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Fuzzy and Exact Necessary Optimality Conditions for a Nonsmooth Bilevel Semi-Infinite Program

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We investigate the so-called nonsmooth bilevel semi-infinite programming problem when the involved functions are nonconvex. This type of problems consists of an infinite number of constraints with arbitrary index sets. To establish the optimality conditions, we rewrite upper estimates of three recently developed subdifferentials of the value functions using two new qualification conditions (CQs), which are weaker than the existing Mangasarian-Fromovitz and Farkas-Minkowski CQs. We point out that the obtained results are new if we take up a finite number of constraints as well.

Keywords: Parametric optimization, semi-infinite programming, optimality conditions, marginal and value functions, generalized differentiation, bilevel programming.

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1. Introduction

We consider nonsmooth bilevel semi-infinite programs with an arbitrary number of inequalities (possibly infinite) at the lower level our treatment makes use of the most recent variational tools. In particular, we consider certain generalized differentiation properties of the value function of the lower level problem. It is worth mentioning that the value function is currently recognized among the most significant approaches for parametric optimization, and proves to be very useful in developing optimality conditions for several problems of optimization, control theory and equilibria, etc; see [14, 19].

More precisely, we study the following class of bilevel semi-infinite programming problems

$$\min_{x,y} F(x,y) \quad \text{s.t. } y \in S(x), \tag{1}$$

where $S(x)$ is a parameter-dependent set of optimal solutions to the following lower-level problem

$$\min_{y} f(x,y) \quad \text{s.t. } g_t(x,y) \le 0 \ \forall t \in T. \tag{2}$$

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The objectives are $F, f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and the constraints are given in terms of $g_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R},\ t \in T$, where T can be any nonempty index set which is not necessarily finite. Furthermore, we do not assume any of the previous functions to be locally Lipschitz. Note that we can readily insert into the upper-level problem (1) any extra convex geometric or functional constraints but we prefer to omit them for the sake of simplicity.

There have been many published works in the last two decades about bilevel optimization problems; see the two monographs [1, 4] for pointers to a vast literature as well as the survey articles [5, 20]. The majority of these works were concerned with necessary optimality conditions. In [21, 22], Ye and Zhu employed the optimal value approach to convert the bilevel program into a scalar-objective optimization problem, and then derived necessary optimality conditions. A similar transformation principle combined with a scalarization technique was used in [10, 11] in order to obtain necessary optimality conditions in the case of a semivectorial bilevel program. In [24], Zemkoho studied the so-called ill-posed bilevel program, in which given some upper-level parameters, the problem admits multiple lower-level solutions. The author established the equivalence between this problem and a certain set-valued optimization problem, which was used to develop optimality conditions. In [7], Dempe *et al.* gave exact and fuzzy/approximate optimality conditions for bilevel programming by applying the exact as well as the approximate extremal principles introduced by Mordukhovich [13, 14].

However, to the best of our knowledge, there are a very few papers that have discussed optimality conditions for the case of bilevel semi-infinite programs. We note that introducing an infinite number of constraints into a bilevel problem were recently studied in [9], where the objectives were stated in terms of the difference of two convex functions.

In this work, we treat the nonconvex case of bilevel semi-infinite programming. Our goal is to rewrite upper estimates of three recent subdifferentials of value functions in the semi-infinite program (2) using generalized differentiation and advanced tools of variational analysis. Furthermore, the obtained results for the value functions will be employed to study the nonsmooth optimization problems (1)-(2) when the involved functions are nonconvex. We also aim to establish new first-order necessary optimality conditions for (1) by exploiting the same technique used in [7, 6].

The paper is structured as follows: Notions and properties from variational analysis that we need are first stated in the next section. Section 3 is devoted to transform our bilevel semi-infinite optimization problem into a single-level optimization problem. In Section 4, we derive upper estimates on three subdifferentials of value functions in the semi-infinite program (2). In Sections 5 and 6, we derive exact and fuzzy necessary optimality conditions by considering cases where all the functions involved are Lipschitz continuous. Finally, we illustrate our main result by providing an example.

2. Preliminaries

We employ the same tools of generalized derivatives and variational analysis as in [14]. Let Θ be a set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. Recall that the domain and graph of Θ are given by

$$\mathrm{dom}\,\Theta := \{z \in \mathbb{R}^n : \Theta(z) \neq \emptyset\},\quad \mathrm{gph}\,\Theta := \{(z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^m : z_2 \in \Theta(z_1)\}.$$

Let S be a subset of $\mathbb{R}^n$. The indicator function δ_S of S and the support function σ_S of S are defined respectively by

$$\delta_S(x) := \begin{cases} 0, & \text{if } x \in S \\ \infty, & \text{otherwise.} \end{cases}$$

and
$$\sigma_S\left(x^*\right) := \sup_{x \in S} \langle x^*, x \rangle \text{ for each } x^* \in \mathbb{R}^n.$$

The Painlevé-Kuratowski outer/upper limit of $\Theta : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at a point $\overline{z}$ is

$$\operatorname{Limsup}_{z \to \overline{z}} \Theta\left(z\right) := \left\{\tau \in \mathbb{R}^m : \exists z_k \to \overline{z},\ \tau_k \to \tau \text{ with } \tau_k \in \Theta\left(z_k\right) \text{ as } k \to \infty\right\}. \quad (3)$$

Given $\overline{z} \in \Xi$ with Ξ is a subset of $\mathbb{R}^n$, the basic/limiting/Mordukhovich normal cone to Ξ at $\overline{z}$ is

$$N\left(\overline{z}; \Xi\right) := \operatorname{Limsup}_{z \to \overline{z}} \widehat{N}\left(z; \Xi\right),$$

where $\widehat{N}\left(\overline{z}; \Xi\right)$ denotes the prenormal/Fréchet normal cone to Ξ at $\overline{z}$:

$$\widehat{N}\left(\overline{z}; \Xi\right) := \left\{\widetilde{z} \in \mathbb{R}^n : \limsup_{z \to \overline{z}(z \in \Xi)} \frac{\langle \widetilde{z}, z - \overline{z} \rangle}{\| z - \overline{z} \|} \leq 0\right\}.$$

The Mordukhovich normal coderivative $D^*\Theta\left(z_1, z_2\right) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of Θ at the point $\left(z_1, z_2\right) \in \text{gph } \Theta$ can be defined as

$$D^*\Theta\left(z_1, z_2\right)\left(\widetilde{z_2}\right) := \left\{\widetilde{z_1} \in \mathbb{R}^n : \left(\widetilde{z_1}, -\widetilde{z_2}\right) \in N\left(\left(z_1, z_2\right); \text{gph } \Theta\right)\right\} \quad \text{for all } \widetilde{z_2} \in \mathbb{R}^m.$$

The Fréchet coderivative of Θ at $\left(z_1, z_2\right) \in$ gph Θ is defined for all $\widetilde{z_2} \in \mathbb{R}^m$ by

$$\widehat{D}^*\Theta\left(z_1, z_2\right)\left(\widetilde{z_2}\right) := \left\{\widetilde{z_1} \in \mathbb{R}^n : \left(\widetilde{z_1}, -\widetilde{z_2}\right) \in \widehat{N}\left(\left(z_1, z_2\right); \text{gph } \Theta\right)\right\}.$$

We can now introduce the Fréchet/viscosity (lower) subdifferential of the function $\Phi : \mathbb{R}^n \to \overline{\mathbb{R}} = (-\infty; +\infty]$ at a point $\overline{z}$ of its domain

$$\widehat{\partial}\Phi(\bar{z}) := \left\{\widetilde{z} \in \mathbb{R}^n \mid \liminf_{z \to \bar{z}} \frac{\Phi(z) - \Phi(\bar{z}) - \langle \widetilde{z}, z - \bar{z} \rangle}{\|z - \bar{z}\|} \geq 0\right\}. \quad (4)$$

Thus one can define the regular upper subdifferential (or superdifferential) of Φ at $\overline{z}$

$$\widehat{\partial}^+\Phi\left(\overline{z}\right) := -\widehat{\partial}\left(-\Phi\right)\left(\bar{z}\right).$$

We have the following striking difference rule for (lower) Fréchet subgradients [17]. If $\Phi_1, \Phi_2 : \mathbb{R}^n \to \overline{\mathbb{R}}$ are finite at $\overline{z}$, then

$$\widehat{\partial}\left(\Phi_1 - \Phi_2\right)\left(\overline{z}\right) \ \subset\ \widehat{\partial}\Phi_1(\overline{z}) - \widehat{\partial}\Phi_2(\overline{z}), \quad (5)$$

providing $\widehat{\partial}\Phi_2(\overline{z}) \neq \emptyset$.

The basic/limiting/Mordukhovich (lower) subdifferential of Φ at $\overline{z}$ can be defined using the outer limit (3) of the regular subgradients (4) as

$$\partial\Phi\left(\overline{z}\right) := \operatorname{Limsup}_{z \to \overline{z}} \widehat{\partial}\Phi\left(z\right).$$

Note that if Φ is convex, then $\partial\Phi(\overline{z})$ will be the same as the classical subdifferential

$$\partial\Phi(\overline{z}) := \{\widetilde{x} \in \mathbb{R}^n \ :\ \Phi\left(z\right) - \Phi\left(\overline{z}\right) \geq \langle \widetilde{x}, z - \overline{z} \rangle,\ \forall\ z \in \mathbb{R}^n\}.$$

Moreover, in this case the Young's equality holds (cf. [23, Theorem 2.4.2(iii)]):

$$\Phi(z) + \Phi^*(z^*) = \langle z^*, z\rangle \text{ if and only if } z^* \in \partial\Phi(z), \tag{6}$$

where Φ^* is the conjugate function of Φ.

Note also that if Φ is locally Lipschitz continuous, then $\partial\Phi(\overline{z})$ is nonempty and compact, and moreover, the Clarke subdifferential $\partial_C\Phi(\overline{z})$ of Φ at $\overline{z}$ is its convex hull, that is

$$\partial_C\Phi(\overline{z}) := \text{co } \partial\Phi(\overline{z}),$$

In what follows, the following important property of the convex hull involving locally Lipschitz functions will be needed:

$$\text{co } \partial\left(-\Phi\right)(\overline{z}) := -\text{co } \partial\Phi(\overline{z}). \tag{7}$$

Now, we define $\text{im } \partial\Phi := \{\mathrm{y}^* \in \mathbb{R}^{\mathrm{n}} \,:\, \mathrm{y}^* \in \partial\Phi(\mathrm{z}) \text{ for some } \mathrm{z} \in \mathbb{R}^{\mathrm{n}}\}$ and

$$\text{dom } \partial\Phi := \{\mathrm{z} \in \mathbb{R}^{\mathrm{n}} : \partial\Phi(\mathrm{z}) \neq \emptyset\}.$$

The next definitions are extremely useful for our investigation.

Definition 2.1. [14] Suppose Ξ_1 and Ξ_2 are two nonempty closed sets in $\mathbb{R}^n$. The set system (Ξ_1, Ξ_2) is said to have $\overline{z} \in \Xi_1 \cap \Xi_2$ as a *local extremal point* if there exist two sequences $\{a_k^1\}$ and $\{a_k^2\}$ of $\mathbb{R}^n$, and some neighborhood V of $\overline{z}$ with $a_k^1 \to 0$ and $a_k^2 \to 0$ as $k \to \infty$ and

$$\left(\Xi_1 - a_k^1\right) \cap \left(\Xi_2 - a_k^2\right) \cap V = \emptyset \text{ for all large } k.$$

In this case $\{\Xi_1, \Xi_2, \overline{x}\}$ is said to be an extremal system in $\mathbb{R}^n$.

Recall that by $\overline{z}$ being locally extremal to Ξ_1 and Ξ_2, we mean both two sets can be locally pushed apart by a small perturbation (translation) of even one of them.

Definition 2.2. [14] We say that an extremal system $\{\Xi_1, \Xi_2, \overline{z}\}$ in $\mathbb{R}^n$ realizes the approximate extremal principle if for each $\epsilon > 0$ there are $z_1 \in \Xi_1 \cap (\overline{z} + \epsilon\mathbb{B}_{\mathbb{R}^n})$, $z_2 \in \Xi_2 \cap (\overline{z} + \epsilon\mathbb{B}_{\mathbb{R}^n})$ and $\widetilde{z} \in \mathbb{R}^n$ with $\| \widetilde{z} \|= 1$ such that

$$\widetilde{z} \in \left(\widehat{N}\left(z_1; \Xi_1\right) + \epsilon\mathbb{B}_{\mathbb{R}^n}\right) \cap \left(-\widehat{N}\left(z_2; \Xi_2\right) + \epsilon\mathbb{B}_{\mathbb{R}^n}\right).$$

We finally some useful definitions and properties of set-valued mappings.

Definition 2.3. Consider $(\bar{x}, \bar{y}) \in gph\ \Theta$. We say that Θ is

- *inner semicompact* at $\bar{x}$ if for each sequence $x_k \to \bar{x}$ with $\Theta(x_k) \neq \emptyset$ for each k, there exists a sequence $y_k \in \Theta(x_k)$ that admits a convergent subsequence.
- *inner semicontinuous* at $(\bar{x}, \bar{y})$ if for each sequence $x_k \to \bar{x}$ there exists a sequence $(y_k)_k$ with $y_k \in \Theta(x_k)$ for all k and $y_k \to \bar{y}$.

Remark 2.4. (i) Θ is inner semicompact at $\bar{z}$ if it is uniformly bounded and takes nonempty values around $\bar{z}$, i.e. there is a neighborhood V of $\bar{z}$ and a bounded set $\Xi \subset \mathbb{R}^m$ such that $\emptyset \neq \Theta(z) \subset \Xi$, for each $z \in V$.

(ii) Θ is inner semicontinuous at (z_1, z_2) in the case where Θ is closed-graph and inner semicompact at z_1 with $\Theta(z_1) = \{z_2\}$.

Now, some amendments are made to the definition of both inner semicontinuity and inner semicompactness so that to make them more appropriate to study marginal functions.

Definition 2.5. Given a function $\mu : \mathbb{R}^n \to \overline{\mathbb{R}}$, we say that the mapping Θ is

- *μ-inner semicontinuous* at $(\bar{x}, \bar{y}) \in \operatorname{gph} \Theta$ if for any sequences $x_k \xrightarrow{\mu} \bar{x}$ there is a sequence $y_k \in \Theta(x_k)$ that admits a convergent subsequence to $\overline{y}$.
- *μ-inner semicompact* at $\bar{x}$ if for any sequence $x_k \xrightarrow{\mu} \bar{x}$ there exists a sequence $y_k \in \Theta(x_k)$ that admits a convergent subsequence.

Here, $x_k \xrightarrow{\mu} \bar{x}$ means that $x_k \to \bar{x}$ with $\mu(x_k) \to \mu(\overline{x})$.

3. The problem and its reformulation

Hereafter, we are mainly concerned by the bilevel semi-infinite programming problem (1) involving functions that are assumed to be locally Lipschitz. By employing the lower-level value function approach together with a suitable "partial calmness" qualification assumption, we can equivalently reformulate the bilevel problem as a one level semi-infinite optimization problem. To proceed, consider the following spaces:

$$\begin{aligned} \mathbb{R}^T &= \{\upsilon = (\upsilon_t)_{t\in T} \mid \upsilon_t \in \mathbb{R} \text{ for each } t \in T\}, \\ \widetilde{\mathbb{R}}^T &= \{\upsilon \in \mathbb{R}^T \mid \upsilon_t \neq 0 \text{ for finitely many } t \in T\}, \\ \widetilde{\mathbb{R}}^T_+ &= \{\upsilon \in \widetilde{\mathbb{R}}^T \mid \upsilon_t \geq 0 \ \text{ for each } \ t \in T\}. \end{aligned} \tag{8}$$

Using the following notation $\operatorname{supp} \upsilon := \{\mathrm{t} \in \mathrm{T} \mid \upsilon_\mathrm{t} \neq 0\}$, one has

$$\forall \mu \in \mathbb{R}^T, \ \forall \upsilon \in \widetilde{\mathbb{R}}^T \quad : \quad \upsilon\mu := \sum_{t\in T} \upsilon_t \mu_t = \sum_{t\in \operatorname{supp} \upsilon} \upsilon_t \mu_t.$$

From now on, we will use the following constructions related to the problems (1)-(2).

- The set-valued perturbed constraint mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$

$$G(z_1) := \{z_2 \in \mathbb{R}^m \mid g_t(z_1, z_2) \leq 0 \ \text{ for each } t \in T\}.$$

- The set of active constraints

$$T(z_1, z_2) := \Big\{t \in T \mid g_t(z_1, z_2) = 0\Big\}. \tag{9}$$

- The set of active constraint multipliers

$$\Upsilon(z_1, z_2) := \Big\{\upsilon \in \widetilde{\mathbb{R}}^T_+ \mid \upsilon_t g_t(z_1, z_2) = 0 \ , \ \forall\, t \in \operatorname{supp} \upsilon\Big\}. \tag{10}$$

- The value function of the lower level problem (2)

$$V(z_1) := \inf\{f(z_1, z_2) \ \mid y \in G(z_1)\}. \tag{11}$$

Applying the value function approach, we reformulate the bilevel problem (1) in its globally equivalent one level problem:

$$\begin{cases} \min\limits_{x,y} \ \ F(x, y) \\ f(x, y) - V(x) \leq 0, \quad g_t(x, y) \leq 0, \ \forall\, t \in T. \end{cases} \tag{12}$$

Next, we consider the partial calmness condition introduced in [21] to reduce the bilevel problem to a one-level optimization problem with infinite constraints. Combined with the optimal value approach, this partial calmness has been broadly investigated in the last years in the context of classical optimistic bilevel programs. Let us take up the perturbed form of (12) linearly parameterized by a real number p:

$$\begin{cases} \min\limits_{x,y} \ F(x,y) \\ f(x,y) - V(x) + p = 0, \ g_t(x,y) \leq 0, \quad \text{for all } t \in T. \end{cases} \tag{13}$$

Definition 3.1. [21]Assume that $\overline{u} = (\overline{x}, \overline{y})$ is a feasible point of (1). The unperturbed problem (1) is said to be *partially calm* at $\overline{u}$ if there exist a constant $\rho > 0$ and a neighborhood $U \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ of $(\overline{x}, \overline{y}, 0)$ verifying

$$F(x,y) - F(\overline{x}, \overline{y}) + \rho \mid p \mid \geq 0 \ \ \forall \ (x,y,p) \in U \ \text{ feasible to } (13). \tag{14}$$

Under the partial calmness condition, the following lemma gives an exact penalization of the initial bilevel semi-infinite program.

Lemma 3.2. *Let $\overline{u} = (\overline{x}, \overline{y})$ be an optimal solution of* (1). *Suppose that* (1) *is partially calm at $\overline{u}$ and the upper-level objective F is lower semi-continuous at $\overline{u}$. Then $\overline{u}$ is a local optimal solution to the penalized problem*

$$\begin{cases} \min\limits_{x,y} \ \frac{1}{\rho} F(x,y) + f(x,y) - V(x) \\ g_t(x,y) \leq 0, \quad \forall \ t \in T, \end{cases} \tag{15}$$

with $\rho > 0$ is the constant mentioned in (14).

Proof. The proof technique is exactly that of [9, Lemma 5]. □

4. A subdifferential estimate for the marginal function

Our main objective here is to obtain an estimate of the subdifferential of the value function (11), and check when it can be locally Lipschitz continuous. Using a variational approach, the sensitivity analysis results established below are expressed via the initial data of (2) and the set of active constraint multipliers (10).

It is worth mentioning that Lemma 3.2 and [18, Theorem 7] provide the basic tools to prove the main result. Our forthcoming discussion relies essentially on the following constraint qualification conditions, which are defined at a given point $\overline{u} = (\overline{x}, \overline{y})$:

- The nonsmooth regular constraint qualification (RCQ):

$$N\Big(\overline{u}; \operatorname{gph} G\Big) = \widehat{N}\Big(\overline{u}; \operatorname{gph} G\Big) = \bigcup_{\upsilon \in \Upsilon(\overline{u})} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{u}) \Big].$$

- The nonsmooth limiting constraint qualification (LCQ):

$$N\Big(\overline{u}; \operatorname{gph} G\Big) \subset \bigcup_{\upsilon \in \Upsilon(\overline{u})} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{u}) \Big].$$

One way to formulate sufficient conditions for regular constraint qualification and limiting constraint qualification conditions is to use of the Farkas-Minkowski constraint qualification (FMCQ). Note that the latter constraint qualification was first introduced by Dinh, Mordukhovich and Nghia in [8]. Here, we introduce the FMCQ for (2).

Definition 4.1. Assume that for all $t \in T$, the constraint g_t in the problem (2) is proper, lower semi-continuous, and convex. We say that the parametric problem (2) satisfies FMCQ if

$$\text{cone}\left(\bigcup_{t\in T} \text{epi } g_t^*\right)$$

is closed in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ with g_t^* is the conjugate function of g_t for all $t \in T$.

We now give a sufficient condition for the regular constraint qualification condition.

Proposition 4.2. *Let $\overline{u} = (\overline{x}, \overline{y})$ be an optimal solution of* (1) *such that* (1) *is partially calm at $\overline{u}$ and the lower-level constraints g_t, $t \in T$ are proper, convex and lower semi-continuous. If FMCQ is valid for the parametric problem* (2)*, then RCQ holds at $\overline{u}$. The converse implication also holds if dom $\sigma_{gph\,G} \subseteq$ im $\partial\delta_{gph\,G}$.*

Proof. Since, g_t, $t \in T$ are convex and the graph of G is defined by

$$\text{gph}\, G = \Big\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \ : \ g_t\,(x, y) \le 0, \ \forall\, t \in T \Big\},$$

we get from [14, Proposition 1.5] that

$$N\big(\overline{u}; \text{gph}\, G\big) = \widehat{N}\big(\overline{u}; \text{gph}\, G\big). \tag{16}$$

On the other hand, we derive from [8, Corollary 3.6] that, under FMCQ, we can estimate the normal cone to gph G as follows

$$N\Big(\overline{u}; \text{gph}\, G\Big) = \bigcup_{\upsilon\in\Upsilon(\overline{u})} \Big[\sum_{t\in \text{supp}\,\upsilon} \upsilon_t \partial g_t\,(\overline{u}) \Big]. \tag{17}$$

By combining (16) and (17), the first result of the proposition is deduced.

Conversely, suppose that RCQ holds at $\overline{u}$ and *dom* $\sigma_{\text{gph}\,G} \subseteq$ *im* $\partial\delta_{\text{gph}\,G}$. Then by [12, Corollary 4.1.(ii)], we need to show only that

$$\textit{epi}\, \sigma_{\text{gph}\,G} = \text{cone}\left(\bigcup_{t\in T} \text{epi } g_t^*\right). \tag{18}$$

Let $\big((\xi^1, \xi^2), \alpha\big) \in \textit{epi}\, \sigma_{\text{gph}\,G}$. Since $(0_{\mathbb{R}^n\times\mathbb{R}^m}, 0_{\mathbb{R}})$ clearly belongs to the right-hand side of (18), we assume without loss of generality that $\big((\xi^1, \xi^2), \alpha\big) = (0_{\mathbb{R}^n\times\mathbb{R}^m}, 0_{\mathbb{R}})$. Now, since $(\xi^1, \xi^2) \in \textit{dom}\, \sigma_{\text{gph}\,G} \subseteq \textit{im}\, \partial\delta_{\text{gph}\,G}$, there exists $(x_0, y_0) \in \text{gph}\, G$ such that $(\xi^1, \xi^2) \in \partial\delta_{\text{gph}\,G}(x_0, y_0) = N((x_0, y_0); \text{gph}\, G)$. The definition of RCQ implies that (ξ^1, ξ^2) can be expressed as

$$(\xi^1, \xi^2) = \sum_{t\in \text{supp}\,\upsilon} \upsilon_t(\xi^1_t, \xi^2_t)$$

for some $\upsilon \in \Upsilon(x_0, y_0)$, $(\xi^1_t, \xi^2_t) \in \partial g_t(x_0, y_0)$, and $\upsilon_t \ge 0$ for each $t \in \text{supp}\,\upsilon$. Note that, from the Young equality (6), we have, $\langle(\xi^1_t, \xi^2_t), (x_0, y_0)\rangle = g^*_t(\xi^1_t, \xi^2_t)$ for each $t \in T$ because $(\xi^1_t, \xi^2_t) \in \partial g_t(x_0, y_0)$ and $g_t(x_0, y_0) = 0$.

On the other hand, since

$$\alpha \geq \langle(\xi^1,\xi^2),(x_0,y_0)\rangle = \sum_{t\in \operatorname{supp}\upsilon} \upsilon_t\langle(\xi^1_t,\xi^2_t),(x_0,y_0)\rangle = \sum_{t\in T}\upsilon_t\langle(\xi^1_t,\xi^2_t),(x_0,y_0)\rangle,$$

there exists a set $\{\alpha_t : t\in T\}$ of real numbers such that

$$\alpha = \sum_{t\in T}\upsilon_t\alpha_t \quad \text{and} \quad g_t^*(\xi^1_t,\xi^2_t) = \langle(\xi^1_t,\xi^2_t),(x_0,y_0)\rangle \leq \alpha_t \quad \text{for all } t\in T.$$

This implies that $\big((\xi^1_t,\xi^2_t),\alpha_t\big)\in \text{epi } g_t^*$ for each t and thus

$$\big((\xi^1,\xi^2),\alpha\big) \in \text{cone}\Big(\bigcup_{t\in T}\text{epi } g_t^*\Big).$$

In consequence we obtain $\text{epi } \sigma_{\text{gph}\,G} \subseteq \text{cone}\Big(\bigcup_{t\in T}\text{epi } g_t^*\Big)$. Since by [12, Remark 4.1] we have $\text{epi } \sigma_{\text{gph}\,G} \supseteq \text{cone}\Big(\bigcup_{t\in T}\text{epi } g_t^*\Big)$, (18) is proved. □

We note that the Farkas-Minkowski constraint qualification may not be verified even if the regular constraint qualification condition holds at $\overline{u}$ as shown in the following example.

Example 4.3. In the parametric problem (2) we consider that the objective function $f : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ and the constraints

$$g_t : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R},\ t\in T = \{\frac{1}{n} \ : \ n = 1,2,...\}\cup\{0,2\},$$

are defined respectively by

$$f(x,y) = (y_1-x)^3 + (y_2-x)^3 \text{ for all } y=(y_1,y_2)\in\mathbb{R}^2 \text{ and } x\in\mathbb{R},$$

$$g_t(x,y) = \begin{cases} -ty_1-(1-t)y_2+x, & t\in T\setminus\{2\}, \\ y_1-y_1^2+y_2-1-4x, & t=2, \end{cases} \quad \forall t\in T.$$

First, let us observe that according to [9, Corollary 3.6] FMCQ is not verified because g_2 is not convex. On the other hand, let $\overline{x}=0$. We have

$$G(\overline{x}) = \{(y_1,y_2)\in\mathbb{R}^2 \ : \ y_1\geq 0,\ y_2\geq 0,\ y_1-y_1^2+y_2-1\leq 0\}$$

and $M(\overline{x}) := \{y\in G(\overline{x}) \ : \ V(\overline{x}) = f(\overline{x},y)\} = \{(0,0)\}$. Let $\overline{y}=(0,0)\in M(\overline{x})$. It is easy to see that

- for every $t\in T$, the function g_t is differentiable,
- the functions $(x,y,t)\mapsto g_t(x,y)$ and $(x,y,t)\mapsto \nabla g_t(x,y)$ are continuous,
- the function $h(x,y)$ defined by $h(x,y)(t) = g_t(x,y)$ for each $t\in T$ is continuously differentiable at $(\overline{x},\overline{y})$ according to [2, Proposition 2.174].

Since all assumption of [9, Theorem 3.2] are verified, then RCQ holds at $(\overline{x},\overline{y})$. □

Next we derive an upper estimate for the Fréchet subdifferential of the marginal function (11) at a given point $\overline{x}$ by using the basic subdifferential of the constraint mappings g_t, $t\in T$, and the upper subdifferential of the lower-level function $\widehat{\partial}^+ f(\overline{x},\overline{y})$, which is supposed to be nonempty for some $\overline{y}$ of the argminimum set

$$S(\overline{x}) = \{y\in G(\overline{x}) \ | \ V(\overline{x}) = f(\overline{x},y)\}. \tag{19}$$

Theorem 4.4. *Suppose that V defined in* (11) *is finite at some $\overline{x} \in$ dom S, and $\overline{y} \in S(\overline{x})$ with $\widehat{\partial}^{+} f(\overline{x}, \overline{y}) \neq \emptyset$. If* RCQ *holds at $\overline{u} = (\overline{x}, \overline{y})$, then*

$$\widehat{\partial} V(\overline{x}) \subset \Big\{ \widetilde{x} \in \mathbb{R}^n \mid (\widetilde{x}, 0) \in \widehat{\partial}^{+} f(\overline{x}, \overline{y}) + \bigcup_{\upsilon \in \Upsilon(\overline{u})} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{u}) \Big] \Big\}. \tag{20}$$

Proof. Let $\overline{y} \in S(\overline{x})$. For every fixed $(\alpha^*, \beta^*) \in \widehat{\partial}^{+} f(\overline{x}, \overline{y})$, we choose an arbitrary subgradient $\widetilde{x} \in \widehat{\partial} V(\overline{x})$. Then, by [18, Theorem 1] we get

$$\widetilde{x} - \alpha^* \in \widehat{D}^* G(\overline{u})(\beta^*).$$

Furthermore, we get from the definition of Fréchet coderivative that

$$\widetilde{x} - \alpha^* \in \widehat{D}^* G(\overline{u})(\beta^*) \iff (\widetilde{x} - \alpha^*, -\beta^*) \in \widehat{N}\big(\overline{u}; \operatorname{gph} G\big). \tag{21}$$

Since the RCQ holds at $(\overline{x}, \overline{y})$, the second inclusion in (21) is equivalent to

$$(\widetilde{x} - \alpha^*, -\beta^*) \in \bigcup_{\upsilon \in \Upsilon(\overline{u})} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{u}) \Big].$$

Hence, we obtain the inclusion

$$(\widetilde{x}, 0) \in (\alpha^*, \beta^*) + \bigcup_{\upsilon \in \Upsilon(\overline{u})} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{u}) \Big].$$

Thus (20) holds true. □

With the inner semicompactness/semicontinuity of S, we can establish verifiable upper estimates for the basic and singular subdifferentials of V in (11). This result is proved by Chuong, Huy and Yao in [3] only when the solution map S is inner-semicontinous. The proof is given below in both cases for the convenience of the reader.

Theorem 4.5. *Assume that $\overline{x} \in$ dom S and f and g_t, $t \in T$, are Lipschitz continuous around $(\overline{x}, \overline{y})$.*

(i) *If S is V-inner semicompact at $\overline{x}$ and the LCQ holds at $(\overline{x}, y)$, for any $y \in S(\overline{x})$, then one has the inclusions*

$$\partial V(\overline{x}) \subset \bigcup_{y \in S(\overline{x})} \Big\{ \alpha^* \in \mathbb{R}^n \mid (\alpha^*, 0) \in \partial f(\overline{x}, y) + \bigcup_{\upsilon \in \Upsilon(\overline{x}, y)} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{x}, y) \Big] \Big\}, \tag{22}$$

$$\partial^\infty V(\overline{x}) \subset \bigcup_{y \in S(\overline{x})} \Big\{ \alpha^* \in \mathbb{R}^n \mid (\alpha^*, 0) \in \bigcup_{\upsilon \in \Upsilon(\overline{x}, y)} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{x}, y) \Big] \Big\}. \tag{23}$$

(ii) *If S is V-inner semicontinuous at $(\overline{x}, \overline{y}) \in$ dom S and the LCQ holds at $(\overline{x}, \overline{y})$, then one has the inclusions*

$$\partial V(\overline{x}) \subset \Big\{ \alpha^* \in \mathbb{R}^n \mid (\alpha^*, 0) \in \partial f(\overline{x}, \overline{y}) + \bigcup_{\upsilon \in \Upsilon(\overline{x}, \overline{y})} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{x}, \overline{y}) \Big] \Big\},$$

$$\partial^\infty V(\overline{x}) \subset \Big\{ \alpha^* \in \mathbb{R}^n \mid (\alpha^*, 0) \in \bigcup_{\upsilon \in \Upsilon(\overline{x}, \overline{y})} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{x}, \overline{y}) \Big] \Big\}.$$

Proof. We will only prove (i) as (ii) can be deduced in a similar way. Since f is Lipschitz continuous around $(\overline{x}, \overline{y})$ and S is V-inner semicompact at $\overline{x}$, then according to [18, Theorem 7 (ii)] we get

$$\partial V(\overline{x}) \subset \bigcup_{y \in S(\overline{x})} \bigcup_{(\widetilde{x}, \widetilde{y}) \in \partial f(\overline{x}, y)} \left\{ \widetilde{x} + D^* G(\overline{x}, y)(\widetilde{y}) \right\}.$$

In taking any $\alpha^* \in \partial V(\overline{x})$ and applying the latter subdifferential description, we can find $y \in S(\overline{x})$ and $(\widetilde{x}, \widetilde{y}) \in \partial f(\overline{x}, y)$ such that

$$\alpha^* - \widetilde{x} \;\in\; D^* G(\overline{x}, y)(\widetilde{y}).$$

The definition of the normal coderivative yields

$$(\alpha^* - \widetilde{x}, -\widetilde{y}) \in N((\overline{x}, y); \operatorname{gph} G). \tag{24}$$

Considering the satisfaction of LCQ at $(\overline{x}, y)$, we can see from (24) that

$$(\alpha^*, 0) \;\in\; (\widetilde{x}, \widetilde{y}) + \bigcup_{\upsilon \in \Upsilon(\overline{x}, y)} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{x}, y) \Big].$$

Hence, the relationship

$$(\alpha^*, 0) \;\in\; \partial f(\overline{x}, y) + \bigcup_{\upsilon \in \Upsilon(\overline{x}, y)} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{x}, y) \Big]$$

for some $y \in S(\overline{x})$ holds true, which means (22) holds true as well. The inclusion (23) can be easily shown owing to the observation that $\partial^\infty f(\overline{x}, y) = \{0\}$, by employing the same steps of proof used to prove (22). □

Also of interest in this section is the Lipschitz continuity of V. It is shown in [18, Example 1(i)] that the value function may not be Lipschitz continuous in the general framework of (11). To address this issue, we use the singular subdifferential estimate of V.

Theorem 4.6. *Let $(\overline{x}, \overline{y}) \in gph\ S$.*

(i) *If S is inner semicompact at $\overline{x}$, LCQ is valid at $(\overline{x}, y)$, for any $y \in S(\overline{x})$, and*

$$\bigcup_{y \in S(\overline{x})} \left\{ \alpha^* \in \mathbb{R}^n \mid (\alpha^*, 0) \;\in\; \bigcup_{\upsilon \in \Upsilon(\overline{x}, y)} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{x}, y) \Big] \right\} = \{0\}, \tag{25}$$

then V is locally Lipschitz around $\overline{x}$.

(ii) *If S is inner semicontinuous at $(\overline{x}, \overline{y})$, LCQ is valid at $(\overline{x}, \overline{y})$, and*

$$\left\{ \alpha^* \in \mathbb{R}^n \mid (\alpha^*, 0) \;\in\; \bigcup_{\upsilon \in \Upsilon(\overline{x}, \overline{y})} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{x}, \overline{y}) \Big] \right\} = \{0\}, \tag{26}$$

then V is locally Lipschitz around $\overline{x}$.

Proof. We will just give the proof of (ii); the proof of (i) can be deduced in a similar way. To justify the semicontinuity of V, pick any sequence x_k converging to $\overline{x}$. The inner semicontinuity of S at $(\overline{x}, \overline{y})$ ensures the existence of a sequence of $y_k \in S(x_k)$ converging to $\overline{y}$.

Since $V(x_k) = f(x_k, y_k)$ for all $k \in \mathbb{N}$, and owing to the continuity of f at $(\overline{x}, \overline{y})$, then by passing to the limit as $k \to \infty$ we get

$$\liminf_{x_k \to \overline{x}} V(x_k) = f(\overline{x}, \overline{y}) = V(\overline{x})$$

Hence, V is semicontinuous at $\overline{x}$.

Under the LCQ and the inner semicontinuity of S, Theorem 4.5 can be employed to get an upper estimate for the singular subdifferential of V as follows

$$\partial^{\infty} V(\overline{x}) \subset \Big\{ \alpha^* \in \mathbb{R}^n \mid (\alpha^*, 0) \in \bigcup_{\upsilon \in \Upsilon(\overline{x}, \overline{y})} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{x}, \overline{y}) \Big] \Big\}.$$

Now, if (26) holds, then $\partial^{\infty} V(\overline{x}) = \{0\}$. Hence combining this fact with the sequential normal epi-compactness property of V, we obtain from [14, Theorem 3.52] that V is Lipschitz continuous around $\overline{x}$. □

5. Fuzzy/approximate necessary optimality conditions

In this and next section, we are mainly interested in the study of necessary optimality conditions for semi-infinite bilevel programs defined in (1)-(2). As mentioned in Section 2, using a suitable "partial calmness" qualification assumption together with the optimal value function of the lower level problem, we can reduce the fully bilevel problem under consideration to a one-level optimization problem with infinitely many constraints.

We begin in this section by providing a fuzzy/approximate necessary optimality conditions for (1) using a variational approach. In particular, we employ the extremal principle developed by Mordukhovich [13]. Before heading to that, let us set

$$C = \operatorname{gph} G,\ \overline{V}(x, y) = V(x) \text{ and } F_0(x, y) = \rho^{-1} F(x, y) + f(x, y) - \overline{V}(x, y),$$

where $\rho > 0$ is the constant in (14). The following result is crucial to prove the next theorem.

Proposition 5.1. *Let $\overline{u} = (\overline{x}, \overline{y})$ be an optimal solution of* (1). *Suppose that* (1) *is partially calm at $\overline{u}$ with $\overline{v} = F(\overline{u})$, and let*

$$\Omega_1 = C \times (-\infty, F(\overline{u})] \quad \text{and} \quad \Omega_2 = \text{gph } F_0. \tag{27}$$

Then, $(\overline{u}, \overline{v})$ is a local extremal point of $\{\Omega_1, \Omega_2\}$.

Proof. Let $\overline{u}$ be given such that all the conditions of the proposition hold true. From Lemma 3.2 we see that $\overline{u}$ is a local minimizer to (15). Suppose by contradiction that $(\overline{u}, \overline{v})$ is not a local extremal point of $\{\Omega_1, \Omega_2\}$. Hence, for every neighborhood U of $(\overline{u}, \overline{v})$ there exists $\epsilon > 0$ such that for all $a \in \mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}}$ one has

$$(\Omega_1 + a) \cap \Omega_2 \cap U \neq \emptyset.$$

Take $a = \left(0_{\mathbb{R}^n \times \mathbb{R}^m}, -\frac{\epsilon}{2}\right) \in \mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}}$. The latter relation gives $(u, F_0(u)) \in U$ with

$$u \in C \quad \text{and} \quad F_0(u) \in F(\overline{u}) - \frac{\epsilon}{2} - \mathbb{R}^+.$$

Consequently, $F_0(u) < F(\overline{u})$, which denies the truth of $\overline{u}$ being a local optimal solution of (15). □

Theorem 5.2. *Let $\overline{u} = (\overline{x}, \overline{y})$ be an optimal solution of (1) such that (1) is partially calm at $\overline{u}$ with $\overline{v} = F(\overline{u})$. Assume that V is locally Lipschitz around $\overline{x}$. Then, for any $\epsilon > 0$, there exist u_0, u_1, $u_2 \in \overline{u} + \epsilon\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}^m}$, v_1, $v_2 \in]F(\overline{u}) - \epsilon, F(\overline{u}) + \epsilon[$ and $\beta^\star_\epsilon \in \mathbb{R}_+ \setminus \{0\}$ such that $u_1 \in C$, $v_1 = F(u_1)$, $v_2 = F_0(u_2)$ and*

$$0 \in \widehat{\partial}(\beta^\star_\epsilon F_0)(u_0) + \widehat{N}(u_1, C) + \epsilon\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}^m}.$$

Proof. Assume that $\overline{u}$ is an optimal solution of (1) and let $\overline{v} = F(\overline{u})$. Suppose that (1) is partially calm at $\overline{u}$. Then Lemma 3.2 implies that $\overline{u}$ is a local minimizer to (15), and hence on the basis of Proposition 5.1, $(\overline{u}, \overline{v})$ is an extremal point of the system (Ω_1, Ω_2) defined by (27). Given $\epsilon > 0$, we choose ϵ' so that

$$\epsilon' < \min\left\{\frac{\epsilon}{2};\ \frac{1}{2(5+3l)};\ \frac{\epsilon}{4+3l};\ \frac{\epsilon}{3+4l};\ \frac{1}{2(1+l)(4+3l)}\right\} \tag{28}$$

with l being the Lipschitz constant of F_0 satisfying $l \leq k_F + k_f + k_V$, where, k_F, k_f, k_V are Lipschitz constants of F, f and V, respectively.

Now, the extremal principle from [14, Theorem 2.10] ensures the existence of

$$u_1, u_2 \in \overline{u} + \epsilon'\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}^m},\ \ v_1, v_2 \in]F(\overline{u}) - \epsilon, F(\overline{u}) + \epsilon[$$

and $(\widetilde{x}, \widetilde{y}) \in \mathbb{R}^{n+m} \times \mathbb{R}$ satisfying $\| (\widetilde{x}, \widetilde{y}) \| = 1$ such that $u_1 \in C$, $v_1 \in (-\infty, F(\overline{u})]$, $v_2 = F_0(u_2)$ and

$$(\widetilde{x}, \widetilde{y}) \in \Big(\widehat{N}((u_1, v_1); \Omega_1) + \epsilon'\mathbb{B}_{\mathbb{R}^{n+m}\times\mathbb{R}}\Big) \bigcap \Big(-\widehat{N}((u_2, v_2); \Omega_2) + \epsilon'\mathbb{B}_{\mathbb{R}^{n+m}\times\mathbb{R}}\Big).$$

This implies that there exist

$$(\alpha^\star_\epsilon, \beta^\star_\epsilon) \in \widehat{N}((u_1, v_1); \Omega_1) \quad \text{and} \quad (v^\star, \mu^\star) \in \widehat{N}((u_2, v_2); \Omega_2) \tag{29}$$

such that
$$(\alpha^\star_\epsilon, \beta^\star_\epsilon) + \epsilon'(a^\star_1, b^\star_1) = (\widetilde{x}, \widetilde{y}) = -(v^\star, \mu^\star) + \epsilon'(a^\star_2, b^\star_2) \tag{30}$$

where, $(a^\star_1, b^\star_1)$, $(a^\star_2, b^\star_2) \in \mathbb{B}_{\mathbb{R}^{n+m}\times\mathbb{R}}$.

On the one hand, we derive from [14, Proposition 1.2] that

$$\widehat{N}((u_1, v_1); \Omega_1) = \widehat{N}(u_1; C) \times \widehat{N}(v_1; (-\infty; F(\overline{u})]). \tag{31}$$

Hence, by (29), $\alpha^\star_\epsilon \in \widehat{N}(u_1; C)$ and $\beta^\star_\epsilon \in \widehat{N}(v_1; (-\infty; F(\overline{u})])$. Furthermore, $\beta^\star_\epsilon \geq 0$. On the other hand, since $(v^\star, \mu^\star) \in \widehat{N}((u_2, v_2); \Omega_2)$, the definition of Fréchet's normal cone yields

$$\langle v^\star, u - u_2\rangle + \langle \mu^\star, v - v_2\rangle - \epsilon' \| (u - u_2, v - v_2) \| \leq 0.$$

for all $(u, v) \in \Omega_2 = \text{gph } F_0$ sufficiently close to (u_2, v_2). Combining the latter inequality with (30) and noting that $v = F_0(u)$ and $v_2 = F_0(u_2)$, we have

$$\langle \alpha^\star_\epsilon, u - u_2\rangle + \beta^\star_\epsilon(F_0(u) - F_0(u_2)) + 3\epsilon'(1 + l) \| u - u_2 \| \geq 0$$

for each u sufficiently close to u_2. Then, u_2 minimizes locally the function

$$\psi(x) = \langle \alpha^\star_\epsilon, u - u_2\rangle + \beta^\star_\epsilon(F_0(u) - F_0(u_2)) + 3\epsilon'(1 + l) \| u - u_2 \|.$$

In applying the fuzzy sum rule [14, Theorem 2.33] to

$$\psi_1(x) = \langle \alpha^\star_\epsilon, u - u_2 \rangle + 3\epsilon'(1+l) \parallel u - u_2 \parallel \text{ and } \psi_2(x) = \beta^\star_\epsilon (F_0(u) - F_0(u_2))$$

and using [14, Proposition 1.107], we find $u_0 \in u_2 + \epsilon' \mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^m}$ such that

$$0 \in \alpha^\star_\epsilon + \widehat{\partial}(\beta^\star_\epsilon F_0)(u_0) + \epsilon' \max\{4+3l; 3+4l\} \mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^m}. \tag{32}$$

Hence, there exist $\xi^\star_\epsilon \in \widehat{\partial}(\beta^\star_\epsilon F_0)(u_0)$ and $\varrho^\star \in \mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^m}$ such that

$$0 = \alpha^\star_\epsilon + \xi^\star_\epsilon + \epsilon'(4+3l)\varrho^\star$$

Since F_0 is Lipschitz continuous around $\overline{u}$, one has from [14, Proposition 1.85] that $\parallel \xi^\star_\epsilon \parallel \leq l\beta^\star_\epsilon$. Hence

$$\parallel \alpha^\star_\epsilon \parallel \leq l\beta^\star_\epsilon + \epsilon'(4+3l). \tag{33}$$

Furthermore, considering the fact that $\parallel (\widetilde{x}, \widetilde{y}) \parallel = 1$, it follows from (30) that

$$\parallel \alpha^\star_\epsilon \parallel + \beta^\star_\epsilon > 1 - \epsilon'. \tag{34}$$

In combining (33) with (34) while taking into account (28), we arrive at

$$\beta^\star_\epsilon > \frac{1}{2(1+l)} > 0. \tag{35}$$

Finally, the combination of (29), (31) and (32), while taking into account (28), ensures the conclusion of our theorem. □

In the next theorem we deduce from Theorem 5.2 a new KKT necessary optimality conditions for (1) in terms of Fréchet subdifferentials.

Theorem 5.3. *Let $\overline{u} = (\overline{x}, \overline{y})$ be an optimal solution of* (1) *such that* (1) *is partially calm at $\overline{u}$. Suppose that V is locally Lipschitz near $\overline{x}$ and assume that the RCQ holds near $\overline{u}$. Then, for any $\epsilon > 0$, there exist $u_1, u_3, u_4 = (x_4, y_4) \in \overline{u} + \epsilon \mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^m}$, $\beta^\star_\epsilon \in \mathbb{R}_+ \setminus \{0\}$ and multipliers $\upsilon = (\upsilon_t) \in \mathbb{R}^T_+$ from the positive cone in* (8) *satisfying*

$$\begin{aligned} 0 \; &\in \beta^\star_\epsilon \Big(\rho^{-1} \, \widehat{\partial} F(u_3) + \widehat{\partial} f(u_4) - \widehat{\partial} V(x_4) \times \{0\} \Big) \\ &+ \sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(u_1) + \epsilon \mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^m}, \end{aligned} \tag{36}$$

$$\upsilon_t g_t(u_1) = 0 \quad \forall t \in T, \tag{37}$$

where $\rho > 0$ is the constant from (14).

Proof. Suppose that $\overline{u} = (\overline{x}, \overline{y})$ satisfies the conditions of the theorem with $\overline{v} = F(\overline{u})$, and let $\epsilon > 0$. Theorem 5.2 gives us

$$u_0, u_1, u_2 \in \overline{u} + \tfrac{\epsilon}{2} \mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^m}, \quad v_1, v_2 \in]F(\overline{u}) - \tfrac{\epsilon}{2}, F(\overline{u}) + \tfrac{\epsilon}{2}[$$

and $\beta^\star_\epsilon \in \mathbb{R}_+ \setminus \{0\}$ with $u_1 \in C$, $v_1 = F(\overline{u})$, $v_2 = F_0(u_2)$ and

$$0 \in \widehat{\partial}(\beta^\star_\epsilon F_0)(u_0) + \widehat{N}(u_1, C) + \frac{\epsilon}{2} \mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^m}. \tag{38}$$

The application of the fuzzy sum rule [14, Theorem 2.33] on $\rho^{-1}\beta_\epsilon^\star F$ and $\beta_\epsilon^\star \left(f-\overline{V}\right)$ permits to find u_3, $u_4 \in u_0 + \frac{\epsilon}{2}\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}^m}$ such that

$$\widehat{\partial}\left(\beta_\epsilon^\star F_0\right)(u_0) \subset \widehat{\partial}\left(\rho^{-1}\beta_\epsilon^\star F\right)(u_3) + \widehat{\partial}\left(\beta_\epsilon^\star f - \beta_\epsilon^\star \overline{V}\right)(u_4) + \frac{\epsilon}{2}\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}^m}. \tag{39}$$

Observe by using the difference rule for regular subdifferentials (see (5)) that (39) becomes

$$\widehat{\partial}\left(\beta_\epsilon^\star F_0\right)(u_0) \subset \widehat{\partial}\left(\rho^{-1}\beta_\epsilon^\star F\right)(u_3) + \widehat{\partial}\left(\beta_\epsilon^\star f\right)(u_4) - \widehat{\partial}\left(\beta_\epsilon^\star \overline{V}\right)(u_4) + \frac{\epsilon}{2}\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}^m}. \tag{40}$$

In combining (38) and (40) while noting the fact that

$$\widehat{N}(u_1; C) = \bigcup_{\upsilon\in\Upsilon(x,y)} \Big[\sum_{t\in\operatorname{supp}\upsilon} \upsilon_t \partial g_t(u_1)\Big]$$

(under RCQ), we see that

$$0 \in \widehat{\partial}\left(\rho^{-1}\beta_\epsilon^\star F\right)(u_3) + \widehat{\partial}\left(\beta_\epsilon^\star f\right)(u_4) - \widehat{\partial}\left(\beta_\epsilon^\star \overline{V}\right)(u_4) + \sum_{t\in\operatorname{supp}\upsilon} \upsilon_t \partial g_t(u_1) + \epsilon\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}^m}$$

for some $\upsilon \in \mathbb{R}_+^T$, with $\upsilon_t g_t(u_1) = 0$ and $u_4 = (x_4, y_4)$. Considering the fact that $\beta_\epsilon^* > 0$, the latter inclusion implies the following one

$$0 \in \beta_\epsilon^\star\Big(\rho^{-1}\ \widehat{\partial}F(u_3) + \widehat{\partial}f(u_4) - \widehat{\partial}V(x_4)\times\{0\}\Big) + \sum_{t\in\operatorname{supp}\upsilon} \upsilon_t \partial g_t(u_1) + \epsilon\mathbb{B}_{\mathbb{R}^n\times\mathbb{R}^m}.$$

In conclusion, we obtain the optimality conditions (36)–(37). □

Note that the Fréchet subdifferential can often be empty at individual points of the domains even for simple nonconvex functions, for example, $\phi(x) = -\mid x\mid$ at $x = 0$. However, we can overcome this difficulty by employing necessary optimality conditions that are similar to those in Theorem 5.3. This is obtained from the upper estimates for the basic subdifferential of the value function under suitable constraint qualification as we will see in the next section.

6. Necessary optimality conditions using the basic subdifferentials

We focus here on necessary optimality conditions for the nonconvex bilevel semi-infinite program (1) in terms of basic subdifferential, in a direct way under LCQ, and this by assuming that the partial calmness is verified for (1). Suppose that the set S in (19) is inner semicompact.

Theorem 6.1. *Suppose that $\overline{u} = (\overline{x}, \overline{y})$ is an optimal solution of (1) such that (1) is partially calm at $\overline{u}$, S is inner semicompact at $\overline{x}$, LCQ is valid at $(\overline{x}, y)$ for any $y \in S(\overline{x})$, and (25) holds. Then there are scalars σ_s, $\gamma^s = (\gamma_t^s) \in \widetilde{\mathbb{R}}_+^T$, vectors $u_s \in \mathbb{R}^n$, $y_s \in S(\overline{x})$, $s = 1, \cdots, n+1$ and multipliers $\upsilon = (\upsilon_t) \in \widetilde{\mathbb{R}}_+^T$, such that*

$$\Big(\sum_{s=1}^{n+1} \sigma_s u_s, 0\Big) \in \rho^{-1}\partial F(\overline{u}) + \partial f(\overline{u}) + \sum_{t\in\operatorname{supp}\upsilon} \upsilon_t \partial g_t(\overline{u})$$

$$\forall s = 1, \cdots, n+1 \quad (u_s, 0) \in \partial f(\overline{u}) + \sum_{t \in \operatorname{supp} \gamma^s} \gamma_t^s \partial g_t(\overline{x}, y_s)$$

$$\forall s = 1, \cdots, n+1, \ \forall t \in T \quad \upsilon_t g_t(\overline{u}) = \gamma_t^s g_t(\overline{u}) = 0 \tag{41}$$

$$\forall s = 1, \cdots, n+1 \quad \sigma_s \geq 0, \quad \sum_{s=1}^{n+1} \sigma_s = 1$$

where $\rho > 0$ is the constant from (14).

Proof. Let $\overline{u}$ verify the assumptions of the theorem. In picking any arbitrary number $\epsilon > 0$ and applying Theorem 5.2, there exist elements $u_0, u_1, u_2 \in \overline{u} + \epsilon\mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^m}$, $v_1, v_2 \in]F(\overline{u}) - \epsilon, F(\overline{u}) + \epsilon[$ and $\beta_\epsilon^\star \in \mathbb{R}_+ \setminus \{0\}$ such that $u_1 \in C$, $v_1 = F(u_1)$, $v_2 = F_0(u_2)$ and

$$0 \in \widehat{\partial}(\beta_\epsilon^\star F_0)(u_0) + \widehat{N}(u_1, C) + \epsilon\mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^m}.$$

By dividing (32) by $\beta_\epsilon^\star$, we set

$$\tilde{\alpha}_\epsilon^\star = \frac{\alpha_\epsilon^\star}{\beta_\epsilon^\star} \in \widehat{N}(u_1; C) \quad \text{and} \quad \tilde{\beta}_\epsilon^\star = \frac{\beta_\epsilon^\star}{\beta_\epsilon^\star} = 1 \in \widehat{N}\left(v_1; \overline{v} - \mathbb{R}^+\right).$$

According to (33) and (35), taking into account (28), we can see that $\| \tilde{\alpha}_\epsilon^\star \| \leq 1 + l$. Hence, the sequences $\tilde{\alpha}_\epsilon^\star$ and $\tilde{\beta}_\epsilon^\star$ are bounded and then they admit two converging subsequences to $\alpha^\star$ and $\beta^\star$. Consequently, by the definition of Mordukhovich normal cone, we get

$$\alpha^\star \in N(\overline{u}; C) \quad \text{and} \quad \beta^\star \in N\left(\overline{v}; \overline{v} - \mathbb{R}^+\right).$$

Now, using the Lipschitz property of F_0, we can apply (32) to get $\xi_\epsilon^\star \in \widehat{\partial}\big(\tilde{\beta}_\epsilon^\star F_0\big)(u_0)$ and $\varrho^\star \in \mathbb{B}_{\mathbb{R}^n \times \mathbb{R}^m}$ such that

$$0 = \tilde{\alpha}_\epsilon^\star + \xi_\epsilon^\star + \epsilon\varrho^\star, \quad \| \xi_\epsilon^\star \| \leq l\tilde{\beta}_\epsilon^\star$$

and
$$\left(\xi_\epsilon^\star, -\tilde{\beta}_\epsilon^\star\right) \in \widehat{N}\left((u_0, F_0(u_0)); gphF_0\right)$$

while taking into account (28). Letting $\epsilon \to 0$, we get the relation

$$(-\alpha^\star, -\beta^\star) \in N\left((\overline{u}, F_0(\overline{u})); gphF_0\right),$$

which means that $-\alpha^\star \in \partial(\beta^\star F_0)(\overline{u}) = \beta^\star \partial(F_0)(\overline{u})$. Therefore

$$0 \in \partial(F_0)(\overline{u}) + N(\overline{u}; C).$$

On the other hand, recalling the fact that

$$N((\overline{x}, \overline{y}); C) \subset \bigcup_{\upsilon \in \Upsilon(\overline{x}, \overline{y})} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{x}, \overline{y}) \Big]$$

(under the LCQ regularity) and using the basic subdifferential sum rule, we find $\upsilon \in \widetilde{\mathbb{R}}_+^T$ satisfying

$$0 \in \rho^{-1}\partial F(\overline{x}, \overline{y}) + \partial f(\overline{x}, \overline{y}) + \partial(-V)(\overline{x}) \times \{0\} + \sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{x}, \overline{y}) \tag{42}$$

and $\upsilon_t g_t(\overline{x}, \overline{y}) = 0$ for each $t \in T$. Since the convex property (7) yields

$$\partial(-V)(\overline{x}) \subset \text{co}\, \partial(-V)(\overline{x}) = -\text{co}\, \partial V(\overline{x}), \tag{43}$$

we obtain from (42) the following

$$0 \in \rho^{-1}\partial F(\overline{x},\overline{y}) + \partial f(\overline{x},\overline{y}) - \operatorname{co}\partial V(\overline{x}) \times \{0\} + \sum_{t\in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{x},\overline{y}). \tag{44}$$

Let us now recall that, according to (i) in Theorem 4.5, since S is inner semicompact at $\overline{x}$ and LCQ holds at $(\overline{x}, y)$, for all $y \in S(\overline{x})$, then we have an upper estimate of the basic subdifferential of V as

$$\partial V(\overline{x}) \subset \bigcup_{y\in S(\overline{x})} \Big\{\alpha^* \in \mathbb{R}^n \mid (\alpha^*, 0) \in \partial f(\overline{x}, y) + \bigcup_{\upsilon \in \Upsilon(\overline{x},y)} \Big[\sum_{t\in \operatorname{supp}\upsilon} \upsilon_t \partial g_t(\overline{x}, y)\Big]\Big\}. \tag{45}$$

Now, on the basis of the classical Carathéodory theorem, we obtain according to (45) and the fact that $u \in \operatorname{co}\partial V(\overline{u})$ the existence of some scalars σ_s and vectors $\gamma^s \in \Upsilon(\overline{x}, y_s)$ and $u_s \in \mathbb{R}^n$, $s = 1, \cdots, n+1$, such that

$$\sum_{s=1}^{n+1} \sigma_s = 1, \quad u = \sum_{s=1}^{n+1} \sigma_s u_s$$

$$\forall s = 1, \cdots, n+1 \quad \sigma_s \geq 0, \quad (u_s, 0) \in \partial f(\overline{u}) + \sum_{t\in \operatorname{supp}\gamma^s} \gamma_t^s \partial g_t(\overline{x}, y_s).$$

From (44), it follows that

$$\Big(\sum_{s=1}^{n+1} \sigma_s u_s, 0\Big) \in \rho^{-1}\partial F(\overline{x},\overline{y}) + \partial f(\overline{x},\overline{y}) + \sum_{t\in \operatorname{supp}\upsilon} \upsilon_t \partial g_t(\overline{x},\overline{y}).$$

Consequently, all the optimality conditions in (41) are fulfilled. □

In the above theorem, when S is strongly inner semicontinuous, we get conditions for (1) involving only the reference optimal solution $\overline{u}$. Indeed, if S has a closed graph and is inner semicompact at $\overline{x}$ such that $S(\overline{x}) = \{\overline{y}\}$, then it is inner semicontinuous at $\overline{u}$.

Theorem 6.2. *Suppose that $\overline{u} = (\overline{x}, \overline{y})$ is an optimal solution of (1) such that (1) is partially calm at $\overline{u}$, S is inner semicontinuous at $\overline{u}$, LCQ is valid at $\overline{u}$, and (26) holds. Then there are scalars σ_s, $\gamma^s = (\upsilon_t^s) \in \widetilde{\mathbb{R}}_+^T$ from the positive cone in (8), vectors $u_s \in \mathbb{R}^n$, $s = 1, \cdots, n+1$ and multipliers $\upsilon = (\upsilon_t) \in \widetilde{\mathbb{R}}_+^T$ satisfying*

$$\Big(\sum_{s=1}^{n+1} \sigma_s u_s, 0\Big) \in \rho^{-1}\partial F(\overline{u}) + \partial f(\overline{u}) + \sum_{t\in \operatorname{supp}\upsilon} \upsilon_t \partial g_t(\overline{u})$$

$$\forall s = 1, \cdots, n+1, \quad (u_s, 0) \in \partial f(\overline{u}) + \sum_{t\in \operatorname{supp}\gamma} \gamma_t^s \partial g_t(\overline{u})$$

$$\forall s = 1, \cdots, n+1, \ \forall t \in T, \quad \upsilon_t g_t(\overline{u}) = \gamma_t^s g_t(\overline{u}) = 0$$

$$\forall s = 1, \cdots, n+1, \quad \sigma_s \geq 0, \ \sum_{s=1}^{n+1} \sigma_s = 1$$

where $\rho > 0$ is the constant from (14).

Proof. As in the previous theorem, it follows from Theorem 4.6 that V is Lipschitz near $\bar{x}$. Furthermore, under the inner semicontinuity of S and the LCQ, Theorem 4.5 ensures the truth of the following upper estimate for the subdifferential of V

$$\partial V(\overline{x}) \subset \left\{\alpha^* \in \mathbb{R}^n \mid (\alpha^*, 0) \in \partial f(\overline{u}) + \bigcup_{\upsilon \in \Upsilon(\overline{u})} \Big[\sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial g_t(\overline{u}) \Big] \right\}. \quad (46)$$

We then combine (42), (43) and (46) to conclude. □

In case the strict differentiability of all functions involved in (1)-(2) holds instead of the local Lipschitz property, we obtain the following result.

Corollary 6.3. *Let $\overline{u} = (\overline{x}, \overline{y})$ be an optimal solution of* (1) *such that* (1) *is partially calm at $\overline{u}$, where F, f and g_t, $t \in T$ are all strict differentiable. Suppose that S is inner semicontinuous at $\overline{u}$, LCQ is valid at $\overline{u}$, and* (26) *holds. Then there are scalars σ_s, multipliers $\upsilon = (\upsilon_t) \in \widetilde{\mathbb{R}}^T_+$, $\gamma^s = (\gamma^s_t) \in \widetilde{\mathbb{R}}^T_+$ from the positive cone in* (8) *and vectors $u_s \in \mathbb{R}^n$, $s = 1, \cdots, n+1$ such that*

$$\sum_{s=1}^{n+1} \sigma_s u_s \in \rho^{-1} \nabla_x F(\overline{u}) + \nabla_x f(\overline{u}) + \sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \nabla_x g_t(\overline{u})$$

$$\rho^{-1} \nabla_y F(\overline{u}) + \nabla_y f(\overline{u}) + \sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \nabla_y g_t(\overline{u}) = 0$$

$$\forall s = 1, \cdots, n+1, \quad u_s = \nabla_x f(\overline{u}) + \sum_{t \in \operatorname{supp} \gamma^s} \gamma^s_t \nabla_x g_t(\overline{u})$$

$$\forall s = 1, \cdots, n+1, \quad \nabla_y f(\overline{u}) + \sum_{t \in \operatorname{supp} \gamma^s} \gamma^s_t \nabla_y g_t(\overline{u}) = 0$$

$$\forall s = 1, \cdots, n+1, \ \forall t \in T, \quad \upsilon_t g_t(\overline{u}) = \gamma^s_t g_t(\overline{u}) = 0$$

$$\forall s = 1, \cdots, n+1, \quad \sigma_s \geq 0, \ \sum_{s=1}^{n+1} \sigma_s = 1$$

where $\rho > 0$ is the constant from (14).

Finally, we present the following result inspired from Theorem 6.2 which treats the case of convexity of all the functions involved in (1)-(2).

Corollary 6.4. *Let $\overline{u} = (\overline{x}, \overline{y})$ be an optimal solution of* (1) *such that* (1) *is partially calm at $\overline{u}$, where F, f and g_t, $t \in T$ are all convex. Suppose that S is inner semicontinuous at $\overline{u}$, LCQ is valid at $\overline{u}$, and* (26) *holds. Then there exist $\upsilon = (\upsilon_t) \in \tilde{\mathbb{R}}^T_+$ and $\gamma = (\gamma_t) \in \tilde{\mathbb{R}}^T_+$ satisfying*

$$0 \in \rho^{-1} \partial_x F(\overline{u}) + [\partial_x f(\overline{u}) - \partial_x f(\overline{u})] + \sum_{t \in T} (\upsilon_t - \gamma_t) \partial_x g_t(\overline{u}),$$

$$0 \in \rho^{-1} \partial_y F(\overline{u}) + \partial_x f(\overline{y}, \overline{y}) + \sum_{t \in T} \upsilon_t \partial_y g_t(\overline{u}),$$

$$0 \in \partial_x f(\overline{y}, \overline{y}) + \sum_{t \in T} \gamma_t \partial_y g_t(\overline{u}),$$

$$\upsilon_t g_t(\overline{u}) = \gamma_t g_t(\overline{u}) = 0 \ \ \forall t \in T,$$

where $\rho > 0$ is the constant from (14).

Proof. First, notice V is locally Lipschitz continuous on the basis of Theorem 4.5. Moreover, Theorem 8 from [9] ensures the convexity of the value function owing to the convexity of the initial functions of this problem.

On the other hand, considering the equality (42), it follows from (46) that there exist vectors $\upsilon = (\upsilon_t), \gamma(\gamma_t) \in \tilde{\mathbb{R}}_+^T$ and $\alpha \in \mathbb{R}^n$ with $\upsilon_t g_t(\overline{u}) = 0$ for all $t \in \operatorname{supp} \upsilon$ and $\gamma_t g_t(\overline{u}) = 0$ for all $t \in \operatorname{supp} \gamma$ such that

$$(\alpha, 0) \in \rho^{-1}(\partial_x F(\overline{u}) \times \partial_y F(\overline{u})) + \partial_x f(\overline{u}) \times \partial_y f(\overline{u}) + \sum_{t \in \operatorname{supp} \upsilon} \upsilon_t \partial_x g_t(\overline{u}) \times \partial_y g_t(\overline{y}, \overline{y}), \quad (47)$$

$$\alpha \in \partial_x f(\overline{u}) + \sum_{t \in \operatorname{supp} \gamma} \gamma_t \partial_x g_t(\overline{u}) \quad (48)$$

and

$$0 \in \partial_y f(\overline{u}) + \sum_{t \in \operatorname{supp} \gamma} \gamma_t \partial_y g_t(\overline{y}, \overline{y})$$

while taking into account the following partial differential relationships

$$\partial F(\overline{u}) \subset \partial_x F(\overline{u}) \times \partial_y F(\overline{u}), \quad \partial f(\overline{u}) \subset \partial_x f(\overline{u}) \times \partial_y f(\overline{u})$$

and

$$\partial g_t(\overline{u}) \subset \partial_x g_t(\overline{u}) \times \partial_y g_t(\overline{u}).$$

The combination of (47) and (48) gives the result. □

In the following example we show that Theorem 6.1 gains in interest if we realize that the follower's objective and constraints functions are not required to be convex as contrary to the results from [9] that require convexity.

Example 6.5. Assume that the upper and lower levels objectives are given by

$$F(x, y) = |x| + |y| \quad \text{and} \quad f(x, y) = -|x| - y^2 + y.$$

The constraints in the lower level are

$$g_1(x, y) = y - 1, \quad g_t(x, y) = \begin{cases} -\frac{1}{t}y, & y < 0, \\ -y, & y \geq 0, \end{cases} \quad \forall t \in T^* = \{2, 3, ...\}.$$

The unique global optimum of the multiobjective bilevel program (1) is $\overline{u} = (0, 0)$.

It is easy to check that the functions F, f and g_t, $t \in T = \{1\} \cup T^* = \mathbb{N} \setminus \{0\}$, are locally Lipschitz at $\overline{u}$ but f and g_t, $t \in T$, are not convex. Note also that the optimal value function $V(x) = -|x|$ is nonconvex too.

The set $S(x)$ of optimal solutions to (2) is $S(x) = \{0, 1\}$, which is inner semicompact at 0. The basic/limiting/Mordukhovich (lower) subdifferentials of F, f, g_1 and g_t, $t \in T$ are given by

$$\partial F(\overline{u}) = [-1, 1] \times [-1, 1], \qquad \partial f(\overline{u}) = \operatorname{co}\{(-1, 1), (1, 1)\},$$

$$\partial g_1(\overline{u}) = \partial g_1(0, 1) = \{(0, 1)\},$$

$$\partial g_t(\overline{u}) = \operatorname{co}\left\{(0, -1), (0, -\frac{1}{t})\right\} \quad \forall t \in T^* \quad \partial g_t(0, 1) = \{(0, -1)\}.$$

By simple calculations we show that $\operatorname{gph} \mathrm{G} = \mathbb{R} \times [0,1]$,

$$\Upsilon(0,0) = \{\upsilon \in \widetilde{\mathbb{R}}^{\mathbb{N}}_{+} : \upsilon_1 = 0\}, \quad \Upsilon(0,1) = \{\upsilon \in \widetilde{\mathbb{R}}^{\mathbb{N}}_{+} : \upsilon_t = 0, \forall t \in T^*\},$$

$$N(\overline{u}, \operatorname{gph} \mathrm{G}) = \{0\} \times \mathbb{R}^- \subseteq \bigcup_{\upsilon \in \Upsilon(0,0)} \Big[\sum_{\mathrm{t} \in \mathrm{T}^*} \upsilon_{\mathrm{t}} \partial \mathrm{g}_{\mathrm{t}}(\overline{\mathrm{u}}) \Big] = \{0\} \times \mathbb{R}^-$$

and

$$N((0,1), \operatorname{gph} \mathrm{G}) = \{0\} \times \mathbb{R}^+ \subseteq \bigcup_{\upsilon \in \Upsilon(0,1)} \Big[\upsilon_1 \partial \mathrm{g}_1(0,1) \Big] = \{0\} \times \mathbb{R}^+.$$

Hence LCQ is valid at $(0, y)$ for any $y \in S(0)$ and (25) holds.

For $\rho = 1$, we can easily deduce that (1) is partially calm at $\overline{u}$. Then for $\sigma_1 = \sigma_2 = \frac{1}{2}$, $u_1 = u_2 = -1$, $y_1 = 0$, $y_2 = 1$, $\upsilon_2 = \gamma_2^1 = \gamma_2^2 = 1$, $\upsilon_t = \gamma_t^1 = \gamma_t^2 = 0$, $\forall t \in \mathbb{N} \setminus \{0, 2\}$, $s \in \{1, 2\}$, the conclusion of Theorem 6.1 hold true.

7. Conclusion

In this work, we have rewritten upper estimates of three recently developed subdifferentials of the value functions using nonsmooth regular and limiting constraint qualification, which are weaker than the existing Mangasarian-Fromovitz and Farkas-Minkowski CQs. We also establish new first-order necessary optimality conditions for bilevel semi-infinite programming problem when the involved functions are nonconvex. For future research, we can obtain similar results for the same problem we studied using weaker subdifferentials such as tangential subdifferentials and directional convexificators.

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