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Inheritance Properties on Cone Continuity for Set-Valued Maps via Scalarization

Premyuda Dechboon
Graduate School of Science and Technology, Niigata University, Niigata, Japan
dechboon@m.sc.niigata-u.ac.jp

Tamaki Tanaka*
Faculty of Science, Niigata University, Niigata, Japan
tamaki@math.sc.niigata-u.ac.jp

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This paper unravels the mechanism by which composite functions of a set-valued map and a scalarization function inherit semicontinuity of parent set-valued maps through several scalarization for sets. Consequently, almost all results given by Sonda, Kuwano and Tanaka in 2010 are verified in general settings where we generalize both concepts of cone-continuity for set-valued maps and ordinary lower and upper semicontinuities for real-valued functions.

Keywords: Set optimization, set relation, set-valued map, scalarization function, semicontinuity, cone continuity, $\preccurlyeq$-semicontinuity

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1. Introduction

A composite function is a function which is the nesting of two or more functions to form a single new function. Such operation frequently preserves several mathematical properties of each nested function. For instance, the composition of increasing functions is again increasing, and the same goes for a decreasing case. The composition of one-to-one functions is always one-to-one. Similarly, the composition of two bijections is also a bijection. Moreover, a composition of continuous maps is continuous on topological spaces. From the view point of vector optimization and set optimization, this kind of inheritance by composite operations is important and useful to prove extended results and to get characterizations of optimal solutions through scalarization. This is a typical approach by which optimization problems with vector-valued or set-valued maps can be easily handled by converting vectors or sets into real numbers; see [2, 3, 4].

Recently, Ike, Liu, Ogata and Tanaka [5] show certain results on the inheritance property of some kinds of continuity of set-valued maps via scalarization functions for sets: if a set-valued map has a kind of continuity (lower continuity or upper

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* Corresponding author.

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continuity; see [3]) then the composition of its set-valued map and a certain scalarization function assures a similar semicontinuity to its scalarization function defined on the family of nonempty subsets of a real topological vector space. Their results are generalizations of results in earlier study by Kuwano, Tanaka and Yamada [11]. However, the statements of inheritance in [5] are confined to four types out of the six set-relations proposed by Kuroiwa, Tanaka and Ha [8]. On the other hand, Sonda, Kuwano, and Tanaka [14] introduce two kinds of continuity with respect to cone, called "cone continuity," for set-valued maps by analogy with semicontinuity for real-valued functions, and they investigate the inheritance properties on cone continuity of parent set-valued maps via scalarization. Therefore, it is interesting to investigate the inheritance of cone continuity for set-valued maps via general scalarization functions for sets in the same manner as [5].

The aim of this paper is to unravel the mechanism by which composite functions of a set-valued map and a scalarization function transmit semicontinuity of parent set-valued maps through several scalarization for sets. Additionally, the paper provides counter examples of the complement part that is the half-missing part of the results of [5]. Moreover, the paper verifies the results of [14] in general settings where we generalize both concepts of cone continuity for set-valued maps and ordinary lower and upper semicontinuities for real-valued functions. Consequently, almost all results in [5, 14] are special cases of corresponding results in the paper.

The organization of the paper is as follows. In Section 2, we generalize the classical notions on semicontinuity of both set-valued maps and scalarization functions by means of representing topological structure in terms of binary relation. Also, we recall some basic concepts on set relations in set optimization and the scalarization scheme for sets in a real vector space such that each scalarization function has order-monotone property for set relation. In Section 3, we provide general results related to generalized continuity, which show inheritance properties on cone continuity of parent set-valued maps via scalarization. In Section 4, we systematically unravel the inheritance mechanism related to lower continuity and upper continuity, respectively. In Section 5, some examples are given to illustrate the reasons why some of the half-missing part of the results cannot be proved.

2. Semicontinuity for set-valued maps and scalarization functions

Let X be a topological space and Y a real topological vector space. Let θ_Y be a zero vector in Y. Denote that $\mathcal{P}(Y)$ is the set of all nonempty subsets of Y. $\mathcal{N}_X(x)$ and $\mathcal{N}_Y(y)$ denote the set of neighborhoods of $x \in X$ and $y \in Y$, respectively. The *topological interior*, *topological closure*, and *complement* of a set $A \in \mathcal{P}(Y)$ are denoted by $\operatorname{int}(A)$, $\operatorname{cl} A$, and A^c, respectively. For given $A, B \in \mathcal{P}(Y)$ and $t \in \mathbb{R}$, the *algebraic sum* $A + B$ and the *scalar multiplication* tA are defined as follows:

$$A + B := \{a + b \mid a \in A,\ b \in B\}, \quad tA := \{ta \mid a \in A\}.$$

In particular, we denote $A+\{y\}$ by $A+y$ and $(-1)A$ by $-A$ for $A\in\mathcal{P}(Y)$ and $y\in Y$.

Throughout the paper, we assume that C is a convex cone in Y with $\operatorname{int}(C) \neq \emptyset$ and $\theta_Y \in C$. Accordingly, we can define a preorder $\leq_C$ on Y induced by C as follows:

$$\text{for } y_1, y_2 \in Y, y_1 \leq_C y_2 \overset{\text{def}}{\iff} y_2 - y_1 \in C.$$

This preorder is compatible with the linear structure of Y:

$$\text{for all } y_1, y_2, y_3 \in Y, \quad y_1 \leq_C y_2 \implies y_1 + y_3 \leq_C y_2 + y_3; \tag{1}$$

$$\text{for all } y_1, y_2 \in Y \text{ and } \ t > 0, \quad y_1 \leq_C y_2 \implies ty_1 \leq_C ty_2. \tag{2}$$

Definition 2.1. $A \in \mathcal{P}(Y)$ is said to be *absorbing* if for every $y \in Y$ there exists $\delta > 0$ such that $ty \in A$ for all $t \in [0, \delta]$. Also $B \in \mathcal{P}(Y)$ is said to be *bounded* if for every $U \in \mathcal{N}_Y(\theta_Y)$ there exists $s > 0$ such that $B \subset tU$ for every $t > s$.

Remark 2.2. Every $V \in \mathcal{N}_Y(\theta_Y)$ is absorbing, and every compact subset K of Y is bounded.

Lemma 2.3. [13] *Let X be a topological vector space. For each A, $B \subset X$, if A is compact and B is closed, then $A + B$ is closed.*

Lemma 2.4. [7] *Let Y be a real topological vector space. Assume that $C \subset Y$ is a convex cone with* $\operatorname{int}(C) \neq \emptyset$ *and let $A \subset Y$. Then* $A + \operatorname{int}(C) = \operatorname{int}(A + C)$.

When we derive ordinary lower and upper continuities of set-valued maps, certain conversions by binary relations on a family of sets are important keystones.

Definition 2.5. For $A, B \in \mathcal{P}(Y)$, we define two binary relations on $\mathcal{P}(Y)$:

$$A \preccurlyeq_1 B \overset{\text{def}}{\iff} A \cap B \neq \emptyset \quad \text{and} \quad A \preccurlyeq_2 B \overset{\text{def}}{\iff} B \subset A.$$

In the beginning, we consider generalizations of semicontinuity for set-valued maps and real-valued functions.

Definition 2.6. Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$ and $C \subset Y$ a convex cone. We say that F is $(\preccurlyeq, C)$-*continuous* at x_0 if

$$\forall W \subset Y, W \text{ open }, W \preccurlyeq F(x_0), \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } W + C \preccurlyeq F(x), \forall x \in V.$$

As special cases, $(\preccurlyeq_1, C)$-continuity and $(\preccurlyeq_2, C)$-continuity coincide with "C-lower continuity" and "C-upper continuity" for set-valued maps, respectively. Indeed, $F : X \to \mathcal{P}(Y)$ is $(\preccurlyeq_1, C)$-continuous at x_0 if and only if

$$\forall W \subset Y,\ W \text{ open},\ W \cap F(x_0) \neq \emptyset,\ \exists V \in \mathcal{N}_X(x_0) \ \text{ s.t. } (W + C) \cap F(x) \neq \emptyset,\ \forall x \in V,$$

that is, F is C-lower continuous at x_0. Similarly, F is $(\preccurlyeq_2, C)$-continuous at x_0 if and only if

$$\forall W \subset Y, W \text{ open }, F(x_0) \subset W, \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } F(x) \subset W + C, \forall x \in V,$$

that is, F is C-upper continuous at x_0; see Definition 2.5.16 of [3].

Remark 2.7. If $C = \{0\}$ then $(\preccurlyeq, C)$-continuity for set-valued maps becomes $\preccurlyeq$-continuity which is defined in Definition 3.2 of [5]. Moreover, $\preccurlyeq_1$-continuity and $\preccurlyeq_2$-continuity coincide with the classical notions of lower continuity and upper continuity for set-valued maps, respectively. See Definition 2.5.1 of [3]; a set-valued map $F : X \to \mathcal{P}(Y)$ is "lower continuous (l.c.) at x_0" if

$$\forall W \subset Y,\ W \text{ open},\ W \cap F(x_0) \neq \emptyset, \exists V \in \mathcal{N}_X(x_0) \ \text{ s.t. } W \cap F(x) \neq \emptyset,\ \forall x \in V,$$

and "upper continuous (u.c.) at x_0" if

$$\forall W \subset Y,\ W \text{ open},\ F(x_0) \subset W,\ \exists V \in \mathcal{N}_X(x_0) \text{ s.t. } F(x) \subset W,\ \forall x \in V.$$

Remark 2.8. In Definition 2.5.16 of [3], the definition of "C-lower continuity" is different from ours. Moreover, the authors use "$F(x) \cap (W - C) \neq \emptyset$" instead of "$F(x) \cap (W + C) \neq \emptyset$." It is, however, well known that lower continuity and upper continuity for singleton set-valued maps (that is, vector-valued functions) coincide with ordinary continuity for single-valued functions. Hence, in case of a singleton set-valued map $F(x) = \{f(x)\}$ where f is some vector-valued function, the two notions of "C-lower continuity" and "C-upper continuity" should coincide because $\{f(x)\} \cap (W + C) \neq \emptyset$ means $f(x) \in W + C$, that is, $\{f(x)\} \subset (W + C)$; this is the notion of "C-continuity" for vector-valued functions introduced by Luc [12, Definition 5.1]. Whenever $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, the notion of C-continuity for vector-valued functions is the same as ordinary lower semicontinuity. As stated in Remark 2.1 of [14], by symbolic interpretation, "C" and "$-C$" correspond to "lower" and "upper" semicontinuities of real-valued functions, respectively. □

Next, we recall the semicontinuity property for real-valued functions and extend it to an extended real-valued function as well as more general case (an extended real-valued set function) where the function is defined on the set of all nonempty subsets of a real topological vector space.

Definition 2.9. Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ and $x_0 \in X$. Then, we say that f is

(i) *lower semicontinuous* at x_0 if
$\forall r < f(x_0), \exists V \in \mathcal{N}_X(x_0)$ s.t. $r < f(x), \forall x \in V$;

(ii) *upper semicontinuous* at x_0 if
$\forall r > f(x_0), \exists V \in \mathcal{N}_X(x_0)$ s.t. $r > f(x), \forall x \in V$.

For an extended real-valued function $f : X \to \mathbb{R} \cup \{\pm\infty\}$, we usually define "lower semicontinuity" (also know as "closedness") for f by the closedness of its epigraph

$$\operatorname{epi} f := \{(x, \alpha) \in X \times \mathbb{R} \,|\, f(x) \leq \alpha\};$$

if $-f$ is lower semicontinuous (closed), we say that f is upper semicontinuous. It is easily seen that f is lower (resp. upper) semicontinuous at every $x \in X$ in the sense of Definition 2.9 if and only if $\operatorname{epi} f$ (resp. $\operatorname{epi}(-f)$) is a closed set in $X \times \mathbb{R}$.

Definition 2.10. Let $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $A_0 \in \mathcal{P}(Y)$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and C a convex cone in Y with $C \neq Y$. Then, we say that φ is

(i) $(\preccurlyeq, C)$-*lower semicontinuous* at A_0 if $\forall r < \varphi(A_0), \exists W \in \mathcal{P}(Y)$, W open, s.t. $W \preccurlyeq A_0$ and $r < \varphi(A), \forall A \in U(W + C, \preccurlyeq)$;

(ii) $(\preccurlyeq, C)$-*upper semicontinuous* at A_0 if $\forall r > \varphi(A_0), \exists W \in \mathcal{P}(Y)$, W open, s.t. $W \preccurlyeq A_0$ and $r > \varphi(A), \forall A \in U(W + C, \preccurlyeq)$,

where $U(V, \preccurlyeq) := \{A \in \mathcal{P}(Y) \,|\, V \preccurlyeq A\}$.

Remark 2.11. When $C = \{0\}$, $(\preccurlyeq, C)$-lower and $(\preccurlyeq, C)$-upper semicontinuities are coincident with $\preccurlyeq$-lower and $\preccurlyeq$-upper semicontinuities, respectively, which are introduced in Definition 3.3 of [5]. In Definition 2.10, we adopt that if $\varphi(A_0) = -\infty$ (resp. $+\infty$) then φ is $(\preccurlyeq, C)$-lower (resp. upper) semicontinuous at A_0.

Now, we recall the concepts of set relations [8] in set optimization and the scalarization scheme [9] for sets related to the set relations.

Definition 2.12. (Set relations, [8]) For $A, B \in \mathcal{P}(Y)$, we define the following six types of binary relations on $\mathcal{P}(Y)$.

(i) $A \leq_C^{(1)} B \overset{\text{def}}{\iff} \forall a \in A,\ \forall b \in B,\ a \leq_C b \iff A \subset \bigcap_{b \in B}(b - C)$;

(ii) $A \leq_C^{(2L)} B \overset{\text{def}}{\iff} \exists a \in A \text{ s.t. } \forall b \in B,\ a \leq_C b \iff A \cap \left(\bigcap_{b \in B}(b - C)\right) \neq \emptyset$;

(iii) $A \leq_C^{(3L)} B \overset{\text{def}}{\iff} \forall b \in B,\ \exists a \in A \text{ s.t. } a \leq_C b \iff B \subset A + C$;

(iv) $A \leq_C^{(2U)} B \overset{\text{def}}{\iff} \exists b \in B \text{ s.t. } \forall a \in A,\ a \leq_C b \iff \left(\bigcap_{a \in A}(a + C)\right) \cap B \neq \emptyset$;

(v) $A \leq_C^{(3U)} B \overset{\text{def}}{\iff} \forall a \in A,\ \exists b \in B \text{ s.t. } a \leq_C b \iff A \subset B - C$;

(vi) $A \leq_C^{(4)} B \overset{\text{def}}{\iff} \exists a \in A,\ \exists b \in B \text{ s.t. } a \leq_C b \iff A \cap (B - C) \neq \emptyset$.

From the definition, we easily obtain the following implications:

$$\begin{cases} A \leq_C^{(1)} B \implies A \leq_C^{(2L)} B \implies A \leq_C^{(3L)} B \implies A \leq_C^{(4)} B; \\ A \leq_C^{(1)} B \implies A \leq_C^{(2U)} B \implies A \leq_C^{(3U)} B \implies A \leq_C^{(4)} B \end{cases} \tag{3}$$

for $A, B \in \mathcal{P}(Y)$.

Proposition 2.13. [5] *Let C' and C be two nonempty convex cones in Y and $d \in Y$. Assume that $C' + (0, +\infty)d \subset C$. Then, for each $j = 1, 2L, 3L, 2U, 3U, 4$, any $A, B \in \mathcal{P}(Y)$, $s, s' \in \mathbb{R}$ with $s' < s$ and $t, t' \in \mathbb{R}$ with $t < t'$,*

$$A \leq_{C'}^{(j)} B + s'd \implies A \leq_C^{(j)} B + sd,$$

and

$$A + t'd \leq_{C'}^{(j)} B \implies A + td \leq_C^{(j)} B.$$

Definition 2.14. [6, 9] For each $j = 1, 2L, 3L, 2U, 3U, 4$, we define

$$I_C^{(j)}(A; V, d) := \inf \left\{ t \in \mathbb{R} \,\middle|\, A \leq_C^{(j)} (V + td) \right\}, \tag{4}$$

$$S_C^{(j)}(A; V, d) := \sup \left\{ t \in \mathbb{R} \,\middle|\, (V + td) \leq_C^{(j)} A \right\}, \tag{5}$$

for any $A, V \in \mathcal{P}(Y)$ and $d \in Y$.

The idea of these scalarization functions is introduced in [9], which originates from the idea of Gerstewitz's (Tammer's) sublinear scalarizing functional in [1]; see [3, 6]. This type of scalarization measures how far a given reference set needs to be moved towards a specific direction to fulfill each set relation between a target set and its moved reference set.

Note that V and d in (4) and (5) are index parameters for scalarization which play key roles as a reference set and a reference direction, respectively.

Proposition 2.15. [5] *Let $A, V \in \mathcal{P}(Y)$ and $d \in Y$. Then we have:*

$$\begin{aligned}
-I_C^{(1)}(-A;-V,d) &= S_C^{(1)}(A;V,d),\\
-I_C^{(2L)}(-A;-V,d) &= S_C^{(2U)}(A;V,d),\\
-I_C^{(3L)}(-A;-V,d) &= S_C^{(3U)}(A;V,d),\\
-I_C^{(2U)}(-A;-V,d) &= S_C^{(2L)}(A;V,d),\\
-I_C^{(3U)}(-A;-V,d) &= S_C^{(3L)}(A;V,d),\\
-I_C^{(4)}(-A;-V,d) &= S_C^{(4)}(A;V,d).
\end{aligned}$$

3. General results related to generalized continuity

In this section, we provide simple but important results for general cases of generalized continuity, that is, we show inheritance properties on $(\preccurlyeq, C)$-semicontinuity for set-valued maps and $(\preccurlyeq, C)$-lower and $(\preccurlyeq, C)$-upper semicontinuities for extended real-valued set functions.

Lemma 3.1. *For $A, B, C \in \mathcal{P}(Y)$, if $B - A \subset C$ then $B \subset A + C$.*

Proof. Let $b \in B$. For any $a \in A$, we have that $b - a = c$ for some $c \in C$ by the assumption. It can be simplified that $b = a + c \in A + C$. That is, $B \subset A + C$. □

Theorem 3.2. *Let $F : X \to \mathcal{P}(Y)$, $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and $C \subset Y$ a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-lower semicontinuous at $F(x_0)$, then $\varphi \circ F$ is lower semicontinuous at x_0.*

Proof. Let $r < \varphi(F(x_0))$. Since φ is $(\preccurlyeq, C)$-lower semicontinuous at $F(x_0)$, then $\exists W \in \mathcal{P}(Y)$, W open, s.t. $W \preccurlyeq F(x_0)$ and $r < \varphi(A), \forall A \in U(W + C, \preccurlyeq)$. By $(\preccurlyeq, C)$-continuity of F at x_0, we can obtain that $\exists V \in \mathcal{N}_X(x_0)$ s.t. $W + C \preccurlyeq F(x), \forall x \in V$. That is, $F(x) \in U(W + C, \preccurlyeq), \forall x \in V$. Therefore, $\varphi \circ F$ is lower semicontinuous at x_0. □

Theorem 3.3. *Let $F : X \to \mathcal{P}(Y)$, $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and $C \subset Y$ a convex cone. If F is $(\preccurlyeq, C)$-continuous at x_0 and φ is $(\preccurlyeq, C)$-upper semicontinuous at $F(x_0)$, then $\varphi \circ F$ is upper semicontinuous at x_0.*

Proof. Let $r > \varphi(F(x_0))$. Since φ is $(\preccurlyeq, C)$-upper semicontinuous at $F(x_0)$, then $\exists W \in \mathcal{P}(Y)$, W open, s.t. $W \preccurlyeq F(x_0)$ and $r > \varphi(A), \forall A \in U(W + C, \preccurlyeq)$. By $(\preccurlyeq, C)$-continuity of F at x_0, we can obtain that $\exists V \in \mathcal{N}_X(x_0)$ s.t. $W + C \preccurlyeq F(x), \forall x \in V$. That is, $F(x) \in U(W + C, \preccurlyeq), \forall x \in V$. Therefore, $\varphi \circ F$ is upper semicontinuous at x_0. □

Remark 3.4. Theorems 3.2 and 3.3 are generalizations and can be reduced to Theorems 3.1 and 3.2 in [5] whenever $C = \{0\}$; see Remarks 2.7 and 2.11. Besides, in case of a singleton function, both theorems generalize a usual inheritance property for a composite function of a set-valued map and a continuous linear functional. If f is C-continuous (vector-valued) function from X to Y then the composite function $\varphi \circ f$ is lower semicontinuous for each $\varphi \in C^+$, where

$$C^+ := \{y^* \in Y^* \,|y^*(y) \geq 0 \;\forall y \in C\}$$

and Y^* denotes the set of all continuous linear functionals on Y.

Similarly, if f is $(-C)$-continuous function from X to Y then the composite function $\varphi \circ f$ is upper semicontinuous for each $\varphi \in C^+$. See Proposition 2.3 in [15].

Proposition 3.5. *Let $\varphi : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $A_0 \in \mathcal{P}(Y)$, $\preccurlyeq$ a binary relation on $\mathcal{P}(Y)$, and $C \subset Y$ a convex cone. Then φ is $(\preccurlyeq, C)$-lower semicontinuous at A_0 if and only if $-\varphi$ is $(\preccurlyeq, C)$-upper semicontinuous at A_0.*

Proof. First, we assume that $(\preccurlyeq, C)$-lower semicontinuous at A_0. Let $r > -\varphi(A_0)$, that is, $-r < \varphi(A_0)$. By the assumption, there exists $W \in \mathcal{P}(Y)$, W open such that $W \preccurlyeq A_0$ and $-r < \varphi(A)$, that is, $r > -\varphi(A)$ for all $A \in U(W + C, \preccurlyeq)$, which implies that $-\varphi$ is $(\preccurlyeq, C)$-upper semicontinuous at A_0. The converse implication can be proved in the same way. □

4. Inherited continuity properties

This section can be considered as the main part of the paper. We separate the results into two subsections depending on the types of binary relations, $\preccurlyeq_1$ and $\preccurlyeq_2$, respectively. As stated in Remark 2.11, these conversions by $\preccurlyeq_1$ and $\preccurlyeq_2$ affect several results which contain valuable outcomes about classical notions of lower continuity and upper continuity for set-valued maps, respectively. In order to embrace classical results as special cases, we consider any convex cone C' which satisfies $\theta_Y \in C'$ and $C' \subset C$. Especially, when $C' = \{0\}$, some results coincide with classical ones.

4.1. Inheritance mechanism related to lower continuity

In this subsection, we systematically unravel the inheritance mechanism related to lower continuity for set-valued maps by using the binary relation $\preccurlyeq_1$. For this purpose, we shall prove that scalarization functions $I_C^{(1)}(\cdot; V, d)$ and $I_C^{(3U)}(\cdot; V, d)$ have $(\preccurlyeq_1, C')$-lower semicontinuity for $C' \subset C$ under some assumptions on $V \in \mathcal{P}(Y)$ and $d \in Y$, and that scalarization functions $I_C^{(2L)}(\cdot; V, d)$ and $I_C^{(4)}(\cdot; V, d)$ have $(\preccurlyeq_1, -C')$-upper semicontinuity for $C' \subset C$ under some assumptions on $V \in \mathcal{P}(Y)$ and $d \in Y$.

By Proposition 2.15, we additionally show that scalarization functions $S_C^{(j)}(\cdot; V, d)$ (for $j = 1, 3L$) and $S_C^{(j)}(\cdot; V, d)$ (for $j = 2U, 4$) are $(\preccurlyeq_1, C')$-upper semicontinuity and $(\preccurlyeq_1, -C')$-lower semicontinuity for $C' \subset C$, respectively. Consequently, we can apply the general results on inheritance properties shown in Section 3 to a composite function of a set-valued map with $(\preccurlyeq_1, C')$-continuity or $(\preccurlyeq_1, -C')$-continuity and either scalarization function $I_C^{(j)}(\cdot; V, d)$ (for $j = 1, 2L, 3U, 4$) or $S_C^{(j)}(\cdot; V, d)$ (for $j = 1, 3L, 2U, 4$).

Lemma 4.1. *Let $V, W \in \mathcal{P}(Y)$, C' and C two nonempty convex cones in Y such that $C' \subset C$, $d \in C$, $\alpha \in \mathbb{R}$. Then*

$$(W + C') \cap \left(\bigcap_{v \in V} (v - C) + \alpha d \right) = \emptyset \quad \Longrightarrow \quad \alpha \leq I_C^{(1)}(A; V, d)$$

for all $A \in \mathcal{P}(Y)$ with $A \cap (W + C') \neq \emptyset$.

Proof. In case of $I_C^{(1)}(A;V,d) = +\infty$, the conclusion is trivial. Then we assume that $I_C^{(1)}(A;V,d) < \infty$. Suppose to the contrary that $\alpha > I_C^{(1)}(A;V,d)$ for some $A \in \mathcal{P}(Y)$ with $A \cap (W + C') \neq \emptyset$. Set $\widehat{t} := I_C^{(1)}(A;V,d)$. If $\widehat{t} = -\infty$, we have $A \leq_C^{(1)} V + td$ for any $t \in \mathbb{R}$ by the definition; otherwise, for any $\varepsilon > 0$, there exists $t \in \mathbb{R}$ such that $\widehat{t} \leq t \leq \widehat{t} + \varepsilon$ and $A \leq_C^{(1)} V + td$. In either case, it follows from Proposition 2.13, that $A \leq_C^{(1)} V + \alpha d$ because $\alpha > \widehat{t}$. That is,

$$A \subset \bigcap_{v \in V} (v - C) + \alpha d.$$

Since $A \cap (W + C') \neq \emptyset$, we get $(W + C') \cap \left(\bigcap_{v \in V}(v - C) + \alpha d\right) \neq \emptyset$, which is a contradiction to the assumption. □

Lemma 4.2. *Let $V, W \in \mathcal{P}(Y)$, C' and C two nonempty convex cones in Y such that $C' \subset C$, $d \in C$, $\alpha \in \mathbb{R}$. Then*

$$(W + C') \cap ((V - C) + \alpha d) = \emptyset \quad \Longrightarrow \quad \alpha \leq I_C^{(3U)}(A;V,d)$$

for all $A \in \mathcal{P}(Y)$ with $A \cap (W + C') \neq \emptyset$.

Proof. In a similar way to the proof of Lemma 4.1, suppose to the contrary that $\alpha > I_C^{(3U)}(A;V,d)$ for some $A \in \mathcal{P}(Y)$ such that $A \cap (W + C') \neq \emptyset$, then we show that $A \leq_C^{(3U)} V + \alpha d$, which implies that

$$A \subset (V - C) + \alpha d.$$

Since $A \cap (W + C') \neq \emptyset$, we get $(W + C') \cap ((V - C) + \alpha d) \neq \emptyset$, which is a contradiction to the assumption. □

Lemma 4.3. *Let $V, W \in \mathcal{P}(Y)$, C' and C two nonempty convex cones in Y such that $C' \subset C$, $d \in C$, $\alpha \in \mathbb{R}$. Then the following statements hold.*

(i) $W + C' \subset \bigcap_{v \in V}(v - C) + \alpha d$
$\Longrightarrow \quad \alpha \geq I_C^{(2L)}(A;V,d)$ *for all $A \in \mathcal{P}(Y)$ with $A \cap (W + C') \neq \emptyset$;*

(ii) $W - C' \subset \bigcap_{v \in V}(v - C) + \alpha d$
$\Longrightarrow \quad \alpha \geq I_C^{(2L)}(A;V,d)$ *for all $A \in \mathcal{P}(Y)$ with $A \cap (W - C') \neq \emptyset$.*

Proof. For (i), in a similar way to the proof of Lemma 4.1, suppose to the contrary that $\alpha < I_C^{(2L)}(A;V,d)$ for some $A \in \mathcal{P}(Y)$ with $A \cap (W + C') \neq \emptyset$, then we have

$$A \cap \left(\bigcap_{v \in V} (v - C) + \alpha d \right) = \emptyset.$$

It follows from the assumption that $A \cap (W + C') = \emptyset$, which is a contradiction. Also, we prove statement (ii) in the same way. □

Lemma 4.4. *Let $V, W \in \mathcal{P}(Y)$, C' and C two nonempty convex cones in Y such that $C' \subset C$, $d \in C$, $\alpha \in \mathbb{R}$. Then the following statements hold:*

(i) $W + C' \subset (V - C) + \alpha d$
$\Longrightarrow \quad \alpha \geq I_C^{(4)}(A; V, d)$ *for all* $A \in \mathcal{P}(Y)$ *with* $A \cap (W + C') \neq \emptyset$*;*

(ii) $W - C' \subset (V - C) + \alpha d$
$\Longrightarrow \quad \alpha \geq I_C^{(4)}(A; V, d)$ *for all* $A \in \mathcal{P}(Y)$ *with* $A \cap (W - C') \neq \emptyset$*.*

Proof. The proof is similar to that of Lemma 4.3. □

Proposition 4.5. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones in Y such that $C' \subset C \neq Y$, $d \in \operatorname{int}(C)$. Then*

$$I_C^{(j)}(\cdot; V, d) \text{ is } (\preccurlyeq_1, C')\text{-lower semicontinuous at } A_0 \text{ for } j = 1, 3U.$$

Proof. In case of $I_C^{(j)}(A_0; V, d) = -\infty$, the conclusion is trivial. Then we assume that $I_C^{(j)}(A_0; V, d) > -\infty$. For each $j = 1, 3U$, let $r < I_C^{(j)}(A_0; V, d)$. Then, we take $\alpha, \beta \in \mathbb{R}$ such that $r < \beta < \alpha < I_C^{(j)}(A_0; V, d)$. We claim that there exists $W \in \mathcal{P}(Y)$, W open, such that

$$W \preccurlyeq_1 A_0 \tag{6}$$

and

$$r < I_C^{(j)}(A; V, d), \forall A \in U(W + C', \preccurlyeq_1). \tag{7}$$

In case $j = 1$, since $\alpha < I_C^{(1)}(A_0; V, d)$, we have

$$A_0 \not\subset \bigcap_{v \in V} (v - C) + \alpha d.$$

Also, it follows from $\beta < \alpha$ and $d \in \operatorname{int}(C)$ that

$$\operatorname{cl} \bigcap_{v \in V} (v - C) + \beta d \subset \bigcap_{v \in V} (v - C) + \alpha d,$$

which implies that there exists $y \in Y$ such that $y \in \left(\operatorname{cl} \bigcap_{v \in V} (v - C) + \beta d\right)^c \cap A_0$. Thus, there exists an open set $W \in \mathcal{P}(Y)$ such that

$$y \in W \subset \left(\operatorname{cl} \bigcap_{v \in V} (v - C) + \beta d\right)^c.$$

Therefore, we get $W \cap A_0 \neq \emptyset$, that is, W satisfies (6). Also, it can be seen that

$$W \cap \left(\bigcap_{v \in V} (v - C) + \beta d\right) = \emptyset,$$

which implies that $(W + C') \cap \left(\bigcap_{v \in V} (v - C) + \beta d\right) = \emptyset$.

It follows from Lemma 4.1 that $\beta \leq I_C^{(1)}(A; V, d)$ for all $A \in \mathcal{P}(Y)$ which satisfy $A \cap (W + C') \neq \emptyset$. Since $r < \beta$, W also satisfies (7). Hence, $I_C^{(1)}(\cdot; V, d)$ is $(\preccurlyeq_1, C')$-lower semicontinuous at A_0. In case $j = 3U$, using Lemma 4.2, we can similarly show that $I_C^{(3U)}(\cdot; V, d)$ is $(\preccurlyeq_1, C')$-lower semicontinuous at A_0. □

Proposition 4.6. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones in Y such that $C' \subset C \neq Y$, $d \in \operatorname{int}(C)$. Then*

$$S_C^{(j)}(\cdot; V, d) \text{ is } (\preccurlyeq_1, C')\text{-upper semicontinuous at } A_0 \text{ for } j = 1, 3L.$$

Proof. By Propositions 2.15, 3.5, and 4.5, the proof is complete. □

Proposition 4.7. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones in Y such that $C' \subset C \neq Y$, $d \in \operatorname{int}(C)$. Then*

$$I_C^{(j)}(\cdot; V, d) \text{ is } (\preccurlyeq_1, -C')\text{-upper semicontinuous at } A_0 \text{ for } j = 2L, 4.$$

Proof. In case of $I_C^{(j)}(A_0; V, d) = +\infty$, the conclusion is trivial. Then we assume that $I_C^{(j)}(A_0; V, d) < +\infty$. For each $j = 2L, 4$, let $r > I_C^{(j)}(A_0; V, d)$. Then, we take $\alpha, \beta \in \mathbb{R}$ such that $r > \beta > \alpha > I_C^{(j)}(A_0; V, d)$. We claim that there exists $W \in \mathcal{P}(Y)$, W open, such that

$$W \preccurlyeq_1 A_0 \tag{8}$$

and

$$r > I_C^{(j)}(A; V, d), \forall A \in U(W - C', \preccurlyeq_1). \tag{9}$$

In case $j = 2L$, define $\widehat{t} := I_C^{(2L)}(A_0; V, d)$; then $\alpha > \widehat{t}$. If $\widehat{t} = -\infty$, we have $A_0 \leq_C^{(2L)} V + td$ for any $t \in \mathbb{R}$ by the definition; otherwise, for any $\varepsilon > 0$, there exists $t \in \mathbb{R}$ such that $\widehat{t} \leq t \leq \widehat{t} + \varepsilon$ and $A_0 \leq_C^{(2L)} V + td$. In either case, it follows from Proposition 2.13, that $A_0 \leq_C^{(2L)} V + \alpha d$ because $\alpha > \widehat{t}$. Thus, it follows that

$$A_0 \cap \left(\bigcap_{v \in V} (v - C) + \alpha d \right) \neq \emptyset.$$

Also, it follows from $\beta > \alpha$ and $d \in \operatorname{int}(C)$ that

$$\bigcap_{v \in V} (v - C) + \alpha d \subset \operatorname{int} \left(\bigcap_{v \in V} (v - C) + \beta d \right)$$

which implies that there exists $y \in Y$ such that $y \in \operatorname{int} \left(\bigcap_{v \in V} (v - C) + \beta d \right) \cap A_0$. Thus, there exists an open set $W \in \mathcal{P}(Y)$ such that

$$y \in W \subset \operatorname{int} \left(\bigcap_{v \in V} (v - C) + \beta d \right).$$

Therefore, we get $W \cap A_0 \neq \emptyset$, that is, W satisfies (8). Also, it can be seen that

$$W \subset \bigcap_{v \in V} (v - C) + \beta d,$$

which implies that $W - C' \subset \bigcap_{v \in V} (v - C) + \beta d$.

It follows from (ii) of Lemma 4.3 that $\beta \geq I_C^{(2L)}(A; V, d)$ for all $A \in \mathcal{P}(Y)$ with $A \cap (W - C') \neq \emptyset$. Since $r > \beta$, W also satisfies (9). Hence, $I_C^{(2L)}(\cdot; V, d)$ is $(\preccurlyeq_1, -C')$-upper semicontinuous at A_0.

In case $j = 4$, using (ii) of Lemma 4.4, we can similarly show that $I_C^{(4)}(\cdot; V, d)$ is $(\preccurlyeq_1, -C')$-upper semicontinuous at A_0. □

Proposition 4.8. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones in Y such that $C' \subset C \neq Y$, $d \in \mathrm{int}\,(C)$. Then*

$$S_C^{(j)}(\cdot; V, d) \text{ is } (\preccurlyeq_1, -C')\text{-lower semicontinuous at } A_0 \text{ for } j = 2U, 4.$$

Proof. By Propositions 2.15, 3.5, and 4.7, the proof is complete. □

Remark 4.9. Propositions 4.5, 4.6, 4.7, and 4.8 explore more general properties of scalarization functions $I_C^{(j)}(\cdot; V, d)$ (for $j = 1, 3U, 2L, 4$) and $S_C^{(j)}(\cdot; V, d)$ (for $j = 1, 3L, 2U, 4$) by the observation from cone continuity, and they are regarded as generalizations of results in [5]. In fact, when $C' = \{0\}$, they are reduced to Propositions 3.1 (1), 3.2 (2), 3.1 (2), 3.2 (1) in [5], respectively.

Theorem 4.10. *Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones in Y such that $C' \subset C \neq Y$ and $d \in \mathrm{int}\,(C)$. If F is $(\preccurlyeq_1, C')$-continuous at x_0, then*

(i) $I_C^{(j)}(F(\cdot); V, d)$ *is lower semicontinuous at x_0, for all $j = 1, 3U$,*

(ii) $S_C^{(j)}(F(\cdot); V, d)$ *is upper semicontinuous at x_0, for all $j = 1, 3L$.*

Proof. By Theorems 3.2, 3.3, and Propositions 4.5, 4.6, the proof is complete. □

Theorem 4.11. *Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones in Y such that $C' \subset C \neq Y$ and $d \in \mathrm{int}\,(C)$. If F is $(\preccurlyeq_1, -C')$-continuous at x_0, then*

(i) $I_C^{(j)}(F(\cdot); V, d)$ *is upper semicontinuous at x_0, for all $j = 2L, 4$,*

(ii) $S_C^{(j)}(F(\cdot); V, d)$ *is lower semicontinuous at x_0, for all $j = 2U, 4$.*

Proof. By Theorems 3.3, 3.2 and Propositions 4.7, 4.8, the proof is complete. □

Remark 4.12. Theorems 4.10 and 4.11 are generalizations of Theorems 3.3 and 3.4 in [5] with respect to $\preccurlyeq_1$-continuity. Moreover, they are generalizations of Theorems 3.1, 3.2, 3.3 and 3.4 in [14]. For the half-missing part, namely $I_C^{(2U)}(\cdot; V, d)$ and $I_C^{(3L)}(\cdot; V, d)$ with respect to $\preccurlyeq_1$, the counter examples are provided in Section 5.

Corollary 4.13. *Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $V \in \mathcal{P}(Y)$, C a nonempty convex cone in Y such that $C \neq Y$ and $d \in \mathrm{int}\,(C)$. If F is lower continuous at x_0, then*

(i) $I_C^{(j)}(F(\cdot); V, d)$ *is lower semicontinuous at x_0, for all $j = 1, 3U$,*

(ii) $I_C^{(j)}(F(\cdot); V, d)$ *is upper semicontinuous at x_0, for all $j = 2L, 4$,*

(iii) $S_C^{(j)}(F(\cdot); V, d)$ *is upper semicontinuous at x_0, for all $j = 1, 3L$,*

(iv) $S_C^{(j)}(F(\cdot); V, d)$ *is lower semicontinuous at x_0, for all $j = 2U, 4$.*

Proof. Set $C' = \{0\}$ in Theorems 4.10 and 4.11. Then the statements follow from Remark 2.7. □

4.2. Inheritance mechanism related to upper continuity

In this subsection, we systematically unravel the inheritance mechanism related to upper continuity for set-valued maps by using the binary relation $\preccurlyeq_2$. For this purpose, we shall prove that scalarization functions $I_C^{(1)}(\cdot; V, d)$, $I_C^{(2U)}(\cdot; V, d)$ and $I_C^{(3U)}(\cdot; V, d)$ have $(\preccurlyeq_2, -C')$-upper semicontinuity for $C' \subset C$ under some assumptions on $V \in \mathcal{P}(Y)$ and $d \in Y$, and that scalarization functions $I_C^{(2L)}(\cdot; V, d)$, $I_C^{(3L)}(\cdot; V, d)$ and $I_C^{(4)}(\cdot; V, d)$ have $(\preccurlyeq_2, C')$-lower semicontinuity for $C' \subset C$ under some assumptions on $V \in \mathcal{P}(Y)$ and $d \in Y$. By Proposition 2.15, we shall additionally show that scalarization functions $S_C^{(j)}(\cdot; V, d)$ (for $j = 1, 2L, 3L$) and $S_C^{(j)}(\cdot; V, d)$ (for $j = 2U, 3U, 4$) are $(\preccurlyeq_2, -C')$-lower semicontinuity and $(\preccurlyeq_2, C')$-upper semicontinuity for $C' \subset C$, respectively. Consequently, we can apply the general results on inheritance properties shown in Section 3 to a composite function of a set-valued map with $(\preccurlyeq_2, C')$-continuity or $(\preccurlyeq_2, -C')$-continuity and either scalarization function $I_C^{(j)}(\cdot; V, d)$ or $S_C^{(j)}(\cdot; V, d)$ (for $j = 1, 2L, 3L, 2U, 3U, 4$).

Lemma 4.14. *Let $A, V, W \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C$ and $d \in C$. Then, for each $j = 1, 2U, 3U$,*

$$A \subset W - C' \implies I_C^{(j)}(A; V, d) \leq I_C^{(j)}(W; V, d).$$

Proof. When $j = 1$, suppose to the contrary that $I_C^{(1)}(A; V, d) > I_C^{(1)}(W; V, d)$. Then, there exists $t \in \mathbb{R}$ such that $I_C^{(1)}(A; V, d) > t > I_C^{(1)}(W; V, d)$. By Proposition 2.13, we obtain that

$$A \not\subset \bigcap_{v \in V}(v - C) + td \quad \text{and} \quad W \subset \bigcap_{v \in V}(v - C) + td$$

which implies that $W - C' \subset \bigcap_{v \in V}(v - C) + td$. By the assumption $A \subset W - C'$, thus, this is a contradiction. Hence, $I_C^{(1)}(A; V, d) \leq I_C^{(1)}(W; V, d)$. Similarly, in case that $j = 2U, 3U$, the proof is complete as desired. □

Lemma 4.15. *Let $A, V, W \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C$ and $d \in C$. Then, for each $j = 2L, 3L, 4$,*

$$A \subset W + C' \implies I_C^{(j)}(A; V, d) \geq I_C^{(j)}(W; V, d).$$

Proof. When $j = 2L$, suppose to the contrary that $I_C^{(2L)}(A; V, d) < I_C^{(2L)}(W; V, d)$. Then, there exists $t \in \mathbb{R}$ such that $I_C^{(2L)}(A; V, d) < t < I_C^{(2L)}(W; V, d)$. By Proposition 2.13, we obtain that

$$A \cap \left(\bigcap_{v \in V}(v - C) + td \right) \neq \emptyset \quad \text{and} \quad W \cap \left(\bigcap_{v \in V}(v - C) + td \right) = \emptyset$$

which implies $(W + C') \cap \left(\bigcap_{v \in V}(v - C) + td \right) = \emptyset$. By the assumption $A \subset W + C'$, thus, this is a contradiction. Hence, $I_C^{(2L)}(A; V, d) \leq I_C^{(2L)}(W; V, d)$. Similarly, in case that $j = 3L, 4$, the proof is complete as desired. □

Proposition 4.16. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int}(C)$. Then*

$$I_C^{(j)}(\cdot; V, d) \text{ is } (\preccurlyeq_2, -C')\text{-upper semicontinuous at } A_0 \text{ for } j = 1, 3U.$$

Proof. For each $j = 1, 3U$, let $r > I_C^{(j)}(A_0; V, d)$. Then, we take $\alpha, \beta \in \mathbb{R}$ such that $r > \beta > \alpha > I_C^{(j)}(A_0; V, d)$. We claim that there exists $W \in \mathcal{P}(Y)$, W open, such that

$$W \preccurlyeq_2 A_0 \tag{10}$$

and
$$r > I_C^{(j)}(A; V, d), \forall A \in U(W - C', \preccurlyeq_2). \tag{11}$$

In case $j = 1$, we have

$$A_0 \subset \bigcap_{v \in V}(v - C) + \alpha d \subset \operatorname{int}\left(\bigcap_{v \in V}(v - C) + \beta d\right).$$

Set $W := \operatorname{int}\left(\bigcap_{v \in V}(v - C) + \beta d\right)$, thus $A_0 \subset W$ which satisfied (10). Moreover, $I_C^{(1)}(W; V, d) \leq \beta < r$. Let $A \in U(W - C', \preccurlyeq_2)$ which implies that $A \subset W - C'$. By Lemma 4.14, $I_C^{(1)}(A; V, d) \leq I_C^{(1)}(W; V, d)$. Hence, $I_C^{(1)}(A; V, d) < r$. That is, (11) holds. Therefore, $I_C^{(1)}(\cdot; V, d)$ is $(\preccurlyeq_2, -C')$ -upper semicontinuous at A_0. In case $j = 3U$, we can similarly show that $I_C^{(3U)}(\cdot; V, d)$ is $(\preccurlyeq_2, -C')$-upper semicontinuous at A_0. □

Proposition 4.17. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int}(C)$. Then*

$$S_C^{(j)}(\cdot; V, d) \text{ is } (\preccurlyeq_2, -C')\text{-lower semicontinuous at } A_0 \text{ for } j = 1, 3L.$$

Proof. By Propositions 2.15, 3.5, and 4.16, the proof is complete. □

Proposition 4.18. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int}(C)$. Then*

$$I_C^{(j)}(\cdot; V, d) \text{ is } (\preccurlyeq_2, C')\text{-lower semicontinuous at } A_0 \text{ for } j = 2L, 4.$$

Proof. For each $j = 2L, 4$, let $r < I_C^{(j)}(A_0; V, d)$. Then, we take $\alpha, \beta \in \mathbb{R}$ such that $r < \beta < \alpha < I_C^{(j)}(A_0; V, d)$. We claim that there exists $W \in \mathcal{P}(Y)$, W open, such that

$$W \preccurlyeq_2 A_0 \tag{12}$$

and
$$r < I_C^{(j)}(A; V, d), \forall A \in U(W + C', \preccurlyeq_2). \tag{13}$$

In case $j = 2L$, we have $A_0 \cap \left(\bigcap_{v \in V}(v - C) + \alpha d\right) = \emptyset$,

that is,
$$A_0 \subset \left(\bigcap_{v \in V}(v - C) + \alpha d\right)^c \subset \operatorname{int}\left(\bigcap_{v \in V}(v - C) + \beta d\right)^c.$$

Set $W := \operatorname{int}\left(\bigcap_{v\in V}(v - C) + \beta d\right)^c$, thus $A_0 \subset W$ which satisfied (12). Moreover, $I_C^{(2L)}(W; V, d) \geq \beta > r$. Let $A \in U(W + C', \preccurlyeq_2)$ which implies that $A \subset W + C'$. By Lemma 4.15, $I_C^{(2L)}(A; V, d) \geq I_C^{(2L)}(W; V, d)$. Hence, $I_C^{(2L)}(A; V, d) > r$. That is, (13) holds. Therefore, $I_C^{(2L)}(\cdot; V, d)$ is $(\preccurlyeq_2, C')$-lower semicontinuous at A_0. In case $j = 4$, we can similarly show that $I_C^{(4)}(\cdot; V, d)$ is $(\preccurlyeq_2, C')$-lower semicontinuous at A_0. □

Proposition 4.19. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int}(C)$. Then*

$S_C^{(j)}(\cdot; V, d)$ is $(\preccurlyeq_2, C')$-upper semicontinuous at A_0 for $j = 2U, 4$.

Proof. By Propositions 2.15, 3.5, and 4.18, the proof is complete. □

Lemma 4.20. *Let $A, V \in \mathcal{P}(Y)$, $C \subset Y$ a convex cone with $\operatorname{int}(C) \neq \emptyset$, and $\theta_Y \in C$. Assume that A is compact. Then, the following statements hold:*

(i) $\operatorname{int}\left(\bigcap_{a\in A}(a + C)\right) \neq \emptyset$;

(ii) *If V is compact then for $d \in \operatorname{int}(C)$, there exists $t_1 \in \mathbb{R}$ such that*

$$\left(\bigcap_{a\in A}(a + C)\right) \cap (V + t_1 d) \neq \emptyset.$$

Moreover if $C \neq Y$, there exists $t_2 \in \mathbb{R}$ such that

$$\left(\bigcap_{a\in A}(a + C)\right) \cap (V + t_2 d) = \emptyset;$$

(iii) $\operatorname{int}\left(\bigcap_{a\in A}(a + C)\right) = \bigcap_{a\in A}(a + \operatorname{int}(C))$;

(iv) *For any $t > 0$, $\bigcap_{a\in A}(a + C) + td \subset \operatorname{int}\left(\bigcap_{a\in A}(a + C)\right)$.*

Proof. By the compactness of A, we have $\bigcap_{a\in A}(a + C) \neq \emptyset$. First we shall prove (i). Since $\operatorname{int}(C) \neq \emptyset$, there is $c_0 \in \operatorname{int}(C)$ such that $c_0 - \operatorname{int}(C)$ is an open neighborhood of θ_Y. By the compactness of A and Remark 2.2, there exists $t > 0$ such that $A \subset t(c_0 - \operatorname{int}(C))$, which implies that $A \subset tc_0 - C$. For each $a \in A$, there is $c_a \in C$ such that $a = tc_0 - c_a$ and then $tc_0 = a + c_a \in a + C$. Therefore, $\bigcap_{a\in A}(a + C) \neq \emptyset$, and then there exists $b \in \bigcap_{a\in A}(a + C)$. Hence, for any $a \in A$, there is $c_a \in C$ such that $b = a + c_a$. For any $c \in C$, we have $b + c = a + c_a + c \in a + C$ for every $a \in A$, which implies that $b + \operatorname{int}(C) \subset b + C \subset \bigcap_{a\in A}(a + C)$. Therefore, $\operatorname{int}\left(\bigcap_{a\in A}(a + C)\right) \neq \emptyset$.

Next, we shall prove (ii). By (i), there exists $b \in \bigcap_{a\in A}(a + C)$. Because of $V \neq \emptyset$, we take $v_0 \in V$. Since $d - C$ is a neighborhood of θ_Y in Y for $d \in \operatorname{int}(C)$, it follows from Remark 2.2 that there is $t_1 \in \mathbb{R}$ such that $b - v_0 \in t_1(d - C)$, which implies that $v_0 + t_1 d \in b + C \subset \bigcap_{a\in A}(a + C)$. Therefore,

$$\left(\bigcap_{a\in A}(a + C)\right) \cap (V + t_1 d) \neq \emptyset.$$

Moreover, since $-d+\operatorname{int}(C), d-\operatorname{int}(C) \in \mathcal{N}_Y(\theta_Y)$, it follows from the compactness of A, V and Remark 2.2 that there are $s_1, s_2 > 0$ such that

$$A \subset s_1(-d+\operatorname{int}(C)) \qquad \text{and} \qquad V \subset s_2(d-\operatorname{int}(C)).$$

Hence, we have $A+s_1d+C \subset \operatorname{int}(C)$ and $V-s_2d \subset -\operatorname{int}(C)$. Since C is a convex cone with $C \neq Y$, $\operatorname{int}(C) \cap (-\operatorname{int}(C)) = \emptyset$. Thus, we have

$$(A+s_1d+C) \cap (V-s_2d) = \emptyset,$$

which implies that
$$(A+C) \cap (V-(s_1+s_2)d) = \emptyset.$$

Therefore,
$$\left(\bigcap_{a\in A}(a+C)\right) \cap (V+t_2d) = \emptyset$$

by putting $t_2 := -(s_1+s_2)$.

Consider (iii), we first let $x \in \operatorname{int}\left(\bigcap_{a\in A}(a+C)\right)$. Then there exists an open neighborhood $U \in \mathcal{N}(\theta_Y)$ such that $x+U \subset \bigcap_{a\in A}(a+C)$. That is, for any $a \in A$, we have $x+U \subset \operatorname{int}(a+C) = a+\operatorname{int}(C)$. Hence, it follows that $x \in \bigcap_{a\in A}(a+\operatorname{int}(C))$. Conversely, let $x \in \bigcap_{a\in A}(a+\operatorname{int}(C))$. For any $a \in A$, since $a \in x-\operatorname{int}(C)$, there exists an open neighborhood $U_a \in \mathcal{N}(\theta_Y)$ such that $-U_a = U_a$ and

$$a+U_a \subset x-C. \tag{14}$$

Since $A \subset \bigcup_{a\in A}\left(a+\frac{1}{3}U_a\right)$ and A is compact, there exist $a_1, \dots, a_m \in A$ such that

$$A \subset \bigcup_{k=1}^{m}\left(a_k+\frac{1}{3}U_{a_k}\right). \tag{15}$$

Set $U_0 := \bigcap_{k=1}^{m}\frac{1}{3}U_{a_k}$, then U_0 is an open neighborhood of θ_Y. Again, since we have $A \subset \bigcup_{a\in A}(a+U_0)$ and A is compact, there exist $\hat{a}_1, \dots, \hat{a}_n \in A$ such that

$$A \subset \bigcup_{i=1}^{n}(\hat{a}_i+U_0). \tag{16}$$

Hence, for any $a \in A$, there is n_0 such that $a \in \hat{a}_{n_0}+U_0$ by (16). Since $\hat{a}_{n_0} \in A$, there is m_0 such that $\hat{a}_{n_0} \in a_{m_0}+\frac{1}{3}U_{a_{m_0}}$ by (15). Therefore,

$$a+U_0 \subset \hat{a}_{n_0}+U_0+U_0 \subset a_{m_0}+\frac{1}{3}U_{a_{m_0}}+2U_0 \subset a_{m_0}+U_{a_{m_0}},$$

which is a subset of $x-C$ by (14). This implies that $x+U_0 \subset a+C$, and then $x+U_0 \subset \bigcap_{a\in A}(a+C)$. Therefore, $x \in \operatorname{int}\left(\bigcap_{a\in A}(a+C)\right)$.

Finally, to prove (iv), we have that, for any $t > 0$,

$$\bigcap_{a\in A}(a+C)+td \subset \bigcap_{a\in A}(a+C)+\operatorname{int}(C).$$

Actually, by Lemma 2.4, $\bigcap_{a\in A}(a+C)+\operatorname{int}(C) = \operatorname{int}\left(\bigcap_{a\in A}(a+C)+C\right)$ which implies that $\bigcap_{a\in A}(a+C)+td \subset \operatorname{int}\left(\bigcap_{a\in A}(a+C)\right)$ because $C+C=C$. □

Remark 4.21. Let $A_0, V \in \mathcal{P}(Y)$ and $d \in \operatorname{int}(C)$. By (ii) of Lemma 4.20, if A_0 and V are compact sets, then $I_C^{(2U)}(A_0; V, d) < +\infty$. Moreover if $C \neq Y$, we have $I_C^{(2U)}(A_0; V, d) \in \mathbb{R}$.

Proposition 4.22. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int}(C)$. Assume that A_0 and V are compact. Then*

$$I_C^{(2U)}(\cdot; V, d) \text{ is } (\preccurlyeq_2, -C')\text{-upper semicontinuous at } A_0.$$

Proof. Denote $r_0 := I_C^{(2U)}(A_0; V, d)$. Due to the compactness of A_0 and V, we have $r_0 \in \mathbb{R}$. Take r and s such that $r > s > r_0$, then

$$\Big(\bigcap_{a \in A_0} (a + C)\Big) \cap (V + sd) \neq \emptyset.$$

Moreover, by (i) of Lemma 4.20, we obtain that $\operatorname{int}\big(\bigcap_{a\in A_0}(a + C)\big) \neq \emptyset$. We can choose $x \in \bigcap_{a\in A_0}(a + C)$ such that $x \in V + sd$, which implies that there exists $v \in V$ such that $x = v + sd$, in other words, $v = x - sd$. Now, we consider $y := v + rd \in V + rd$. That is, $y = x - sd + rd = x + (r - s)d$. Observe that $r - s > 0$, and by the assumption that $d \in \operatorname{int}(C)$, we obtain that $(r - s)d \in \operatorname{int}(C)$. Thus $x + (r - s)d \in \bigcap_{a\in A_0}(a + C) + (r - s)d$. From (iv) of Lemma 4.20, it implies that $y = x + (r - s)d \in \operatorname{int}\big(\bigcap_{a\in A_0}(a + C)\big)$. Hence,

$$\operatorname{int}\Big(\bigcap_{a \in A_0} (a + C)\Big) \cap (V + rd) \neq \emptyset.$$

Let $\varepsilon \in \mathbb{R}$ such that $r - s > \varepsilon > 0$, and choose $z := y - \varepsilon d$. Define $W := z - \operatorname{int}(C)$, which is open. Note that $z = y - \varepsilon d = v + rd - \varepsilon d = x - sd + rd - \varepsilon d = x + (r - s - \varepsilon)\, d$ where $x \in \big(\bigcap_{a\in A_0}(a + C)\big) \cap (V + sd)$. Since $r - s - \varepsilon > 0$ and $d \in \operatorname{int}(C)$, we have $x + (r - s - \varepsilon)\, d \in \operatorname{int}\big(\bigcap_{a\in A_0}(a + C)\big)$. That is,

$$z \in \operatorname{int}\big(\textstyle\bigcap_{a\in A_0}(a + C)\big) = \bigcap_{a\in A_0}(a + \operatorname{int}(C))$$

by (iii) of Lemma 4.20. Moreover, we obtain that, for any $a \in A_0$, $z \in a + \operatorname{int}(C)$ which implies that $a \in z - \operatorname{int}(C)$. Hence, $A_0 \subset W$.

Last, we claim that $r > I_C^{(2U)}(A; V, d)$ for all $A \in U(W - C', \preccurlyeq_2)$ which implies that $A \subset W - C'$. Note that $W - C' = W$ because $W = z - \operatorname{int}(C)$ and $C' \subset C$. Then, whenever $A \subset W - C'$, we have $A \subset z - \operatorname{int}(C)$. Hence, for any $a \in A$, $z \in a + \operatorname{int}(C)$, which implies that $z \in \big(\bigcap_{a\in A}(a + C)\big)$. Therefore $\bigcap_{a\in A}(a + C) \neq \emptyset$. Moreover, since $z = v + (r - \varepsilon)\, d \in V + (r - \varepsilon)\, d$,

$$\Big(\bigcap_{a\in A}(a + C)\Big) \cap (V + (r - \varepsilon)\, d) \neq \emptyset.$$

Therefore, $I_C^{(2U)}(A; V, d) \leq r - \varepsilon < r$. The proof is complete. □

Proposition 4.23. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int}(C)$. Assume that A_0 and V are compact. Then*

$$S_C^{(2L)}(\cdot; V, d) \text{ is } (\preccurlyeq_2, -C')\text{-lower semicontinuous at } A_0.$$

Proof. By Propositions 2.15, 3.5, and 4.22, the proof is complete. □

Lemma 4.24. *Let $A, V \in \mathcal{P}(Y)$, $C \subset Y$ a convex cone with $\operatorname{int}(C) \neq \emptyset$, and $\theta_Y \in C$. If A and V are compact then for $d \in \operatorname{int}(C)$, there exists $t_1 \in \mathbb{R}$ such that*

$$V + t_1 d \subset A + C.$$

Moreover if $C \neq Y$, there exists $t_2 \in \mathbb{R}$ such that $(V + t_2 d) \cap (A + C) = \emptyset$.

Proof. Since $-d + \operatorname{int}(C), d - \operatorname{int}(C) \in \mathcal{N}_Y(\theta_Y)$, it follows from the compactness of A, V and Remark 2.2 that there are $s_1, s_2 > 0$ such that

$$A \subset s_1 (d - \operatorname{int}(C)) \qquad \text{and} \qquad V \subset s_2 (-d + \operatorname{int}(C)).$$

Since C is a convex cone, we have $C \subset A - s_1 d + C$ and $V + s_2 d \subset \operatorname{int}(C)$. Hence, we get $V + s_2 d \subset A - s_1 d + C$, which implies that $V + t_1 d \subset A + C$ by putting $t_1 := s_1 + s_2$. Similarly there are $u_1, u_2 > 0$ such that

$$A \subset u_1 (-d + \operatorname{int}(C)) \qquad \text{and} \qquad V \subset u_2 (d - \operatorname{int}(C)).$$

Hence, we have $A + u_1 d + C \subset \operatorname{int}(C)$ and $V - u_2 d \subset -\operatorname{int}(C)$. Since C is a convex cone with $C \neq Y$, $\operatorname{int}(C) \cap (-\operatorname{int}(C)) = \emptyset$. Therefore, we have

$$(A + u_1 d + C) \cap (V - u_2 d) = \emptyset,$$

which implies that $(A + C) \cap (V + t_2 d) = \emptyset$ by putting $t_2 := -(u_1 + u_2)$. □

Remark 4.25. Let $A_0, V \in \mathcal{P}(Y)$ and $d \in \operatorname{int}(C)$. By Lemma 4.24, if A_0 and V are compact sets, then $I_C^{(3L)}(A_0; V, d) < +\infty$. Moreover if $C \neq Y$, we have $I_C^{(3L)}(A_0; V, d) \in \mathbb{R}$.

Proposition 4.26. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int}(C)$. Assume that A_0 and V are compact. Then*

$$I_C^{(3L)}(\cdot; V, d) \textit{ is } (\preccurlyeq_2, C')\textit{-lower semicontinuous at } A_0.$$

Proof. Denote $r_0 := I_C^{(3L)}(A_0; V, d)$. Due to the compactness of A_0 and V, we have $r_0 \in \mathbb{R}$. Take r and ε_0 such that $r < r + \varepsilon_0 < r_0$, then $V + (r + \varepsilon)d \not\subset A_0 + C$ for all $0 < \varepsilon < \varepsilon_0$. We set $W := A_0 - \frac{\varepsilon_0}{2} d + \operatorname{int}(C)$, which is open. Then $A_0 \subset W$ and $W + C' = W$. Now, we show that $r < I_C^{(3L)}(A; V, d)$ for all $A \in U(W + C', \preccurlyeq_2)$ which implies that $A \subset W + C'$. Suppose to the contrary that there exists $A \subset W + C'$ such that $V + (r + \varepsilon)d \subset A + C$ for any $\varepsilon > 0$. For all $v \in V$ and $0 < \varepsilon \leq \frac{\varepsilon_0}{2}$, we have $v + (r + \varepsilon)d \in A + C \subset W + C' + C = W + C \subset A_0 - \frac{\varepsilon_0}{2} d + C$. Therefore, $v + (r + \varepsilon_0)\, d \in A_0 + C$. This implies that $I_C^{(3L)}(A_0; V, d) \leq r + \varepsilon_0 < r_0$, which contradicts with $r_0 = I_C^{(3L)}(A_0; V, d)$. This completes the proof. □

Proposition 4.27. *Let $A_0, V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int}(C)$. Assume that A_0 and V are compact. Then*

$$S_C^{(3U)}(\cdot; V, d) \textit{ is } (\preccurlyeq_2, C')\textit{-upper semicontinuous at } A_0.$$

Proof. By Propositions 2.15, 3.5, and 4.26, the proof is complete. □

Theorem 4.28. *Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int}(C)$. If F is $(\preccurlyeq_2, -C')$-continuous at x_0, then*

(i) $I_C^{(j)}(F(\cdot); V, d)$ *is upper semicontinuous at x_0, for all $j = 1, 3U$,*

(ii) $S_C^{(j)}(F(\cdot); V, d)$ *is lower semicontinuous at x_0, for all $j = 1, 3L$.*

Proof. By Theorems 3.2, 3.3 and Propositions 4.16, 4.17, the proof is complete. □

Theorem 4.29. *Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int}(C)$. If F is $(\preccurlyeq_2, C')$-continuous at x_0, then*

(i) $I_C^{(j)}(F(\cdot); V, d)$ *is lower semicontinuous at x_0, for all $j = 2L, 4$,*

(ii) $S_C^{(j)}(F(\cdot); V, d)$ *is upper semicontinuous at x_0, for all $j = 2U, 4$.*

Proof. By Theorems 3.2, 3.3 and Propositions 4.18, 4.19, the proof is complete. □

Theorem 4.30. *Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $V \in \mathcal{P}(Y)$, C' and C two nonempty convex cones such that $C' \subset C \neq Y$ and $d \in \operatorname{int}(C)$. Assume that $F(x_0)$ and V are compact.*

(i) *If F is $(\preccurlyeq_2, -C')$-continuous at x_0, then*
$I_C^{(2U)}(F(\cdot); V, d)$ *is upper semicontinuous at x_0, and*
$S_C^{(2L)}(F(\cdot); V, d)$ *is lower semicontinuous at x_0.*

(ii) *If F is $(\preccurlyeq_2, C')$-continuous at x_0, then*
$I_C^{(3L)}(F(\cdot); V, d)$ *is lower semicontinuous at x_0, and*
$S_C^{(3U)}(F(\cdot); V, d)$ *is upper semicontinuous at x_0.*

Proof. By Theorems 3.2, 3.3 and Propositions 4.22, 4.23, 4.26, and 4.27, the proof is complete. □

Remark 4.31. To compare with classical results in [5] and [14], Theorems 4.28 and 4.29 are generalizations of Theorems 3.3 and 3.4 in [5] with respect to $\preccurlyeq_2$-continuity. They appear similar aspects to Theorems 3.1, 3.2, 3.3 and 3.4 in [14]. Moreover, certain inheritance mechanism related to upper and lower continuity for set-valued maps can be examined for scalarization functions $I_C^{(j)}(\cdot; V, d)$ when $j = 2U, 3L$ and $S_C^{(j)}(\cdot; V, d)$ when $j = 2L, 3U$. Therefore, Theorem 4.30 solves some of the half-missing parts of results in [5] by compactness assumptions.

Corollary 4.32. *Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $V \in \mathcal{P}(Y)$, C a nonempty convex cone in Y such that $C \neq Y$ and $d \in \operatorname{int}(C)$. If F is upper continuous at x_0, then*

(i) $I_C^{(j)}(F(\cdot); V, d)$ *is upper semicontinuous at x_0, for all $j = 1, 3U$,*

(ii) $I_C^{(j)}(F(\cdot); V, d)$ *is lower semicontinuous at x_0, for all $j = 2L, 4$,*

(iii) $S_C^{(j)}(F(\cdot); V, d)$ *is lower semicontinuous at x_0, for all $j = 1, 3L$,*

(iv) $S_C^{(j)}(F(\cdot); V, d)$ *is upper semicontinuous at x_0, for all $j = 2U, 4$.*

Proof. Set $C' = \{0\}$ in Theorems 4.28 and 4.29. Then the statements follow from Remark 2.7. □

Corollary 4.33. *Let $F : X \to \mathcal{P}(Y)$, $x_0 \in X$, $V \in \mathcal{P}(Y)$, C a nonempty convex cones such that $C \neq Y$ and $d \in \operatorname{int}(C)$. Assume that $F(x_0)$ and V are compact. If F is upper continuous at x_0, then*

(i) $I_C^{(2U)}(F(\cdot); V, d)$ *is upper semicontinuous at* x_0,

(ii) $I_C^{(3L)}(F(\cdot); V, d)$ *is lower semicontinuous at* x_0,

(iii) $S_C^{(2L)}(F(\cdot); V, d)$ *is lower semicontinuous at* x_0,

(iv) $S_C^{(3U)}(F(\cdot); V, d)$ *is upper semicontinuous at* x_0.

Proof. Set $C' = \{0\}$ in Theorem 4.30 and then the statements follow from Remark 2.7. □

5. Examples

In this section, we provide some examples which, for comact sets $A_0, V \in \mathcal{P}(Y)$, a nonempty convex cone C and $d \in \operatorname{int}(C)$, the scalarization functions $I_C^{(2U)}(\cdot; V, d)$ and $I_C^{(3L)}(\cdot; V, d)$ are neither $(\preccurlyeq_1, C)$-lower semicontinuous nor $(\preccurlyeq_1, C)$-upper semicontinuous at A_0. Also, the same goes for $S_C^{(2L)}(\cdot; V, d)$ and $S_C^{(3U)}(\cdot; V, d)$.

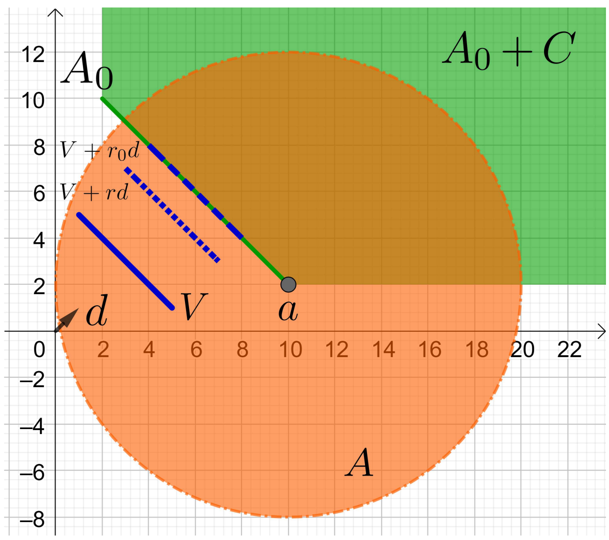

Figure 1: $I_C^{(3L)}(\cdot; V, d)$ is not $(\preccurlyeq_1, C)$-lower semicontinuous at A_0

Example 5.1. Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $A_0 = \{(x, y) : x + y = 12 \text{ and } 2 \leq x \leq 10\}$, $V = \{(x, y) : x + y = 6 \text{ and } 1 \leq x \leq 5\}$ and $d = (1, 1)$. Consider $r_0 := I_C^{(3L)}(A_0; V, d)$. Since the lines $x + y = 12$ and $x + y = 6$ are parallel, the distance between these lines is $3\sqrt{2}$. By $\|d\| = \sqrt{2}$, we can move the set V towards the direction d to satisfy $V + kd \subset A_0 + C$ when $k \geq \frac{3\sqrt{2}}{\sqrt{2}} = 3$, and hence we get $r_0 = 3$. Take $r = 2 < r_0$ and an open set W with $W \cap A_0 \neq \emptyset$, that is, there is $a \in A_0$ such that $a \in W$. Now, we take $A(\varepsilon) = B(a, \varepsilon)$, which is a ball around a with radius $\varepsilon > 0$. For any $\varepsilon > 0$

and convex cone $C' \subset C$, $A(\varepsilon) \cap (W \pm C') \neq \emptyset$. Set $\varepsilon = 10$ and $A := A(10)$, then we have $V + rd \subset A \subset A + C$. Hence $I_C^{(3L)}(A; V, d) < r$ as shown in Figure 1. Therefore $I_C^{(3L)}(\cdot; V, d)$ is not $(\preccurlyeq_1, C')$-lower semicontinuous at A_0. Additionally, this example can illustrate that $I_C^{(3L)}(\cdot; V, d)$ is not $(\preccurlyeq_1, -C')$-lower semicontinuous at A_0. Besides, $S_C^{(3U)}(\cdot; -V, d)$ is neither $(\preccurlyeq_1, C')$-upper semicontinuous nor $(\preccurlyeq_1, -C')$-upper semicontinuous at $-A_0$. □

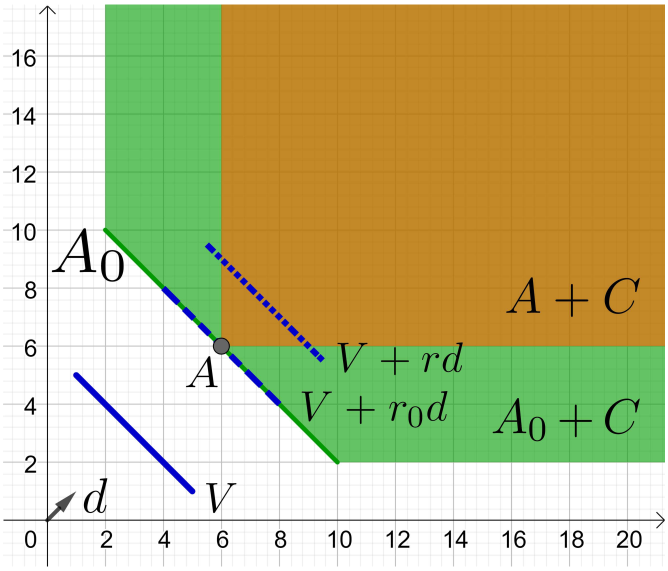

Figure 2: $I_C^{(3L)}(\cdot; V, d)$ is not $(\preccurlyeq_1, C)$-upper semicontinuous at A_0

Example 5.2. With the same settings of Example 5.1, we consider $(\preccurlyeq_1, \pm C)$-upper semicontinuity for $I_C^{(3L)}(\cdot; V, d)$. Note that $r_0 := I_C^{(3L)}(A_0; V, d) = 3$. Take $r \in (3, 5)$ and an open set W with $W \cap A_0 \neq \emptyset$. That is, we can consider $a \in A_0$ such that $a \in W$. Now, we set $A = \{a\}$, which implies that $A \cap (W \pm C') \neq \emptyset$ for any convex cone $C' \subset C$. However, $V + rd \not\subset A + C = \{a\} + \mathbb{R}^2_+$. Then $I_C^{(3L)}(A; V, d) \geq 5 > r$ as shown in Figure 2. Hence $I_C^{(3L)}(\cdot; V, d)$ is not $(\preccurlyeq_1, C')$-upper semicontinuous at A_0. Additionally, this example can illustrate that $I_C^{(3L)}(\cdot; V, d)$ is not $(\preccurlyeq_1, -C')$-upper semicontinuous at A_0. Besides, $S_C^{(3U)}(\cdot; -V, d)$ is neither $(\preccurlyeq_1, C')$-lower semicontinuous nor $(\preccurlyeq_1, -C')$-lower semicontinuous at $-A_0$.

Example 5.3. Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $A_0 = \{(x, y) : x + y = 12 \text{ and } 4 \leq x \leq 8\}$, $V = \{(x, y) : x + y = 6 \text{ and } 1 \leq x \leq 5\}$ and $d = (1, 1)$. Consider $r_0 := I_C^{(2U)}(A_0; V, d)$. We observe that $\bigcap_{a \in A_0}(a + C) = \{(x, y) : x \geq 8 \text{ and } y \geq 8\} = (8, 8) + \mathbb{R}^2_+$. Then we can move the set V towards the direction d to satisfy the condition $\left(\bigcap_{a \in A_0}(a + C)\right) \cap (V + kd) \neq \emptyset$ when $k \geq 5$; hence we get $r_0 = 5$. Take $r \in (3, 5)$ and an open set W with $W \cap A_0 \neq \emptyset$, that is, there is $a \in A_0$ such that $a \in W$. Now, we take $A = \{a\}$, which implies that $A \cap (W \pm C') \neq \emptyset$ for any convex cone $C' \subset C$. Also, $\bigcap_{a \in A}(a + C) = a + C$ and $(a + C) \cap (V + rd) \neq \emptyset$.

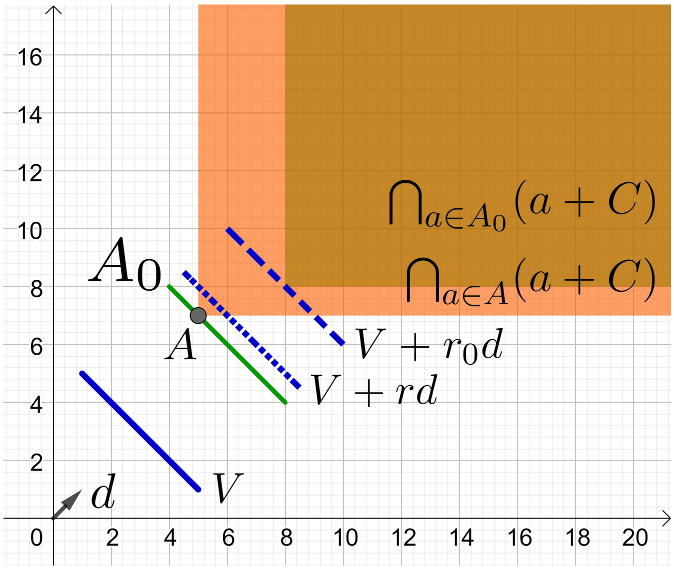

Figure 3: $I_C^{(2U)}(\cdot;V,d)$ is not $(\preccurlyeq_1,C)$-lower semicontinuous at A_0

Hence $I_C^{(2U)}(A;V,d)=3<r$ as shown in Figure 3. Therefore $I_C^{(2U)}(\cdot;V,d)$ is not $(\preccurlyeq_1,C')$-lower semicontinuous at A_0. Additionally, this example can illustrate that $I_C^{(2U)}(\cdot;V,d)$ is not $(\preccurlyeq_1,-C')$-lower semicontinuous at A_0. Besides, $S_C^{(2L)}(\cdot;-V,d)$ is neither $(\preccurlyeq_1,C')$-upper semicontinuous nor $(\preccurlyeq_1,-C')$-upper semicontinuous at $-A_0$.

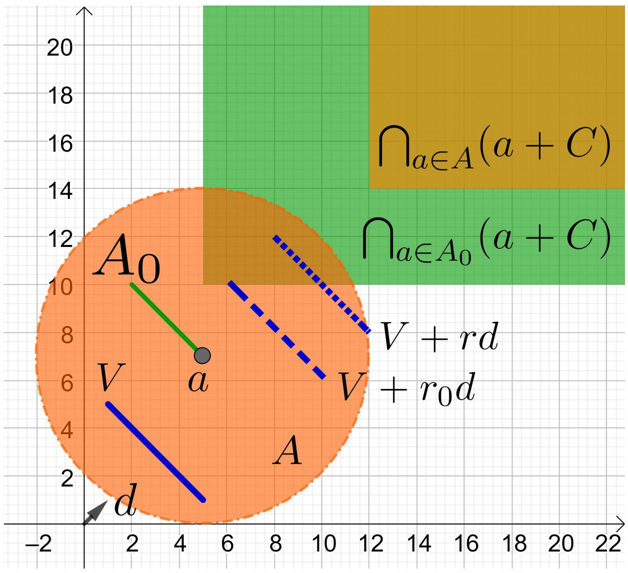

Figure 4: $I_C^{(2U)}(\cdot;V,d)$ is not $(\preccurlyeq_1,C)$-upper semicontinuous at A_0

Example 5.4. Let $Y=\mathbb{R}^2$, $C=\mathbb{R}^2_+$, $A_0=\{(x,y):x+y=12 \text{ and } 2\le x\le 5\}$, $V=\{(x,y):x+y=6 \text{ and } 1\le x\le 5\}$ and $d=(1,1)$. Consider $r_0:=I_C^{(2U)}(A_0;V,d)$. We observe that $\bigcap_{a\in A_0}(a+C)=\{(x,y):x\ge 5 \text{ and } y\ge 10\}=(5,10)+\mathbb{R}^2_+$. Then we can move the set V towards the direction d to satisfy the condition $\left(\bigcap_{a\in A_0}(a+C)\right)\cap(V+kd)\ne\emptyset$ when $k\ge 5$; hence we get $r_0=5$. Take $r>r_0$

and an open set W with $W \cap A_0 \neq \emptyset$, that is, there is $a \in A_0$ such that $a \in W$. In the same way as Example 5.1, we take $A(\varepsilon) = B(a, \varepsilon)$. For any $\varepsilon > 0$ and convex cone $C' \subset C$, $A(\varepsilon) \cap (W \pm C') \neq \emptyset$. Set $\varepsilon = r$ and $A := A(r)$, then we have $\left(\bigcap_{a \in A}(a + C)\right) \cap (V + rd) = \emptyset$. Hence $I_C^{(2U)}(A; V, d) > r$ as shown in Figure 4. Therefore $I_C^{(2U)}(\cdot; V, d)$ is not $(\preccurlyeq_1, C')$-upper semicontinuous at A_0. Additionally, this example can illustrate that $I_C^{(2U)}(\cdot; V, d)$ is not $(\preccurlyeq_1, -C')$-upper semicontinuous at A_0. Besides, $S_C^{(2L)}(\cdot; -V, d)$ is neither $(\preccurlyeq_1, C')$-lower semicontinuous nor $(\preccurlyeq_1, -C')$-lower semicontinuous at $-A_0$.

6. Conclusions

The classical semicontinuity of both scalarization functions and set-valued maps is extended to general concepts with respect to binary relations and cones in a topological vector space, called $(\preccurlyeq, C)$-lower (upper) semicontinuity and $(\preccurlyeq, C)$-continuity, respectively; $(\preccurlyeq, C)$-l.s.c., $(\preccurlyeq, C)$-u.s.c., and $(\preccurlyeq, C)$-conti. shortly in Tables 1, 2, 3.

As special cases for scalarization functions, the paper explores unified types defined by $I_C^{(j)}(\cdot; V, d)$ and $S_C^{(j)}(\cdot; V, d)$, for each $j = 1, 2L, 3L, 2U, 3U, 4$, as a class of sublinear-like scalarization. These scalarization functions are characterized by continuity properties as in Tables 1 and 2, respectively. In the tables, $V \in \mathcal{P}(Y)$, $d \in \operatorname{int}(C)$, C' and C are nonempty convex cones in Y such that $C' \subset C \neq Y$, and the tables have marks ✓, $*$, ✗ with proposition (Prop.) or example (Ex.) numbers.

- ✓ means that the property is satisfied;
- $*$ means that the property is satisfied under some compactness assumptions;
- ✗ means that the property is not satisfied even if compactness assumptions hold.

Accordingly, we can summarize inheritance properties on cone continuity for set-valued maps via scalarization in Table 3; the generalized continuity of parent set-valued maps is inherited to composite functions with scalarization functions. In the table, mark $(*)$ means that the property is satisfied under some compactness assumptions.

Table 1: Continuity properties of $I_C^{(j)}(\cdot; V, d)$ for each $j = 1, 2L, 3L, 2U, 3U, 4$.

	$(\preccurlyeq_1, C')$-l.s.c	$(\preccurlyeq_2, C')$-l.s.c	$(\preccurlyeq_1, -C')$-u.s.c	$(\preccurlyeq_2, -C')$-u.s.c
$I_C^{(1)}(\cdot; V, d)$	✓ (Prop. 4.5)			✓ (Prop. 4.16)
$I_C^{(2L)}(\cdot; V, d)$		✓ (Prop. 4.18)	✓ (Prop. 4.7)	
$I_C^{(3L)}(\cdot; V, d)$	✗ (Ex. 5.1)	$*$ (Prop. 4.26)	✗ (Ex. 5.2)	
$I_C^{(2U)}(\cdot; V, d)$	✗ (Ex. 5.3)		✗ (Ex. 5.4)	$*$ (Prop. 4.22)
$I_C^{(3U)}(\cdot; V, d)$	✓ (Prop. 4.5)			✓ (Prop. 4.16)
$I_C^{(4)}(\cdot; V, d)$		✓ (Prop. 4.18)	✓ (Prop. 4.7)	

Table 2: Continuity properties of $S_C^{(j)}(\cdot; V, d)$ for each $j = 1, 2L, 3L, 2U, 3U, 4$.

	$(\preceq_1, -C')$-l.s.c	$(\preceq_2, -C')$-l.s.c	$(\preceq_1, C')$-u.s.c	$(\preceq_2, C')$-u.s.c
$S_C^{(1)}(\cdot; V, d)$		✓ (Prop. 4.17)	✓ (Prop. 4.6)	
$S_C^{(2L)}(\cdot; V, d)$	✗ (Ex. 5.4)	$*$ (Prop. 4.23)	✗ (Ex. 5.3)	
$S_C^{(3L)}(\cdot; V, d)$		✓ (Prop. 4.17)	✓ (Prop. 4.6)	
$S_C^{(2U)}(\cdot; V, d)$	✓ (Prop. 4.8)			✓ (Prop. 4.19)
$S_C^{(3U)}(\cdot; V, d)$	✗ (Ex. 5.2)		✗ (Ex. 5.1)	$*$ (Prop. 4.27)
$S_C^{(4)}(\cdot; V, d)$	✓ (Prop. 4.8)			✓ (Prop. 4.19)

Table 3: Continuity properties of the composite functions.

F	$(\preceq_1, C')$ -conti.	$(\preceq_2, C')$ -conti.	$(\preceq_1, -C')$ -conti.	$(\preceq_2, -C')$ -conti.	l.c. $(C' = \{0\})$	u.c. $(C' = \{0\})$
$I_C^{(1)} \circ F$	l.s.c.	-	-	u.s.c.	l.s.c.	u.s.c.
$I_C^{(2L)} \circ F$	-	l.s.c.	u.s.c.	-	u.s.c.	l.s.c.
$I_C^{(3L)} \circ F$	-	l.s.c. $(*)$	-	-	-	l.s.c. $(*)$
$I_C^{(2U)} \circ F$	-	-	-	u.s.c. $(*)$	-	u.s.c. $(*)$
$I_C^{(3U)} \circ F$	l.s.c.	-	-	u.s.c.	l.s.c.	u.s.c.
$I_C^{(4)} \circ F$	-	l.s.c.	u.s.c.	-	u.s.c.	l.s.c.
$S_C^{(1)} \circ F$	u.s.c.	-	-	l.s.c.	u.s.c.	l.s.c.
$S_C^{(2L)} \circ F$	-	-	-	l.s.c. $(*)$	-	l.s.c. $(*)$
$S_C^{(3L)} \circ F$	u.s.c.	-	-	l.s.c.	u.s.c.	l.s.c.
$S_C^{(2U)} \circ F$	-	u.s.c.	l.s.c.	-	l.s.c.	u.s.c.
$S_C^{(3U)} \circ F$	-	u.s.c. $(*)$	-	-	-	u.s.c. $(*)$
$S_C^{(4)} \circ F$	-	u.s.c.	l.s.c.	-	l.s.c.	u.s.c.

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