3, we study convergence conditions for the problem $(\text{SIP})^\alpha$. We first give the following important result.

Proposition 4.1. *Let* A *be a nonempty closed subset of* X *and* $\{F_n, F\}_n \subset \mathcal{L}[A, Y]$ *be such that* $\sup_{x\in A} H(F_n(x), F(x)) \to 0$. *Assume that* F *is continuous and compact-valued on* A. *If for any sequences* $\{x_n\}, \{\bar{x}_n\} \subset A$ *with* $x_n \to x, \bar{x}_n \to \bar{x}$, *and* $F_n(\bar{x}_n) \preceq_C^\alpha F_n(x_n)$, *then* $F(\bar{x}) \preceq_C^\alpha F(x)$.

Proof. By the similarity, we only discuss the proof for the case of $\alpha = u$. Let an arbitrary element $z \in F(\bar{x})$. For any $\varepsilon > 0$, we have $F(\bar{x}) \cap B(z, 2^{-1}\varepsilon) \neq \emptyset$. From the lower semicontinuity of F at $\bar{x}$ and $\bar{x}_n \to \bar{x}$ we can pick up a sequence $\{z_n\}$ with

$$z_n \in F(\bar{x}_n) \cap B(z, 2^{-1}\varepsilon) \text{ for } n \text{ large enough.} \tag{2}$$

On the other hand, we have

$$H(F_n(\bar{x}_n), F(\bar{x}_n)) \leq \sup_{x\in A} H(F_n(x), F(x)) \to 0,$$

and hence there is a sequence $\{\bar{z}_n\}$ with $\bar{z}_n \in F_n(\bar{x}_n)$ satisfying

$$d(\bar{z}_n, z_n) \leq \frac{\varepsilon}{2} \text{ for } n \text{ large enough.} \tag{3}$$

From (2) and (3), we obtain

$$d(\bar{z}_n, z) \leq d(\bar{z}_n, z_n) + d(z_n, z) \leq \varepsilon,$$

and thus $\bar{z}_n \to z$ as $n \to \infty$. By $F_n(\bar{x}_n) \preceq_C F_n(x_n)$, we get $F_n(\bar{x}_n) \subset F_n(x_n) - C$, so there is $\bar{y}_n \in F_n(x_n)$ satisfying $\bar{z}_n \in \bar{y}_n - C$. Moreover, it follows from

$$H(F_n(x_n), F(x_n)) \leq \sup_{x\in A} H(F_n(x), F(x)) \to 0$$

that

$$H(F_n(x_n), F(x_n)) \leq \frac{\rho}{2}$$

for all $\rho > 0$ for n sufficiently large. Consequently, there exists a sequence $\{y_n\}$ with $y_n \in F(x_n)$ such that for n sufficiently large

$$d(\bar{y}_n, y_n) \leq \frac{\rho}{2}. \tag{4}$$

From the upper semicontinuity of F at x_0, we can assume that $\{y_n\}$ admits a subsequence (still denoted $\{y_n\}$ for simplification) converging to some $\bar{y} \in F(x_0)$, and hence for n large enough, $d(y_n, \bar{y}) \leq 2^{-1}\rho$. This together with (4) gives

$$d(\bar{y}_n, \bar{y}) \leq d(\bar{y}_n, y_n) + d(y_n, \bar{y}) \leq \rho,$$

that is, $\{\bar{y}_n\}$ converges to $\bar{y}$. Combining this with the facts $\bar{z}_n \to z, \bar{z}_n \in \bar{y}_n - C$, and the closedness of C, we come to $z \in \bar{y} - C$. As a result, $F(\bar{x}) \subset F(x_0) - C$, and so (5) holds true. □

Applying Theorem 3.4 and Proposition 4.2, we establish upper convergence conditions for efficient solutions to the problem $(\text{SIP})^\alpha$ which is having the converse property.

Theorem 4.2. *Let* $\alpha \in \{u, l\}$ *and* $p_0 = (h_0, F_0) \in \mathbb{P}$. *Assume that all assumptions of Theorem* 3.4 *are satisfied. Assume further that* F_0 *is continuous and compact-valued on* A, *and the problem* $(\text{SIP})_p^\alpha$ *has the* α*-converse property. Then, for every sequence* $\{p_n\} \subset \mathbb{P}$ *converging to* p_0 *and* $x_n \in \text{Eff}^\alpha(p_n)$ *there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *converging to some* $x \in \text{Eff}^\alpha(p_0)$.

Proof. We will give the proof for the case $\alpha = l$, the proof for the other case is quite similar. For every sequence $p_n = (h_n, F_n) \in \mathbb{P}$ converging to p_0, let an arbitrary sequence $\{x_n\}$ be such that $x_n \in \text{Eff}^l(p_n)$. According to Lemma 2.5 and Theorem 3.4, the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converges to some $x \in M(h_0)$. Suppose that there exists an element $\hat{x} \in M(h_0)$ satisfying $F_0(\hat{x}) \preceq_C^l F_0(x)$, we show that

$$F_0(x) \preceq_C^l F_0(\hat{x}). \tag{5}$$

Obviously, (5) holds trivially when $x = \hat{x}$, so we consider the case of $x \neq \hat{x}$. From the lower semicontinuity of M at h_0, there exists a sequence $\{\hat{x}_n\}, \hat{x}_n \in M(h_n)$, converging $\hat{x}$. Because $F_0(\hat{x}) \preceq_C^l F_0(x), p_n \to p_0$, and the problem $(\text{SIP})_{p_0}^\alpha$ has α-converse property, there exist subsequences (still denoted $\{F_{n_k}\}, \{x_{n_k}\}, \{\hat{x}_{n_k}\}$ for simplification) satisfying

$$F_{n_k}(\hat{x}_{n_k}) \preceq_C^l F_{n_k}(x_{n_k}), \ \forall k. \tag{6}$$

From $x_{n_k} \in \text{Eff}^l(p_{n_k})$ and (6), we get

$$F_{n_k}(x_{n_k}) \preceq_C^l F_{n_k}(\hat{x}_{n_k}), \ \forall k.$$

Applying Proposition 4.1, we conclude that the relation (5) holds true. Therefore, $x \in \text{Eff}^l(p_0)$. The proof is complete. □

We give an example to illustrate the applicability of Theorem 4.2.

Example 4.3. Let $X = Y = Z = \mathbb{R}^2, \Omega = \mathbb{R}$ and $C = K = \mathbb{R}_+^2$. Let $T = [0, 1]$, and

$$A = \{(x_1, x_2) \in [-1, 3] \times [0, 2] \mid x_1 + x_2 \geq 1\}.$$

Let $h, h_n : A \times T \to \mathbb{R}^2$, $F, F_n : A \rightrightarrows \mathbb{R}^2$ be respectively defined by:
For all $x = (x_1, x_2) \in A$ and $t \in T$,

$$h(x,t) = (x_1^2 - 2x_1 - t, t - x_2 - 1),\ h_n(x,t) = \left(x_1^2 - 2x_1 - t - \frac{1}{n}, t - \frac{n^2(x_2+1) - 3n}{n^2+n}\right).$$

$$F(x) = \{(z_1, z_2) \in \mathbb{R}^2 \mid (z_1 - x_1)^2 + (z_2 - x_2)^2 \leq 1\},$$

$$F_n(x) = \left\{(z_1, z_2) \in \mathbb{R}^2 \mid \left(z_1 - x_1 + \frac{1}{n}\right)^2 + \left(z_2 - x_2 - \frac{1}{n}\right)^2 \leq 1\right\}.$$

It is obvious that all assumptions of Theorem 4.2 are fulfilled, so the conclusion of Theorem 4.2 is true. Indeed, direct computations give us

$$M(h) = \{(x_1, x_2) \in A \mid 0 \leq x_1 \leq 2\},$$

$$M(h_n) = \left\{(x_1, x_2) \in A \mid 1 - \sqrt{\frac{n+1}{n}} \leq x_1 \leq 1 + \sqrt{\frac{n+1}{n}}\right\},$$

$$\mathrm{Eff}^l(p) = \{(x_1, x_2) \in A \mid 0 \leq x_1 \leq 1, x_2 + x_1 = 1\},$$

and
$$\mathrm{Eff}^l(p_n) = \left\{(x_1, x_2) \in A \mid 1 - \sqrt{\frac{n+1}{n}} \leq x_1 \leq 1, x_2 + x_1 = 1\right\}.$$

It is worth noting that the weak solution set of the problem $(\mathrm{SIP})_p^l$ is defined by

$$\mathrm{WEff}^l(p) = \mathrm{Eff}^l(p) \cup (]1, 2] \times \{0\}),$$

and hence $\mathrm{Eff}^l(p) \neq \mathrm{WEff}^l(p)$.

Remark 4.4. As far as we know, there is no work with contributions to convergence conditions for efficient solutions to the problem $(\mathrm{SIP})^\alpha$, and hence for the aim of giving a comparison of our obtained results with existing ones concerning convergence conditions for solutions to set optimization problems, we now consider the case of which the problem $(\mathrm{SIP})^\alpha$ reduces to the set optimization with abstract constraints, say (SOP).

(a) Motivated by the open question concerning the stability properties of solutions in the decision space given by Gutiérrez et al. in [19], Karuna and Lalitha [26] utilized the strict quasi-convexity to establish upper convergence conditions for efficient solutions to the problem (SOP) via the corresponding property of the weakly efficient solutions. When set-valued objective maps are not perturbed, the problem $(\mathrm{SIP})^\alpha$ reduces to the (SOP) considered in [19, 26], then Theorem 4.2 provides upper convergence conditions for efficient solutions to (SOP) in which the efficient solution sets are different from the weak ones, and hence the obtained result in Theorem 4.2 is a new and good answer to the question in [19].

(b) Recently, Karuna and Lalitha [27] investigated the stability of solution sets to the perturbed problem (SOP) by perturbing both the objective and constraint maps. Therein, the Authors had to impose an assumption related to the compactness of $\mathrm{cl}\left(\cup_{n\in\mathbb{N}}(\cup_{x\in S}(F_n(x)))\right)$ to establish the upper convergence for weak solutions

(Theorem 5.4 in [27]), and then they used the strict quasi-convexity, which makes efficient and weakly efficient solution sets are coincident, to formulate upper convergence conditions for efficient solutions. Therefore, the approach and obtained result in Theorem 4.2 are new and different from the corresponding ones in [27].

By using the strong domination property of the problem $(\mathrm{SIP})^\alpha$, we next provide the lower convergence condition for efficient solutions of the reference problem.

Theorem 4.5. *Let* $\alpha \in \{u, l\}$ *and* $p_0 = (h_0, F_0) \in \mathbb{P}$. *Assume that all assumptions of Theorem* 3.4 *are satisfied, and assume further that* F_0 *is continuous and compact-valued on* A *and the problem* $(\mathrm{SIP})^\alpha_{p_0}$ *has the strong domination property. Then, for every* $\{p_n\} \subset \mathbb{P}, p_n \to p_0$, *and* $x \in \mathrm{Eff}^\alpha(p_0)$ *there exist* $x_n \in \mathrm{Eff}^\alpha(p_n)$ *such that* $\{x_n\}$ *has a subsequence* $\{x_{n_k}\}$ *converging to* z *with* $z \sim^\alpha_{F_0} x$.

Proof. We discuss the case $\alpha = u$. For every $\{p_n\} \subset \mathbb{P}, p_n \to p_0$, let $x \in \mathrm{Eff}^u(p_0)$ be arbitrary. Since M is l.s.c. at h_0, there exists a sequence $\{x_n\}$ with $x_n \in M(h_n)$ and $x_n \to x$. By the strong domination property of the problem $(\mathrm{SIP})^\alpha_{p_0}$, there is a subsequence $\{z_{n_k}\}, z_{n_k} \in \mathrm{Eff}^u(p_{n_k})$ such that $F_{n_k}(z_{n_k}) \preceq^u_C F_{n_k}(x_{n_k})$. On the other hand, M is u.s.c. and compact-valued at h_0, so we can assume that the subsequence $\{z_{n_k}\}$ converges to some $z \in M(h_0)$. This together with Proposition 4.1 implies that $F_0(z) \preceq^u_C F_0(x)$. Consequently, $z \sim^u_{F_0} x$. This brings the proof to its end. □

Remark 4.6. Based on the strong domination property, Karuna and Lalitha [27] have studied lower convergence conditions for efficient solutions to the problem (SOP) under a key assumption related to the compactness of $\mathrm{cl}\,(\cup_{n\in\mathbb{N}}(\cup_{x\in S}(F_n(x))))$ (Theorem 6.6 in [27]). Herein, although we do not use this assumption as in Theorem 6.6, the conclusion of Theorem 4.5 is similar to that of Theorem 6.6, and hence Theorem 4.5 is an improvement version of Theorem 6.6.

The following example shows that Theorem 4.5 is applicable.

Example 4.7. Let $X = \Omega = \mathbb{R}, Y = Z = \mathbb{R}^2$, $C = K = \mathbb{R}^2_+$, $T = [0, 1]$, and $A = [-2, 2]$. Let $h, h_n : A \times T \to \mathbb{R}^2$, $F, F_n : A \rightrightarrows \mathbb{R}^2$ be respectively defined by:
For all $x \in A$ and $t \in T$,

$$\begin{aligned} h(x,t) &= (t + x^2 - x - 3, x^2 - x - t), \\ h_n(x,t) &= \left(t + x^2 - x - 3 + \frac{1}{n}, \left(x - \frac{1}{n}\right)\left(x - \frac{n+2}{n+1}\right) - t\right), \\ F(x) &= \left\{(z_1, z_2) \in \mathbb{R}^2 \mid (z_1 - x)^2 + (z_2 - x)^2 \le (x+3)^2\right\}, \\ F_n(x) &= \left\{(z_1, z_2) \in \mathbb{R}^2 \mid (z_1 - x)^2 + (z_2 - x)^2 \le (x+3)^2 - \frac{1}{n}\right\}. \end{aligned}$$

Obviously, all assumption of Theorem 4.5 are fulfilled, so that the conclusion of Theorem 4.5 is true. Indeed, direct computations give us

$$M(h) = [0, 1], \quad M(h_n) = \left[\frac{1}{n}, \frac{n+2}{n+1}\right],$$

$$\mathrm{Eff}^l(p) = \{0\}, \mathrm{Eff}^u(p) = \{1\},$$

and

$$\mathrm{Eff}^l(p_n) = \left\{\frac{1}{n}\right\}, \mathrm{Eff}^u(p_n) = \left\{\frac{n+2}{n+1}\right\}.$$

To close this section, we propose a weaker version of the notions of strict quasicovexity for set-valued mappings introduced by Xu and Li [40], Karuna and Lalitha [25], and employ it to investigate the lower convergence condition for efficient solutions.

Definition 4.8. Let L be a nonempty subset of X. A set-valued map $F : L \rightrightarrows Y$ is said to be *α-C-strictly quasiconvexlike* at $\bar{x} \in L$ if for any $x \in L \setminus \{\bar{x}\}$ with $F(x) \preceq^{\alpha}_{C} F(\bar{x})$, then there exists $y \in L$ such that $F(y) \prec^{\alpha}_{C} F(\bar{x})$,

Corollary 4.9. *For $\alpha \in \{l, u\}$ and $p_0 = (h_0, F_0) \in \mathbb{P}$. Assume that all assumptions of Theorem* 4.5 *are satisfied and assume further that F_0 is α-C-strictly quasiconvexlike on $M(h_0)$. Then, for every $\{p_n\} \subset \mathbb{P}, p_n \to p_0$, and $x \in \mathrm{Eff}^{\alpha}(p_0)$ there exist $x_n \in \mathrm{Eff}^{\alpha}(p_n)$ such that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to x.*

Proof. Let $x \in \mathrm{Eff}^{\alpha}(p_0)$ be taken arbitrary. According to Theorem 4.5, there exist $x_n \in \mathrm{Eff}^{\alpha}(p_n)$ such that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to z with $z \sim^{\alpha}_{F_0} x$. Suppose that $z \neq x$. Since F_0 is α-C-strictly quasi-convexlike at z there exists $y \in M(h_0)$ such that $F_0(y) \prec^{\alpha}_{C} F_0(z)$. This is impossible as $z \in \mathrm{Eff}^{\alpha}(p_0)$. Therefore, $z = x$, and the conclusion follows. ☐

Remark 4.10. The authors in [26, 27] established the lower convergence condition for efficient solutions to set optimization problems by using assumptions related to different strict quasi-convexity types of the objective mapping. In Corollary 4.9, we obtain the lower convergence of efficient solution sets under the assumption of cone strict quasi-convex-likeness, which is a weaker assumption. Moreover, we do not require conditions guaranteeing the convexity of the constraint sets, which is considered as a key condition in [26, 27].

5. Conclusions

In this work we have studied perturbed set-valued optimization problems under functional perturbations of both objective and constraint maps. The efficient solutions to reference problems have been discussed by means of both upper and lower types set ordered relations. Sufficient conditions for the upper and lower convergence for efficient solutions of the given problems have been provided without requiring the convexity of constraint sets, and hence our approach can be applied to class of non-convex abstract set-optimization problems.

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