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A Support Function Approach for Characterizations of Convex Normal Cone

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We study the normal cone to a convex set given by a class of convex inequalities, by the robust version of Farkas' lemma for convex functions, and by a support function approach. Some special cases are given.

Keywords: Normal cone, Farkas' lemma, support function.

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1. Introduction

Let Ω be a closed convex subset in $\mathbb{R}^n$ and $x \in \Omega$, then $N_\Omega(x)$ is called the *normal cone* to Ω at x, where

$$N_\Omega(x) := \{\xi \in \mathbb{R}^n \colon \langle \xi, y - x\rangle \le 0, \text{ for all } y \in \Omega\}.$$

Observe that if $x \in \operatorname{int}\Omega$, then $N_\Omega(x) = \{0\}$, which is a trivial cone. In what follows, we will consider the case $N_\Omega(x) \neq \{0\}$, that is, $x \in \operatorname{bd}\Omega$ (where $\operatorname{bd}\Omega$ stands for the boundary of Ω). Note that $x \in \operatorname{bd}\Omega$ is a necessary and sufficient condition for $N_\Omega(x) \neq \{0\}$, even for the case that Ω is a nonconvex set; see, e.g. [16, Prop. 1.2].

It is well known that the computation of the normal cone $N_\Omega(x)$ plays a major role in the derivation of the celebrated Karush–Kuhn–Tucker (KKT) optimality conditions for convex optimization problems; see, e.g., [8, Chapter 3]. In this note, we are interested in answering the following question:

(Q) "*If the set Ω is explicitly given by a system of convex inequalities, then what does the formula of $N_\Omega(x)$ under a weaker condition (as compared with the Slater-type condition) look like?*"

Indeed, if one aims at deriving the KKT optimality conditions in terms of the subdifferentials of the objective function as well as the constraint functions at an optimal solution to a *constrained convex* optimization problem, then the equivalent characterization of the normal cone in terms of the subdifferentials of the *constraint* functions will be extremely important and also highly necessary.

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In this work, we consider the set $\Omega \subset \mathbb{R}^n$, given by a system of convex inequalities, in a unified form as follows,

$$\Omega := \{x \in \mathbb{R}^n \colon g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m\}, \tag{1}$$

where $g_i : \mathbb{R}^n \times \mathbb{R}^{q_i} \to \mathbb{R}$, $i = 1, \dots, m$ are continuous functions such that for each v_i, $g_i(\cdot, v_i)$ are proper lower semicontinuous convex, here $v_i \in \mathbb{R}^{q_i}$ is an uncertain parameter, which belongs to $\mathcal{V}_i \subset \mathbb{R}^{q_i}$ for each $i = 1, \dots, m$, and we usually call $\mathcal{V}_i$ an *uncertainty set* in robust optimization; see, e.g., [2, 3, 4, 5, 6, 7].

The set Ω as defined in (1) is a rather rich unified set, which covers (at least) the following two cases: (i) if $\mathcal{V}_i$ is a singleton for each $i = 1, \dots, m$, then Ω reduces to (for convenience),

$$\Omega_1 := \{x \in \mathbb{R}^n \colon g_i(x) \leq 0, i = 1, \dots, m\}, \tag{2}$$

which is a set explicitly given by *finitely* many proper lower semicontinuous convex functions; (ii) if $m = 1$, then Ω reduces to (for convenience),

$$\Omega_2 := \{x \in \mathbb{R}^n \colon g(x, v) \leq 0, \forall v \in \mathcal{V}\}, \tag{3}$$

which is a set explicitly given by *infinitely* many proper lower semicontinuous convex functions.

We will show in this note the representation of the normal cone $N_\Omega(\bar{x})$ (where Ω is given by (1)) under a weaker condition (comparing with the Slater-type condition). Precisely, the main result is stated as follows.

Theorem 1.1. (Representation of the Normal Cone) *Let $\bar{x} \in \Omega$, where Ω is defined in* (1). *Suppose that the cone*

$$\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \left(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i) \right)^*$$

is closed and convex. Then the following statements are equivalent:

(i) $\xi \in N_\Omega(\bar{x})$;

(ii) *there exist $\bar{\lambda}_i \geq 0$ and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \dots, m$ such that*

$$\xi \in \sum_{i=1}^m \bar{\lambda}_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}) \quad \textit{and} \quad \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0.$$

Actually, Theorem 1.1 can be proved by employing the robust version of Farkas' lemma for convex functions ([14, Theorem 2.4]) directly; see Method I in Section 3.

Nevertheless, we will provide a new approach due to Rockafellar ([17, Theorem 13.5]) to revisit Theorem 1.1, and this is also the main finding of our work. Besides, as by-products, the representations of the normal cones $N_{\Omega_1}(\bar{x})$ and $N_{\Omega_2}(\bar{x})$ will also be investigated under weaker conditions (also comparing with the Slater-type conditions), where Ω_1 and Ω_2 are given by (2) and (3), respectively.

The rest of the paper is organized as follows. Section 2 contains some basic definitions from convex analysis. Section 3 is devoted to the proof of the main result, Theorem 1.1. In Section 4, we pay our attention for some special cases of the considered convex set. Conclusions are stated in Section 5.

2. Preliminaries

In this section, we overview briefly some notions of convex analysis widely used in the sequel; see [8, 11, 17] for more details. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with the inner product $\langle\cdot,\cdot\rangle$ and the associated Euclidean norm $\|\cdot\|$. The non-negative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$.

We say that a set Ω in $\mathbb{R}^n$ is *convex* whenever $\mu a_1 + (1-\mu)a_2 \in \Omega$ for all $\mu \in [0,1]$, $a_1, a_2 \in \Omega$. For a given set $\Omega \subset \mathbb{R}^n$, we denote the *closure*, the *convex hull*, and the *conical hull* generated by Ω, by $\operatorname{cl}\Omega$, $\operatorname{co}\Omega$, and $\operatorname{cone}\Omega$, respectively. Let f be an extended real-valued function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, +\infty]$. Here, f is said to be *proper* if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. We say $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is *lower semicontinuous* at $\bar{x} \in \mathbb{R}^n$ if $\liminf_{x\to\bar{x}} f(x) \geq f(\bar{x})$. We denote the domain and epigraph of function f, by $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x,r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$, respectively. A function f is said to be *convex* if and only if its epigraph $\operatorname{epi} f$ is a convex set, and f is said to be *concave* whenever $-f$ is convex.

Definition 2.1. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lower semicontinuous proper convex function. The (*convex*) *subdifferential* of f at $x \in \operatorname{dom} f$ is defined by

$$\partial f(x) = \{\xi \in \mathbb{R}^n : \langle \xi, y - x\rangle \leq f(y) - f(x), \text{ for all } y \in \mathbb{R}^n\}.$$

Set $\partial f(x) = \emptyset$, if $x \notin \operatorname{dom} f$. □

Recall also that, for $\epsilon \geq 0$, the *ϵ-subdifferential* of f at $x \in \operatorname{dom} f$ is defined as

$$\partial_\epsilon f(x) = \{\xi \in \mathbb{R}^n : \langle \xi, y - x\rangle - \epsilon \leq f(y) - f(x), \text{ for all } y \in \mathbb{R}^n\}.$$

As is also well-known, for any proper convex function f on $\mathbb{R}^n$, its conjugate function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $f^*(\xi) = \sup\{\langle \xi, x\rangle - f(x) : x \in \mathbb{R}^n\}$ for any $\xi \in \mathbb{R}^n$. Moreover, let δ_C be the *indicator function* with respect to a closed convex subset Ω of $\mathbb{R}^n$, that is, $\delta_\Omega(x) = 0$ if $x \in \Omega$, and $\delta_\Omega(x) = +\infty$ if $x \notin \Omega$. Furthermore, for a set $\Omega \subset \mathbb{R}^n$, the *support function* $\sigma_\Omega : \mathbb{R}^n \to \overline{\mathbb{R}}$, to Ω at $y \in \mathbb{R}^n$ is defined as $\sigma_\Omega(y) := \sup_{x\in\Omega}\langle y, x\rangle$. In addition, the convex hull of an arbitrary collection of functions $\{f_t : t \in T\}$ with T being an arbitrary index set is denoted by $\operatorname{co}\bigcup_{t\in T} f_t$, which is also the convex hull of the pointwise infimum of the collection denoted by $\operatorname{co}\inf_{t\in T} f_t$, that is the greatest convex function f (not necessarily proper) on $\mathbb{R}^n$ such that $f(x) \leq f_t(x)$ for every $x \in \mathbb{R}^n$ for every $t \in T$; see [17, Section 5] and [8, Section 2.3] for more details.

The following proposition is the conjugacy of a max-function; see, e.g., [11, Theorem 2.4.7] and [8, Proposition 2.116].

Proposition 2.2. (Conjugacy of a max-function) *Consider the proper lower semicontinuous convex functions $\phi_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, where $i = 1, 2, \ldots, m$. Then define $\phi(x) = \max\{\phi_1(x), \ldots, \phi_m(x)\}$ and $p = \min\{m, n+1\}$.*
Then for every $\xi \in \operatorname{dom}\phi^* = \operatorname{co}\bigcup_{i=1}^m \operatorname{dom}\phi_i^*$, there exist $\xi_i \in \operatorname{dom}\phi_i^*$ and $\lambda_i \geq 0$, $i = 1, \ldots, p$, with $\sum_{i=1}^p \lambda_i = 1$ such that

$$\phi^*(\xi) = \sum_{i=1}^p \lambda_i \phi_i^*(\xi_i) \quad \text{and} \quad \xi = \sum_{i=1}^p \lambda_i \xi_i.$$

The epigraphical conditions for the operations of the conjugate functions are stated as below; see [8, Theorem 2.123 (i),(ii)] for more details.

Lemma 2.3. *Consider again the proper lower semicontinuous convex functions* $\phi_i\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, *where* $i = 1, \ldots, m$. *Then* $\operatorname{epi}(\phi_1 + \ldots + \phi_m)^* = \operatorname{cl}(\operatorname{epi}\phi_1^* + \ldots + \operatorname{epi}\phi_m^*)$.

Lemma 2.4. *Consider a family of proper lower semicontinuous convex functions* $\phi_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i \in I$, *where* I *is an arbitrary index set. Then*

$$\operatorname{epi}(\sup_{i\in I} \phi_i)^* = \operatorname{cl}\operatorname{co}\bigcup_{i\in I} \operatorname{epi}\phi_i^*.$$

Lemma 2.5. (see [13, Lemma 2.1]) *Let* $\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a proper lower semicontinuous convex function. If* $x \in \operatorname{dom}\phi$, *then*

$$\operatorname{epi}\phi^* = \bigcup_{\epsilon\geq 0} \{(\xi, -\phi(x) + \epsilon + \langle \xi, x\rangle) \colon \xi \in \partial_\epsilon \phi(x)\}.$$

As we mentioned above, for a given set $\Omega \subset \mathbb{R}^n$, its closure is denoted by $\operatorname{cl}\Omega$. Here, we introduce the notion of the *closure* for a given function $\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$; see [12, Definition 1.2.4].

Definition 2.6. The *closure* (or *lower semicontinuous hull*) of a function ϕ is the function $\operatorname{cl}\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\operatorname{cl}\phi(x) := \liminf_{y\to x} \phi(y) \quad \text{for all } x \in \mathbb{R}^n,$$

or equivalently

$$\operatorname{epi}(\operatorname{cl}\phi) := \operatorname{cl}(\operatorname{epi}\phi). \qquad \square$$

Finally, note that the standard operation of non-negative *left scalar multiplication* is $(\lambda\phi)(x) = \lambda\phi(x)$. In fact, there is also a useful operation of *right scalar multiplication*, which was firstly introduced by Rockafellar [17, Section 5]. Below, we recall the right scalar multiplication, and a related theorem that will play a significant role in deriving Theorem 1.1.

Definition 2.7. Let $\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper convex function and $\lambda \geq 0$. The *right scalar multiplication* denoted by $\phi\lambda$, is defined as

$$(\phi\lambda)(x) = \inf\{\mu\colon (x, \mu) \in \lambda(\operatorname{epi}\phi)\}. \qquad \square$$

Observe that

$$(\phi\lambda)(x) = \begin{cases} \lambda\phi(\lambda^{-1}x), & \lambda > 0, \\ \delta_{\{0\}}(x), & \lambda = 0. \end{cases}$$

Now, let $\phi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper convex function and $\lambda \geq 0$. Define a function ψ as

$$\psi(x) = \inf\{(\phi\lambda)(x)\colon \lambda \geq 0\}.$$

Then following [17, Section 5], we call ψ the *positively homogeneous convex function generated by* ϕ.

Theorem 2.8. [17, Theorem 13.5] *Let* $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a proper lower semicontinuous convex function. The support function of the set* $\{x \in \mathbb{R}^n \colon g(x) \leq 0\}$ *is then* $\operatorname{cl}\phi$, *where* ϕ *is the positively homogeneous convex function generated by* g^*. *Dually, the closure of the positively homogeneous convex function* φ *generated by* g *is the support function of the set* $\{x^* \in \mathbb{R}^n \colon g^*(x^*) \leq 0\}$.

3. Proof of Theorem 1.1

3.1. Proof of (i)⇒(ii) in Theorem 1.1

In order to establish the KKT-type necessary optimality condition for nonlinear optimization problems, a suitable constraint qualification (CQ) is needed as usual. Among others, in this work, we will assume the following CQ condition holds.

Definition 3.1. [14] We say the *robust characteristic cone constraint qualification* (RCCCQ for short) holds for (1) if the robust characteristic cone

$$D := \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \Big(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \Big)^*$$

is closed and convex.

Remark 3.2. The Slater condition (saying that there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}, v_i) < 0$, for all $v_i \in \mathcal{V}_i$, $i = 1, \ldots, m$), together with the compactness of each set $\mathcal{V}_i$, implies that the cone D is closed (see [14, Proposition 3.2]). If in addition, for all $i = 1, \ldots, m$, the function $g_i(x, \cdot)$ is concave for each x, and each set $\mathcal{V}_i$ of $\mathbb{R}^q$ is convex, then convexity of the cone D will also be guaranteed (see [14, Proposition 2.3]); roughly speaking, the RCCCQ in Definition 3.1 is a rather weak one (actually known as the weakest one; see [14]). □

Method I: *A proof from the viewpoint of the robust version of Farkas' lemma for convex functions.* Actually, it is a natural way to show [(i) ⇒ (ii)] in Theorem 1.1 by the robust version of Farkas' lemma, and also it can be proved easily (see [15]). For completement, we provide the proof in the following.

Proof. Let $\xi \in N_\Omega(\bar{x})$, then by definition

$$\begin{aligned} N_\Omega(\bar{x}) &= \{\xi \in \mathbb{R}^n \colon \langle \xi, x - \bar{x} \rangle \leq 0,\ \forall x \in \Omega\} \\ &= \{\xi \in \mathbb{R}^n \colon \langle -\xi, \bar{x} \rangle \leq \langle -\xi, x \rangle,\ \forall x \in \Omega\}. \end{aligned}$$

In other words, $\bar{x}$ is an optimal solution to the following problem,

$$\min \langle -\xi, x \rangle \quad \text{subject to} \quad x \in \Omega.$$

Now, consider $h(x) := \langle -\xi, x - \bar{x} \rangle$ with $\bar{x}$ being the optimal solution, one has $h(x) \geq 0$ for all $x \in \Omega$. By the robust version of Farkas' Lemma for convex functions (see [14, Theorem 2.4]), it yields that the following two statements are equivalent:

(I) $\quad x \in \Omega \Rightarrow h(x) \geq 0$;

(II) $\quad (0, 0) \in \operatorname{epi} h^* + \operatorname{cl} \operatorname{co} \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \right)^*.$

Since the RCCCQ holds true for the system (1), it follows from (II) that

$$(0, 0) \in \operatorname{epi} h^* + \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \right)^*,$$

so there exist $\bar{v}_i \in \mathcal{V}_i$ and $\bar{\lambda}_i \geq 0$, $i = 1, \ldots, m$ such that

$$(0,0) \in \operatorname{epi} h^* + \operatorname{epi} \Big(\sum_{i=1}^{m} \bar{\lambda}_i g_i\left(\cdot, \bar{v}_i\right) \Big)^*. \tag{4}$$

To proceed,
$$\operatorname{epi} h^* = \ \{-\xi\} \times [\langle -\xi, \bar{x} \rangle, +\infty). \tag{5}$$

By considering (5) in mind, and it follows from (4) that, there exists $(-\xi, r) \in \operatorname{epi} h^*$ with $r \geq \langle -\xi, \bar{x} \rangle$ such that $(\xi, -r) \in \ \operatorname{epi} \big(\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot, \bar{v}_i)\big)^*$, so,

$$\Big(\sum_{i=1}^{m} \bar{\lambda}_i g_i\left(\cdot, \bar{v}_i\right) \Big)^* (\xi) \leq -r \leq \langle \xi, \bar{x} \rangle.$$

By definition,
$$\langle \xi, x \rangle - \sum_{i=1}^{m} \bar{\lambda}_i g_i(x, \bar{v}_i) \leq \langle \xi, \bar{x} \rangle, \ \forall x \in \mathbb{R}^n. \tag{6}$$

Particularly, letting $x = \bar{x}$ in (6) entails that $\sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \geq 0$, which along with $\bar{x} \in \Omega$ and $\bar{\lambda}_i \geq 0$, $i = 1, \ldots, m$, implies that $\sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0$. Now, from (6), we have

$$\sum_{i=1}^{m} \bar{\lambda}_i g_i(x, \bar{v}_i) - \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \geq \langle \xi, x - \bar{x} \rangle, \ \forall x \in \mathbb{R}^n,$$

which, combining with the Moreau–Rockafellar sum rule [17, Theorem 23.8], yields that

$$\xi \in \sum_{i=1}^{m} \bar{\lambda}_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}).$$

Thus, the proof follows. □

Method II: *An alternative proof by a new approach due to Theorem* 2.8. This is also our main finding of the present work, and it is worth emphasizing that, this approach does *not* rely on the robust version of Farkas' lemma for convex functions.

Proof. Note that the set Ω can be equivalently written as follows:

$$\begin{aligned} \Omega &= \{x \in \mathbb{R}^n \colon g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \ldots, m\} \\ &= \big\{x \in \mathbb{R}^n \colon \sup_{v_i \in \mathcal{V}_i} g_i(x, v_i) \leq 0, i = 1, \ldots, m\big\} \\ &= \big\{x \in \mathbb{R}^n \colon \max_{i=1,\ldots,m} \sup_{v_i \in \mathcal{V}_i} g_i(x, v_i) \leq 0\big\}. \end{aligned}$$

For convenience, let $\hat{g}_i(x) = \sup_{v_i \in \mathcal{V}_i} g_i(x, v_i)$, and $G(x) = \max_{i=1,\ldots,m} \hat{g}_i(x)$.

Now, by the definition of normal cone, we have

$$N_\Omega(\bar{x}) = \{\xi \in \mathbb{R}^n \colon \langle \xi, x - \bar{x} \rangle \leq 0, \ \forall x \in \Omega\} = \big\{\xi \in \mathbb{R}^n \colon \sup_{x \in \Omega} \langle \xi, x \rangle \leq \langle \xi, \bar{x} \rangle\big\}. \tag{7}$$

Since the support function $\sup_{x \in \Omega} \langle \xi, x \rangle$ is the closure of the positively homogeneous convex function h generated by G^*, which is the conjugate function of G. According to Theorem 2.8, it follows from (7) that

$$N_\Omega(\bar{x}) = \{\xi \in \mathbb{R}^n \colon (\operatorname{cl} h)(\xi) \leq \langle \xi, \bar{x} \rangle\}, \tag{8}$$

where
$$\begin{aligned} h(\xi) &= \inf_{\lambda\geq 0}(G^*\lambda)(\xi) \\ &= \inf_{\lambda\geq 0}\begin{cases} \lambda G^*(\frac{\xi}{\lambda}), & \lambda>0 \\ \delta_{\{0\}}(\xi), & \lambda=0 \end{cases} && (9) \\ &= \inf_{\lambda\geq 0}\begin{cases} (\lambda G)^*(\xi), & \lambda>0 \\ \delta_{\{0\}}(\xi), & \lambda=0 \end{cases} && (10) \\ &= \inf_{\lambda\geq 0}(\lambda G)^*(\xi), && (11) \end{aligned}$$

here, the implication [(9) $\Rightarrow$ (10)] holds due to [8, Proposition 2.103 (iii)].

Let $\xi\in N_\Omega(\bar{x})$, then from (8), we have

$$\begin{aligned} (\xi,\langle\xi,\bar{x}\rangle) \in \operatorname{epi}\operatorname{cl} h &= \operatorname{cl}\operatorname{epi} h && \text{(by Definition 2.6)} \\ &= \operatorname{cl}\operatorname{epi}\Big(\inf_{\lambda\geq 0}(\lambda G)^*\Big) && (12) \\ &= \operatorname{cl}\bigcup_{\lambda\geq 0}\operatorname{epi}(\lambda G)^* && (13) \\ &= \operatorname{cl}\bigcup_{\lambda_i\geq 0}\operatorname{epi}\Big(\sum_{i=1}^m \lambda_i\hat{g}_i\Big)^* && (14) \\ &= \bigcup_{v_i\in\mathcal{V}_i,\lambda_i\geq 0}\operatorname{epi}\Big(\sum_{i=1}^m \lambda_i g_i(\cdot,v_i)\Big)^*. && (15) \end{aligned}$$

Below, we will show the above relations successively.

- [(12) $\Rightarrow$ (13)]: This is due to [8, page 70].
- [(13) $\Rightarrow$ (14)]: Remember that we have denoted $G(x) := \max_{i=1,\ldots,m}\hat{g}_i(x)$, which is a convex max-function. Let us first show

$$\bigcup_{\lambda\geq 0}\operatorname{epi}(\lambda G)^* \subset \bigcup_{\lambda_i\geq 0}\operatorname{epi}\Big(\sum_{i=1}^m \lambda_i\hat{g}_i\Big)^*.$$

Indeed, for any $(a,\alpha)\in\bigcup_{\lambda\geq 0}\operatorname{epi}(\lambda G)^*$, there exists $\hat{\lambda}\geq 0$ such that

$$(a,\alpha)\in\operatorname{epi}\big(\hat{\lambda}G\big)^*. \tag{16}$$

Thanks to Proposition 2.2, there exist $a_i\in\operatorname{dom}(\hat{\lambda}\hat{g}_i)^*$ and $t_i\geq 0$, $i=1,\ldots,p$ with $\sum_{i=1}^p t_i = 1$ such that

$$(\hat{\lambda}G)^*(a) = \sum_{i=1}^p t_i\Big(\hat{\lambda}\hat{g}_i\Big)^*(a_i) \quad \text{and} \quad a=\sum_{i=1}^p t_i a_i.$$

By definition of conjugate functions, it yields from (16) that

$$\begin{aligned} \alpha \geq \Big(\hat{\lambda}G\Big)^*(a) &= \sum_{i=1}^p t_i\Big(\hat{\lambda}\hat{g}_i\Big)^*(a_i) = \sum_{i=1}^p t_i\Big[\sup_{x\in\mathbb{R}^n}\Big\{\langle x,a_i\rangle - \Big(\hat{\lambda}\hat{g}_i\Big)(x)\Big\}\Big] \\ &\geq \sup_{x\in\mathbb{R}^n}\Big\{\Big\langle x,\sum_{i=1}^p t_i a_i\Big\rangle - \sum_{i=1}^p t_i\hat{\lambda}\hat{g}_i(x)\Big\} = \sup_{x\in\mathbb{R}^n}\Big\{\langle x,a\rangle - \sum_{i=1}^p \hat{\lambda}_i\hat{g}_i(x)\Big\}, \end{aligned}$$

where the last equality is obtained by letting $\hat{\lambda}_i := t_i\hat{\lambda}$; moreover, since $t_i \geq 0$, $\hat{\lambda} \geq 0$, we have $\hat{\lambda}_i \geq 0$, for each $i = 1, \ldots, p$. On the other hand, as we have $p = \min\{m, n+1\}$, we get

$$\sum_{i=1}^{p} \hat{\lambda}_i \hat{g}_i(x) = \sum_{i=1}^{m} \hat{\lambda}_i \hat{g}_i(x),$$

by letting $\lambda_i = 0$, $i \geq p+1$ if $p < m$. Consequently, by definition of conjugate functions again, we have

$$\alpha \geq \left(\sum_{i=1}^{m} \hat{\lambda}_i \hat{g}_i\right)^* (a);$$

in other words, $(a, \alpha) \in \operatorname{epi}\Big(\sum_{i=1}^{m} \hat{\lambda}_i \hat{g}_i\Big)^* \subset \bigcup_{\lambda_i \geq 0} \operatorname{epi}\Big(\sum_{i=1}^{m} \lambda_i \hat{g}_i\Big)^*.$

Now, we will show $\bigcup_{\lambda_i \geq 0} \operatorname{epi}\Big(\sum_{i=1}^{m} \lambda_i \hat{g}_i\Big)^* \subset \bigcup_{\lambda \geq 0} \operatorname{epi}(\lambda G)^*.$

Take any $(b, \beta) \in \bigcup_{\lambda_i \geq 0} \operatorname{epi}\left(\sum_{i=1}^{m} \lambda_i \hat{g}_i\right)^*$, there exist $\bar{\lambda}_i \geq 0$, $i = 1, \ldots, m$ such that

$$\begin{aligned}
\beta \geq \Big(\sum_{i=1}^{m} \bar{\lambda}_i \hat{g}_i\Big)^*(b) &= \sup_{x \in \mathbb{R}^n} \Big\{\langle b, x\rangle - \sum_{i=1}^{m} \bar{\lambda}_i \hat{g}_i(x)\Big\} \geq \sup_{x \in \mathbb{R}^n} \Big\{\langle b, x\rangle - \sum_{i=1}^{m} \bar{\lambda}_i G(x)\Big\} \\
&= \sup_{x \in \mathbb{R}^n} \big\{\langle b, x\rangle - \bar{\lambda} G(x)\big\} \qquad (17) \\
&= \big(\bar{\lambda} G\big)^* (b), \qquad (18)
\end{aligned}$$

where (17) is obtained by letting $\bar{\lambda} := \sum_{i=1}^{m} \bar{\lambda}_i$. Note that $\bar{\lambda} \geq 0$ as $\bar{\lambda}_i \geq 0$ for each $i = 1, \ldots, m$. To proceed, it follows from (18) that

$$(b, \beta) \in \operatorname{epi}\big(\bar{\lambda} G\big)^* \subset \bigcup_{\lambda \geq 0} \operatorname{epi}(\lambda G)^*.$$

Finally, by taking closures of both sides, we get the implication [(13) ⇒ (14)].

- [(14) ⇒ (15)]: Note that (as we have denoted above) $\hat{g}_i(x) = \sup_{v_i \in \mathcal{V}_i} g_i(x, v_i)$, $i = 1, \ldots, m$. We then start from (14):

$$\begin{aligned}
&\operatorname{cl} \bigcup_{\lambda_i \geq 0} \operatorname{epi}\Big(\sum_{i=1}^{m} \lambda_i \hat{g}_i\Big)^* \\
= \; &\operatorname{cl} \bigcup_{\lambda_i \geq 0} \Big[\operatorname{cl}\Big(\sum_{i=1}^{m} \operatorname{epi}(\lambda_i \hat{g}_i)^*\Big)\Big] \qquad \text{(based on Lemma 2.3)} \\
= \; &\operatorname{cl} \bigcup_{\lambda_i \geq 0} \Big[\operatorname{cl}\Big(\sum_{i=1}^{m} \operatorname{epi}\Big(\sup_{v_i \in \mathcal{V}_i} \lambda_i g_i(\cdot, v_i)\Big)^*\Big)\Big] \\
= \; &\operatorname{cl} \bigcup_{\lambda_i \geq 0} \Big[\operatorname{cl}\Big(\sum_{i=1}^{m} \operatorname{cl}\operatorname{co} \bigcup_{v_i \in \mathcal{V}_i} \operatorname{epi}(\lambda_i g_i(\cdot, v_i))^*\Big)\Big] \qquad (19)
\end{aligned}$$

$$= \operatorname{cl} \bigcup_{\lambda_i \geq 0} \Big[\operatorname{cl} \operatorname{co} \Big(\sum_{i=1}^{m} \bigcup_{v_i \in \mathcal{V}_i} \operatorname{epi} \big(\lambda_i g_i(\cdot, v_i)\big)^* \Big) \Big] \tag{20}$$

$$= \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \Big(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \Big)^*, \tag{21}$$

where (19) follows from Lemma 2.4, (20) is based on the fact $\operatorname{cl}(\operatorname{cl}\operatorname{co} A + \operatorname{cl}\operatorname{co} B) = \operatorname{cl}\operatorname{co}(A+B)$; see, e.g., [1, Exercise 6.6 (iii)]. Now it suffices to show the last equality (21). Indeed, for any (fixed) $\bar{\lambda}_i \geq 0$ and $\bar{v}_i \in \mathcal{V}_i$, by Lemma 2.3, we have

$$\begin{aligned} \operatorname{epi} \Big(\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot, \bar{v}_i) \Big)^* &= \operatorname{cl} \sum_{i=1}^{m} \operatorname{epi} \big(\bar{\lambda}_i g_i(\cdot, \bar{v}_i)\big)^* \\ &\subset \bigcup_{\lambda_i \geq 0} \Big[\operatorname{cl} \operatorname{co} \Big(\sum_{i=1}^{m} \bigcup_{v_i \in \mathcal{V}_i} \operatorname{epi} \big(\lambda_i g_i(\cdot, v_i)\big)^* \Big) \Big] \\ &\subset \operatorname{cl} \bigcup_{\lambda_i \geq 0} \Big[\operatorname{cl} \operatorname{co} \Big(\sum_{i=1}^{m} \bigcup_{v_i \in \mathcal{V}_i} \operatorname{epi} \big(\lambda_i g_i(\cdot, v_i)\big)^* \Big) \Big], \end{aligned}$$

which implies

$$\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \Big(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \Big)^* \subset \operatorname{cl} \bigcup_{\lambda_i \geq 0} \Big[\operatorname{cl} \operatorname{co} \Big(\sum_{i=1}^{m} \bigcup_{v_i \in \mathcal{V}_i} \operatorname{epi} \big(\lambda_i g_i(\cdot, v_i)\big)^* \Big) \Big].$$

Now we show the other way around. Indeed, for any $\bar{\lambda}_i \geq 0$, we obtain

$$\bigcup_{v_i \in \mathcal{V}_i} \operatorname{epi} \big(\bar{\lambda}_i g_i(\cdot, v_i)\big)^* \subset \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \Big(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \Big)^*.$$

Since the RCCCQ holds, i.e., the cone

$$\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \Big(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \Big)^*$$

is closed convex, so

$$\sum_{i=1}^{m} \bigcup_{v_i \in \mathcal{V}_i} \operatorname{epi}(\bar{\lambda}_i g_i(\cdot, v_i))^* \subset \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \Big(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \Big)^*,$$

which implies

$$\bigcup_{\lambda_i \geq 0} \Big[\operatorname{cl} \operatorname{co} \Big(\sum_{i=1}^{m} \bigcup_{v_i \in \mathcal{V}_i} \operatorname{epi} \big(\lambda_i g_i(\cdot, v_i)\big)^* \Big) \Big] \subset \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \operatorname{epi} \Big(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \Big)^*,$$

which still holds by taking closures of both sides. In conclusion, the implication $[(14) \Rightarrow (15)]$ follows.

Finally, it follows from (15) that there exist $\bar{\lambda}_i \geq 0$ and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ with

$$\left(\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot, \bar{v}_i)\right)^*(\xi) \leq \langle \xi, \bar{x} \rangle.$$

Then, the proof follows by a similar argument as shown in Method I. □

Remark 3.3. It seems that the Method II is more complicated and/or difficult than Method I. However, let us point out the fact that Method I employs the robust version of Farkas' lemma for convex functions, which is just invoked from [14, Theorem 2.4], while Method II provides a new view by directly dealing with the support function $\sup_{x\in\Omega}\langle \xi, x\rangle$. Besides, in the proof of Method II, the implications [(13) ⇒ (14)] and [(14) ⇒ (15)] are of their own interest, and both of them show more operations for epigraphical rules by conjugate functions. □

3.2. Proof of (ii)⇒(i) in Theorem 1.1

Proof. Let $\bar{x} \in \Omega$, if there exist $\bar{\lambda}_i \geq 0$ and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ such that

$$\xi \in \sum_{i=1}^{m} \bar{\lambda}_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}) \ \text{ and } \ \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0,$$

then $\xi = \sum_{i=1}^{m} \bar{\lambda}_i \xi_i$ for some $\xi_i \in \partial g_i(\cdot, \bar{v}_i)(\bar{x})$, $i = 1, \ldots, m$.
Furthermore, one has for each $i = 1, \ldots, m$,

$$g_i(x, \bar{v}_i) - g_i(\bar{x}, \bar{v}_i) \geq \langle \xi_i, x - \bar{x} \rangle, \ \forall x \in \mathbb{R}^n.$$

Multiplying $\bar{\lambda}_i$ (as $\bar{\lambda}_i \geq 0$) in the above inequalities and summing them yield that for all $x \in \mathbb{R}^n$,

$$\sum_{i=1}^{m} \bar{\lambda}_i g_i(x, \bar{v}_i) - \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \geq \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, x - \bar{x} \rangle = \langle \xi, x - \bar{x} \rangle,$$

which, in particular for $x \in \Omega$, implies $0 \geq \langle \xi, x - \bar{x} \rangle$, that is, $\xi \in N_\Omega(\bar{x})$. □

4. Special cases

In this section, we present the representations of the normal cones to two special sets, namely, the set Ω_1 given by finitely many convex functions (2), and Ω_2 given by infinitely many convex functions (3).

Proposition 4.1. *Let $\bar{x} \in \Omega_1$, where Ω_1 is defined in* (2). *Suppose that the convex cone*

$$\bigcup_{\lambda_i \geq 0} \operatorname{epi} \left(\sum_{i=1}^{m} \lambda_i g_i\right)^*$$

is closed. Then $\xi \in N_{\Omega_1}(\bar{x})$ if and only if there exist $\bar{\lambda}_i \geq 0$, $i = 1, \ldots, m$ such that

$$\xi \in \sum_{i=1}^{m} \bar{\lambda}_i \partial g_i(\bar{x}) \ \text{ and } \ \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}) = 0.$$

Remark 4.2. The Proposition 4.1 can be proved by similar arguments of the proof of Theorem 1.1. We therefore omit it here and leave it to the reader. Besides, the closed-type constraint qualification adopted in Proposition 4.1 can be seen in [8, Theorem 2.123 (iii)], and the Slater-type condition ensures it; see [8, Proposition 7.12]. □

We close this section by considering the second case, i.e., Ω_2 defined in (3), in which the set $\mathcal{V}$ can be seen as *any* (infinite) index set. Before we go ahead, let us recall some notations frequently occurred in *semi-infinite* optimization; see, e.g., [10]. Let $\mathbb{R}^{\mathcal{V}}$ be the *product space* of $\lambda = (\lambda_v \in \mathbb{R} \colon v \in \mathcal{V})$ and

$$\mathbb{R}^{|\mathcal{V}|} := \left\{\lambda \in \mathbb{R}^{\mathcal{V}} \colon \lambda_v \neq 0 \text{ for finitely many } v \in \mathcal{V}\right\}.$$

Denote by $\mathbb{R}_+^{|\mathcal{V}|}$ the *positive cone* in $\mathbb{R}^{|\mathcal{V}|}$, where

$$\mathbb{R}_+^{|\mathcal{V}|} := \left\{\lambda \in \mathbb{R}^{|\mathcal{V}|} \colon \lambda_v \geq 0, \forall v \in \mathcal{V}\right\}.$$

For a given $z \in \mathbb{R}^{\mathcal{V}}$ and $\lambda \in \mathbb{R}^{|\mathcal{V}|}$, we also define the *supporting set* of λ as the set $\mathrm{supp}\lambda = \{v \in \mathcal{V} \colon \lambda_v \neq 0\}$, and thus we have

$$\langle \lambda, z\rangle = \sum_{v\in\mathcal{V}} \lambda_v z_v = \sum_{v\in \mathrm{supp}\lambda} \lambda_v z_v.$$

Finally, following [9], we define the set of *active constraint multipliers* as

$$A(\bar{x}) := \left\{\lambda \in \mathbb{R}_+^{|\mathcal{V}|} \colon \lambda_v g(\bar{x}, v) = 0, \quad \text{for all} \quad v \in \mathrm{supp}\lambda\right\}.$$

The forthcoming result presents the representation of the normal cone to Ω_2. Due to its own structure, we now give the proof by using Method II. Nevertheless, one could show it by Method I, and we leave it to the reader.

Proposition 4.3. *Let $\bar{x} \in \Omega_2$, where Ω_2 is defined in* (3)*. Suppose that the set*

$$\mathrm{cone}\,\mathrm{co} \bigcup_{v\in\mathcal{V}} \mathrm{epi}\, g^*(\cdot, v) \tag{22}$$

is closed. Then $\xi \in N_{\Omega_2}(\bar{x})$ if and only if there exist $\lambda \in A(\bar{x})$ such that

$$\xi \in \sum_{v\in supp\lambda} \lambda_v \partial g(\cdot, v)(\bar{x}).$$

Proof. Since the "if" part holds obviously, we therefore suffer to show the "only if" part. Note that

$$\begin{aligned}
\Omega_2 &:= \{x \in \mathbb{R}^n \colon g(x, v) \leq 0, \forall v \in \mathcal{V}\} \\
&= \left\{x \in \mathbb{R}^n \colon \sup_{v\in\mathcal{V}} g(x, v) \leq 0\right\} \\
&= \left\{x \in \mathbb{R}^n \colon \hat{g}(x) \leq 0\right\}. \qquad \text{(by letting } \hat{g}(x) = \sup\nolimits_{v\in\mathcal{V}} g(x, v))
\end{aligned}$$

To proceed, let $\xi \in N_{\Omega_2}(\bar{x})$, by definition, $\langle \xi, x - \bar{x}\rangle \leq 0$ for all $x \in \Omega_2$, which is equivalent to

$$\sup_{x\in\Omega_2} \langle \xi, x\rangle \leq \langle \xi, \bar{x}\rangle. \tag{23}$$

Now, by Theorem 2.8 and a similar argument as shown in (11), it yields that

$$\sup_{x\in\Omega_2} \langle \xi, x\rangle = \operatorname{cl} \inf_{\mu\geq 0} (\mu\hat{g})^* (\xi).$$

This, combining with (23), entails that

$$\begin{aligned}
(\xi, \langle \xi, \bar{x}\rangle) \in\ & \operatorname{epi} \operatorname{cl} \inf_{\mu\geq 0} (\mu\hat{g})^* = \operatorname{cl} \bigcup_{\mu\geq 0} \operatorname{epi}(\mu\hat{g})^* && \text{(by (13))}\\
=\ & \operatorname{cl} \operatorname{cone} \operatorname{epi} \hat{g}^* = \operatorname{cl} \operatorname{cone} \operatorname{cl} \operatorname{co} \bigcup_{v\in\mathcal{V}} \operatorname{epi} g^*(\cdot, v) && \text{(by Lemma 2.4)}\\
=\ & \operatorname{cone} \operatorname{co} \bigcup_{v\in\mathcal{V}} \operatorname{epi} g^*(\cdot, v), && (24)
\end{aligned}$$

where (24) holds due to the assumption (22).

By invoking Lemma 2.5, we have for each $v \in \mathcal{V}$,

$$\operatorname{epi} g^*(\cdot, v) = \bigcup_{\epsilon\geq 0} \{(\xi, -g(\bar{x}, v) + \epsilon + \langle \xi, \bar{x}\rangle) \colon \xi \in \partial_\epsilon g(\cdot, v)(\bar{x})\},$$

it then follows from (24) that for each $v \in \mathcal{V}$, there exist $\lambda_v \geq 0$, $\epsilon_v \geq 0$ and

$$\xi_v \in \partial_{\epsilon_v} g(\cdot, v)(\bar{x}) \tag{25}$$

such that

$$\begin{aligned}
(\xi, \langle \xi, \bar{x}\rangle) &= \sum_{v\in\mathcal{V}} \lambda_v (\xi_v, -g(\bar{x}, v) + \epsilon_v + \langle \xi_v, \bar{x}\rangle) && (26)\\
&= \sum_{v\in\operatorname{supp}\lambda} \lambda_v (\xi_v, -g(\bar{x}, v) + \epsilon_v + \langle \xi_v, \bar{x}\rangle). && (27)
\end{aligned}$$

Therefore,
$$\xi = \sum_{v\in\operatorname{supp}\lambda} \lambda_v \xi_v, \tag{28}$$

and
$$\langle \xi, \bar{x}\rangle = \sum_{v\in\operatorname{supp}\lambda} \lambda_v (-g(\bar{x}, v) + \epsilon_v + \langle \xi_v, \bar{x}\rangle). \tag{29}$$

Combining (28) and (29) yields

$$\sum_{v\in\operatorname{supp}\lambda} \lambda_v g(\bar{x}, v) = \sum_{v\in\operatorname{supp}\lambda} \lambda_v \epsilon_v,$$

which, together with $\bar{x} \in \Omega_2$ implies that $\epsilon_v = 0$ for each $v \in \operatorname{supp}\lambda$ and $\lambda \in A(\bar{x})$.

Thereby, the proof follows from (25) and (28) as

$$\xi = \sum_{v\in\operatorname{supp}\lambda} \lambda_v \xi_v \in \sum_{v\in\operatorname{supp}\lambda} \lambda_v \partial g(\cdot, v)(\bar{x}),$$

for some $\lambda \in A(\bar{x})$. □

5. Conclusions

In this paper, we provide two methods to show the representation of the normal cone to a convex set Ω at a reference point, one is based on the robust version of Farkas' lemma for convex functions, another is based on a new approach due to [17, Theorem 13.5]. We mention here that the second method seems a rather new one, and it is worth paying much more attention. Interestingly, Suzuki [18] studied some characterizations of the normal cone in terms of the robust version of Farkas' lemma, while by using a support function approach, it will be an interesting topic to prove the characterizations in [18].

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