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A Ky Fan Minimax Inequality Approach to Implicit Obstacle Problems of Fractional Laplacian Type Involving a Generalized Gradient Operator

Ouayl Chadli
Department of Mathematics, University of Central Florida, U.S.A.
and: Ibn Zohr University, Agadir, Morocco
ochadli@gmail.com

Ram N. Mohapatra*
Department of Mathematics, University of Central Florida, U.S.A.
ram.mohapatra@ucf.edu

Abdellatif Koukkous
Department of Economics, Ibn Zohr University, Agadir, Morocco
a.koukkous@uiz.ac.ma

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We study the existence of solutions of a constrained obstacle problem driven by a generalized fractional Laplace operator and involving Clarke's generalized gradient, where the constraint depends on the unknown state variable. We use a method based on the *Ky Fan minimax inequality* approach and recent developments in that theory. Our results are new and improve considerably recent results in literature.

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1. Introduction

We study the existence of solutions of the following implicit obstacle problem of the fractional Laplacian type involving a Clarke's generalized gradient. Precisely, we consider the following problem:

Problem $(\mathcal{P})$: Find $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\begin{aligned} &(\mathcal{L}_K u)(x) + \partial j(x, u(x)) + \xi(x) \ni f(x) && \text{in } \Omega, \\ &u = 0 && \text{in } \Omega^c, \\ &\Big(\int_{\mathbb{R}^{2N}} (u(x) - u(y))^2 K(x - y)\, dxdy \Big)^{\frac{1}{2}} \le U(u), \end{aligned} \tag{1}$$

where Ω is an open bounded set in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $f \in L^2(\Omega)$, $\Omega^c = \mathbb{R}^N \setminus \Omega$ and $U : L^2(\Omega) \to \mathbb{R}$ is a given function.

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* Corresponding author.

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Here, $\mathcal{L}_K$ represents a nonlocal operator defined by

$$\mathcal{L}_K u(x) := -\int_{\mathbb{R}^N} [u(x+y) + u(x-y) - 2u(x)]K(y)\, dy \quad \text{for a.e.} \quad x \in \mathbb{R}^N, \qquad (2)$$

where $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a function satisfying some appropriate conditions which will be specified later. The operator $\partial j(x, \cdot)$ denotes the Clarke's generalized gradient of a locally Lipschitz function $j(x, \cdot) : \mathbb{R} \to \mathbb{R}$; and $\xi \in N_{C(u)}(u) \subset L^2(\Omega)$, where $N_{C(u)}(\cdot)$ is the normal cone of the set

$$C(u) := \{v \in L^2(\Omega) : v = 0 \text{ in } \Omega^c, L(v) \leq U(u)\}$$

at the point $u \in C(u)$, with $L(v) := \left(\int_{\mathbb{R}^{2N}} (v(x) - v(y))^2 K(x-y)\, dxdy\right)^{\frac{1}{2}}$.

Problem $(\mathcal{P})$ has been studied recently by Motreanu et al. [28], and by Migórski et al. [24] without the condition

$$\left(\int_{\mathbb{R}^{2N}} (u(x) - u(y))^2 K(x-y)\, dxdy\right)^{\frac{1}{2}} \leq U(u)$$

in the particular case where $\mathcal{L}_K$ is the fractional Laplace operator defined by

$$(-\Delta)^s u(x) := -\int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}}\, dy,$$

i.e. $K(y) = |y|^{-(N+2s)}$ for all $x \in \mathbb{R}^N \setminus \{0\}$, where $s \in (0,1)$ and $N > 2s$.

The generalized fractional Laplace operator $\mathcal{L}_K$ in problem $(\mathcal{P})$ is characterized by its nonlocality coming from the fact that the values of the solution are not just point wise but are in the whole space $\mathbb{R}^N$. This special feature of the operator $\mathcal{L}_K$ makes partial differential equations involving nonlocal operators more appropriate to describe complex systems in various fields such as finance, population dynamics, image processing, game theory, phase transition, obstacle problems etc., see e.g. [1, 3, 4, 5, 6, 22, 29, 41] and the references therein.

Different methods have been used to study the existence of solutions of systems driven by nonlocal operators and involving a Clarke's generalized gradient. For instance, Ten [38] and Xi et al. [39] used nonsmooth critical point theory to obtain multiplicity of weak solutions of hemivariational inequalities driven by the fractional Laplacian operator $(-\Delta)^s$. Liu and Tan [21], Liu et al. [20] and Motreanu et al. [28] employed a surjectivity result on pseudomonotone perturbations of maximal monotone operators. The theory of hemivariational inequalities has been introduced and studied since the 1980s by Panagiotopoulos [30, 31, 32] as a generalization of variational inequalities with the aim to model many problems coming from mechanics and economics when the energy functionals are non-convex, it is based on the Clarke's subgradient for locally Lipschitz functionals, see [33] and the references therein.

In this paper, we use a different method to study the existence of solutions of systems governed by nonlocal operators than the ones quoted in the references above. It is based on the *Ky Fan minimax inequality* approach and recent developments in that theory. The main advantage of this method is that it helped us to reduce considerably the assumptions on the data of the problem in comparison with previous studies.

By *Ky Fan minimax inequality*, we mean the problem of finding $u \in C$ such that

$$\Phi(u,v) \geq 0 \quad \text{for all } v \in C, \tag{3}$$

where C is a nonempty, closed and convex subset of a Banach space X, and $\Phi : C \times C \to \mathbb{R}$ is a real-valued bifunction such that $\Phi(v,v) = 0$ for all $v \in C$. First existence results for the inequality (3) were obtained by Ky Fan [11]. This inequality appears to be a cornerstone in nonlinear analysis since it plays an important role in many fields, such as variational inequalities, game theory, mathematical economics, optimization theory, and fixed point theory. Nowadays, it is widely known within the literature as *equilibrium problem*, see [2], where further particular cases have been considered. Mosco [27] and Joly–Mosco [14] introduced a general formulation of the inequality (3) described by the sum of two bifunctions which they called *implicit variational problem*. It is currently known in the literature by *mixed equilibrium problem* and it consists to find $u \in C$ such that

$$\Phi(u,v) + \Psi(u,v) \geq 0 \quad \text{for all } v \in C, \tag{4}$$

where $\Phi, \Psi : C \times C \to \mathbb{R}$ are two real-valued bifunctions with $\Phi(v,v) = \Psi(v,v) = 0$ for all $v \in C$ and satisfying separate properties. The constraint

$$\left(\int_{\mathbb{R}^{2N}} (u(x) - u(y))^2 K(x-y) \, dxdy \right)^{\frac{1}{2}} \leq U(u)$$

in problem $(\mathcal{P})$ depends on the unknown state variable u, therefore for our study we shall use a mixed quasi-equilibrium problem formulation: Find $u \in C(u)$ such that

$$\Phi(u,v) + \Psi(u,v) \geq 0 \quad \text{for all } v \in C(u). \tag{5}$$

The existence of solutions of problem (5) relies on a fixed point theorem which requires, among other properties, the convexity of the solution set of the problem (4), considered here as a variational selection of (5). In literature, this property is usually obtained by assuming some monotonicity conditions on the data of the problem and a Minty's formulation, see e.g. [15, 16, 17] and the references therein. In comparison with a recent study in [28] on the existence of solutions of problem $(\mathcal{P})$, the approach developed in this paper leads us to relax considerably the assumptions on the data of the problem and to avoid any monotonicity condition, especially the one obtained by adding a specific assumption on $j(x,\cdot)$ usually called *relaxed monotonicity condition.*

The paper is organized as follows: In Section 2, we give some concepts and preliminary results that will be needed for the development of our results as well as some basic facts from the theory of nonlocal operators. In Section 3, we formulate problem $(\mathcal{P})$ as a Ky Fan minimax inequality problem and we study its existence by introducing an auxiliary problem. The existence of solutions of the auxiliary problem is studied by means of a variational selection procedure and the Kluge's fixed point theorem. We end the paper by some comments and comparisons with recent results in literature on the existence of solutions of systems governed by nonlocal operators showing the interest of the approach developed.

2. Preliminaries

Let X be a Banach space and X^* be its *topological dual.* By $\langle \cdot, \cdot \rangle_X$ we denote the duality pairing between X^* and X. The norms of X and X^* are denoted respectively by $\|\cdot\|_X$ and $\|\cdot\|_{X^*}$. For each subset M of X, we denote by $\text{conv}(M)$ the *convex hull* of M and by M^c the *complementary set* of M. We shall denote by $\mathcal{F}(M)$ the family of all finite subsets of M and by $\bar{\mathbb{B}}_X(x_0, R)$ the *closed ball* in X of center x_0 and radius R. For a set-valued mapping $A : X \to 2^{X^*}$, we denote by $\mathcal{D}(A) := \{x \in X : A(x) \neq \emptyset\}$ the *domain* of A, and by $\mathcal{G}(A) := \{(x, x^*) \in X \times X^* : x \in \mathcal{D}(A) \text{ and } x^* \in A(x)\}$ the *graph* of A. For a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$, we use the standard notations $u_n \to u$ and $u_n \rightharpoonup u$ to denote, respectively, the *strong* convergence and the *weak* convergence of $\{u_n\}_{n\in\mathbb{N}}$ to u. Let $\mathcal{J} : X \to 2^{X^*}$ be the duality mapping defined by

$$\mathcal{J}(x) = \left\{x^* \in X^* : \langle x^*, x\rangle_X = \|x^*\|^2_{X^*} \text{ and } \|x^*\|_{X^*} = \|x\|_X\right\}.$$

By Hahn-Banach theorem, $\mathcal{J}(x) \neq \emptyset$ for any $x \in X$. We assume that the space X has been renormed, so that X and its dual space are locally uniformly convex. Therefore, the duality mapping $\mathcal{J}$ is single-valued, continuous, see [40].

Next, we briefly survey some notions on nonlinear operators. For more details, we refer to the book by Zeidler [40].

Definition 2.1. A mapping $A : \mathcal{D}(A) \subset X \longrightarrow X^*$ is said to be

(i) *monotone*, if for any $u, v \in \mathcal{D}(A)$, the inequality $\langle A(u) - A(v), u - v\rangle_X \geq 0$ holds;

(ii) *maximal monotone*, if and only if A is monotone and $\langle u^* - A(v), u - v\rangle_X \geq 0$ for all $v \in \mathcal{D}(A)$ implies $u \in \mathcal{D}(A)$ and $u^* = A(u)$.

Definition 2.2. A mapping $A : X \to X^*$ is said to be

(i) *pseudomonotone* in the sense of Brézis, if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ with $x_n \rightharpoonup x$ in X and $\limsup_{n\to+\infty}\langle A(x_n), x_n - x\rangle_X \leq 0$, we have $\liminf\limits_{n\to+\infty}\langle A(x_n), x_n - y\rangle_X \geq \langle A(x), x - y\rangle_X$, for all $y \in X$;

(ii) *demicontinuous*, if $x_n \to x$ in X implies $A(x_n) \rightharpoonup A(x)$ in X^*;

(iii) *hemicontinuous* (respectively, *upper hemicontinuous*), if for all $x, y, z \in X$, the functional $t \mapsto \langle A(x + ty), z\rangle_X$ is continuous (respectively, upper semicontinuous) on $[0, 1]$.

Now, we recall the concepts mentioned above for real-valued bifunction considered within the literature in the recent years. Most of these notions were inspired by similar monotonicity/continuity concepts defined for operators acting from a topological vector space to its dual space, see for instance [2, 12].

Definition 2.3. Let $\Phi : C \times C \longrightarrow \mathbb{R}$ be a real-valued bifunction, where C is a nonempty, closed and convex subset of X. Then Φ is said to be

(i) *monotone*, if for all $x, y \in C$ we have $\Phi(x, y) + \Phi(y, x) \leq 0$;

(ii) *pseudomonotone* in the sense of Brézis (for short, **B**-*pseudomonotone*), if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset C$ such that $x_n \rightharpoonup x$ in X and $\liminf\limits_{n\to+\infty} \Phi(x_n, x) \geq 0$, we have $\limsup\limits_{n\to+\infty} \Phi(x_n, y) \leq \Phi(x, y)$, $\forall y \in C$;

(iii) *hemicontinuous* (respectively, *upper hemicontinuous*), if, for all $x, y, z \in C$, the functional $t \in [0, 1] \mapsto \Phi(tx + (1 - t)y, z)$ is continuous (respectively, upper semicontinuous) on [0,1].

Remark 2.4. (a) Let $A : X \to X^*$ be a pseudomonotone operator in the sense of Brézis. Then the bifunction Φ defined by $\Phi(x, y) = \langle A(x), y - x\rangle_X$ is **B**-pseudomonotone, as can easily be seen.

(b) If the bifunction $\Phi : C \times C \to \mathbb{R}$ is *Ky Fan hemicontinuous* (see [37, Definition 2.1] and [23, Definition 2.2] for operators), i.e. for each $y \in C$ the function $\Phi(\cdot, y)$ is upper semicontinuous with respect to the weak topology $\sigma(X, X^*)$ of X, then it is **B**-pseudomonotone. The converse is not true, this has been shown by Steck [37] in a counter example to a claim in a paper by Sadeqi and Paydar [35] on the equivalence of the two properties for operators.

(c) If $\Phi, \Psi : C\times C \to \mathbb{R}$ are two real-valued **B**-pseudomonotone bifunctions such that $\Phi(x, x) \leq 0$ and $\Psi(x, x) \leq 0$ for all $x \in C$, then $\Phi + \Psi$ is also **B**-pseudomonotone, see [9]. ☐

Now, we recall the concept of maximal monotonicity for bifunctions.

Definition 2.5. [2] Let C be a nonempty, closed and convex subset of X and let $\Phi : C \times C \to \mathbb{R}$ be a real-valued bifunction such that $\Phi(x, x) = 0$ for all $x \in C$. Φ is said to be *maximal monotone*, if for every $x \in C$ and for every convex function $\varphi : C \to \mathbb{R}$ satisfying $\varphi(x) = 0$, we have

$$\Phi(y, x) \leq \varphi(y), \quad \text{for all } y \in C \quad \Rightarrow \quad 0 \leq \Phi(x, y) + \varphi(y), \quad \text{for all } y \in C.$$

Remark 2.6. The notion of maximal monotonicity of bifunctions was introduced by Blum and Oettli [2] as an attempt to extend to bifunctions the notion of maximal monotonicity of operators. Another notion of maximal monotone bifunction has been considered by Hadjisavvas and Khatibzadeh [13]. For a comparison between these two notions and some related properties, we refer to [13] and the references therein. For comments, examples and properties of maximal monotone bifunctions, we refer to [7, 8]. ☐

In the sequel, we shall need the following characterization of maximal monotone bifunctions, see [8, Lemma 2.4].

Proposition 2.7. *Let C be a nonempty, closed and convex subset of X and let $\Phi : C \times C \to \mathbb{R}$ be a real-valued bifunction such that $\Phi(x, x) \geq 0$ for all $x \in C$. Suppose that Φ is upper hemicontinuous and convex with respect to the second argument. Then it is maximal monotone.*

We recall the following recent result on the existence of solutions of the generalized Ky Fan minimax inequality (4), which will paly a significant role in the proof of the main result of this paper.

Theorem 2.8. [8, Theorem 2.3] *Let C be a nonempty, closed and convex subset of a reflexive Banach space X, and $\Phi, \Psi : C \times C \to \mathbb{R}$ be two real-valued bifunctions such that $\Phi(x, x) = \Psi(x, x) = 0$ for all $x \in C$. Let $\mathcal{J} : X \longrightarrow X^*$ be the duality mapping. Suppose that*

(i) Φ *is monotone and maximal monotone;*

(ii) Ψ *is* ***B****-pseudomonotone;*

(iii) Φ *is weakly lower semicontinuous with respect to the second argument;*

(iv) *For each* $N \in \mathcal{F}(C)$, *the function* $x \in C \mapsto \Psi(x,y)$ *is upper semicontinuous on* $\operatorname{conv}(N)$ *for each* y *in* C*;*

(v) Φ *and* Ψ *are convex with respect to the second argument;*

(vi) *(Coercivity) there exist a nonempty weakly compact subset* W *such that for each* $\varepsilon > 0$ *(small enough) there exists a weakly compact convex subset* B_ε *of* C *such that, for each* $x \in C \setminus W$ *there exists* $y \in B_\varepsilon$ *satisfying the inequality* $\Psi(x,y) + \varepsilon\langle \mathcal{J}(x), y - x\rangle < \Phi(y,x)$.

Then, there exists $x \in C$ *such that* $\Phi(x,y) + \Psi(x,y) \geq 0$ *for all* $y \in C$*. Furthermore, the solution set is weakly compact.*

Remark 2.9. (a) If C is assumed to be bounded in X, then the coercivity assumption (vi) in Theorem 2.8 can be dropped.

(b) The coercivity condition (vi) in Theorem 2.8 is satisfied if we suppose that there exists $y_0 \in C$ such that

$$\frac{\Psi(x,y_0) + \varepsilon\langle \mathcal{J}(x), y_0 - x\rangle_X}{\|x - y_0\|_X} \to -\infty \text{ uniformly in } \varepsilon > 0 \text{ if } \|x - y_0\|_X \to +\infty. \tag{6}$$

For the proof, see [8, Remark 2.5 (c)]. □

For the convenience of the reader, we recall here some basic tools from nonsmooth analysis.

Definition 2.10. [10] Let $\psi : X \to \mathbb{R}$ be a locally Lipschitz function, i. e. for every $u \in X$ there exists a constant $l_u > 0$ and a neighbourhood $\mathcal{O}$ of u such that $|\psi(v) - \psi(w)| \leq l_u \|v - w\|_X$ for all $v, w \in \mathcal{O}$. The *Clarke's generalized directional derivative* of ψ at the point u in the direction v is defined by

$$\psi^0(u;v) := \limsup_{w \to u,\ t \downarrow 0} \frac{\psi(w + tv) - \psi(w)}{t}.$$

The *generalized Clarke gradient* $\partial\psi : X \to 2^{X^*}$ of $\psi : X \to \mathbb{R}$ at $u \in X$ is defined by

$$\partial\psi(u) := \{u^* \in X^* : \psi^0(u;v) \geq \langle u^*, v\rangle,\ \forall v \in X\}.$$ □

The next two propositions provide important properties of the Clarke's generalized directional derivative and the generalized gradient.

Proposition 2.11. [10, Proposition 2.1.1] *Let* $\psi : X \to \mathbb{R}$ *be a locally Lipschitz functional of constant* l_u *near the point* $u \in X$*. Then,*

(i) *The function* $v \in X \mapsto \psi^0(u;v)$ *is finite, positively homogeneous, sub-additive and satisfies* $|\psi^0(u;v)| \leq l_u\, \|v\|_X$*;*

(ii) $\psi^0(u;v)$ *is upper semicontinuous as a function of* (u,v)*.*

Proposition 2.12. [10, Proposition 2.1.2] *Let* $\psi : X \to \mathbb{R}$ *be a locally Lipschitz functional of constant* l_u *near the point* $u \in X$*. Then,*

(i) $\partial\psi(u)$ *is a convex, weak* compact subset of* X^* *and*

$$\|u^*\|_{X^*} \leq l_u \quad \text{for all } u^* \in \partial\psi(u);$$

(ii) *For each $v \in X$, one has*

$$\psi^0(u; v) = \max\{\langle u^*, v\rangle : u^* \in \partial\psi(u)\}.$$

We recall the Kluge's fixed point theorem [18] that will be needed in the sequel.

Theorem 2.13. *Let $\mathcal{C}$ be a nonempty, closed and convex subset of a reflexive Banach space Z and $\mathbb{T} : \mathcal{C} \to 2^{\mathcal{C}}$ be a set-valued mapping. Suppose that for every $w \in \mathcal{C}$, the set $\mathbb{T}(w)$ is nonempty, closed and convex, and that the graph of $\mathbb{T}$ is sequentially weakly closed. If $\mathbb{T}(\mathcal{C})$ is bounded, then the set-valued mapping $\mathbb{T}$ has at least one fixed point in $\mathcal{C}$.*

We recall the following continuity property introduced by Mosco [26].

Definition 2.14. Let K and D be nonempty closed and convex subsets of Banach spaces X and Y, respectively. A set-valued mapping $F : K \to 2^D$ is said to be *M-continuous* if the following conditions hold:

(i) For $v_n \in F(u_n)$ with $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$, we have $v \in F(u)$.

(ii) For any sequence $\{u_n\}_{n\in\mathbb{N}} \subset K$ with $u_n \rightharpoonup u$, and for each $v \in F(u)$, there exists $\{v_n\}_{n\in\mathbb{N}}$ such that $v_n \in F(u_n)$ and $v_n \to v$. □

We end this section by recalling some basic facts from the theory of nonlocal operators that will be used throughout the paper.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $s \in (0, 1)$ and 2^*_s the fractional critical exponent defined by

$$2^*_s := \begin{cases} \frac{2N}{N-2s}, & \text{if } 2s < N, \\ +\infty, & \text{if } 2s \geq N. \end{cases}$$

Let $\mathcal{B}$ be the subset of $\mathbb{R}^{2N}$ defined by $\mathcal{B} := (\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)$ and let us set $\mathcal{Q} = \mathbb{R}^{2N} \setminus \mathcal{B}$. In the rest of the paper, we will always suppose that the function $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ satisfies the following assumption:

[$\mathbf{A}_K$] (i) $\mu(\cdot)K(\cdot) \in L^1(\mathbb{R}^N)$, where $\mu(x) = \min\{1, |x|^2\}$;

(ii) $\exists c > 0$ such that $K(x) \geq c|x|^{-(N+2s)}$, $\forall x \in \mathbb{R}^N \setminus \{0\}$;

(iii) $K(x) = K(-x)$, $\forall x \in \mathbb{R}^N \setminus \{0\}$.

We consider the function space

$$W := \{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} \mid u|_\Omega \in L^2(\Omega) \text{ and } (u(x) - u(y))^2 K(x - y) \in L^2(\mathcal{B})\},$$

where $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ satisfies assumption [A_K] and $u|_\Omega$ means the restriction of u to Ω. The space W is endowed with the norm

$$\|u\|_W := \|u\|_{L^2(\Omega)} + \left(\int_{\mathcal{B}} (u(x) - u(y))^2 K(x - y)\,dy\,dx\right)^{\frac{1}{2}} \text{ for all } u \in W,$$

see e.g. [1, 34] for more details. We shall work on the following subspace W_0 of W which highlights the generalized Dirichlet boundary condition in problem $(\mathcal{P})$

$$W_0 := \{u \in W : u = 0 \quad \text{a.e. in } \Omega^c\}.$$

Below, we present useful properties of the space W_0.

Proposition 2.15. [1, 36] *Let* $s \in (0,1)$, $N > 2s$ *and* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with Lipschitz boundary. Then,*

(i) W_0 *is a Hilbert space endowed with the inner product*

$$\langle u, v\rangle_{W_0} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} [u(x)-u(y)][v(x)-v(y)]K(x-y)\, dxdy \quad \text{for all } u, v \in W_0;$$

(ii) *For* $p \in [1, 2_s^*]$, *there exists a constant* $c_p > 0$ *such that*

$$\|u\|_{L^p(\mathbb{R}^N)} \le c_p\, \|u\|_{W_0} \quad \text{for all } u \in W_0;$$

(iii) W_0 *is compactly embedded into* $L^p(\mathbb{R}^N)$ *for* $p \in [1, 2_s^*)$.

From Proposition 2.15(ii), we deduce that the norm $\|\cdot\|_{W_0}$ on W_0 associated to the inner product $\langle\cdot,\cdot\rangle_{W_0}$ and defined by

$$\|u\|_{W_0} := \left(\int_{\mathcal{B}} [u(x) - u(y)]^2 K(x-y)\, dxdy\right)^{\frac{1}{2}}$$

is equivalent to the norm induced by $\|\cdot\|_W$. In addition, it is important to mention that W_0 is dense in $L^2(\Omega)$ since $C_0^\infty(\Omega)$ is dense in W_0, see e.g. [36].

3. Existence of solutions

In this section, we study the existence of solutions of problem $(\mathcal{P})$ by using an auxiliary problem which existence of solutions will be investigated by means of recent results on mixed equilibrium problems. We consider the following assumptions on the data of problem $(\mathcal{P})$.

$[\mathbf{A}_j]$ $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that

- (i) the function $j(\cdot, t)$ is measurable on Ω for each $t \in \mathbb{R}$ and there exists $z(\cdot) \in L^2(\Omega)$ such that $j(\cdot, z(\cdot)) \in L^1(\Omega)$;
- (ii) $j(x, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $x \in \Omega$;
- (iii) There exist $\beta(\cdot) \in L^2(\Omega)$, with $\beta(x) \ge 0$, and a constant $\alpha > 0$ such that

$$|\xi| \le \beta(x) + \alpha|s| \text{ for all } \xi \in \partial j(x,s) \text{ and a.e. } x \in \Omega.$$

$[\mathbf{A}_f]$ $f \in L^2(\Omega)$.

$[\mathbf{A}_U]$ $U : L^2(\Omega) \to (0, +\infty)$ is a continuous mapping.

Further, define the function $J : L^2(\Omega) \to \mathbb{R}$ by

$$J(u) = \int_\Omega j(x, u(x))dx \text{ for all } u \in L^2(\Omega). \tag{7}$$

We have the following result, see e.g. [25, Theorem 3.47].

Proposition 3.1. *Suppose that assumption* $[\mathbf{A}_j]$ *holds. Then the functional* J *defined in* (7) *satisfies the following properties*

(i) J *is locally Lipschitz and there exists a constant* $c_J > 0$ *such that*

$$\begin{cases} J^0(u; v) \le c_J\, (1 + \|u\|_{L^2(\Omega)})\|v\|_{L^2(\Omega)}, & \text{for all } u, v \in L^2(\Omega); \\ \|\zeta\|_{L^2(\Omega)} \le c_J\, (1 + \|u\|_{L^2(\Omega)}), & \text{for all } \zeta \in \partial J(u). \end{cases} \tag{8}$$

(ii) *For all* $u, v \in L^2(\Omega)$ *we have*

$$J^0(u;v) \le \int_\Omega j^0(x, u(x); v(x))\ dx;$$

In the sequel, we define the set-valued mapping $C : W_0 \to 2^{W_0}$ by

$$C(u) = \{v \in W_0 : \|v\|_{W_0} \le U(u)\}. \tag{9}$$

Remark that the set $C(u)$ is exactly the one considered in problem $(\mathcal{P})$ and it is a nonempty, closed, convex and bounded subset of W_0, moreover we have $\mathbf{0}_{W_0} \in C(u)$ for each $u \in W_0$.

Proposition 3.2. *If assumption* $[\mathbf{A}_U]$ *is satisfied, then the set-valued mapping* $C : W_0 \to 2^{W_0}$ *defined by* (9) *is M-continuous.*

Proof. Let $v_n \in C(u_n)$ such that $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$. Let us verify that $v \in C(u)$. Since W_0 is compactly embedded in $L^2(\Omega)$, it follows that $u_n \to u$ in $L^2(\Omega)$. Hence, from $[\mathbf{A}_U]$ we deduce that $U(u_n) \to U(u)$. On the other hand, as $v_n \in C(u_n)$, we get $\|v_n\|_{W_0} \le U(u_n)$. Therefore, $\|v\|_{W_0} \le \liminf \|v_n\|_{W_0} \le \lim U(u_n) = U(u)$ which implies that $v \in C(u)$, and hence condition (i) of Definition 2.14 is satisfied. Now, let $\{u_n\}_{n\in\mathbb{N}} \subset W_0$ such that $u_n \rightharpoonup u$ and let $v \in C(u)$. As W_0 is compactly embedded in $L^2(\Omega)$, we have $u_n \to u$ in $L^2(\Omega)$. Set $v_n := \frac{U(u_n)}{U(u)}v$. We have $v_n \in C(u_n)$ and $\|v_n - v\| = |\frac{U(u_n)-U(u)}{U(u)}|\|v\|$. It follows from $[\mathbf{A}_U]$ that $\lim \|v_n - v\| = 0$ and hence condition (ii) of Definition 2.14 holds. □

Now, let $u \in W_0$ such that the relation given by problem $(\mathcal{P})$ holds. Multiply both terms of the relation $(\mathcal{L}_K u)(x) + \partial j(x, u(x)) + \xi(x) \ni f(x)$ by $v(x) - u(x)$ and integrate over Ω, where $v \in C(u)$ be an arbitrary element, we obtain

$$\int_\Omega (\mathcal{L}_K u)(x)[v(x) - u(x)]dx + \int_\Omega \eta(x)[v(x) - u(x)]dx + \int_\Omega \xi(x)[v(x) - u(x)]dx$$
$$= \int_\Omega f(x)[v(x) - u(x)]dx \tag{10}$$

where $\xi \in N_{C(u)}(u)$ and $\eta : \Omega \to \mathbb{R}$ is a function such that $\eta(x) \in \partial j(x, u(x))$ for a.e. $x \in \Omega$. From the definition of W_0, we get

$$\int_\Omega (\mathcal{L}_K u)(x)[v(x) - u(x)]dx = \int_{\mathbb{R}^N} (\mathcal{L}_K u)(x)[v(x) - u(x)]dx. \tag{11}$$

On the other hand, Proposition 2.12 (ii) implies that

$$\int_\Omega \eta(x)[v(x) - u(x)]dx \le \int_\Omega j^0(x, u(x); v(x) - u(x))dx, \tag{12}$$

and as $\xi \in N_{C(u)}(u)$ we obtain

$$\int_\Omega \xi(x)[v(x) - u(x)]dx \le 0. \tag{13}$$

Therefore, from relations (10)–(11) problem $(\mathcal{P})$ becomes

Problem 3.3. *Find $u \in C(u)$ such that*

$$\begin{aligned}\int_{\mathbb{R}}(\mathcal{L}_K u)(x)[v(x)-u(x)]dx + \int_{\Omega} j^0(x,u(x);v(x)-u(x))dx \\ \geq \int_{\Omega} f(x)[v(x)-u(x)]dx \qquad \textit{for all } v \in C(u)\end{aligned} \tag{14}$$

Assuming that the hypothesis $[\mathbf{A}_j]$ holds, then from Proposition 3.1 (ii) we get

$$\int_{\Omega} j^0(x,u(x);v(x)-u(x))\ dx \geq J^0(u;v-u),$$

where $J(\cdot)$ is the functional given in (7). Therefore, the solvability of problem (3.3) is obtained by the one of the following auxiliary problem

Problem 3.4. *Find $u \in C(u)$ such that, for all $v \in C(u)$,*

$$\int_{\mathbb{R}}(\mathcal{L}_K u)(x)[v(x)-u(x)]dx + J^0(u;v-u) \geq \int_{\Omega} f(x)[v(x)-u(x)]dx \tag{15}$$

Let us define the linear operator $A : W_0 \to W_0^*$ by

$$\langle A(u), v\rangle_{W_0} := \int_{\mathbb{R}^N}(\mathcal{L}_K u)(x)v(x)dx \qquad \text{for } u, v \in W_0,$$

then the auxiliary problem (3.4) can be written as the following:
Find $u \in C(u)$ such that, for all $v \in C(u)$,

$$\langle A(u), v-u\rangle_{W_0} + J^0(u;v-u) \geq \langle f, v-u\rangle_{L^2(\Omega)}. \tag{16}$$

The operator A satisfies the following properties.

Proposition 3.5. *If $[\mathbf{A}_K]$ holds, then the linear operator $A : W_0 \to W_0^*$ is monotone and continuous.*

Proof. Let $u, v \in W_0$, then from the definition of $\mathcal{L}_K u$ we derive that

$$\langle A(u), v\rangle_{W_0} = -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[u(x+y)+u(x-y)-2u(x)]v(x)K(y)\ dydx.$$

Hence,

$$\begin{aligned}\langle A(u), v\rangle_{W_0} = &-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[u(x+y)-u(x)]v(x)K(y)dydx \\ &-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[u(x-y)-u(x)]v(x)K(y)\ dydx.\end{aligned}$$

By using $[\mathbf{A}_K]$(iii) and a change of variable, we obtain

$$\langle A(u), v\rangle_{W_0} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[u(x)-u(y)][v(x)-v(y)])K(x-y)\ dydx = \langle u, v\rangle_{W_0}$$

It follows that

$$\langle A(u), u\rangle_{W_0} = \|u\|_{W_0}^2 \quad \text{and} \quad \|A(u)\|_{W_0^*} = \|u\|_{W_0} \qquad \text{for all } u \in W_0. \tag{17}$$

Hence, the first equality in (17) implies that the operator A is (strongly) monotone, and the second one implies that A continuous since it is linear and bounded. $\square$

Now, we consider the bifunctions $\Phi, \Psi : W_0 \times W_0 \to \mathbb{R}$ defined for $u, v \in W_0$ by

$$\Phi(u,v) := \langle A(u), v-u\rangle_{W_0} + \langle f, u-v\rangle_{L^2(\Omega)} \textit{ and } \Psi(u,v) := J^0(u; v-u).$$

In order to study the existence of solutions of the problem (3.4), we consider the following parameterized family of mixed equilibrium problems:

Problem 3.6. *Given $z \in W_0$, find $u \in C(z)$ such that*

$$\Phi(u,v) + \Psi(u,v) \geq 0 \qquad \textit{for all } v \in C(z). \tag{18}$$

We shall denote by $\mathbb{S}(z)$ the solution set of the mixed equilibrium problem (3.6) and by $\mathbb{S} := \bigcup_{z\in W_0} \mathbb{S}(z)$.

Theorem 3.7. *Suppose that the assumptions $[\mathbf{A}_K]$ and $[\mathbf{A}_j]$ are satisfied. Then $\mathbb{S}(z) \neq \emptyset$ for any $z \in W_0$. Furthermore, if $1 - c_J c_2^2 > 0$, then*

$$\mathbb{S} := \bigcup_{z\in W_0} \mathbb{S}(z) \subset \bar{\mathbb{B}}_{W_0}(\boldsymbol{0}_{W_0}, R_0), \quad \textit{where} \quad R_0 = \frac{(\|f\|_{L^2(\Omega)} + c_J)c_2}{1 - c_J c_2^2}.$$

Proof. We shall apply Theorem 2.8 with $C = C(z)$ and $X = W_0$, where W_0 is endowed with the week topology. Note that $C(z)$ is a weakly compact subset of W_0 since it is a closed, convex and bounded subset of W_0. Therefore, by Remark 2.9(a) we only need to verify that conditions (i) to (v) of Theorem 2.8 are satisfied. First, it is easy to see that conditions (iii) and (v) of Theorem 2.8 hold. On the other hand, form Proposition 3.5 we have that the operator A is monotone and continuous, this implies that the bifunction Φ is monotone and (upper) hemicontinuous. Consequently, by using Proposition 2.7 we deduce that Φ is maximal monotone, and hence condition (i) of Theorem 2.8 is satisfied. Now, let us verify that the bifunction Ψ is **B**-pseudomonotone. To this aim, thanks to Remark 2.4 (b), it suffices to show that Ψ is Ky Fan hemicontinuous, that is $\Psi(\cdot, v)$ is weakly upper semicontinuous for each fixed v in $C(z)$. Let $\{u_n\}_{n\in\mathbb{N}} \subset C(z)$ such that $u_n \rightharpoonup u \in C(z)$ and let us verify that $\limsup_{n\to+\infty} \Psi(u_n, v) \leq \Psi(u, v)$ for all $v \in C(z)$. As $u_n \rightharpoonup u$ in W_0 and W_0 is compactly embedded in $L^2(\Omega)$ (see Proposition 2.15 (iii)), we deduce that $u_n \to u$ in $L^2(\Omega)$ for a subsequence. By using Proposition 2.11 (iii), we deduce that $\limsup_{n\to+\infty} J^0(u_n; v - u_n) \leq J^0(u; v - u)$ and hence condition (ii) of Theorem 2.8 holds. Condition (iv) of Theorem 2.8 follows from the Ky Fan hemicontinuity of Ψ. Therefore, all the conditions of Theorem 2.8 are satisfied. We conclude that $\mathbb{S}(z) \neq \emptyset$ for any $z \in W_0$.

Now, suppose that $1 - c_J c_2^2 > 0$. Let $u \in \mathcal{S}$, then there exists $z \in W_0$ such that $u \in \mathbb{S}(z)$. It follows that

$$\Phi(u,v) + \Psi(u,v) \geq 0 \qquad \text{for all } v \in C(z).$$

As $\mathbf{0}_{W_0} \in C(z)$, we get $\Phi(u, \mathbf{0}_{W_0}) + \Psi(u, \mathbf{0}_{W_0}) \geq 0$.

Hence we get $-\langle A(u), u\rangle_{W_0} + \langle f, u\rangle_{L^2(\Omega)} + J^0(u; -u) \geq 0$.

Accordingly, by using (8) and (17) we obtain

$$\|u\|_{W_0}^2 \leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + c_J(1 + \|u\|_{L^2(\Omega)})\|u\|_{L^2(\Omega)}.$$

Hence, from Proposition 2.15 (iii) we obtain the following estimate

$$\|u\|^2_{W_0} \leq \|f\|_{L^2(\Omega)} c_2 \|u\|_{W_0} + c_J(1 + c_2\|u\|_{W_0}) c_2 \|u\|_{W_0}.$$

This implies that $\|u\|_{W_0} \leq \dfrac{c_2[\|f\|_{L^2(\Omega)} + c_J]}{1 - c_2^2 c_J} = R_0$. Thus, $\mathbb{S} \subset \bar{\mathbb{B}}_{W_0}(\mathbf{0}_{W_0}, R_0)$.

The proof is complete. □

The following theorem establishes the existence of solutions of problem $(\mathcal{P})$.

Theorem 3.8. *Suppose that the assumptions* $[\mathbf{A}_K]$, $[\mathbf{A}_j]$ *and* $[\mathbf{A}_U]$ *are satisfied. If* $1 - c_J c_2^2 > 0$, *then problem* $(\mathcal{P})$ *has at least one solution.*

Proof. Following Theorem 3.7, we know that $\mathbb{S}(z) \neq \emptyset$ for any $z \in W_0$ and that $\mathbb{S} := \bigcup_{z \in W_0} \mathbb{S}(z) \subset \bar{\mathbb{B}}_{W_0}(\mathbf{0}_{W_0}, R_0)$, where $R_0 = \frac{(\|f\|_{L^2(\Omega)} + c_J)c_2}{1 - c_J c_2^2}$.

Let $\mathcal{C} := W_0 \times \bar{\mathbb{B}}_{L^2(\Omega)}(0, M_0)$, where $M_0 := \frac{c_J[1 + \|f\|_{L^2(\Omega)} c_2^2]}{1 - c_J c_2^2}$, and let us consider the set-valued mapping

$$\mathbb{T} : \mathcal{C} \to 2^{\mathcal{C}}, \quad (z, \varphi) \mapsto \mathbb{T}(z, \varphi) = (u, \mathbb{Q}(u))$$

where u is the (unique) solution of the following variational inequality problem:

Find $u \in C(z)$ such that for all $v \in C(z)$ we have

$$\langle A(u), v - u \rangle_{W_0} + \int_\Omega \varphi(x)[v(x) - u(x)]dx \geq \int_\Omega f(x)[v(x) - u(x)]dx, \tag{19}$$

and $\mathbb{Q} : W_0 \to 2^{L^2(\Omega)}$ is the set-valued mapping defined for some $R > 0$ by

$$\mathbb{Q}(u) = \begin{cases} \partial J(u), & \text{if } \|u\|_{W_0} \leq R \\ \partial J(\frac{R}{\|u\|_{W_0}} u), & \text{if } \|u\|_{W_0} > R. \end{cases} \tag{20}$$

The solvability of the variational inequality (19) is obtained by using Theorem 2.8 with $C := C(z)$, $X := W_0$ and an appropriate choice of the bifunctions Φ and Ψ. The uniqueness of the solution of the problem (19) is obtained by using relation (17).

Now, let z and φ be arbitrary elements respectively in W_0 and $\bar{\mathbb{B}}_{L^2(\Omega)}(0, M_0)$, and let u be the unique solution of the variational inequality (19) associated to z and φ. Since $\mathbf{0}_{W_0} \in C(z)$, we obtain from (19)

$$-\langle \varphi, u \rangle_{L^2(\Omega)} + \langle f, u \rangle_{L^2(\Omega)} \geq \langle A(u), u \rangle_{W_0}.$$

It follows $\|u\|^2_{W_0} \leq [\|f\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}]\|u\|_{L^2(\Omega)} \leq c_2[\|f\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}]\|u\|_{W_0}$,

which implies $\|u\|_{W_0} \leq c_2[\|f\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}] \leq c_2[\|f\|_{L^2(\Omega)} + M_0]$. (21)

Replacing M_0 by its value in the previous inequality, we obtain

$$\|u\|_{W_0} \leq R_0. \tag{22}$$

We choose R in (20) such that $R \geq R_0$. Hence, $\mathbb{Q}(u) = \partial J(u)$. Let $\zeta \in \mathbb{Q}(u)$, then from (8) and (22) we get

$$\|\zeta\|_{L^2(\Omega)} \leq c_J[1 + \|u\|_{L^2(\Omega)}] \leq c_J[1 + c_2\|u\|_{W_0}] \leq c_J[1 + c_2 R_0].$$

Remark that $c_J[1+c_2R_0] = M_0$, it follows that $\mathbb{Q}(u) \subset \bar{\mathbb{B}}_{L^2(\Omega)}(0, M_0)$. Consequently,

$$\mathbb{T}(\mathcal{C}) \subset \bar{\mathbb{B}}_{W_0}(\mathbf{0}_{W_0}, R_0) \times \bar{\mathbb{B}}_{L^2(\Omega)}(0, M_0), \tag{23}$$

and hence $\mathbb{T}(\mathcal{C})$ is bounded. Let us verify that the set-valued mapping $T : \mathcal{C} \to 2^{\mathcal{C}}$ has a fixed point. To this aim, we shall use the Kluge's fixed point theorem (Theorem 2.13). Note that $\mathcal{C}$ is a nonempty, closed and convex subset of $W_0 \times L^2(\Omega)$. Moreover, for each $w = (z, \varphi) \in \mathcal{C}$, $\mathbb{T}(w)$ is a nonempty, closed and convex subset of $W_0 \times L^2(\Omega)$ since $\mathbb{Q}(u) = \partial J(u)$ is a closed and convex subset of $L^2(\Omega)$ (see Proposition 2.12(i)).

Now, let show that the graph of $\mathbb{T}$ is weakly sequentially closed. Let $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ such that $w_n = (z_n, \varphi_n) \rightharpoonup (z, \varphi)$ and let $\theta_n \in \mathbb{T}(w_n)$ such that $\theta_n = (u_n, \zeta_n) \rightharpoonup (u, \zeta)$ where u_n is the unique solution of the variational inequality (19) associated to z_n and φ_n, and $\zeta_n \in \mathbb{Q}(u_n) = \partial J(u_n)$. Let us verify that u is the unique solution of the variational inequality (19) associated to z and φ, and $\zeta \in \mathbb{Q}(u) = \partial J(u)$. As $\theta_n = (u_n, \zeta_n) \in T(z_n, \varphi_n)$, then $u_n \in C(z_n)$ and

$$\langle A(u_n), v - u_n\rangle_{W_0} + \int_\Omega \varphi_n(x)[v(x) - u_n(x)]dx \geq \int_\Omega f(x)[v(x) - u_n(x)]dx \tag{24}$$

for all $v \in C(z_n)$. Since $u_n \rightharpoonup u$ in W_0 and W_0 is compactly embedded in $L^2(\Omega)$, we have that $u_n \rightharpoonup u$ in $L^2(\Omega)$. Let $v \in C(z)$, then from Proposition 3.2 and Definition 2.14(ii), there exists $v_n \in C(z_n)$ such that $v_n \to v$ in $L^2(\Omega)$. For $v = v_n$ in (24), we get

$$\langle A(u_n), v_n - u_n\rangle_{W_0} + \langle \varphi_n, v_n - u_n\rangle \geq \langle f, v_n - u_n\rangle. \tag{25}$$

By considering the limit in the previous inequality and taking into consideration of the continuity of A (see Proposition 3.5), we obtain

$$\langle A(u), v - u\rangle_{W_0} + \langle \varphi, v - u\rangle \geq \langle f, v - u\rangle,$$

which implies that u is a solution of the variational inequality problem (19) associated to z and φ. On the other hand, from $\zeta_n \in \mathbb{Q}(u_n) = \partial J(u_n)$ with $\zeta_n \rightharpoonup \zeta$ and $u_n \rightharpoonup u$, we obtain that $\zeta \in \partial J(u)$. Hence $(u, \zeta) \in \mathbb{T}(z, \varphi)$ and therefore the graph of $\mathbb{T}$ is weakly sequentially closed. Accordingly, by the Kluge's fixed point theorem (Theorem 2.13), there exists $(u, \varphi) \in \mathcal{C}$ such that $(u, \varphi) \in \mathbb{T}(u, \varphi)$. This implies that u is a solution of problem $(\mathcal{P})$, which completes the proof of the theorem. □

4. Conclusion

In recent work by Motreanu et al. [28], the solvability of problem $(\mathcal{P})$ has been obtained by using a specific condition on the locally Lipschitz function $j(x, \cdot)$, known in literature by the *relaxed monotonicity* condition for $j(x, \cdot)$, see e.g. [25]. The relaxed monotonicity condition reads as follows: There exists a constant $m_j \geq 0$ such that for all $s_1, s_2 \in \mathbb{R}$ and a.e. $x \in \Omega$, we have

$$(s_1^* - s_2^*)(s_1 - s_2) \geq -m_j|s_1 - s_2|^2, \ \forall s_1^* \in \partial j(x, s_1), \forall s_2^* \in \partial j(x, s_2).$$

This condition is equivalent to the following inequality:

$$\begin{aligned}&\text{for all } s_1, s_2 \in \mathbb{R} \text{ and a.e. } x \in \Omega\\ &j^0(x, s_1; s_2 - s_1) + j^0(x, s_2; s_1 - s_2) \le m_j|s_1 - s_2|^2 \text{ for a.e. } x \in \Omega.\end{aligned}$$

This condition has been considered in [28] in order to guaranty a monotonicity property used to show, by means of a Minty's technique, the convexity of the solution set of a variational selection of problem $(\mathcal{P})$.

The approach developed in our paper helped us to avoid this condition as well as to present an alternative to some ambiguities and inaccuracies in the proof of [28, Theorem 6.1]. Our method can also be used to drop the relaxed monotonicity condition used for the solvability of systems driven by nonlocal operators and involving a Clarke's generalized gradient in [19, 20].

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