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Some Qualitative Properties of Matrix Two-Person Games

Duong Thi Kim Huyen
Department of Mathematics, Hanoi Pedagogical University 2, Vinh Phuc, Vietnam
duongthikimhuyen@hpu2.edu.vn

Nguyen Dong Yen
Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi, Vietnam
ndyen@math.ac.vn

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This paper studies some qualitative aspects of matrix two-person games. The continuity of the characteristic sets and of the value of the games are established under suitable conditions. The upper semicontinuity of the characteristic sets and the density of the games having a unique optimal strategy for one player are obtained for matrix games of dimension 2×2. For further investigations, four open questions are formulated.

Keywords: Matrix two-person game, mixed strategy, linear program, characteristic set map, the value of the game function, characteristic number, density.

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1. Introduction

In two-person zero-sum games, one player's gains are the other's losses and vice versa. A two-person zero-sum game in matrix form is defined by a matrix, whose elements are the payoffs to the first player – those for the second player are the negative of these values. Such games have the origin in the pioneering paper [12] of von Neumann and the landmark book [13] of von Neumann and Morgenstern. Briefly called *matrix two-person games*, the games have been discussed by many authors (see, e.g., Aubin [1, Subsect. 6.5.4] and Aubin and Ekeland [2, Sect. 2 in Chapter 6]). We refer to Barron [4, Chapters 1 and 2] for a nice introduction to matrix two-person games and Kuhn [8] for a comprehensive exposition of the history of the game theory together with its basic concepts, results, and some illustrative applications.

Despite the fact that the solution methods for matrix two-person games have been studied in great details (see, e.g., [4, Chapter 2]), there are only a few papers addressing qualitative properties of the games. For symmetric matrix two-person games, Stanford [11, Proposition 2] has proposed an algebraic procedure to test the uniqueness of a given optimal strategy of one player and given a characterization of non-unique solutions. Recently, using the concept of minimax variational inequality in [5] and Robinson's results on generalized equations [9], Huyen and Yao [6] have studied the structure of the solution set of a general matrix two-person game and its stability with respect to a small change of the payoffs matrix. Then, focusing on the extreme points of the solution sets of matrix two-person games, Huyen, Yao and Yen [7] have defined the notions of characteristic sets and characteristic numbers, which allow ones to look deeper into the structure of the games. In [7], upper bounds for the characteristic numbers have been obtained and a geometric construction for describing the optimal strategy set of each player of a game has been given.

The qualitative analyses of matrix two-person games of [6] and [7] are the basis for our investigations in this paper. We will obtain some theorems on the continuity of the characteristic set maps, the value of the game function, and the density of the games having a unique optimal strategy for one player. To the best of our knowledge, there are no similar results in the existing literature on matrix two-person games. It is worth to stress that hard open questions remain in this topic.

In Section 2, we recall the notations and auxiliary results needed for our subsequent study. Section 3 is devoted to the continuity of the characteristic sets and of the value of the game. In Section 4, we investigate the upper semicontinuity of the characteristic sets for matrix games of dimension 2 × 2. The density of the games of dimension 2 × 2 having a unique optimal strategy for one player is proved in Section 5. The last section presents some concluding remarks and open questions.

2. Notations and auxiliary results

The closed unit ball (resp., the closed ball centered at a with radius $\delta > 0$) in an Euclidean space $\mathbb{R}^k$ is denoted by $\bar{B}_{\mathbb{R}^k}$ (resp., $\bar{B}(a,\delta)$). The corresponding open balls are abbreviated respectively to $B_{\mathbb{R}^k}$ and $B(a,\delta)$. As usual, $\mathbb{R}^k_+$ stands for the nonnegative orthant of $\mathbb{R}^k$ and the scalar product of $x^1, x^2 \in \mathbb{R}^k$ is denoted by $\langle x^1, x^2\rangle$. For a subset $\Omega \subset \mathbb{R}^k$, by $\overline{\Omega}$, $\operatorname{int}\Omega$, and $\partial\Omega$ we denote the *closure*, the *interior*, and the *boundary* of Ω, respectively. The distance from x to Ω is $d(x,\Omega) := \inf_{y\in\Omega}\|x-y\|$, where $\inf\emptyset = +\infty$ by convention. For any subsets Ω_1, Ω_2 of $\mathbb{R}^k$, the *excess* of Ω_1 over Ω_2, denoted by $e(\Omega_1,\Omega_2)$, is defined by setting $e(\Omega_1,\Omega_2) = \sup_{x\in\Omega_1} d(x,\Omega_2)$. The *Hausdorff distance* between Ω_1 and Ω_2 is given by $h(\Omega_1,\Omega_2) = \max\{e(\Omega_1,\Omega_2), e(\Omega_2,\Omega_1)\}$.

Let $G : \mathbb{R}^{k_1} \rightrightarrows \mathbb{R}^{k_2}$ be a set-valued map between two Euclidean spaces. As in [3], we say that G is *upper semicontinuous* (usc) at $u \in \mathbb{R}^{k_1}$ if for every open set $V \subset \mathbb{R}^{k_2}$ satisfying $G(u) \subset V$ there exists a neighborhood U of u, such that $G(u') \subset V$ for all $u' \in U$. If for every open set $V \subset \mathbb{R}^{k_2}$ satisfying $G(u) \cap V \neq \emptyset$ there exists a neighborhood U of $u \in \mathbb{R}^{k_1}$, such that $G(u') \cap V \neq \emptyset$ for all $u' \in U$, then we say that G is *lower semicontinuous* (lsc) at u. If G is both usc and lsc at u, then G is said to be *continuous* at u.

The norm in the product space $\mathbb{R}^{k_1+k_2} = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ is defined by setting

$$\|(u,v)\| = \left(\|u\|^2 + \|v\|^2\right)^{1/2} \qquad \forall (u,v) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}. \tag{1}$$

Since the following elementary result will play an important role in our subsequent arguments, a detailed proof is provided for clarity.

Lemma 2.1. *Let* Ω_1 *and* $\tilde{\Omega}_1$ *(resp.,* Ω_2 *and* $\tilde{\Omega}_2$*) be subsets of* $\mathbb{R}^{k_1}$ *(resp.,* $\mathbb{R}^{k_2}$*). Then,*

$$\frac{1}{\sqrt{2}}\Big[e\big(\tilde{\Omega}_1,\Omega_1\big)+e\big(\tilde{\Omega}_2,\Omega_2\big)\Big]\le e\big(\tilde{\Omega}_1\times\tilde{\Omega}_2,\Omega_1\times\Omega_2\big)\le e\big(\tilde{\Omega}_1,\Omega_1\big)+e\big(\tilde{\Omega}_2,\Omega_2\big) \quad (2)$$

and

$$\frac{1}{2\sqrt{2}}\Big[h\big(\tilde{\Omega}_1,\Omega_1\big)+h\big(\tilde{\Omega}_2,\Omega_2\big)\Big]\le h\big(\tilde{\Omega}_1\times\tilde{\Omega}_2,\Omega_1\times\Omega_2\big)\le h\big(\tilde{\Omega}_1,\Omega_1\big)+h\big(\tilde{\Omega}_2,\Omega_2\big). \quad (3)$$

Proof. For any $(\tilde{u},\tilde{v})\in\tilde{\Omega}_1\times\tilde{\Omega}_2$ and $(u,v)\in\Omega_1\times\Omega_2$, by using (1) we have

$$\|(\tilde{u},\tilde{v})-(u,v)\|=\left(\|\tilde{u}-u\|^2+\|\tilde{v}-v\|^2\right)^{1/2}.$$

Hence, $\frac{1}{\sqrt{2}}\left[\|\tilde{u}-u\|+\|\tilde{v}-v\|\right]\le\|(\tilde{u},\tilde{v})-(u,v)\|\le\|\tilde{u}-u\|+\|\tilde{v}-v\|.$

Taking the infimum of each expression in these inequalities on $(u,v)\in\Omega_1\times\Omega_2$ gives

$$\frac{1}{\sqrt{2}}\left[d\left(\tilde{u},\Omega_1\right)+d\left(\tilde{v},\Omega_2\right)\right]\le d\left((\tilde{u},\tilde{v}),\Omega_1\times\Omega_2\right)\le d\left(\tilde{u},\Omega_1\right)+d\left(\tilde{v},\Omega_2\right).$$

Now, taking supremum of each expression in the last inequalities on $(\tilde{u},\tilde{v})\in\tilde{\Omega}_1\times\tilde{\Omega}_2$, we obtain (2). Then, by the symmetry we have

$$\frac{1}{\sqrt{2}}\Big[e\big(\Omega_1,\tilde{\Omega}_1\big)+e\big(\tilde{\Omega}_2,\Omega_2\big)\Big]\le e\big(\Omega_1\times\Omega_2,\tilde{\Omega}_1\times\tilde{\Omega}_2\big)\le e\big(\Omega_1,\tilde{\Omega}_1\big)+e\big(\Omega_2,\tilde{\Omega}_2\big). \quad (4)$$

Now, combining the second inequality in (2) and the second inequality in (4) with the definition of Hausdorff distance, we get the second inequality in (3). Finally, from the first inequality in (2), the first inequality in (4), and the definition of Hausdorff distance, we obtain

$$\frac{1}{\sqrt{2}}h\big(\tilde{\Omega}_1,\Omega_1\big)\le h\big(\tilde{\Omega}_1\times\tilde{\Omega}_2,\Omega_1\times\Omega_2\big)$$

and

$$\frac{1}{\sqrt{2}}h\big(\tilde{\Omega}_2,\Omega_2\big)\le h\big(\tilde{\Omega}_1\times\tilde{\Omega}_2,\Omega_1\times\Omega_2\big).$$

Adding these inequalities and dividing the resulted inequality by 2, we get the first inequality in (3). □

Following [4, Chapter 1] and [6], we now recall the concept of matrix two-person game. Let P_1 and P_2 be two players. Suppose that P_1 (resp., P_2) makes a choice from n possible strategies (resp., m possible strategies). If P_1 chooses a strategy $i\in I$ and P_2 chooses a strategy $j\in J$, where $I:=\{1,...,n\}$ and $J:=\{1,...,m\}$, then they play the game. In this *zero-sum game*, whatever one player wins, the other loses. Namely, if a_{ij} is the *payoff* of P_1, then the payoff of P_2 is $-a_{ij}$. Both players want to *maximize* their individual payoffs. The $n\times m$ matrix $A=(a_{ij})$ is called the *payoff matrix* or the *game matrix.*

If player P_2 has chosen a strategy $j_*\in J$, then a strategy $i_*\in I$ satisfying $a_{ij_*}\le a_{i_*j_*}$ for all $i\in I$ is best for player P_1. Similarly, if player P_1 has chosen a strategy $i_*\in I$, then a strategy $j_*\in J$ satisfying $-a_{i_*j_*}\ge -a_{i_*j}$ for all $j\in J$ is best for player P_2. The last condition can be rewritten equivalently as $a_{i_*j_*}\le a_{i_*j}$ for all $j\in J$. These observations are the motivations for the next fundamental definition.

Definition 2.2. (See [4, Definition 1.1.2]) A pair $(i_*, j_*) \in I \times J$ is called a *saddle point in pure strategies* of the game if

$$a_{ij_*} \leq a_{i_*j_*} \leq a_{i_*j} \quad \forall (i,j) \in I \times J. \tag{5}$$

In general, there may not exist any pair $(i_*, j_*) \in I \times J$ fulfilling (5). In other words, the game may not have any solution in pure strategies. In this case, the players choose their strategies randomly, i.e., they play the game randomly. To get the best possible individual average payoff after a series of games, each player has to abide by a mixed strategy. The latter is defined as follows.

Suppose that the players play the game many times. A *mixed strategy* [4, Definition 1.13] for P_1 is a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with

$$x_i \geq 0 \ \ \forall i \in I, \quad \sum_{i \in I} x_i = 1. \tag{6}$$

Denote by S_n the set of $x = (x_1, \ldots, x_n)$ satisfying (6). The choice of a mixed strategy $x \in S_n$ of P_1 means that this player chooses strategy $i \in I$ with the probability x_i. Similarly, a mixed strategy for P_2 is a vector $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ with

$$y_j \geq 0 \ \ \forall j \in J, \quad \sum_{j \in J} y_j = 1. \tag{7}$$

Let S_m be the set of $y = (y_1, \ldots, y_m)$ satisfying (7). The choice of a mixed strategy $y \in S_m$ of P_2 means that the player chooses strategy $j \in J$ with the probability y_j.

If P_1 abides by a mixed strategy $x \in S_n$ and P_2 abides by a mixed strategy $y \in S_m$, then the expected payoff to P_1 of the game is $x^T Ay$. Put $f(x,y) = x^T Ay$. A *saddle point* in mixed strategies (see [4, Definition 1.3.3]) is a pair $(\bar{x}, \bar{y})$ from $S_n \times S_m$ satisfying

$$f(x, \bar{y}) \leq f(\bar{x}, \bar{y}) \leq f(\bar{x}, y) \qquad \forall x \in S_n, \ \forall y \in S_m.$$

As shown by von Neumann [12] (see also [4, Theorem 1.3.4]), such a saddle point is always available. The set of saddle points in mixed strategies of the game is denoted by $S(A)$. For a saddle point $(\bar{x}, \bar{y}) \in S(A)$, the first component $\bar{x}$ (resp., the second component $\bar{y}$) is said to be an *optimal strategy* of player P_1 (resp., of player P_2). Note that the values

$$\min_{y \in S_m} \max_{x \in S_n} f(x,y) \quad \text{and} \quad \max_{x \in S_n} \min_{y \in S_m} f(x,y)$$

coincide. As in [4, Theorem 1.3.4], the just mentioned common value is called the *value of the game* and it is denoted by $v(A)$. For any $(\bar{x}, \bar{y}) \in S(A)$, one has $f(\bar{x}, \bar{y}) = v(A)$.

Theorem 2.3. (See [6, Theorem 3.1]) *The saddle point set in mixed strategies of a matrix two-person game is a nonempty compact polyhedral convex set. The set of optimal strategies of each player is also nonempty compact polyhedral convex.*

Given a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times m}$, we consider the linear program

$$\begin{cases} \text{Minimize} \ \ \displaystyle\sum_{i\in I} p_i \\ \text{subject to} \ \ \displaystyle\sum_{i\in I} a_{ij}p_i \geq 1, \ j \in J, \ \ p_i \geq 0, \ i \in I. \end{cases} \tag{8}$$

The dual problem of (8) is the linear program

$$\begin{cases} \text{Maximize} \ \ \displaystyle\sum_{j\in J} q_j \\ \text{subject to} \ \ \displaystyle\sum_{j\in J} a_{ij}q_j \leq 1, \ i \in I, \ \ q_j \geq 0, \ j \in J. \end{cases} \tag{9}$$

The problem of solving a matrix two-person game is reduced to solving the above pair of linear programs in the following way.

Theorem 2.4. (See, e.g., [4, pp. 83–85] and [7, Theorem 3.2]) *For every $n \times m$ matrix A, one can find a positive constant μ such that if A_μ is the matrix obtained from A by adding a positive number μ to every element a_{ij}, then $v(A_\mu) > 0$. With such a number μ, the following properties hold for the linear programs* (8) *and* (9) *where a_{ij} stands for the (i, j) element of A_μ for any $i \in I$ and $j \in J$:*

(a) *If $\bar p = (\bar p_1, \ldots, \bar p_n)$ is an optimal solution of* (8) *with the corresponding optimal value $\bar\alpha$, then the vector $\frac{1}{\bar\alpha}\,\bar p$ is an optimal strategy of player P_1 and $\frac{1}{\bar\alpha} - \mu$ is the value of the game defined by A. Moreover, every optimal strategy of player P_1 can be obtained in this way.*

(b) *If $\bar q = (\bar q_1, \ldots, \bar q_m)$ is an optimal solution of* (9) *with the corresponding optimal value $\bar\beta$, then the vector $\frac{1}{\bar\beta}\,\bar q$ is an optimal strategy of player P_2 and $\frac{1}{\bar\beta} - \mu$ is the value of the game defined by A. In addition, every optimal strategy of player P_2 can be obtained in this way.*

In connection with Theorem 2.4, it is assumed that for a given $n \times m$ matrix A, a constant $\mu > 0$ has been chosen so that the matrix A_μ obtained from A by adding a positive number μ to every element a_{ij} has the property $v(A_\mu) > 0$, where $v(A_\mu)$ is the value of the game described by A_μ. *In the sequel, the linear programs* (8) *and* (9) *will be considered with $A = A_\mu$.* Thus, both problems have solutions, the corresponding optimal values $\bar\alpha$ and $\bar\beta$ are positive, and $\bar\beta = \bar\alpha$.

Following [7], we denote the *optimal strategy set* of player P_1 (resp., of player P_2) by $S_1(A)$ (resp., $S_2(A)$). The set of extreme points of $S_1(A)$ (resp., $S_2(A)$) is abbreviated to $E_1(A)$ (resp., $E_2(A)$).

Theorem 2.5. (See [7, Theorem 3.3]) *One has $S_1(A) = \frac{1}{\bar\alpha}\mathcal{P}$, $S_2(A) = \frac{1}{\bar\beta}\mathcal{D}$,*

$$S(A) = S_1(A) \times S_2(A), \tag{10}$$

where $\mathcal{P}$ and $\bar\alpha > 0$ (resp., $\mathcal{D}$ and $\bar\beta > 0$) are the solution set and the optimal value of (8) *(resp., of* (9)*). In addition,*

$$E(A) = E_1(A) \times E_2(A). \tag{11}$$

Formula (10) shows that the saddle point set in mixed strategies of a matrix two-person game has a decomposition property. Namely, $S(A)$ is the product of the optimal strategy sets of the two players. Since $S(A)$ is a nonempty compact polyhedral convex set by Theorem 2.3, from (10) it follows that both sets $S_1(A)$ and $S_2(A)$ are nonempty compact polyhedral convex.

Definition 2.6. (See [7]) The sets $E(A)$, $E_1(A)$, and $E_2(A)$ are said to be *characteristic sets* of the matrix two-person game defined by matrix A. The numbers of elements of $E(A)$, $E_1(A)$, $E_2(A)$ are denoted respectively by $\chi(A)$, $\chi_1(A)$, $\chi_2(A)$, and are called *characteristic numbers* of the game.

It has been noted in [7, Remark 3.7] that, for any matrix $A \in \mathbb{R}^{n\times m}$, the numbers $\chi(A)$, $\chi_1(A)$, $\chi_2(A)$ are positive integers and the equality $\chi(A) = \chi_1(A)\chi_2(A)$ holds.

According to [7, Remark 3.7], the number $\chi(A)$ is a *measure of complexity* of the game defined A: the game is complex if $\chi(A)$ is large. Similarly, the number $\chi_1(A)$ (resp., $\chi_2(A)$) is a measure of complexity of the game from the point of view of player P_1 (resp., player P_2). If $\chi(A) = 1$ (in this situation, one has $\chi_1(A) = \chi_2(A) = 1$), then the game can be regarded as a simple one.

Following [7], we put $\Delta = \mathrm{co}\,\{a_{1.}, a_{2.}, \ldots, a_{n.}\}$, where

$$a_{1.} = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1m} \end{bmatrix}, \; a_{2.} = \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix}, \ldots, \; a_{n.} = \begin{bmatrix} a_{n1} \\ a_{12} \\ \vdots \\ a_{nm} \end{bmatrix}$$

are the column vectors of A^T, and build the "carriage"

$$C(\alpha) := \frac{1}{\alpha}J_m + \mathbb{R}^m_+ \quad (\alpha > 0),$$

where J_m stands for the column vector consisting of m numbers 1. Consider the conditions

$$\Delta \cap (\mathrm{int}\, C(\bar{\alpha})) = \emptyset, \tag{12}$$

$$\Delta \cap (\partial C(\bar{\alpha})) \neq \emptyset. \tag{13}$$

Theorem 2.7. (See [7, Theorem 5.4]) *The couple of conditions* (12) *and* (13) *is necessary and sufficient for* $\bar{\alpha} > 0$ *to be the optimal value of* (8). *In addition, if* $n = m$, *the matrix* A *is nonsingular, and* $\bar{\alpha} > 0$ *is the optimal value of* (8), *then the solution set of* (8) *and the optimal strategy set of player* P_1 *are respectively computed by the formulas*

$$\mathcal{P} = \bar{\alpha}(A^T)^{-1}\,[\Delta \cap (\partial C(\bar{\alpha}))] \tag{14}$$

and

$$S_1(A) = (A^T)^{-1}\,[\Delta \cap (\partial C(\bar{\alpha}))]\,. \tag{15}$$

We will need the next fundamental stability property of matrix two-person games.

Theorem 2.8. (See [6, Theorem 4.1]) *The saddle point set map in mixed strategies $S(\cdot)$ of the matrix two-person game is locally upper Lipschitz everywhere, i.e., for any matrix $A \in \mathbb{R}^{n\times m}$, there exist constants $\varepsilon_A > 0$ and $\ell_A > 0$ such that*

$$S(\tilde{A}) \subset S(A) + \ell_A \|\tilde{A} - A\| \bar{B}_{\mathbb{R}^n \times \mathbb{R}^m}$$

for any $\tilde{A} \in \mathbb{R}^{n\times m}$ satisfying $\|\tilde{A} - A\| < \varepsilon_A$.

3. Continuity of the characteristic sets and of the value of the game

The first result of this section gives a sufficient condition for the continuity of the characteristic sets with respect to the perturbation of the matrix A.

Theorem 3.1. *If $S(A)$ is a singleton, then the characteristic set maps*

$$E(\cdot) : \mathbb{R}^{n\times m} \rightrightarrows \mathbb{R}^{n+m}, \quad E_1(\cdot) : \mathbb{R}^{n\times m} \rightrightarrows \mathbb{R}^{n}, \quad E_2(\cdot) : \mathbb{R}^{n\times m} \rightrightarrows \mathbb{R}^{m}$$

are continuous at A. Moreover, there exist constants $\varepsilon_A > 0$ and $\ell_A > 0$ such that

$$h(E(\tilde{A}), E(A)) \le \ell_A \|\tilde{A} - A\| \tag{16}$$

and

$$h(E_1(\tilde{A}), E_1(A)) \le 2\sqrt{2}\, \ell_A \|\tilde{A} - A\|, \quad h(E_2(\tilde{A}), E_2(A)) \le 2\sqrt{2}\, \ell_A \|\tilde{A} - A\| \tag{17}$$

for every matrix $\tilde{A} \in \mathbb{R}^{n\times m}$ satisfying $\|\tilde{A} - A\| < \varepsilon_A$.

Proof. By Theorem 2.8, we can find constants $\varepsilon_A > 0$ and $\ell_A > 0$ such that the inclusion

$$S(\tilde{A}) \subset S(A) + \ell_A \|\tilde{A} - A\| \bar{B}_{\mathbb{R}^n \times \mathbb{R}^m} \tag{18}$$

holds for any $\tilde{A} \in \mathbb{R}^{n\times m}$ satisfying $\|\tilde{A} - A\| < \varepsilon_A$. Clearly, (18) implies that

$$e(S(\tilde{A}), S(A)) \le \ell_A \|\tilde{A} - A\|. \tag{19}$$

By our assumption, $S(A)$ has just one element. Hence, $E(A) = S(A)$ and from (19) it follows that $e(S(A), S(\tilde{A}))) \le \ell_A \|\tilde{A} - A\|$. Moreover, since $E(\tilde{A}) \subset S(\tilde{A})$ and $E(A)$ is a singleton, by (19) we have

$$e(E(\tilde{A}), E(A)) \le \ell_A \|\tilde{A} - A\| \tag{20}$$

and

$$e(E(A), E(\tilde{A})) \le \ell_A \|\tilde{A} - A\|. \tag{21}$$

Combining (20) with (21), we obtain the inequality (16) for every matrix $\tilde{A} \in \mathbb{R}^{n\times m}$ satisfying $\|\tilde{A} - A\| < \varepsilon_A$.

For any matrix $\tilde{A} \in \mathbb{R}^{n\times m}$ with $\|\tilde{A} - A\| < \varepsilon_A$, by Theorem 2.5 one has

$$E(A) = E_1(A) \times E_2(A), \quad E(\tilde{A}) = E_1(\tilde{A}) \times E_2(\tilde{A}).$$

So, applying the second assertion of Lemma 2.1 to the sets $E_1(A)$, $E_1(\tilde{A})$, $E_2(A)$, $E_2(\tilde{A})$ gives

$$\begin{aligned} &\frac{1}{2\sqrt{2}} \left[h\left(E_1(\tilde{A}), E_1(A)\right) + h\left(E_2(\tilde{A}), E_2(A)\right)\right] \\ &\le h\left(E(\tilde{A}), E(A)\right) \le h\left(E_1(\tilde{A}), E_1(A)\right) + h\left(E_2(\tilde{A}), E_2(A)\right). \end{aligned} \tag{22}$$

From the first inequality in (22) and the estimate (16) we get the estimate

$$h\left(E_1(\tilde{A}), E_1(A)\right) + h\left(E_2(\tilde{A}), E_2(A)\right) \le 2\sqrt{2}\,\ell_A\|\tilde{A} - A\|,$$

which clearly implies (17).

Now, it remains to show that the fulfillment of (16) and (17) for every matrix $\tilde{A} \in \mathbb{R}^{n\times m}$ satisfying $\|\tilde{A} - A\| < \varepsilon_A$ implies the desired continuity of the set-valued maps $E(\cdot), E_1(\cdot)$ and $E_2(\cdot)$ at A.

Suppose that $E(A) = \{\bar{z}\}$, where $\bar{z} = (\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$. Let $W \subset \mathbb{R}^n \times \mathbb{R}^m$ be any open set containing $E(A)$. Choose $\rho > 0$ as small as $\bar{B}(\bar{z}, \rho) \subset W$ and put

$$\varepsilon_1 = \min\left\{\varepsilon_A, \frac{\rho}{\ell_A}\right\}.$$

Then, for any matrix $\tilde{A} \in \mathbb{R}^{n\times m}$ satisfying $\|\tilde{A} - A\| < \varepsilon_1$, one has $\|\tilde{A} - A\| < \varepsilon_A$ and $\ell_A\|\tilde{A} - A\| < \rho$. So, from (16) we get the inequality $h(E(\tilde{A}), E(A)) < \rho$, which assures that $E(\tilde{A}) \subset \bar{B}(\bar{z}, \rho) \subset W$. Thus, the usc property of $E(\cdot)$ at A has been established. Now, let $W \subset \mathbb{R}^n \times \mathbb{R}^m$ be any open set having a nonempty intersection with $E(A)$. Since $E(A) = \{\bar{z}\}$, there exists $\rho > 0$ such that $\bar{B}(\bar{z}, \rho) \subset W$. As what has been done above, we can find a constant $\varepsilon_1 > 0$ such that the inclusions $E(\tilde{A}) \subset \bar{B}(\bar{z}, \rho) \subset W$ hold for any matrix $\tilde{A} \in \mathbb{R}^{n\times m}$ satisfying $\|\tilde{A} - A\| < \varepsilon_1$. Since $E(\tilde{A}) \neq \emptyset$ by the celebrated theorem of John von Neumann [12] (see also Theorem 2.3), this implies that $E(\tilde{A}) \cap W \neq \emptyset$ for every matrix $\tilde{A} \in \mathbb{R}^{n\times m}$ satisfying $\|\tilde{A} - A\| < \varepsilon_1$. The lsc property of $E(\cdot)$ at A and, therefore, the desired continuity of $E(\cdot)$ at A, have been proved. The fact that both set-valued maps $E_1(\cdot)$ and $E_2(\cdot)$ are continuous at A can be derived from the validity of (17) for every matrix $\tilde{A} \in \mathbb{R}^{n\times m}$ satisfying $\|\tilde{A} - A\| < \varepsilon_A$ in a similar way.

The proof is complete. □

Remark 3.2. One referee observed that for a compact polyhedral convex set S and an one-point set $\{p\}$, we always have

$$h\big(S, \{p\}\big) = \max\left\{\sup_{x\in S}\|x - p\|, \inf_{x\in S}\|p - x\|\right\} = e\big(S, \{p\}\big) = e\big(E(S), \{p\}\big),$$

where $E(S)$ denotes the finite set of extreme points of S (for the last equality, see, e.g., [7, Lemma 2.3]). This property clearly shows equivalence between (16) and the upper Lipschitz relation in Theorem 2.8 in the setting of Theorem 3.1 and therefore can help to shorten the proof of the theorem. Arguing in this way, one can see why the condition $S(A)$ is a singleton comes into use. □

The requirement that $S(A)$ is a singleton in the assumption of Theorem 3.1 seems to be too strict. However, as it will be shown in the forthcoming example, the assertions of that theorem may not hold if $S(A)$ has more than one element.

Example 3.3. Let $n = m = 2$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. By solving the linear program (8) and the dual program (9) we get

$$S_1(A) = \{p = (p_1, p_2) \mid p_1 \ge 0,\ p_2 \ge 0,\ p_1 + p_2 = 1\}$$

and $$S_2(A) = \{q = (q_1, q_2) \mid q_1 \ge 0,\ q_2 \ge 0,\ q_1 + q_2 = 1\}$$

Hence, $E_1(A) = \{(0,1),(1,0)\}$ and $E_2(A) = \{(0,1),(1,0)\}$. So, thanks to (11), we have $E(A) = \{(0,1),(1,0)\} \times \{(0,1),(1,0)\}$. Thus, $E(A)$ is a set of four isolated points in $\mathbb{R}^4$. For each $\varepsilon \in (0,1)$, we put $A_\varepsilon = \begin{bmatrix} 1-\varepsilon & 1+2\varepsilon \\ 1+\varepsilon & 1-2\varepsilon \end{bmatrix}$. The problem (8), where A_ε plays the role of A, becomes

$$\begin{cases} \text{Minimize } \ p_1 + p_2 \ \text{ subject to} \\ (1-\varepsilon)p_1 + (1+\varepsilon)p_2 \geq 1, \quad (1+2\varepsilon)p_1 + (1-2\varepsilon)p_2 \geq 1, \quad p_1 \geq 0, \ p_2 \geq 0. \end{cases}$$

The corresponding dual problem is

$$\begin{cases} \text{Maximize } \ q_1 + q_2 \ \text{ subject to} \\ (1-\varepsilon)p_1 + (1+2\varepsilon)q_2 \leq 1, \quad (1+\varepsilon)p_1 + (1-2\varepsilon)q_2 \leq 1, \quad q_1 \geq 0, \ q_2 \geq 0. \end{cases}$$

Solving these linear programs yields

$$S_1(A_\varepsilon) = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\} \ \text{ and } \ S_2(A_\varepsilon) = \left\{\left(\frac{2}{3}, \frac{1}{3}\right)\right\}.$$

Hence, $E_1(A_\varepsilon) = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$, $E_2(A_\varepsilon) = \left\{\left(\frac{2}{3}, \frac{1}{3}\right)\right\}$. By Theorem 2.5, we have

$$E(A_\varepsilon) = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\} \times \left\{\left(\frac{2}{3}, \frac{1}{3}\right)\right\}.$$

To show that the set-valued map $E_1(\cdot) : \mathbb{R}^{2\times 2} \rightrightarrows \mathbb{R}^2$ is neither usc nor lsc at A, we choose $V = B\left((1,0), \frac{1}{3}\right) \cup B\left((0,1), \frac{1}{3}\right)$. Clearly, $E_1(A) \subset V$ and $E_1(A) \cap V \neq \emptyset$. Note that $\lim\limits_{\varepsilon \to 0^+} A_\varepsilon = A$ and $E_1(A_\varepsilon)$ contains only the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ for every number $\varepsilon \in (0,1)$. So, one cannot find any neighborhood U of A such that $E_1(\tilde{A}) \subset V$ for all $\tilde{A} \in U$ or $E_1(\tilde{A}) \cap V \neq \emptyset$ for all $\tilde{A} \in U$. Arguing similarly, one can show that the set-valued maps $E_2(\cdot) : \mathbb{R}^{2\times 2} \rightrightarrows \mathbb{R}^2$ and $E(\cdot) : \mathbb{R}^{2\times 2} \rightrightarrows \mathbb{R}^4$ are neither usc nor lsc at A.

Remark 3.4. Clearly, $S_n \subset \bar{B}_{\mathbb{R}^n}$ for all $x \in S_n$. Hence, we have $\|x\| \leq 1$ for all $x \in S_n$. Similarly, $\|y\| \leq 1$ for all $y \in S_m$. □

The second result of this section is about the continuity of the value of the game w.r.t. the perturbation of A. It will play an important role in our investigations of matrix two-person games having multiple saddle points in mixed strategies.

Theorem 3.5. *For any matrix $A \in \mathbb{R}^{n\times m}$, there are constants $\varepsilon_A > 0$ and $\ell_A > 0$ such that*

$$|v(\tilde{A}) - v(A)| \leq (1 + 2\ell_A \|A\|)\|\tilde{A} - A\| \tag{23}$$

for every matrix $\tilde{A} \in \mathbb{R}^{n\times m}$ satisfying $\|\tilde{A} - A\| < \varepsilon_A$. Thus, the value of the game function $\tilde{A} \mapsto v(\tilde{A})$ is locally upper Lipschitz on the space $\mathbb{R}^{n\times m}$ of $n \times m$ matrices.

Proof. Similarly as in the proof of Theorem 3.1, by Theorem 2.8 we can find constants $\varepsilon_A > 0$ and $\ell_A > 0$ such that (19) holds for all $\tilde{A} \in \mathbb{R}^{n\times m}$ satisfying $\|\tilde{A} - A\| < \varepsilon_A$. Fix any matrix $\tilde{A} \in \mathbb{R}^{n\times m}$ with $\|\tilde{A} - A\| < \varepsilon_A$ and recall that both sets $S(\tilde{A})$ and $S(A)$ are nonempty and compact (see Theorem 2.3).

To obtain the inequality (23), take any $(\tilde{x},\tilde{y}) \in S(\tilde{A})$. By (19) we can find some point $(\bar{x},\bar{y}) \in S(A)$ such that

$$\|(\tilde{x},\tilde{y}) - (\bar{x},\bar{y})\| \le \ell_A \|\tilde{A} - A\|. \tag{24}$$

Note that $v(\tilde{A}) = \tilde{x}^T \tilde{A}\tilde{y}$, $v(A) = \bar{x}^T A \bar{y}$, and

$$\tilde{x}^T \tilde{A}\tilde{y} - \bar{x}^T A\bar{y} = \tilde{x}^T(\tilde{A} - A)\tilde{y} + (\tilde{x} - \bar{x})^T A\tilde{y} + \bar{x}^T A(\tilde{y} - \bar{y}).$$

Hence, by (24) and Remark 3.4 we have the following estimates

$$\begin{aligned}
|v(\tilde{A}) - v(A)| &= |\tilde{x}^T(\tilde{A} - A)\tilde{y} + (\tilde{x} - \bar{x})^T A\tilde{y} + \bar{x}^T A(\tilde{y} - \bar{y})| \\
&\le \|\tilde{x}\|\|\tilde{A} - A\|\|\tilde{y}\| + \|\tilde{x} - \bar{x}\|\|A\|\|\tilde{y}\| + \|\bar{x}\|\|A\|\|\tilde{y} - \bar{y}\| \\
&\le \|\tilde{A} - A\| + \ell_A\|\tilde{A} - A\|\|A\| + \ell_A\|A\|\|\tilde{A} - A\| \\
&= (1 + 2\ell_A\|A\|)\|\tilde{A} - A\|.
\end{aligned}$$

Thus, (23) holds for any $\tilde{A} \in \mathbb{R}^{n\times m}$ satisfying $\|\tilde{A} - A\| < \varepsilon_A$. □

Remark 3.6. The continuity property of v described by (23) is also called the *local calmness* of v at A, where A is a fixed matrix. □

4. Upper semicontinuity of the characteristic sets

As shown by Example 3.3, the characteristic set maps can be discontinuous in general. So, in light of Theorem 2.8, it is of interest to find some sufficient conditions for the usc property of these maps.

To prove a result on the usc property of the characteristic set maps for a class of matrix two-person games by using Theorem 2.4, we first establish the next lemma.

Lemma 4.1. *For every nonsingular $n \times n$ matrix $A = (a_{ij})$, one can find a constant $\bar{\mu} > 0$ such that, for any $\mu \in [\bar{\mu}, +\infty)$, all the entries of the matrix $A_\mu := (a_{ij} + \mu)$ are positive and* $\det A_\mu \neq 0$.

Proof. Put $g(\mu) = \det A_\mu$ for all $\mu \in \mathbb{R}$. Clearly, $g(\mu)$ is a polynomial whose degree does not exceed n. Since $g(0) = \det A \neq 0$ by our assumption, the constant term of $g(\mu)$ is nonzero. Hence, the polynomial $g(\mu)$ is not identically null. Therefore, it can have at most n real roots. Hence, if $\bar{\mu} > 0$ is chosen large enough, then all the entries of the matrix $A_\mu = (a_{ij} + \mu)$ are positive and $\det A_\mu \neq 0$ for every $\mu \in [\bar{\mu}, +\infty)$. □

Theorem 4.2. *If $A \in \mathbb{R}^{2\times 2}$ is a nonsingular matrix, then the characteristic set maps $E_1(\cdot) : \mathbb{R}^{2\times 2} \rightrightarrows \mathbb{R}^2$, $E_2(\cdot) : \mathbb{R}^{2\times 2} \rightrightarrows \mathbb{R}^2$ and $E(\cdot) : \mathbb{R}^{2\times 2} \rightrightarrows \mathbb{R}^4$ are upper semicontinuous at A. Moreover, these maps are upper Lipschitzian at A in the sense that there exist constants $\varepsilon_1 > 0$ and $\ell_1 > 0$ such that*

$$e(E_1(\tilde{A}), E_1(A)) \le \ell_1\|\tilde{A} - A\|, \tag{25}$$

$$e(E_2(\tilde{A}), E_2(A)) \le \ell_1\|\tilde{A} - A\|, \tag{26}$$

$$e(E(\tilde{A}), E(A)) \le \ell_1\|\tilde{A} - A\| \tag{27}$$

for any matrix $\tilde{A} \in \mathbb{R}^{2\times 2}$ satisfying $\|\tilde{A} - A\| < \varepsilon_1$.

Proof. By Theorem 2.8 we can find constants $\varepsilon > 0$ and $\ell > 0$ such that the inequality

$$e(S(\tilde{A}), S(A)) \le \ell \|\tilde{A} - A\| \tag{28}$$

holds for any matrix $\tilde{A} \in \mathbb{R}^{2\times 2}$ satisfying $\|\tilde{A} - A\| < \varepsilon$. In addition, according to Theorem 2.5,

$$S(A) = S_1(A) \times S_2(A), \quad S(\tilde{A}) = S_1(\tilde{A}) \times S_2(\tilde{A}).$$

Therefore, we can apply the first assertion of Lemma 2.1 to the sets $S_1(A)$, $S_1(\tilde{A})$, $S_2(A)$, $S_2(\tilde{A})$ to get

$$\begin{aligned} &\frac{1}{\sqrt{2}} \left[e\left(S_1(\tilde{A}), S_1(A)\right) + e\left(S_2(\tilde{A}), S_2(A)\right)\right] \\ &\le e\left(S(\tilde{A}), S(A)\right) \le e\left(S_1(\tilde{A}), S_1(A)\right) + e\left(S_2(\tilde{A}), S_2(A)\right). \end{aligned} \tag{29}$$

Combining the first inequality in (29) with the estimate (28) we obtain

$$e\big(S_1(\tilde{A}), S_1(A)\big) + e\big(S_2(\tilde{A}), S_2(A)\big) \le \sqrt{2}\ell \|\tilde{A} - A\|,$$

which implies that

$$e\big(S_1(\tilde{A}), S_1(A)\big) \le \sqrt{2}\ell \|\tilde{A} - A\|, \;\; e\big(S_2(\tilde{A}), S_2(A)\big) \le \sqrt{2}\ell \|\tilde{A} - A\|. \tag{30}$$

Thus, the inequalities in (30) are valid for any matrix $\tilde{A} \in \mathbb{R}^{2\times 2}$ with $\|\tilde{A} - A\| < \varepsilon$.

We are now in a position to establish the usc property of the characteristic set map $E_1(\cdot)$ at A.

First, let us consider the case where all the entries of the matrix A are positive. Under the assumptions made, the problem (8) becomes

$$\begin{cases} \text{Minimize } \; p_1 + p_2 \; \text{ subject to} \\ a_{11}p_1 + a_{21}p_2 \ge 1, \;\; a_{12}p_1 + a_{22}p_2 \ge 1, \;\; p_1 \ge 0, \; p_2 \ge 0 \end{cases} \tag{31}$$

with $a_{ij} > 0$ for all $(i, j) \in I \times J$ and $I = J = \{1, 2\}$. Denote the solution set and the optimal value of (31), respectively, by $\mathcal{P}$ and $\bar{\alpha} > 0$. Applying Theorems 2.4 and 2.5, we have $v(A) = \dfrac{1}{\bar{\alpha}}$ and $S_1(A) = v(A)\mathcal{P}$. If (31) has a unique solution, i.e., $S_1(A)$ is a singleton, then using some arguments of the proof of Theorem 3.1 and the first inequality in (30), which is valid for any matrix $\tilde{A} \in \mathbb{R}^{2\times 2}$ with$\|\tilde{A} - A\| < \varepsilon$, we can show that $E_1(\cdot)$ is usc at A. Hence, *in what follows we will assume that $S_1(A)$ is not a singleton.*

Let C stand for the constraint set of (31). Clearly, C is an unbounded two-dimensional polyhedral convex set. Since $S_1(A)$ is not a singleton and $S_1(A) = v(A)\mathcal{P}$, the solution set $\mathcal{P}$ of the linear program (31) has more than one element. As $\mathcal{P}$ is a closed face of C, which is compact by Theorem 2.3, we can assert that $\mathcal{P}$ must coincide with one of the following ‘candidates to be an one-dimensional compact face of C’:

$$\begin{aligned} F_1 &:= \left\{p = (p_1, p_2) \in \mathbb{R}^2_+ \mid a_{11}p_1 + a_{21}p_2 = 1, a_{12}p_1 + a_{22}p_2 \ge 1\right\}, \\ F_2 &:= \left\{p = (p_1, p_2) \in \mathbb{R}^2_+ \mid a_{11}p_1 + a_{21}p_2 \ge 1, \; a_{12}p_1 + a_{22}p_2 = 1\right\}. \end{aligned}$$

(It may happen that one of these sets is empty.) By renumbering the columns of A, we may assume that $\mathcal{P} = F_1$. Let $a_{.j}$, $j \in J$, stand for the j-th column of A. Setting $c = (1,1) \in \mathbb{R}^2$, we can rewrite the objective function of (31) as $\langle c, p \rangle$. Since this function is constant on the polyhedral convex set F_1, which has more than one element, there exists $\gamma > 0$ such that $a_{.1} = \gamma c = (\gamma, \gamma)$. Consider the line segments

$$L_1 := \left\{p = (p_1, p_2) \in \mathbb{R}^2_+ \mid a_{11}p_1 + a_{21}p_2 = 1\right\},$$
$$L_2 := \left\{p = (p_1, p_2) \in \mathbb{R}^2_+ \mid a_{12}p_1 + a_{22}p_2 = 1\right\}.$$

Since A is nonsingular, the situation $L_2 = L_1$ is excluded. Hence, one of the following situations must occur:

(a) $L_2 \cap L_1 = \emptyset$;

(b) L_2 and L_1 have one common end-point;

(c) L_2 and L_1 have one common point, say $\bar{u} = (\bar{u}_1, \bar{u}_2)$, belonging to the interior of $\mathbb{R}^2$.

Let $\varepsilon' \in (0, \varepsilon)$ be such that for any matrix $\tilde{A} = (\tilde{a}_{ij}) \in \mathbb{R}^{2\times 2}$ with $\|\tilde{A} - A\| < \varepsilon'$ one has $\tilde{a}_{ij} > 0$ for all $(i,j) \in I \times J$. The linear program of the form (8) related to such a matrix $\tilde{A}$ reads as follows

$$\begin{cases} \text{Minimize } \ p_1 + p_2 \ \text{ subject to} \\ \tilde{a}_{11}p_1 + \tilde{a}_{21}p_2 \geq 1, \ \ \tilde{a}_{12}p_1 + \tilde{a}_{22}p_2 \geq 1, \ \ p_1 \geq 0, \ p_2 \geq 0. \end{cases} \tag{32}$$

By Theorems 2.4 and 2.5, if $\tilde{\mathcal{P}}$ and $\tilde{\alpha} > 0$ are, respectively, the solution set and the optimal value of (31), then $v(\tilde{A}) = \frac{1}{\tilde{\alpha}}$ and $S_1(\tilde{A}) = v(\tilde{A})\tilde{\mathcal{P}}$. Let us denote the constraint set of (32) by $\tilde{C}$ and set

$$\tilde{L}_1 := \left\{p = (p_1, p_2) \in \mathbb{R}^2_+ \mid \tilde{a}_{11}p_1 + \tilde{a}_{21}p_2 = 1\right\},$$
$$\tilde{L}_2 := \left\{p = (p_1, p_2) \in \mathbb{R}^2_+ \mid \tilde{a}_{12}p_1 + \tilde{a}_{22}p_2 = 1\right\}.$$

If the situation (a) occurs, then one has

$$\mathcal{P} = F_1 = L_1 = \left\{p = (p_1, p_2) \in \mathbb{R}^2_+ \mid a_{11}p_1 + a_{21}p_2 = 1\right\},$$

where L_1 is a line segment orthogonal to c. Hence, from the formula $S_1(A) = v(A)\mathcal{P}$ we have

$$E_1(A) = v(A)\left\{\left(\frac{1}{a_{11}}, 0\right), \left(0, \frac{1}{a_{21}}\right)\right\} = v(A)\left\{\left(\frac{1}{\gamma}, 0\right), \left(0, \frac{1}{\gamma}\right)\right\}. \tag{33}$$

Moreover, by taking a smaller number $\varepsilon' > 0$ if necessary, for any matrix $\tilde{A} = (\tilde{a}_{ij})$ with $\|\tilde{A} - A\| < \varepsilon'$ we will have $\tilde{L}_2 \cap \tilde{L}_1 = \emptyset$ and

$$\tilde{C} = \left\{p = (p_1, p_2) \in \mathbb{R}^2_+ \mid \tilde{a}_{11}p_1 + \tilde{a}_{21}p_2 \geq 1\right\}.$$

It follows that

$$\tilde{\mathcal{P}} = \begin{cases} \left[\left(\frac{1}{\tilde{a}_{11}}, 0\right), \left(0, \frac{1}{\tilde{a}_{21}}\right)\right] & \text{if } \tilde{a}_{21} = \tilde{a}_{11} \\ \left\{\left(\frac{1}{\tilde{a}_{11}}, 0\right)\right\} & \text{if } \tilde{a}_{21} < \tilde{a}_{11} \\ \left\{\left(0, \frac{1}{\tilde{a}_{21}}\right)\right\} & \text{if } \tilde{a}_{21} > \tilde{a}_{11}. \end{cases}$$

Consequently, the formula $S_1(\tilde{A}) = v(\tilde{A})\tilde{\mathcal{P}}$ yields

$$E_1(\tilde{A}) = \begin{cases} v(\tilde{A})\big\{\big(\frac{1}{\tilde{a}_{11}}, 0\big), \big(0, \frac{1}{\tilde{a}_{21}}\big)\big\} & \text{if } \tilde{a}_{21} = \tilde{a}_{11} \\ v(\tilde{A})\big\{\big(\frac{1}{\tilde{a}_{11}}, 0\big)\big\} & \text{if } \tilde{a}_{21} < \tilde{a}_{11} \\ v(\tilde{A})\big\{\big(0, \frac{1}{\tilde{a}_{21}}\big)\big\} & \text{if } \tilde{a}_{21} > \tilde{a}_{11} \end{cases} \tag{34}$$

for any matrix $\tilde{A} = (\tilde{a}_{ij})$ with $\|\tilde{A} - A\| < \varepsilon'$. For every $i \in I$, one has

$$\begin{aligned} \left|\frac{v(\tilde{A})}{\tilde{a}_{i1}} - \frac{v(A)}{a_{i1}}\right| &= \left|\frac{a_{i1}v(\tilde{A}) - \tilde{a}_{i1}v(A)}{\tilde{a}_{i1}a_{i1}}\right| \\ &= \frac{\big|\big(a_{i1}v(\tilde{A}) - a_{i1}v(A)\big) + \big(a_{i1}v(A) - \tilde{a}_{i1}v(A)\big)\big|}{\tilde{a}_{i1}a_{i1}} \\ &\le \frac{a_{i1}|v(\tilde{A}) - v(A)| + v(A)|a_{i1} - \tilde{a}_{i1}|}{\tilde{a}_{i1}a_{i1}} \end{aligned}$$

and $|a_{i1} - \tilde{a}_{i1}| \le \|\tilde{A} - A\|$. So, thanks to (33), (34), using Theorem 3.5 we can find constants $\varepsilon_1 \in (0, \varepsilon')$ and $\ell_1 > 0$ such that (25) holds for any matrix $\tilde{A} \in \mathbb{R}^{2\times 2}$ satisfying $\|\tilde{A} - A\| < \varepsilon_1$.

Slightly modifying the above arguments, we can show that if the situation (b) or the situation (c) the situation occurs, then there also exist constants $\varepsilon_1 \in (0, \varepsilon')$ and $\ell_1 > 0$ such that (25) holds for any matrix $\tilde{A} \in \mathbb{R}^{2\times 2}$ satisfying $\|\tilde{A} - A\| < \varepsilon_1$. (One can reduce $\varepsilon_1 > 0$ and enlarge $\ell_1 > 0$ if necessary.)

Renumbering the players and applying the obtained result, we can establish (26). To get (27) from (25) and (26), one can apply the second inequality of (2) to the sets $E_1(\tilde{A}), E_1(A), E_2(\tilde{A}), E_2(A)$. (Again, one can enlarge $\ell_1 > 0$ if necessary.)

Since $E_1(A)$ (resp., $E_2(A)$, $E(A)$) is a compact set, from the validity of (25) (resp., (26), (27)) for any matrix $\tilde{A} \in \mathbb{R}^{2\times 2}$ satisfying $\|\tilde{A} - A\| < \varepsilon_1$ one can easily show that $E_1(.)$ (resp., $E_2(.)$, $E(.)$) is usc at A.

To deal with the case where A has some non-positive entries, it suffices to apply Lemma 4.1 and observe that the transformation

$$\tilde{A} = (\tilde{a}_{ij}) \mapsto \tilde{A}_\mu = (\tilde{a}_{ij} + \mu) \qquad (\mu \in \mathbb{R}), \tag{35}$$

denoted by Φ_μ, does not change the norm of the difference of two arbitrary matrices. Namely, for any matrices $\tilde{A}$ and $\hat{A}$ from $\mathbb{R}^{n\times m}$, where n and m are positive integers, one has $\|\Phi_\mu(\tilde{A}) - \Phi_\mu(A)\| = \|\tilde{A}_\mu - \hat{A}_\mu\| = \|\tilde{A} - \hat{A}\|$. □

Remark 4.3. One referee noted that the first estimate (reps., the second estimate) in (17) can be derived from the inequalities in formula (30) under a less restrictive condition: $S_1(A)$ *is a singleton* (resp., $S_2(A)$ *is a singleton*). □

5. Density of the games having a unique optimal strategy for one player

It is reasonable to conjecture that *the set of games having a unique optimal strategy for one player, say,* P_1*, is dense in the space of matrices* $\mathbb{R}^{n\times m}$. Our first attempt was successful in the case $n = m = 2$.

Theorem 5.1. *For $n = 2$, the set $\left\{\tilde{A} \in \mathbb{R}^{n\times n} \mid \chi_1(\tilde{A}) = 1\right\}$ is dense in $\mathbb{R}^{n\times n}$.*

Proof. Let $A \in \mathbb{R}^{n\times n}$ be an arbitrary matrix. To prove the theorem, we have to show that for any $\rho > 0$ there exists $\tilde{A} \in \mathbb{R}^{n\times n}$ with $\|\tilde{A}-A\| < \rho$ such that $\chi_1(\tilde{A}) = 1$. Consider the parametric matrix $A+\varepsilon E$, where E denotes the unit matrix of order n and $\varepsilon \in \mathbb{R}$ is a parameter. Clearly, $\det(A + \varepsilon E)$ is a polynomial of order n of the variable ε. Since the polynomial can have at most n distinct roots, we can find $\varepsilon \in (0, 2^{-1}\rho)$ such that $\det(A+\varepsilon E) \neq 0$. As $\|(A+\varepsilon E)-A\| < 2^{-1}\rho$, the proof will be completed if we can show that there exists $\tilde{A} \in \mathbb{R}^{n\times n}$ with $\|\tilde{A} - (A + \varepsilon E)\| < 2^{-1}\rho$ such that $\chi_1(\tilde{A}) = 1$. Thus, by replacing A with $A + \varepsilon E$ (if necessary), we may assume from the beginning that $\det A \neq 0$.

According to Lemma 4.1, there exists $\mu > 0$ such that all the entries of the matrix $A_\mu = (a_{ij} + \mu)$ are positive and $\det A_\mu \neq 0$. If we can find a matrix $\hat{A} \in \mathbb{R}^{n\times n}$ with $\|\hat{A} - A_\mu\| < \rho$ such that $\chi_1(\hat{A}) = 1$, then applying the inverse of the transformation Φ_μ defined by (35) to $\hat{A}$ gives the desired matrix. Indeed, setting $\tilde{A} = \Phi_{-\mu}(\hat{A})$, we have

$$\|\tilde{A} - A\| = \|\Phi_\mu(\hat{A}) - \Phi_\mu(A_\mu)\| = \|\hat{A} - A_\mu\| < \rho$$

and $\chi_1(\tilde{A}) = \chi_1(\Phi_\mu(\tilde{A})) = \chi_1(\hat{A}) = 1$ because

$$S(\tilde{A}) = S(\Phi_\mu(\tilde{A})) = S\left(\Phi_\mu\big(\Phi_{-\mu}(\hat{A})\big)\right) = S(\hat{A})$$

(see also Theorem 2.5 and Definition 2.6). Therefore, it remains to prove the next result.

Claim. *For $n = 2$, given any nonsingular matrix $A \in \mathbb{R}^{n\times n}$ having positive entries and a constant $\rho > 0$, one can find a matrix $\hat{A} \in \mathbb{R}^{n\times n}$ with $\|\hat{A} - A\| < \rho$ such that $\chi_1(\hat{A}) = 1$.*

To prove the claim, we apply Theorem 2.5 to get $S_1(A) = \frac{1}{\bar{\alpha}}\mathcal{P}$, where $\mathcal{P}$ and $\bar{\alpha} > 0$ are respectively the solution set and the optimal value of (8). Thus, if $\mathcal{P}$ is a singleton, then $S_1(A)$ is also a a singleton, and we have $\chi_1(A) = 1$. Hence, in this case it suffices to take $\hat{A} = A$. Suppose now that $\mathcal{P}$ has more than one element. We have to show the existence of a slight perturbation of A which yields a linear program having a unique solution.

Under the hypotheses of the claim, by Theorem 2.7 we see that the conditions (12), (13) are satisfied and the equalities (14), (15) hold true.

Recall [10, p. 162] that a *face* of a convex set C is a subset C' of C such that every closed line segment in C with a relative interior point in C' has both endpoints in C'. We refer to [10, p. 2] for the definition of dimension of a convex set. The *relative interior* of a convex set C, which is denoted by $\operatorname{ri} C$, is defined [10, p. 44] as the interior which results when C is regarded as a subset of its affine hull.

To proceed furthermore, we will rely on Theorem 2.7. First, note that the set

$$C(\bar{\alpha}) = \frac{1}{\bar{\alpha}}J_2 + \mathbb{R}^2_+$$

has the following two *faces of dimension* 1:

$$C_j(\bar{\alpha}) := \left\{ z = (z_1, z_2) \in \mathbb{R}^2 \ : \ z_j = \frac{1}{\bar{\alpha}},\ z_{j'} \geq \frac{1}{\bar{\alpha}} \ \text{for} \ j' \neq j \right\} \quad (j \in J),$$

where $J = \{1, 2\}$. Clearly,

$$\operatorname{ri} C_j(\bar{\alpha}) := \left\{ z = (z_1, z_2) \in \mathbb{R}^2 \ : \ z_j = \frac{1}{\bar{\alpha}},\ z_{j'} > \frac{1}{\bar{\alpha}} \ \text{for} \ j' \neq j \right\} \quad (j \in J).$$

If $\Delta \cap [\operatorname{ri} C_j(\bar{\alpha})]$ is empty for every $j \in J$, then from (12) and (13) we can deduce that

$$\Delta \cap (\partial C(\bar{\alpha})) = \left\{ \frac{1}{\bar{\alpha}} J_2 \right\}.$$

Hence, by (14) we see that $\mathcal{P}$ is a singleton. But, this contradicts the standing assumption that $\mathcal{P}$ has more than one element. Thus, the must exits $j_0 \in J$ such that $\Delta \cap [\operatorname{ri} C_{j_0}(\bar{\alpha})] \neq \emptyset$. Hence, applying [7, Proposition 5.6], we have $\Delta \cap (\partial C(\bar{\alpha})) = \Delta \cap C_{j_0}(\bar{\alpha})$. Since Δ is the line segment joining $a_{1.}$ with $a_{2.}$ and $C_{j_0}(\bar{\alpha})$ is a closed half-line, the condition $\Delta \cap [\operatorname{ri} C_{j_0}(\bar{\alpha})] \neq \emptyset$ implies that one of the end-points of Δ must belong to $\operatorname{ri} C_{j_0}(\bar{\alpha})$. Without loss of generality, we may assume that $j_0 = 1$. Moreover, by renumbering the indexes if necessary, we may admit that $a_{2.} \in \operatorname{ri} C_1(\bar{\alpha})$ and $a_{12} < a_{22}$. Therefore, one has

$$a_{1.} = \begin{bmatrix} \bar{\alpha}^{-1} \\ a_{12} \end{bmatrix}, \quad a_{2.} = \begin{bmatrix} \bar{\alpha}^{-1} \\ a_{22} \end{bmatrix}, \quad a_{22} > \frac{1}{\bar{\alpha}}, \quad a_{12} \in (0, a_{22}). \tag{36}$$

(It may happen that $a_{12} \in (0, \bar{\alpha}^{-1})$.) Let $A(\varepsilon) \in \mathbb{R}^{2\times 2}$ be the matrix whose transpose matrix is

$$A(\varepsilon)^T = \begin{bmatrix} \bar{\alpha}^{-1} & \bar{\alpha}^{-1} + \varepsilon \\ a_{12} & a_{22} \end{bmatrix}. \tag{37}$$

Clearly, for any $\rho > 0$, one has $\|A_\varepsilon - A\| < \rho$ if $\varepsilon < \rho$. The linear program corresponding to the matrix two-person game defined by $A(\varepsilon)$ is

$$\begin{cases} \text{Minimize} \ \ p_1 + p_2 \ \ \text{subject to} \\ \bar{\alpha}^{-1} p_1 + (\bar{\alpha}^{-1} + \varepsilon) p_2 \geq 1, \quad a_{12} p_1 + a_{22} p_2 \geq 1, \quad p_1 \geq 0,\ p_2 \geq 0. \end{cases} \tag{38}$$

Denote the solution set and the optimal value of (38), respectively, by $\mathcal{P}_\varepsilon$ and α_ε. One has $\alpha_\varepsilon > 0$. Let $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \mathcal{P}_\varepsilon$ be given arbitrarily. Clearly, the inequalities in (38) are satisfied with $(p_1, p_2) = (\bar{p}_1, \bar{p}_2)$, i.e.,

$$\begin{cases} \bar{\alpha}^{-1} \bar{p}_1 + (\bar{\alpha}^{-1} + \varepsilon) \bar{p}_2 \geq 1 \\ a_{12} \bar{p}_1 + a_{22} \bar{p}_2 \geq 1 \\ \bar{p}_1 \geq 0,\ \bar{p}_2 \geq 0. \end{cases} \tag{39}$$

Since $\bar{p}_1 + \bar{p}_2 = \alpha_\varepsilon$, from the first inequality in (39) it follows that $\bar{\alpha}^{-1} \alpha_\varepsilon \geq 1 - \varepsilon \bar{p}_2$. Hence,

$$\bar{\alpha}^{-1} \geq \alpha_\varepsilon^{-1} - \varepsilon \left(\alpha_\varepsilon^{-1} \bar{p}_2 \right) \geq \alpha_\varepsilon^{-1} - \varepsilon, \tag{40}$$

where the second inequality follows from the fact that $\alpha_\varepsilon^{-1} \bar{p}_2 \in [0, 1]$. From (40) we can deduce that

$$\bar{\alpha}_\varepsilon^{-1} \leq \bar{\alpha}^{-1} + \varepsilon. \tag{41}$$

Thanks to the strict inequality (36), we can find $\bar{\varepsilon} \in (0, \rho)$ such that

$$\frac{a_{22}}{\bar{\alpha}^{-1} + \varepsilon} \geq 1 \quad (\forall \varepsilon \in (0, \bar{\varepsilon})). \tag{42}$$

For any $\varepsilon \in (0, \bar{\varepsilon})$, using (42), we can easily verify that $p = (p_1, p_2)$ with $p_1 := 0$ and $p_2 := \dfrac{1}{\bar{\alpha}^{-1} + \varepsilon}$ is a feasible vector of (38). Thus, one has $\alpha_\varepsilon \leq \dfrac{1}{\bar{\alpha}^{-1} + \varepsilon}$ or, equivalently, $\alpha_\varepsilon^{-1} \geq \bar{\alpha}^{-1} + \varepsilon$. Combining this with (41) yields $\alpha_\varepsilon^{-1} = \bar{\alpha}^{-1} + \varepsilon$. Hence, looking back to (40), we see that for any $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \mathcal{P}_\varepsilon$ the following must hold true

$$\bar{\alpha}^{-1} = \alpha_\varepsilon^{-1} - \varepsilon \left(\alpha_\varepsilon^{-1} \bar{p}_2\right) = \alpha_\varepsilon^{-1} - \varepsilon,$$

This clearly forces $\bar{p}_2 = \alpha_\varepsilon$ and $\bar{p}_1 = 0$. We have thus proved that $\mathcal{P}_\varepsilon$ is a singleton for every $\varepsilon \in (0, \bar{\varepsilon})$. Now, applying Theorem 2.7 for the game defined by $A(\varepsilon)$, we can conclude that $S_1(A_\varepsilon)$ is a singleton. So, the assertion of our claim is fulfilled with $\hat{A} := A_\varepsilon$.

The proof of the theorem is completed. □

As an illustration for the result in Theorem 5.1, let us apply the method of perturbation suggested in the above proof to the matrix two-person game in [7, Example 6.2].

Example 5.2. Let $n = m = 2$ and $A = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 3 \end{bmatrix}$. Then one has $a_{1.} = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}$ and $a_{2.} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The linear program (8) has the optimal value $\bar{\alpha} = 1$ and the solution set $\{p = (p_1, p_2) \geq 0 \, : \, p_1 + p_2 = 1\}$. Thus, $\chi_1(A) = 2$. For this game, $\Delta \cap [\operatorname{ri} C_1(\bar{\alpha})]$ is nonempty. The relationship between the simplex $\Delta = \operatorname{co}\{a_{1.}, a_{2.}\}$ and the carriage $C(\bar{\alpha}) = J_2 + \mathbb{R}^2_+$ is shown in Figure 5.1. The result $\chi_1(A) = 2$ can also be obtained by applying Theorem 2.7. In accordance with (37), the transpose of the matrix $A(\varepsilon)$ of the perturbed game is

$$A(\varepsilon)^T = \begin{bmatrix} 1 & 1 + \varepsilon \\ \frac{3}{2} & 3 \end{bmatrix}.$$

So, the linear program (38) becomes

$$\begin{cases} \text{Minimize } \ p_1 + p_2 \ \text{ subject to} \\ p_1 + (1 + \varepsilon)p_2 \geq 1, \ \ \frac{3}{2}p_1 + 3p_2 \geq 1, \ \ p_1 \geq 0, \ p_2 \geq 0. \end{cases} \tag{43}$$

The largest number $\bar{\varepsilon} > 0$ satisfying the condition (42) is $\bar{\varepsilon} := 2$. For any $\varepsilon \in (0, \bar{\varepsilon})$, the solution set of (43) is

$$\mathcal{P}_\varepsilon = \left\{ \left(0, \frac{1}{1 + \varepsilon}\right) \right\}.$$

Hence, $\alpha_\varepsilon = \dfrac{1}{1 + \varepsilon}$. This result comes in full agreement with the formula $\alpha_\varepsilon^{-1} = \bar{\alpha}^{-1} + \varepsilon$ which was obtained in the proof of Theorem 5.1.

Remark 5.3. There is a hope that the perturbation method in the proof of Theorem 5.1 can be applied to the case where $n = m \geq 3$, provided that the set

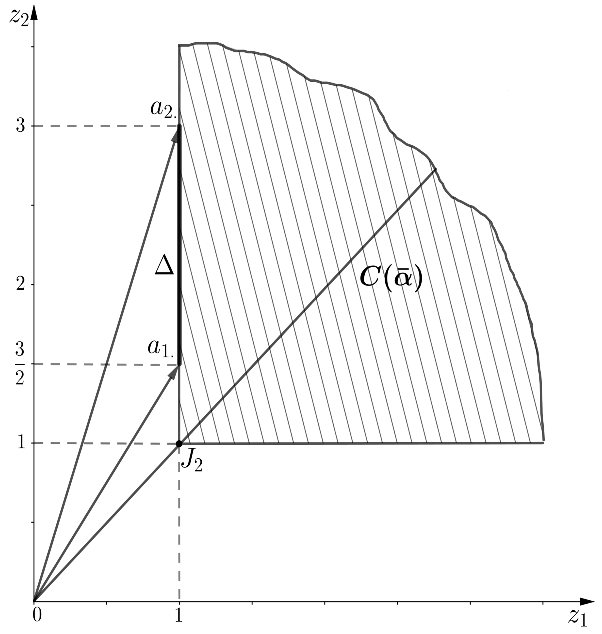

Figure 5.1: The relationship between Δ and $C(\bar{\alpha})$ of Example 5.2

$\Delta \cap (\partial C(\bar{\alpha}))$ contains at least one of the vectors $a_{1.}, a_{2.}, \ldots, a_{n.}$, say, $a_{i_0.}$. Namely, in this situation, one can try to perturb $a_{i_0.}$ in a suitable way and let other vectors in the system unchanged. □

Remark 5.4. In the case where $n = m \geq 3$, it may happen that the set $\Delta \cap (\partial C(\bar{\alpha}))$ does not contain any vectors from the system $a_{1.}, a_{2.}, \ldots, a_{n.}$. In that situation, it is still unclear to us what kind of perturbation can be applied to A. □

6. Conclusions

We have studied some qualitative properties of matrix two-person games. Several results on the continuity of the characteristic sets and of the value of the games, the upper semicontinuity of the characteristic sets, the density of the games having a unique optimal strategy for one player have been established. We think that the following questions are interesting and deserve further investigations.

Question 1: *Whether the upper semicontinuity and the upper Lipschitzian property in the conclusions of Theorem* 4.2 *are still valid if* A *is a nonsingular matrix from* $\mathbb{R}^{n\times n}$ *with* $n \geq 3$*?*

Question 2: *Is it true that the set* $\{\tilde{A} \in \mathbb{R}^{n\times m} \mid \chi_1(\tilde{A}) = 1\}$ *is dense in* $\mathbb{R}^{n\times m}$*?*

Question 3: *Is it true that the set* $\{\tilde{A} \in \mathbb{R}^{n\times m} \mid \chi_1(\tilde{A}) = 1\}$ *is open in* $\mathbb{R}^{n\times m}$*?*

Question 4: *Is it true that, for any matrix* $A \in \mathbb{R}^{n\times m}$*, each one of the sets*

$$\Omega(A) := \{\tilde{A} \in \mathbb{R}^{n\times m} \mid \chi(\tilde{A}) \leq \chi(A)\}, \quad \Omega_1(A) := \{\tilde{A} \in \mathbb{R}^{n\times m} \mid \chi_1(\tilde{A}) \leq \chi_1(A)\},$$
$$\Omega_2(A) := \{\tilde{A} \in \mathbb{R}^{n\times m} \mid \chi_2(\tilde{A}) \leq \chi_2(A)\}$$

contains A *in its interior?*

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