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A Concave-Convex Ky Fan Minimax Inequality

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In the form most frequently used in scalar generalizations of the celebrated Ky Fan minimax inequality, some topological hypotheses and one convexity condition are assumed. In this work we introduce a Ky Fan minimax inequality where the convexity assumption is replaced by a concavity-convexity requirement: a slightly restrictive concavity property on one variable and a new convexity condition, necessary for the validity of such an inequality, in the another one. Our result is different from the aforementioned extensions of the Ky Fan minimax inequality and moreover avoids their vectorial setting.

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1. Minimax, Ky Fan minimax

Let X and Y be nonempty sets and let $f : X \times Y \longrightarrow \mathbb{R}$ be a scalar function. We can consider either the corresponding *minimax inequality* – in fact it is an equality, since the opposite inequality always holds –

$$\inf_{y\in Y} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} \inf_{y\in Y} f(x,y) \tag{MMI}$$

or, when $X = Y$, the *Ky Fan minimax inequality*, which strictly speaking is not such an inequality, and asserts that

$$\inf_{x\in X} f(x,x) \leq \sup_{x\in X} \inf_{y\in X} f(x,y)\,. \tag{KFI}$$

Very simple and well-known examples, such as the function $f : \{0,1\} \times \{0,1\} \longrightarrow \mathbb{R}$ defined at each $x, y = 0, 1$ by

$$f(x,y) := \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases},$$

and that we write as

$$\begin{array}{cccc} y & & & \\ 1 & \mathbf{0} & \mathbf{1} & \\ 0 & \mathbf{1} & \mathbf{0} & \\ & 0 & 1 & x \end{array},$$

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show that both the minimax inequality (MMI) and the Ky Fan minimax inequality (KFI) are not always satisfied. For this reason, a *minimax theorem* or *inequality* is understood to be a statement where, under certain hypotheses on X, Y or f, (MMI) is valid, in the same way that a *Ky Fan minimax inequality* or *theorem* guarantees the validity of (KFI), provided that X or f satisfy some additional assumptions.

The connection between these two kinds of inequalities for $X = Y$ is clear. Taking into account that, without a need for any hypothesis, it always holds that

$$\inf_{x\in X} f(x,x) \leq \inf_{y\in X} \sup_{x\in X} f(x,y),$$

then

$$\text{(MMI)} \;\Rightarrow\; \text{(KFI)},$$

although the converse is not true, as proven by the function $f : \{0,1\}\times\{0,1\} \longrightarrow \mathbb{R}$ give for each $x, y = 0, 1$ as

$$f(x,y) := \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases},$$

or in data form,

$$\begin{array}{cccc} y & & & \\ 1 & \mathbf{1} & \mathbf{0} & \\ 0 & \mathbf{0} & \mathbf{1} & \\ & 0 & 1 & x \end{array} \quad . \tag{1}$$

However, an important difference between these inequalities is that (KFI) can be strict: take

$$f(x,y) := 1 - x\,, \quad x, y \in [0,1]\,.$$

We illustrate both kinds of inequalities with two representative statements, although an overview of the subject can be found, for instance, in [29, 9, 26, 24, 14, 27, 6, 23, 28, 17, 2] and [8, 4, 1, 7, 31, 16, 5, 15, 20]. But first, let us mention that they usually impose a sort of generalized convexity for the involved function, on one or two of its variables. Let us also emphasize that we say that a function $f : X \times Y \longrightarrow \mathbb{R}$ satisfies a certain property on its first variable when for all $y \in Y$, the function $f(\cdot, y) : X \longrightarrow \mathbb{R}$, defined for each $x \in X$ by $f(\cdot, y) := f(x, y)$, also satisfies such a property. Analogously a concept on the second variable is defined. For example, if X is a nonempty set and Y is a nonempty convex subset of some vector space, a function $f : X \times Y \longrightarrow \mathbb{R}$ is *quasi-convex* on Y (see [29, 11, 10, 13, 19]) when $\{y \in Y : \; f(x,y) \leq b\}$ is convex, whenever $x \in X$ and $b \in \mathbb{R}$. Dually, a function $f : X \times Y \longrightarrow \mathbb{R}$, X being a nonempty convex subset of certain vector space and Y a nonempty set, is *quasi-concave* on X if for all $y \in Y$ and for all $a \in \mathbb{R}$, the set $\{x \in X : \; f(x,y) \geq a\}$ is convex.

For the minimax type, strictly speaking, let us mention the minimax theorem of M. Sion, [26, Corollary 3.3]. Note that its topological hypotheses allows us to replace "inf" by "min".

Theorem 1.1. *Let X and Y be nonempty convex subsets of some topological vector spaces, such that Y is compact, and let $f : X \times Y \longrightarrow \mathbb{R}$ be a function which is*

(i) *upper semicontinuous and quasi-concave on X and*
(ii) *lower semicontinuous and quasi-convex on Y.*

Then: $$\min_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X}\min_{y\in Y} f(x,y)\ .$$

Regarding the Ky Fan minimax inequality, the ideal example is, of course, the seminal inequality due to K. Fan, [8, Theorem 1]. Now "sup" becomes "max" thanks to the topological assumptions.

Theorem 1.2. *Suppose that X is a nonempty compact and convex subset of a certain topological vector space and that $f : X \times X \longrightarrow \mathbb{R}$ is a function which is*

(i) *upper semicontinuous on its first variable and*
(ii) *quasi-convex on its second variable.*

Then: $$\inf_{x\in X} f(x,x) \le \max_{x\in X}\inf_{y\in X} f(x,y)\ .$$

In the subsequent sections we introduce a quite general convexity concept for a function, *inf-diagonal convexity*, Definition 2.1, which is necessary for the Ky Fan minimax inequality (KFI) to hold. The main result is stated in Theorem 3.1, a minimax inequality of Ky Fan type where, in addition to suitable and standard topological hypotheses, a not very restrictive concavity condition on the first variable is imposed, as well as inf-diagonal convexity on the second one. This result is different from the classical Ky Fan inequality [8, Theorem 1] and from most of its scalar extensions, and does not require a vectorial setting.

2. Several concepts of generalized convexity

Focusing exclusively on minimax inequalities of the Ky Fan type, we consider some general notions of convexity and their connection with this kind of inequality. One of them, included in the next definition, is necessary in order that the Ky Fan minimax inequality (KFI) holds, as shown in Lemma 2.2 below. We agree that, given $m \ge 1$, $\mathbf{t}$ denotes the vector $(t_1, \dots, t_m) \in \mathbb{R}^m$ and that Δ_m is the *unit simplex* in $\mathbb{R}^m$, that is, $\mathbf{t} \in \Delta_m$ means

$$\mathbf{t} \in \mathbb{R}^m, \quad t_1, \dots, t_m \ge 0 \ \text{ and } \ \sum_{j=1}^m t_j = 1.$$

Definition 2.1. Let X be a nonempty set. A function $f : X \times X \longrightarrow \mathbb{R}$ is *inf-diagonally convex* on its second variable provided that

$$\left.\begin{array}{l} m \ge 1,\ \mathbf{t} \in \Delta_m \\ y_1, \dots, y_m \in X \end{array}\right\} \Rightarrow \inf_{x\in X} f(x,x) \le \sup_{x\in X} \sum_{j=1}^m t_j f(x,y_j).$$

This concept of convexity not only extends the usual one, but also generalizes the one given in [31, Definition 2.5], in a linear context: if X is a nonempty convex subset of a (locally convex topological) vector space, a function $f : X \times X \longrightarrow \mathbb{R}$ is said to be *diagonally convex* on its second variable when

$$\left.\begin{array}{r} m \geq 1,\ \mathbf{t} \in \Delta_m \\ y_1, \dots, y_m \in X \end{array}\right\} \Rightarrow f\left(\sum_{k=1}^m t_k y_k, \sum_{k=1}^m t_k y_k\right) \leq \sum_{j=1}^m t_j f\left(\sum_{k=1}^m t_k y_k, y_j\right).$$

The inf-diagonal convexity, which –as we have mentioned– is necessary for the validity of the inequality (KFI), has a certain parallel with a not very restrictive concept of convexity, which is also necessary for (MMI). It is the so-called *infsup-convexity*: for a nonempty set X, a function $f : X \times X \longrightarrow \mathbb{R}$ (in fact, it makes sense for functions defined in the product of two different nonempty sets) is *infsup-convex* on its second variable when

$$\left.\begin{array}{r} m \geq 1,\ \mathbf{t} \in \Delta_m \\ y_1, \dots, y_m \in X \end{array}\right\} \Rightarrow \inf_{y \in X} \sup_{x \in X} f(x,y) \leq \sup_{x \in X} \sum_{j=1}^m t_j f(x, y_j).$$

In a similar way, f is said to be *supinf-concave* on its first variable if

$$\left.\begin{array}{r} n \geq 1,\ \mathbf{s} \in \Delta_n \\ x_1, \dots, x_n \in X \end{array}\right\} \Rightarrow \inf_{y \in X} \sum_{i=1}^n s_i f(x_i, y) \leq \sup_{x \in X} \inf_{y \in X} f(x,y).$$

Let us recall that they were considered for the first time, as far as the author is aware, by A. Stefanescu as a hypothesis of [27, Theorem 4.1] without a concrete nomenclature, although some particular case ([14, p. 653]) arose in a natural way when dealing with equilibrium problems. We should point out that these are necessary conditions for the validity of the minimax inequality (MMI) (see [22, Lemma 4.1]) and that they are natural convexity properties in the minimax framework, as was shown in [21, Theorem 2.20]. In order to contextualize this notion of convexity –we restrict ourselves to the generalizations of convexity, since those for concavity are analogous– let us mention that K. Fan introduced the following generalization of convexity (see [9, p. 42]) in a nonvectorial setting, different from quasi-convexity: f is said to be *convexlike* on its second variable (once again it can be extended to functions defined in the cartesian product of two different sets) provided that

$$\left.\begin{array}{r} 0 \leq t \leq 1 \\ y_1, y_2 \in X \end{array}\right\} \Rightarrow \begin{array}{l} \text{there exists } y_0 \in X \text{ such that for all } x \in X \\ f(x, y_0) \leq t f(x, y_1) + (1-t) f(x, y_2)\ . \end{array}$$

More generally, one can consider the concept of convexity introduced in [21, Definition 2.1], motivated by finding quite general minimax inequalities: f is *sup-convexlike* on its second variable as soon as

$$\left.\begin{array}{r} m \geq 1,\ \mathbf{t} \in \Delta_m \\ y_1, \dots, y_m \in X \end{array}\right\} \Rightarrow \begin{array}{l} \text{there exists } y_0 \in X \text{ such that} \\ \sup_{x \in X} f(x, y_0) \leq \sup_{x \in X} \sum_{j=1}^m t_j f(x, y_j)\ , \end{array}$$

which in view of [21, Corollary 2.3 and Example 2.4] is strictly weaker than convexlikeness. This last notion of sup-convexity has an advantage over infsup-convexity (which is more general, see [22, Example 3.1]): one is under no obligation to compute $\inf_{y\in X}\sup_{x\in X} f(x,y)$.

The relationship between these concepts of convexity and the minimax inequality of Ky Fan (KFI) is reflected in the following result:

Lemma 2.2. *Let X be a nonempty set and let $f : X\times X \longrightarrow \mathbb{R}$ be a function. Then the following implications are valid:*

$$\text{(i)} \;\Rightarrow\; \text{(ii)} \;\Leftarrow\; \text{(iii)}\ ,$$

where

(i) *f satisfies the Ky Fan minimax inequality* (KFI),
(ii) *f is inf-diagonally convex on its second variable, and*
(iii) *f is infsup-convex on its second variable.*

Proof. It is clear that (ii) $\Leftarrow$ (iii), because

$$\begin{aligned}\inf_{x\in X} f(x,x) &\le \inf_{y\in X}\sup_{x\in X} f(x,y)\\ &\le \inf_{\substack{m\ge 1,\ \mathbf{t}\in\Delta_n\\ y_1,\dots,y_m\in X}} \sup_{x\in X}\sum_{j=1}^m t_j f(x,y_j) \qquad \text{(infsup-convexity)}\ .\end{aligned}$$

For (i) $\Rightarrow$ (ii), observe first the well-known fact that if X is a nonempty set and $f : X\times X \longrightarrow \mathbb{R}$ is any function, then given $n\ge 1$, $x_1,\dots,x_n\in X$, $\mathbf{s}\in\Delta_n$ and $m\ge 1$, $y_1,\dots,y_m\in X$, $\mathbf{t}\in\Delta_m$, there holds

$$\min_{j=1,\dots,m}\sum_{i=1}^n s_i f(x_i,y_j) \le \max_{i=1,\dots,n}\sum_{j=1}^m t_j f(x_i,y_j)\ ,$$

since

$$\begin{aligned}\min_{j=1,\dots,m}\sum_{i=1}^n s_i f(x_i,y_j) &\le \sum_{j=1}^m t_j\left(\sum_{i=1}^n s_i f(x_i,y_j)\right) = \sum_{i=1}^n s_i\left(\sum_{j=1}^m t_j f(x_i,y_j)\right)\\ &\le \max_{i=1,\dots,n}\sum_{j=1}^m t_j f(x_i,y_j)\ .\end{aligned}$$

In particular,

$$\sup_{\substack{n\ge 1,\ \mathbf{s}\in\Delta_n\\ x_1,\dots,x_n\in X}} \inf_{y\in X}\sum_{i=1}^n s_i f(x_i,y) \le \inf_{\substack{m\ge 1,\ \mathbf{t}\in\Delta_n\\ y_1,\dots,y_m\in X}} \sup_{x\in X}\sum_{j=1}^m t_j f(x,y_j), \tag{2}$$

and therefore

$$\begin{aligned}\inf_{x\in X} f(x,x) &\le \sup_{x\in X}\inf_{y\in X} f(x,y) \qquad \text{(by (KFI))}\\ &\le \sup_{\substack{n\ge 1,\ \mathbf{s}\in\Delta_n\\ x_1,\ldots,x_n\in X}} \inf_{y\in X}\sum_{i=1}^{n} s_i f(x_i,y)\\ &\le \inf_{\substack{m\ge 1,\ \mathbf{t}\in\Delta_n\\ y_1,\ldots,y_m\in X}} \sup_{x\in X}\sum_{j=1}^{m} t_j f(x,y_j) \qquad \text{(by (2))},\end{aligned}$$

hence f is inf-diagonally convex on its second variable. □

Now we show, through easy examples, the optimality of the connections established in Lemma 2.2.

Example 2.3. (f is inf-diagonally convex on its second variable $\not\Rightarrow$ (KFI).) It suffices to take the nonempty set $X := \{0,1,2,3\}$ and the function f defined in $X \times X$ as

$$\begin{array}{c|cccc} y & & & & \\ 3 & \mathbf{-1} & \mathbf{-1} & \mathbf{0} & \mathbf{0}\\ 2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ 1 & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1}\\ 0 & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1}\\ & 0 & 1 & 2 & 3 \quad x\end{array} .$$

It is obvious that f does not satisfy the Ky Fan minimax inequality (KFI):

$$\sup_{x\in X}\inf_{y\in X} f(x,y) = -1 < 0 = \inf_{x\in X} f(x,x) ,$$

but it is inf-diagonally convex on its second variable since

$$\min_{y\in X} \frac{f(2,y)+f(3,y)}{2} = 0 ,$$

hence

$$0 \le \sup_{\substack{n\ge 1,\ \mathbf{s}\in\Delta_n\\ x_1,\ldots,x_n\in X}} \inf_{y\in X}\sum_{i=1}^{n} s_i f(x_i,y)$$

and thus

$$\inf_{x\in X} f(x,x) \le \sup_{\substack{n\ge 1,\ \mathbf{s}\in\Delta_n\\ x_1,\ldots,x_n\in X}} \inf_{y\in X}\sum_{i=1}^{n} s_i f(x_i,y),$$

which according to inequality (2) implies

$$\inf_{x\in X} f(x,x) \le \inf_{\substack{m\ge 1,\ \mathbf{t}\in\Delta_n\\ y_1,\ldots,y_m\in X}} \sup_{x\in X}\sum_{j=1}^{m} t_j f(x,y_j) ,$$

i.e., f is inf-diagonally convex on its second variable. □

Let us observe that the fact that f is inf-diagonally convex on its second variable $\not\Rightarrow$ f is infsup-convex on its second variable, follows from Example 3.2 below.

Example 2.4. ((KFI) $\not\Rightarrow$ f is infsup-convex on its second variable.) Let $X := \{0,1\}$ and let $f : X \times X \longrightarrow \mathbb{R}$ be the function defined by the data (1), for which the inequality (KFI) is valid, but

$$\max_{x \in X} \frac{f(x,0) + f(x,1)}{2} \;=\; \frac{1}{2} \;<\; 1 \;=\inf_{y \in X} \sup_{x \in X} f(x,y) \;. \qquad \square$$

Finally:

Example 2.5. (f is infsup-convex on its second variable $\not\Rightarrow$ (KFI).) Take $f : (0,1) \times (0,1) \longrightarrow \mathbb{R}$ given by

$$f(x,y) := \begin{cases} 1, & \text{if } 0 < x \le y < 1 \\ 0, & \text{if } 0 < y < x < 1 \end{cases}$$

and notice three elementary facts. The first one is that

$$\inf_{x \in X} f(x,x) \;=\; 1 \;=\; \inf_{y \in X} \sup_{x \in X} f(x,y) \;;$$

the second one, that

$$\sup_{x \in X} \inf_{x \in X} f(x,y) = 0 \;;$$

and the third one is that whenever $m \ge 1$, $\mathbf{t} \in \Delta_m$ and $y_1, \dots, y_m \in (0,1)$, necessarily we arrive at

$$\sup_{x \in (0,1)} \sum_{j=1}^{m} t_j f(x, y_j) = 1 \;,$$

because $0 \le f \le 1$ and if $x_0 := \min\{y_1, \dots, y_m\}$, then

$$\sum_{j=1}^{m} t_j f(x_0, y_j) = 1 \;. \qquad \square$$

3. The Ky Fan inequality

The purpose of this section is to establish a Ky Fan minimax theorem different from the original one, [8, Theorem 1], under suitable topological hypotheses and convexity assumptions which do not involve any vectorial structure. In fact, unlike [8, Theorem 1] and most of its (scalar) extensions, we impose a concavity and a convexity condition on the first and second variable, respectively, the latter being weaker than quasi-convexity under the involved topological assumptions.

Let us first notice the following well-known fact (at least when X_1 below is empty: see the proofs of [9, Theorem 2], [3, Theorem A] or [24, Theorem 3.1]), which immediately follows from the finite intersection property (the details can be found in [21, Lemma 2.8]): if X is a nonempty compact topological space and f :

$X \times X \longrightarrow \mathbb{R}$ is an upper semicontinuous function on its first variable and $\alpha \in \mathbb{R}$, then

$$\alpha \leq \max_{x\in X} \inf_{y\in X} f(x,y)$$

if, and only if, there exists a finite subset X_1 of X such that

$$\left.\begin{array}{c} X_1 \subset X_0 \subset X \\ \emptyset \neq X_0 \text{ finite} \end{array}\right\} \Rightarrow \alpha \leq \max_{x\in X} \min_{y\in X_0} f(x,y).$$

In particular we deduce, under those topological assumptions, that

$$\inf_{x\in X} f(x,x) \leq \max_{x\in X} \inf_{y\in X} f(x,y)$$

as soon as there exists a finite subset X_1 of X in such a way that

$$\left.\begin{array}{c} X_1 \subset X_0 \subset X \\ \emptyset \neq X_0 \text{ finite} \end{array}\right\} \Rightarrow \inf_{x\in X} f(x,x) \leq \max_{x\in X} \min_{y\in X_0} f(x,y).$$

For a nonempty subset B of a set A and a function $h : A \longrightarrow \mathbb{R}$, we write $h_{|B}$ to denote the restriction of h to B.

Theorem 3.1. *Let X be a nonempty compact topological space and let $f : X \times X \longrightarrow \mathbb{R}$ be a function which is upper semicontinuous on its first variable and satisfies:*

(i) *there exists a finite subset X_1 of X such that*

$$\left.\begin{array}{c} X_1 \subset X_0 \subset X \\ \emptyset \neq X_0 \textit{ finite} \end{array}\right\} \Rightarrow f_{|X\times X_0} \textit{ is supinf-concave on its first variable, and}$$

(ii) *f is inf-diagonally convex on its second variable.*

Then: $\quad \inf_{x\in X} f(x,x) \leq \max_{x\in X} \inf_{y\in X} f(x,y)$.

Proof. If $\inf_{x\in X} f(x,x) = -\infty$ there is nothing to prove. Otherwise, the remark before the theorem guarantees us that it suffices to show, for the finite subset X_1 of X given in (i), that

$$\left.\begin{array}{c} X_1 \subset X_0 \subset X \\ \emptyset \neq X_0 \text{ finite} \end{array}\right\} \Rightarrow \inf_{x\in X} f(x,x) \leq \max_{x\in X} \min_{y\in X_0} f(x,y). \tag{3}$$

So let X_0 be a nonempty finite subset of X that contains X_1, which we can write as $X_0 = \{y_1, \ldots, y_m\}$. Let us denote by C the convex hull of the subset of $\mathbb{R}^m$ $\{(f(x,y_1), \ldots, f(x,y_m)) : \ x \in X\}$ and apply the Sion minimax theorem, Theorem 1.1, to the function $f : C \times \Delta_m \longrightarrow \mathbb{R}$ defined for each $(\mathbf{z}, \mathbf{t}) \in C \times \Delta_m$ as

$$f(\mathbf{z}, \mathbf{t}) := \sum_{j=1}^{m} t_j z_j \, .$$

Therefore we arrive at

$$\min_{\mathbf{t}\in\Delta_m}\sup_{\mathbf{z}\in C}\sum_{j=1}^m t_j z_j \le \sup_{\mathbf{z}\in C}\min_{\mathbf{t}\in\Delta_m}\sum_{j=1}^m t_j z_j \,,$$

equivalently,

$$\min_{\mathbf{t}\in\Delta_m}\sup_{\mathbf{z}\in C}\sum_{j=1}^m t_j z_j \le \sup_{\mathbf{z}\in C}\min_{j=1,\ldots,m} z_j \,,$$

that is,

$$\min_{\mathbf{t}\in\Delta_m}\sup_{\mathbf{z}\in C}\sum_{j=1}^m t_j z_j \le \sup_{\substack{n\ge 1,\ \mathbf{s}\in\Delta_n\\ x_1,\ldots,x_n\in X}}\min_{j=1,\ldots,m}\sum_{i=1}^n s_i f(x_i,y_j) \,,$$

and since, in view of the supinf-concavity of f, when restricted to $X\times X_0$, on X,

$$\sup_{\substack{n\ge 1,\ \mathbf{s}\in\Delta_n\\ x_1,\ldots,x_n\in X}}\min_{j=1,\ldots,m}\sum_{i=1}^n s_i f(x_i,y_j) = \sup_{x\in X}\min_{j=1,\ldots,m} f(x,y_j) \,,$$

then we have derived the existence of a $\mathbf{u}\in\Delta_m$ with

$$\max_{x\in X}\sum_{j=1}^m u_j f(x,y_j) \le \max_{x\in X}\min_{j=1,\ldots,m} f(x,y_j) \,. \tag{4}$$

But besides, f is inf-diagonally convex on its second variable, so

$$\inf_{x\in X} f(x,x) \le \max_{x\in X}\sum_{j=1}^m u_j f(x,y_j) \,. \tag{5}$$

Therefore, combining (4) and (5) yields (3), which completes the proof. □

Let us point out that the existence of a $\mathbf{u}\in\Delta_m$ satisfying (4) is not only necessary, but also sufficient, for the supinf-concavity of $f_{|X\times X_0}$ on its first variable, as shown in [22, Proposition 4.1 and Remark 4.1]. In [22] (see also [21, Proposition 2.9]) the proof of this fact relies on the Hahn–Banach theorem for real finite dimensional spaces; and here on Sion's minimax inequality, Theorem 1.1, which are equivalent results (see for instance [12, §2]). In fact, we have included the proof of the validity of (4) for some $\mathbf{u}\in\Delta_m$ for two reasons. First, for the sake of completeness; secondly to highlight the aforementioned equivalence. Let us emphasize that the sort of result given in (4) is motivated by the concave case stated by S. Simons in [25, Lemma 9].

Let us recall that condition (i) in Theorem 3.1 is a type of concavity, actually being stronger than the supinf-concavity of f on its first variable (see [21, Lemma 2.13 and Example 2.14] and [22, Lemma 4.3]). In addition, the converse of Theorem 3.1 is not true (consider the function $f:\{0,1\}\times\{0,1\}\longrightarrow\mathbb{R}$ defined by (1)).

Let us also observe that Theorem 3.1 does not follow from the minimax theorem [21, Theorem 2.15] and the fact that (MMI) implies (KFI). To this end, in the following example we give a function under the hypotheses of Theorem 3.1 but not under those in [21, Theorem 2.15].

Example 3.2. Let X be the compact set $[0,1]$ (usual topology) and define the function $f : X \times X \longrightarrow \mathbb{R}$ for each $(x,y) \in X \times X$ by

$$f(x,y) := \begin{cases} x, & \text{if } 0 \leq y \leq 0.5 \\ 1-x & \text{if } 0.5 < y \leq 1 \end{cases},$$

which is continuous on its first variable. The minimax theorem [21, Theorem 2.15] does not apply to f. Indeed, the minimax inequality (MMI) fails, since

$$\max_{x\in X} \min_{y \in X} f(x,y) \;=\; \max_{x\in X} \min\{x, 1-x\} \;=\; 0.5\,,$$

while

$$\min_{y\in X} \max_{x \in X} f(x,y) \;=\; \min_{y\in X} 1 \;=\; 1\,.$$

Let us check that, nevertheless, f satisfies the assumptions of Theorem 3.1. Condition (i) holds with $X_1 := \{0,1\}$: if $X_0 = \{y_1 := 0, y_2 := 1, y_3, \dots, y_m\}$ is contained in X; then given $n \geq 1$, $\mathbf{s} \in \Delta_n$ and $x_1, \dots, x_n \in X$ we have that

$$\begin{aligned} \min_{j=1,\dots,m} \sum_{i=1}^{n} s_i f(x_i, y_j) &= \min \left\{ \sum_{i=1}^{n} s_i f(x_i, 0), \sum_{i=1}^{n} s_i f(x_i, 1) \right\} \\ &= \min \left\{ \sum_{i=1}^{n} s_i x_i, 1 - \sum_{i=1}^{n} s_i x_i \right\} \\ &\leq 0.5 \;= \max_{x\in X} \min_{j=1,\dots,m} f(x, y_j). \end{aligned}$$

Finally, hypothesis (ii) in Theorem 3.1, inf-diagonal convexity of f on its second variable, is also satisfied, because $f \geq 0$ and $\inf_{x\in X} f(x,x) = 0$. □

Going back to the notion of diagonally convex function given in [31, Definition 2.5], a generalization of the Ky Fan minimax inequality is stated, in terms of the following weaker concept ([31, Definition 2.1]): if X is a nonempty convex subset of a real linear space, a function $f : X \times X \longrightarrow \mathbb{R}$ is *diagonally quasi-convex* on its second variable provided that

$$\left.\begin{array}{l} m \geq 1,\ \mathbf{t} \in \Delta_m \\ y_1, \dots, y_m \in X \end{array}\right\} \Rightarrow f\left(\sum_{k=1}^{m} t_k y_k, \sum_{k=1}^{m} t_k y_k \right) \leq \max_{j=1,\dots,m} f\left(\sum_{k=1}^{m} t_k y_k, y_j \right).$$

It is clear that a function is diagonally quasi-convex on its second variable as soon as it is quasi-convex on such a variable. The result in question, [31, Corollary 2.12], reads as follows: Suppose that X is a nonempty compact and convex subset of a certain real vector space and that $f : X \times X \longrightarrow \mathbb{R}$ is a function that is upper semicontinuous on its first variable and diagonally quasi-convex on its second variable. Then

$$\inf_{x \in X} f(x,x) \le \max_{x \in X} \inf_{y \in X} f(x,y) \ .$$

Let us notice that, since a vectorial setting is required for the diagonal convexity,

$$\left\{ \begin{array}{c} \text{inf-diagonal convexity} \\ \text{on second variable} \end{array} \right\} \not\Rightarrow \left\{ \begin{array}{c} \text{diagonal quasi-convexity} \\ \text{on second variable} \end{array} \right\} .$$

And conversely, Example 3.3 below shows that

$$\left\{ \begin{array}{c} \text{diagonal quasi-convexity} \\ \text{on second variable} \end{array} \right\} \not\Rightarrow \left\{ \begin{array}{c} \text{inf-diagonal convexity} \\ \text{on second variable} \end{array} \right\} .$$

What is more, it is proven in this example that

$$\left\{ \begin{array}{c} \text{quasi-convexity} \\ \text{on second variable} \end{array} \right\} \not\Rightarrow \left\{ \begin{array}{c} \text{inf-diagonal convexity} \\ \text{on second variable} \end{array} \right\} .$$

In particular, it follows that

$$\left\{ \begin{array}{c} \text{quasi-convexity} \\ \text{on second variable} \end{array} \right\} \not\Rightarrow \left\{ \begin{array}{c} \text{infsup-convexity} \\ \text{on second variable} \end{array} \right\} .$$

(the same example shows that "quasi-convexity on first variable $\not\Rightarrow$ supinf-concavity on first variable").

Example 3.3. (f quasi-convex on its second variable $\not\Rightarrow$ f is inf-diagonally convex on its second variable.) Let $h : [0,1] \longrightarrow \mathbb{R}$ be the function defined as

$$h(x) := \left\{ \begin{array}{rl} 1, & \text{if } 0 \le x \le 0.5 \\ -1, & \text{if } 0.5 < x \le 1 \end{array} \right. ,$$

and let $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ be the function given for each $0 \le x, y \le 1$ by

$$f(x,y) := h(x)\, h(y) \, .$$

It is clear that f is quasi-convex on its second variable, because it is monotone on it, but it is not inf-diagonally convex on its second variable, since

$$\inf_{x \in [0,1]} f(x,x) = 1 \, , \quad \text{while} \quad \sup_{x \in [0,1]} \frac{f(x,0) + f(x,1)}{2} = 0 \ . \qquad \square$$

However, according to [31, Corollary 2.12] and the fact that, in view of Lemma 2.2 the inf-diagonal convexity is a necessary condition for (KFI), we deduce that under the compactness of X and the upper semicontinuity of f on its first variable,

f is inf-diagonally convex on its second variable provided that it is diagonally quasi-convex on that variable. The same can be said about the diagonal convexity hypothesis in [15], despite the fact that it is more general than the inf-diagonal convexity, above all when used in the corresponding minimax Ky Fan type inequality ([15, Theorem 1], [30, Theorem 3.1]) falls on it. And we could continue, but the idea is always similar: if a result asserts that for a nonempty compact set X and a function $f : X \times X \longrightarrow \mathbb{R}$ which is upper semicontinuous on its first variable and satisfies any additional assumption (or assumptions), there holds

$$\inf_{x \in X} f(x,x) \leq \max_{x \in X} \inf_{y \in X} f(x,y)\,,$$

then, in view of Lemma 2.2, f is necessary inf-diagonally convex on its second variable. Notice that such a result cannot dispense with the additional assumption, as shown by the function of (1).

We finally show that Theorem 3.1 and [31, Corollary 2.12] are independent statements. Indeed, on the one hand, [31, Corollary 2.12] requires a linear framework, unlike Theorem 3.1. On the other hand:

Example 3.4. Consider the function $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ defined as

$$f(x,y) := \begin{cases} 1, & \text{if } (x,y) = (0,1) \text{ or } x = 1 \text{ and } 0 \leq y < 1, \\ 0, & \text{otherwise}. \end{cases}$$

By endowing the real interval $[0,1]$ with its usual topology, it becomes a nonempty compact topological space and f is an upper semicontinuous function on its first variable. Furthermore, f is quasi-convex on its second variable (it is monotone), hence it is diagonally quasi-convex. Therefore [31, Corollary 2.12] applies. However, the concavity assumption (i) in Theorem 3.1 is not satisfied: it suffices to take any finite subset X_0 of $[0,1]$ containing 0 and 1, let us say, $X_0 = \{y_1 = 0, y_2 = 1, \ldots, y_m\}$. Then, for $n = 2$, $\mathbf{s} = (0.5, 0.5)$ and $x_1 = 0$, $x_2 = 1$, we have that

$$\min_{j=1,\ldots,m} \frac{f(0,y_j) + f(1,y_j)}{2} = \frac{1}{2}\,, \quad \text{although} \quad \max_{x \in [0,1]} \min_{j=1,\ldots,m} f(x,y_j) = 0\,. \qquad \square$$

Let us notice that this same reasoning shows that the Ky Fan minimax inequality [8, Theorem 1] and Theorem 3.1 are equally independent.

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