

# Connectedness in Set Optimization via a Nonlinear Scalarization

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We consider the sets of minimal solutions and weak minimal solutions to a set optimization problem with respect to the lower and upper set less relations. Via a nonlinear scalarization based on the oriented distance, we establish that they are connected under the assumption of strictly natural quasi cone-convexity.

*Keywords:* Set optimization, connectedness, weak l-minimal solution, lower set less relation, scalarization.

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## 1. Introduction

In recent years, much attention has been paid on set-valued optimization problems due to their extensive applications in many fields as, for example, mathematical economics, optimal control, differential inclusions, viability theory, game theory, well posedness, fuzzy programming or duality in vector optimization (see, for instance, [6, 7, 9, 15, 22] and the references there).

Scalarization is an important method in vector and set optimization since it allows us to transform a set problem into an appropriate scalar problem. In this method, minimal solutions of the given problem are characterized through minimal solutions of the associated scalar problem. The most popular nonlinear scalarizations are the Gerstewitz function [3] and the oriented distance of Hiriart-Urruty [16]. Several extensions of oriented distance function in set optimization have been considered in the literature [2, 6, 8, 19, 20, 23, 24, 29]. Ha [8] introduced one of these extensions of sup-inf type and Jiménez et al. [19] another similar extension. In [8, 17, 19, 20, 21] it is characterized the (weak) lower and upper set less relations, optimality conditions are obtained for set optimization problems with respect to these order relations and many properties of the scalarization such as finiteness, monotonicity, cone-invariance, continuity and convexity, etc., were established.

One of the important questions in vector and set optimization is to investigate properties of the solutions sets such as closedness or connectedness. This latter is important because it provides a possibility of continuously moving from a solution

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to another. Connectedness of the efficient solution set or weak efficient solution set or (Henig or super efficient) proper efficient solution set for vector optimization problems or vector equilibrium problems has been intensively investigated (see, for instance, [1, 5, 27, 28, 30]). In the last years, in set optimization, connectedness of the (weak) l-minimal solution set has been studied in several papers [10, 11, 13].

Gong [5] studied the connectedness of the efficient sets in convex vector optimization with set-valued maps in normed spaces; in a Pareto problem with a vector valued map, Sun [28] proved that the efficient solution set is connected if one component is strongly quasiconvex, and Qiu and Yang [27] established the connectedness of the Henig proper and super efficient solution sets under the condition that the objective function be a cone-arcwise connected set-valued mapping. Recently, in set optimization, Han, Wang and Huang [13] stated the arcwise connectedness of the sets of (weak) l-minimal solutions under strictly natural quasi cone-convexity with a compact valued objective; Han [10] investigated the connectedness of the set of p-minimal solutions, and finally, Han [11] established the connectedness and arcwise connectedness of the sets of approximate (weak) l-minimal solutions by utilizing a set extension of the Gerstewitz function under cone-convexity or (strictly) natural quasi cone-convexity.

Based on the above papers, in this paper, by means of scalarizations based on the oriented distance introduced in [19], we establish the connectedness of the (weak) solution sets with respect to the lower and upper set less relations. Properties as convexity, continuity and monotonicity stated in [17, 19, 20, 21] play an important role in the proof of our results. We also establish the closedness of the weak l-minimal set and u-minimal set under weak assumptions.

## 2. Preliminaries

Let  $X, Y$  be real normed vector spaces. Given a subset  $A \subset Y$ , we denote the *interior* of  $A$  by  $\text{int } A$ . It is said that  $A$  is *solid* if  $\text{int } A \neq \emptyset$ . Let  $\mathcal{P}_0(Y)$  be the set of all nonempty subsets of  $Y$ . For every  $A, B \in \mathcal{P}_0(Y)$  and  $\lambda \in \mathbb{R}$ , we denote

$$A + B = \{y_1 + y_2 : y_1 \in A, y_2 \in B\}, \quad \lambda A = \{\lambda y : y \in A\}.$$

Throughout this paper,  $K \subset Y$  is a *solid pointed convex cone* satisfying the condition  $K \cap (-K) = \{0\}$ . We consider in  $Y$  the following relations:

$$\forall x, y \in Y, \quad x \leq_K y \Leftrightarrow y - x \in K, \quad \text{and} \quad x \ll_K y \Leftrightarrow y - x \in \text{int } K.$$

Given a subset  $A \subset Y$  it is said that  $A$  is *K-proper* if  $A + K \neq Y$ , *K-closed* if  $A + K$  is closed, *K-bounded* if for each neighbourhood  $U$  of zero in  $Y$  there exists some positive number  $t$  such that  $A \subset tU + K$  and  $A$  is *K-compact* if any cover of  $A$  of the form  $\{U_\alpha + K : \alpha \in I, U_\alpha \text{ are open}\}$  admits a finite subcover. Recall that every *K-compact* set is *K-closed* and *K-bounded* (see [26, Chapter 1, Proposition 3.3]), and every *K-bounded* set is also *K-proper* (see [7, Theorem 3.1]).

Let  $S$  be an arbitrary set and  $F : S \rightrightarrows Y$  be a set-valued map. If  $N$  denotes a property of sets on  $Y$ , it is said that  $F$  is *N-valued* on  $S$  if  $F(x)$  has the property  $N$  for all  $x \in S$ .

Next, we recall the so-called lower and upper set less order relations [25, 15] and the corresponding strict relations.

Let  $A, B \in \mathcal{P}_0(Y)$ , we consider the following set relations:

- $A \leq_K^l B \Leftrightarrow \forall b \in B, \exists a \in A : a - b \in -K \Leftrightarrow B \subset A + K$ .
- $A \ll_K^l B \Leftrightarrow \forall b \in B, \exists a \in A : a - b \in -\text{int } K \Leftrightarrow B \subset A + \text{int } K$ .
- $A \leq_K^u B \Leftrightarrow \forall a \in A, \exists b \in B : a - b \in -K \Leftrightarrow A \subset B - K$ .
- $A \ll_K^u B \Leftrightarrow \forall a \in A, \exists b \in B : a - b \in -\text{int } K \Leftrightarrow A \subset B - \text{int } K$ .

When  $A$  and  $B$  are singletons,  $A = \{a\}$  and  $B = \{b\}$ , and  $j \in \{l, u\}$  we have  $\{a\} \leq_K^j \{b\}$  if and only if  $a \leq_K b$ , and  $\{a\} \ll_K^j \{b\}$  if and only if  $a \ll_K b$ .

We define  $A \sim^j B \Leftrightarrow A \leq_K^j B$  and  $B \leq_K^j A$  for  $j \in \{l, u\}$ .

We collect in the following lemma some basic and well-known properties (see [18]).

**Lemma 2.1.** *Let  $A, B, C \in \mathcal{P}_0(Y)$ .*

- (i)  $A \leq_K^l B \Leftrightarrow A + K \leq_K^l B \Leftrightarrow A \leq_K^l B + K \Leftrightarrow A + K \leq_K^l B + K$ .
- (ii)  $\leq_K^l$  is reflexive and transitive.
- (iii)  $A \leq_K^l B \Rightarrow tA \leq_K^l tB$  for all  $t \geq 0$  and  $A + C \leq_K^l B + C$ .
- (iv)  $A \ll_K^l B \Rightarrow A \leq_K^l B$ .
- (v)  $A \ll_K^l B$  and  $B \leq_K^l C$  (or  $A \leq_K^l B$  and  $B \ll_K^l C$ )  $\Rightarrow A \ll_K^l C$ .
- (vi)  $A \sim^l B \Leftrightarrow A + K = B + K$ .
- (vii)  $A \ll_K^l B \Leftrightarrow A + K \ll_K^l B \Leftrightarrow A \ll_K^l B + K \Leftrightarrow A + K \ll_K^l B + K$ .
- (viii) *The above properties (i)-(vii) are valid for  $\leq_K^u$  changing “l” to “u” and “+” to “-”.*

Next, the oriented distance of Hiriart-Urruty [16] is introduced. Recall that the distance of  $y \in Y$  to a set  $A$  is given by  $d(y, A) = \inf_{x \in A} \|x - y\|$ , being  $+\infty$  if  $A = \emptyset$ .

**Definition 2.2.** Let  $A \subset Y$ . The oriented distance  $D(\cdot, A) : Y \rightarrow \mathbb{R} \cup \{\pm\infty\}$  is defined as follows:  $D(y, A) = d(y, A) - d(y, Y \setminus A)$ . □

We assume throughout that  $D(y, \emptyset) = +\infty$  and  $D(y, Y) = -\infty$ . Then we have  $D(y, A) \leq d(y, A)$  and  $D(y, A) = d(y, A)$  if  $\text{int } A = \emptyset$ .

**Definition 2.3.** [8, 19] Let  $A, B \in \mathcal{P}_0(Y)$ . Then, the following two functions from  $\mathcal{P}_0(Y) \times \mathcal{P}_0(Y)$  into  $\mathbb{R} \cup \{\pm\infty\}$  are defined by

$$\mathbb{D}^{si}(A, B) = \sup_{b \in B} \inf_{a \in A} D(a - b, -K), \quad \text{and} \quad \widehat{\mathbb{D}}^{si}(A, B) = \sup_{a \in A} \inf_{b \in B} D(a - b, -K). \quad \square$$

**Definition 2.4.** ([4, Definition 2.5.16]) Let  $x_0 \in X$ . A set-valued map  $F : X \rightrightarrows Y$  is said to be

- (i) *K-upper continuous (K-u.c.)* at  $x_0$  if for any open set  $U$  with  $F(x_0) \subset U$ , there exists a neighbourhood  $V$  of  $x_0$  such that  $F(x) \subset U + K$  for all  $x \in V$ .

- (ii) *K-lower continuous* (*K-l.c.*) at the point  $x_0$  if for every  $y \in F(x_0)$  and for any neighbourhood  $U$  of  $y$ , there exists a neighbourhood  $V$  of  $x_0$  such that  $F(x) \cap (U - K) \neq \emptyset$  for all  $x \in V$ .

We say that  $F$  is *K-continuous* at  $x_0$  if  $F$  is *K-u.c.* and *K-l.c.* at  $x_0$ . We say that  $F$  is *K-u.c.* (resp., *K-l.c.*, *K-continuous*) on  $S \subset X$  if  $F$  is *K-u.c.* (resp., *K-l.c.*, *K-continuous*) at each point in  $S$ .  $\square$

When  $K = \{0\}$ , we say that  $F$  is *u.c.* (resp., *l.c.*, *continuous*) at  $x_0$  instead of  $F$  is  $\{0\}$ -*u.c.* (resp.,  $\{0\}$ -*l.c.*,  $\{0\}$ -*continuous*) at  $x_0$ . Similarly, when we say  $F$  is *u.c.*, *l.c.* or *continuous* on  $S$ .

The following implications are obvious: *u.c.*  $\Rightarrow$  *K-u.c.*, and *l.c.*  $\Rightarrow$  *K-l.c.*.

**Definition 2.5.** [13, 25] Let  $j \in \{l, u\}$  and let  $S$  be a nonempty convex subset of  $X$ . A set-valued map  $F : X \rightrightarrows Y$  is said to be:

- (i)  $\leq_K^j$ -*convex* on  $S$  if, for all  $x_1, x_2 \in S$  and for all  $t \in [0, 1]$ , one has
- $$F(tx_1 + (1-t)x_2) \leq_K^j tF(x_1) + (1-t)F(x_2). \quad (1)$$
- (ii) *strictly*  $\leq_K^j$ -*convex* on  $S$  if, for all  $x_1, x_2 \in S$  with  $x_1 \neq x_2$  and for all  $t \in (0, 1)$ , one has
- $$F(tx_1 + (1-t)x_2) \ll_K^j tF(x_1) + (1-t)F(x_2). \quad (2)$$
- (iii) *natural quasi*  $\leq_K^j$ -*convex* on  $S$  if, for all  $x_1, x_2 \in S$  and for any  $t \in (0, 1)$ , there exists  $\lambda \in [0, 1]$  such that
- $$F(tx_1 + (1-t)x_2) \leq_K^j \lambda F(x_1) + (1-\lambda)F(x_2).$$
- (iv) *strictly natural quasi*  $\leq_K^j$ -*convex* on  $S$  if, for all  $x_1, x_2 \in S$  with  $x_1 \neq x_2$  and for any  $t \in (0, 1)$ , there exists  $\lambda \in [0, 1]$  such that
- $$F(tx_1 + (1-t)x_2) \ll_K^j \lambda F(x_1) + (1-\lambda)F(x_2).$$

**Remark 2.6.** (i) For  $j = l$ , condition (1) (resp., (2)) is equivalent to

$$tF(x_1) + (1-t)F(x_2) \subset F(tx_1 + (1-t)x_2) + K$$

$$\left[ \text{resp., } tF(x_1) + (1-t)F(x_2) \subset F(tx_1 + (1-t)x_2) + \text{int } K \right],$$

and many authors say that  $F$  is *K-convex* (resp., *strictly K-convex*). Similarly if  $F$  is *natural quasi*  $\leq_K^l$ -*convex* or *strictly natural quasi*  $\leq_K^l$ -*convex*.

(ii) For  $j = u$ , condition (1) (resp., (2)) is equivalent to

$$F(tx_1 + (1-t)x_2) \subset tF(x_1) + (1-t)F(x_2) - K$$

$$\left[ \text{resp., } F(tx_1 + (1-t)x_2) \subset tF(x_1) + (1-t)F(x_2) - \text{int } K \right],$$

and many authors say that  $F$  is  $(-K)$ -*concave* (resp., *strictly*  $(-K)$ -*concave*). Similarly if  $F$  is *natural quasi*  $\leq_K^u$ -*convex* or *strictly natural quasi*  $\leq_K^u$ -*convex*.

(iii) When  $F = f : X \rightarrow Y$  is single-valued, as  $\leq_K^l \equiv \leq_K^u \equiv \leq_K$  on singleton sets, we will say that  $f$  is (strictly)  $\leq_K$ -*convex* or (strictly) *natural*  $\leq_K$ -*convex*, i.e., we omit the superscript  $l$  or  $u$ .

(iv) When  $Y = \mathbb{R}$  and  $K = \mathbb{R}_+$ , we will say that a function  $\psi : X \rightarrow \mathbb{R}$  is (strictly) concave, (strictly) natural quasi concave instead of (strictly)  $\mathbb{R}_+$ -concave, (strictly) natural quasi  $\mathbb{R}_+$ -concave. Similarly for the convexity notions.

(v) For  $j \in \{l, u\}$ , it is well-known that each  $\leq_K^j$ -convex (resp., strictly  $\leq_K^j$ -convex) set-valued map is natural (resp., strictly natural) quasi  $\leq_K^j$ -convex, but the converse is false. It is also clear that every  $\leq_K^l$ -convex set-valued map is  $K$ -convex valued.  $\square$

Next, inspired by Proposition 3.5 in [14], we study the natural quasi  $K$ -convexity of some particular set-maps, but first, a note that will shorten our proofs.

**Lemma 2.7.** *Let  $A \in \mathcal{P}_0(Y)$ .*

- (i)  $tA + (1 - t)A \leq_K^l A$  for all  $t \in [0, 1]$ .
- (ii) If  $A$  is  $K$ -convex, then  $tA + (1 - t)A + K = A + K$  for all  $t \in [0, 1]$ , i.e.,  $tA + (1 - t)A \sim^l A$ .
- (iii) If  $A$  is  $(-K)$ -convex, then  $tA + (1 - t)A - K = A - K$  for all  $t \in [0, 1]$ , i.e.,  $tA + (1 - t)A \sim^u A$ .

**Proof.** Part (i) says that  $A \subset tA + (1 - t)A + K$ , which is evident. To prove part (ii), the inclusion “ $\supset$ ” is obvious and so, we only have to see the inclusion “ $\subset$ ”. Let  $a_1, a_2 \in A$  and  $k \in K$ , then  $ta_1 + (1 - t)a_2 + k = t(a_1 + k) + (1 - t)(a_2 + k) \in A + K$  because  $A + K$  is convex by assumption. The proof of part (iii) is identical.  $\square$

For  $a, b \in Y$ , we denote  $[a, b] := \{y \in Y : y = ta + (1 - t)b, t \in [0, 1]\}$  (segment of extremes  $a$  and  $b$ ), and if  $a \leq_K b$  then  $[a, b]_K := (a + K) \cap (b - K)$  (a box).

We say that  $A \in \mathcal{P}_0(Y)$  is *radiant* if  $tA \subset A$  for all  $t \in [0, 1]$ , and  $A$  is  $K$ -radiant if  $A + K$  is radiant. It is obvious that if  $A$  is radiant, then  $A$  is  $K$ -radiant (the converse is false). Finally, let us observe that for a convex set  $A$  one has that  $A$  is radiant if and only if  $0 \in A$ .

**Proposition 2.8.** *Let  $A \in \mathcal{P}_0(Y)$  and  $S$  be a nonempty convex subset of  $X$ .*

- (i) If  $A$  is  $K$ -convex and  $f : X \rightarrow Y$  is (resp., strictly) natural quasi  $\leq_K$ -convex on  $S$ , then  $F(x) := f(x) + A$  is (resp., strictly) natural quasi  $\leq_K^l$ -convex on  $S$ .
- (ii) If  $A$  is  $K$ -convex and  $K$ -radiant and  $r : X \rightarrow \mathbb{R}$  is natural quasi concave on  $S$  with  $r(x) \geq 0$  for all  $x \in S$ , then  $F(x) := r(x)A$  is natural quasi  $\leq_K^l$ -convex on  $S$ .
- (iii) If  $q \in Y$  and  $f : X \rightarrow Y$  is (resp., strictly) natural quasi  $\leq_K$ -convex on  $S$ , then  $F(x) := [f(x), f(x) + q]$  is (resp., strictly) natural quasi  $\leq_K^l$ -convex on  $S$ .
- (iv) If  $f : X \rightarrow Y$  is (resp., strictly) natural quasi  $\leq_K$ -convex on  $S$  and  $g : X \rightarrow Y$  satisfies  $f(x) \leq_K g(x)$  for all  $x \in S$ , then  $F(x) := [f(x), g(x)]_K$  is (resp., strictly) natural quasi  $\leq_K^l$ -convex on  $S$ .
- (v) If  $A$  is  $K$ -convex and  $K$ -radiant,  $F : X \rightrightarrows Y$  is  $\leq_K^l$ -convex on  $S$  and  $r : X \rightarrow \mathbb{R}$  is concave on  $S$  with  $r(x) \geq 0$  for all  $x \in S$ , then  $G(x) := F(x) + r(x)A$  is  $\leq_K^l$ -convex.

**Proof.** (i) Let  $x_1, x_2 \in S$  and  $t \in (0, 1)$ . As  $f$  is natural quasi  $\leq_K$ -convex on  $S$ , there exists  $\lambda \in [0, 1]$  such that  $f(tx_1 + (1 - t)x_2) \leq_K \lambda f(x_1) + (1 - \lambda)f(x_2)$ , that is

$$\lambda f(x_1) + (1 - \lambda)f(x_2) \in f(x_t) + K, \tag{3}$$

where  $x_t = tx_1 + (1 - t)x_2$ . As  $A$  is  $K$ -convex, we obtain from Lemma 2.7(ii) that  $\lambda A + (1 - \lambda)A + K = A + K$ . Adding to (3) we obtain

$$\lambda f(x_1) + (1 - \lambda)f(x_2) + \lambda A + (1 - \lambda)A + K \subset f(x_t) + K + A + K,$$

or equivalently,  $\lambda(f(x_1) + A) + (1 - \lambda)(f(x_2) + A) + K \subset f(x_t) + A + K$ , which is equivalent to  $F(x_t) \leq_K^l \lambda F(x_1) + (1 - \lambda)F(x_2) + K$ , and using Lemma 2.1(i), this is equivalent to  $F(x_t) \leq_K^l \lambda F(x_1) + (1 - \lambda)F(x_2)$ . So we have proved that  $F$  is natural quasi  $\leq_K^l$ -convex on  $S$ .

If  $f$  is strictly natural quasi  $\leq_K$ -convex, the proof is similar: we simply apply that  $K + \text{int } K = \text{int } K$  and Lemma 2.1(vii) instead of  $K + K = K$  and Lemma 2.1(i).

(ii) Let  $x_1, x_2 \in S$  and  $t \in (0, 1)$ . As  $r$  is natural quasi concave on  $S$ , there exists  $\lambda \in [0, 1]$  such that

$$\lambda r(x_1) + (1 - \lambda)r(x_2) \leq r(tx_1 + (1 - t)x_2). \quad (4)$$

Let  $s_1 = \lambda r(x_1) \geq 0$ ,  $s_2 = (1 - \lambda)r(x_2) \geq 0$  and  $s_0 = s_1 + s_2$ . We claim that

$$s_1 A + s_2 A + K = s_0 A + K. \quad (5)$$

Indeed, if  $s_0 = 0$ , there is nothing to prove since then  $s_1 = s_2 = 0$ . Thus, let  $s_0 > 0$ . As  $A$  is  $K$ -convex, by Lemma 2.7(ii) we have  $\frac{s_1}{s_0} A + \frac{s_2}{s_0} A + K = A + K$ , and from here we deduce that (5) holds.

Now, let  $\beta = r(tx_1 + (1 - t)x_2) \geq 0$ . From (4), one has  $s_0 \leq \beta$ . We claim that  $s_0 A + K \subset \beta A + K$ . Indeed, if  $\beta = 0$  then  $s_0 = 0$  and it is obvious. If  $\beta > 0$ , then  $\frac{s_0}{\beta} \in [0, 1]$  and as  $A$  is  $K$ -radiant, it results that  $\frac{s_0}{\beta}(A + K) \subset A + K$ . In consequence,  $s_0 A + K \subset \beta A + K$ . Besides, in view of (5), we get

$$s_1 A + s_2 A + K = s_0 A + K \subset \beta A + K.$$

Now, from here, it results

$$\lambda r(x_1)A + (1 - \lambda)r(x_2)A + K \subset r(tx_1 + (1 - t)x_2)A + K.$$

This is equivalent to  $F(x_t) \leq_K^l \lambda F(x_1) + (1 - \lambda)F(x_2) + K$ , which is equivalent to  $F(x_t) \leq_K^l \lambda F(x_1) + (1 - \lambda)F(x_2)$  by Lemma 2.1(i). Thus  $F$  is natural quasi  $\leq_K^l$ -convex.

(iii) This follows from part (i) since  $F(x) = f(x) + [0, q]$  and the segment  $A = [0, q]$  is a convex set of  $Y$  for all  $q \in Y$ , and therefore, it is a  $K$ -convex set.

(iv) First, assume that  $f$  is natural quasi  $\leq_K$ -convex. Let us note that  $F(x) + K = f(x) + K$  and so from part (i) it follows that  $F(x) + K$  is natural quasi  $\leq_K^l$ -convex on  $S$ . Finally, as  $G = F + K$  is natural quasi  $\leq_K^l$ -convex, we derive that  $F$  is natural quasi  $\leq_K^l$ -convex as one can be easily checked by using the definition and Lemma 2.1(i).

The same reasoning is valid if  $f$  is strictly natural quasi  $\leq_K$ -convex, we use Lemma 2.1(vii) instead of Lemma 2.1(i).

(v) The mapping  $H(x) = r(x)A$  is  $\leq_K^l$ -convex on  $S$ , which is proved as in part (ii). Now  $G(x) = F(x) + H(x)$  is  $\leq_K^l$ -convex on  $S$ .  $\square$

Part (v) improves Proposition 3.5(c) in [14], where it is assumed that  $A$  is the unit closed ball in  $Y$ .

**Remark 2.9.** (a) If  $f : \mathbb{R} \rightarrow Y$  is  $(\leq, \leq_K)$ -increasing on a convex set  $S \subset \mathbb{R}$ , i.e.,  $[x_1, x_2 \in S, x_1 \leq x_2 \Rightarrow f(x_1) \leq_K f(x_2)]$ , then  $f$  is natural quasi  $\leq_K$ -convex on  $S$ . The same result is valid if  $f$  is  $(\leq, \leq_K)$ -decreasing.

This is easy to check. For example, if  $f$  is  $(\leq, \leq_K)$ -increasing then (3) is satisfied for all  $t \in (0, 1)$  with  $\lambda = 0$ , and with  $\lambda = 1$  if  $f$  is  $(\leq, \leq_K)$ -decreasing.

(b) If  $f_1, f_2 : X \rightarrow Y$  are natural quasi  $\leq_K$ -convex then  $f_1 + f_2$  is not necessarily natural quasi  $\leq_K$ -convex. This can easily be checked. Consider  $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f_1(x) = -x^2$ ,  $f_2(x) = 2x$ ,  $S = [0, 2]$ . Then  $f_1, f_2$  are natural quasi convex on  $S$  (let us observe that  $f_1$  is decreasing on  $S$ ), however  $f(x) = f_1(x) + f_2(x) = -x^2 + 2x$  is not natural quasi convex.

(c) Proposition 2.8(iv) is true for any set-valued map  $F : X \rightrightarrows Y$  satisfying the condition  $f(x) \in F(x) \subset f(x) + K$  for all  $x \in S$ .

(d) A similar result to Proposition 2.8 can be stated for  $\leq_K^u$  changing “ $\leq_K^l$ ” to “ $\leq_K^u$ ” and requiring  $A$  is  $(-K)$ -convex in part (i),  $A$  is  $(-K)$ -convex and  $(-K)$ -radiant in parts (ii) and (v), and  $g$  is (resp., strictly) natural quasi  $\leq_K$ -convex on  $S$  instead of  $f$  in part (iv). □

The level sets of a natural quasi  $\leq_K^j$ -convex set-valued map are convex sets as the next result shows. For a mapping  $F : X \rightrightarrows Y$ , a nonempty set  $S \subset X$  and  $A \in \mathcal{P}_0(Y)$ , we denote

$$Lev_j(F, A, S) := \{x \in S : F(x) \leq_K^j A\} \quad \text{for } j \in \{l, u\}.$$

**Proposition 2.10.** *Assume that  $S$  is convex and  $F$  is natural quasi  $\leq_K^j$ -convex on  $S$ . Then we have*

- (i) *if  $j = l$ , then  $Lev_l(F, A, S)$  is convex for all  $A \in \mathcal{P}_0(Y)$ .*
- (ii) *if  $j = u$ , then  $Lev_u(F, A, S)$  is convex for all  $(-K)$ -convex set  $A \in \mathcal{P}_0(Y)$ .*

*In particular, if  $f : X \rightarrow Y$  is a natural quasi  $\leq_K$ -convex function on  $S$ , then the set  $Lev_{\leq_K}(f, y, S) := \{x \in S : f(x) \leq_K y\}$  is convex for all  $y \in Y$ .*

**Proof.** (i) If  $j = l$ . Let  $x_1, x_2 \in Lev_l(F, A, S)$  and  $t \in (0, 1)$ . As  $F$  is natural quasi  $\leq_K^l$ -convex on  $S$ , there exists  $\lambda \in [0, 1]$  such that

$$F(tx_1 + (1 - t)x_2) \leq_K^l \lambda F(x_1) + (1 - \lambda)F(x_2). \tag{6}$$

As  $x_1, x_2 \in Lev_l(F, A, S)$ , one has  $F(x_1) \leq_K^l A$  and  $F(x_2) \leq_K^l A$ .

Multiplying the first inequality by  $\lambda$  and the second one by  $(1 - \lambda)$  and adding, using (6) and taking into account Lemma 2.7(i), it results

$$F(tx_1 + (1 - t)x_2) \leq_K^l \lambda F(x_1) + (1 - \lambda)F(x_2) \leq_K^l \lambda A + (1 - \lambda)A \leq_K^l A.$$

This implies that  $tx_1 + (1 - t)x_2 \in Lev_l(F, A, S)$ .

(ii) If  $j = u$ , the proof is similar but we use Lemma 2.7(iii) instead of Lemma 2.7(i). □

For a function  $\psi : X \rightarrow \mathbb{R}$  and a nonempty set  $S \subset X$ , we define

$$Sol(\psi, S) := \{x_0 \in S : \psi(x) \geq \psi(x_0), \forall x \in S\}.$$

**Proposition 2.11.** *If  $S$  is convex and  $\psi : X \rightarrow \mathbb{R}$  is natural quasi convex on  $S$ , then  $\text{Sol}(\psi, S)$  is a convex set.*

**Proof.** If  $x_0 \in \text{Sol}(\psi, S)$ , then it is clear that  $\text{Lev}_{\leq}(\psi, \psi(x_0), S) = \text{Sol}(\psi, S)$ , and so the conclusion follows from Proposition 2.10.  $\square$

**Proposition 2.12.** *If  $S$  is convex and  $\psi : X \rightarrow \mathbb{R}$  is strictly natural quasi convex on  $S$ , then  $\text{Sol}(\psi, S)$  is empty or a singleton.*

**Proof.** By contradiction, assume that there exist  $x_1, x_2 \in \text{Sol}(\psi, S)$ , with  $x_1 \neq x_2$ . Then  $\psi(x_1) = \psi(x_2)$ . Choose  $t \in (0, 1)$ , as  $S$  is convex we have  $tx_1 + (1 - t)x_2 \in S$ . Now, as  $\psi$  is strictly natural quasi convex on  $S$ , there exists  $\lambda \in [0, 1]$  such that

$$\psi(tx_1 + (1 - t)x_2) < \lambda\psi(x_1) + (1 - \lambda)\psi(x_2) = \psi(x_1),$$

which contradicts that  $x_1 \in \text{Sol}(\psi, S)$ .

This is a consequence from the fact that the graph of a strictly natural quasi convex function does not contain any horizontal segment, i.e., for all  $a, b \in S$  with  $a \neq b$ , and  $\psi(a) = \psi(b) = \alpha$ , one has  $[(a, \psi(a)), (b, \psi(b))]$  is not contained in the graph of  $\psi$  since  $\psi(ta + (1 - t)b) < \alpha$  for all  $t \in (0, 1)$ .  $\square$

Let  $S$  be a non-empty subset of  $X$ . For  $j \in \{l, u\}$  we consider the following set optimization problem

$$(\text{SOP}) \quad \leq_K^j - \min F(x) \quad \text{subject to } x \in S.$$

In the following, we recall some concepts of minimal solution using the set criterion and the set relations stated above (see, for instance, [15, 18, 20]).

**Definition 2.13.** Let  $x_0 \in S$ . It is said that  $x_0$  is a

- (i)  $\leq_K^j$ -minimal (or  $j$ -minimal) solution of (SOP), denoted  $x_0 \in \text{Min}_j(F, S)$ , if  $F(x) \leq_K^j F(x_0)$  for some  $x \in S$  implies  $F(x_0) \leq_K^j F(x)$ ,
- (ii) weak  $\leq_K^j$ -minimal (or weak  $j$ -minimal) solution of (SOP), denoted  $x_0 \in \text{wMin}_j(F, S)$ , if  $F(x) \ll_K^j F(x_0)$  for some  $x \in S$  implies  $F(x_0) \ll_K^j F(x)$ .

We also define the sets of strict  $\leq_K^j$ -minimal solutions and strict weak  $\leq_K^j$ -minimal solution as follows:

$$\begin{aligned} \text{sMin}_j(F, S) &:= \{x_0 \in S : F(x) \not\leq_K^j F(x_0), \forall x \in S \setminus \{x_0\}\}, \\ \text{swMin}_j(F, S) &:= \{x_0 \in S : F(x) \not\ll_K^j F(x_0), \forall x \in S \setminus \{x_0\}\} \end{aligned}$$

We associate to problem (SOP) two scalarizations:  $\phi$  for  $\leq_K^l$  and  $\widehat{\phi}$  for  $\leq_K^u$  as follows: Let  $\phi, \widehat{\phi} : X \times X \rightarrow \mathbb{R}$  be defined by

$$\phi(x, y) := \mathbb{D}^{si}(F(x), F(y)) \quad \text{and} \quad \widehat{\phi}(x, y) := \widehat{\mathbb{D}}^{si}(F(x), F(y)), \quad \forall x, y \in X.$$

When we consider  $\phi$  (resp.,  $\widehat{\phi}$ ) we will assume that  $F$  is  $K$ -bounded valued (resp.,  $(-K)$ -bounded valued), so  $\phi(x, y) \in \mathbb{R}$  (resp.,  $\widehat{\phi}(x, y) \in \mathbb{R}$ ) (see [21, Theorem 3.13]).

We define

$$\text{Sol}(\phi(\cdot, y), S) := \{x_0 \in S : \phi(x, y) \geq \phi(x_0, y), \forall x \in S\}, \quad \forall y \in S.$$

**Lemma 2.14.** [20, Proposition 4.15(i)] *Assume that  $A_1, A_2, B \in \mathcal{P}_0(Y)$ ,  $A_2$  and  $B$  are  $K$ -compact and  $K$  is solid. If  $A_1 \ll_K^l A_2$ , then  $\mathbb{D}^{si}(A_1, B) < \mathbb{D}^{si}(A_2, B)$ .*

**Lemma 2.15.** [20, Theorem 5.7 and Remark 5.8] *Suppose that  $K$  is solid,  $x_0 \in S$  and  $F(x_0)$  is  $K$ -compact. Then the following statements are equivalent:*

- (a)  $x_0 \in \text{wMin}_l(F, S)$ .
- (b)  $x_0 \in \text{swMin}_l(F, S)$ .
- (c)  $F(x) \not\ll_K^l F(x_0)$  for all  $x \in S$ .
- (d)  $\mathbb{D}^{si}(F(x), F(x_0)) \geq \mathbb{D}^{si}(F(x_0), F(x_0))$ , for all  $x \in S$ .

**Lemma 2.16.** [4, Proposition 2.5.8(ii)] *Let  $\emptyset \neq S \subset X$  and  $F : X \rightrightarrows Y$  be upper-continuous on  $S$ . If  $S$  is connected and  $F(x)$  is nonempty and connected for all  $x \in S$ , then  $F(S)$  is connected.*

### 3. Closedness and connectedness

In this section, first we study the closedness of the sets of weak  $\leq_K^j$ -minimal solutions of (SOP), and then we apply our results to obtain connectedness of the sets of  $\leq_K^j$ -minimal solution and weak  $\leq_K^j$ -minimal solutions to set optimization problem (SOP) for  $j \in \{l, u\}$ .

**Proposition 3.1.** *If  $S$  is closed and  $F : X \rightrightarrows Y$  is  $K$ -u.c. on  $S$ , then  $\text{swMin}_l(F, S)$  is closed. If, in addition,  $F$  is  $K$ -compact valued, then  $\text{wMin}_l(F, S)$  is closed.*

**Proof.** Let  $(x_n) \subset \text{swMin}_l(F, S)$  with  $x_n \rightarrow x_0$ . As  $S$  is closed, one has  $x_0 \in S$ . By contradiction, assume that  $x_0 \notin \text{swMin}_l(F, S)$ . Then there exists  $\hat{x} \in S \setminus \{x_0\}$  such that  $F(\hat{x}) \ll_K^l F(x_0)$ , that is,  $F(x_0) \subset F(\hat{x}) + \text{int } K$ . Since  $U = F(\hat{x}) + \text{int } K$  is an open set that contains to  $F(x_0)$  and  $F$  is  $K$ -u.c. at  $x_0$ , there exists a neighbourhood  $V$  of  $x_0$  such that

$$F(x) \subset F(\hat{x}) + \text{int } K + K = F(\hat{x}) + \text{int } K, \quad \forall x \in S \cap V. \tag{7}$$

Since  $x_n \rightarrow x_0$ , there is  $n_0$  such that  $x_n \in S \cap V$  for all  $n \geq n_0$ , and consequently, in view of (7),  $F(x_n) \subset F(\hat{x}) + \text{int } K$ , which means that  $F(\hat{x}) \ll_K^l F(x_n)$  for all  $n$  large enough. This contradicts the fact that  $x_n \in \text{swMin}_l(F, S)$ .

The second part, follows from Lemma 2.15 since  $\text{swMin}_l(F, S) = \text{wMin}_l(F, S)$  whenever  $F$  is  $K$ -compact valued. □

This proposition improves Proposition 2.3 in [12] where it is assumed that  $F$  is u.c. with compact values.

**Proposition 3.2.** *If  $S$  is closed and  $F : X \rightrightarrows Y$  is  $(-K)$ -l.c. and  $(-K)$ -compact valued on  $S$ , then  $\text{wMin}_u(F, S)$  is closed.*

**Proof.** Let  $(x_n) \subset \text{wMin}_u(F, S)$  with  $x_n \rightarrow x_0$ . As  $S$  is closed, one has  $x_0 \in S$ .

On the one hand, as  $F$  is  $(-K)$ -l.c. and  $(-K)$ -compact valued on  $S$ , the function  $\hat{\phi}_A : X \rightarrow \mathbb{R}$ , defined by  $\hat{\phi}_A(x) = \widehat{\mathbb{D}}^{si}(A, F(x))$  for all  $x \in X$ , is upper semicontinuous (u.s.c.) on  $S$  by Theorem 5(ii-a) in [17] for each  $(-K)$ -compact set  $A$  (we apply this theorem to the pair  $(G, F)$ , where  $G$  is the constant set-valued map  $G(x) = A$  for all  $x \in X$ ).

Let us observe that  $\widehat{\phi}_A(x) \in \mathbb{R}$  by [21, Theorem 3.13(v)] since  $A$  is  $(-K)$ -bounded because it is  $(-K)$ -compact and  $F(x)$  is  $(-K)$ -proper because it is  $(-K)$ -compact. On the other hand, by Theorem 5.15 in [20], assuming that  $F$  is  $(-K)$ -compact valued on  $S$ , one has that

$$z_0 \in \text{wMin}_u(F, S) \iff \widehat{\mathbb{D}}^{si}(F(x), F(z_0)) \geq 0 \quad \forall x \in S, \quad (8)$$

because  $\widehat{\mathbb{D}}^{si}(F(z_0), F(z_0)) = 0$  by [20, Theorem 3.20(iii)]. Hence, as  $x_n \in \text{wMin}_u(F, S)$ , we derive that

$$\widehat{\phi}_{F(x)}(x_n) = \widehat{\mathbb{D}}^{si}(F(x), F(x_n)) \geq 0, \quad \forall x \in S, \forall n \in \mathbb{N}. \quad (9)$$

Therefore, as  $\widehat{\phi}_{F(x)}(\cdot)$  is u.s.c. on  $S$ , the set  $E_x := \{y \in S : \widehat{\phi}_{F(x)}(y) \geq 0\}$  is closed. As  $x_n \in E_x$  for all  $n$  by (9) and as  $x_n \rightarrow x_0$ , it follows that  $x_0 \in E_x$ , and this is true for all  $x \in S$ . This means that  $\widehat{\phi}_{F(x)}(x_0) \geq 0$  for all  $x \in S$ . Now, by applying (8) with  $z_0 = x_0$ , we conclude that  $x_0 \in \text{wMin}_u(F, S)$ .  $\square$

This result improves [12, Proposition 2.4], where it is assumed that  $F$  is l.c. with compact values. Let us observe that the proof of Proposition 3.2 is very different from the one of Proposition 3.1, where we have used a scalarization while in the second one, it is a direct proof.

From now on we focus on connectedness, starting with some previous results.

**Lemma 3.3.** *Assume that  $S$  is a convex set, and  $F$  is strictly natural quasi  $\leq_K^l$ -convex on  $S$  and  $K$ -compact valued. Then*

$$\text{swMin}_l(F, S) = \text{wMin}_l(F, S) = \text{sMin}_l(F, S) = \text{Min}_l(F, S).$$

**Proof.** The first equality follows from Lemma 2.15 since  $F$  is  $K$ -compact valued. The inclusion  $\text{sMin}_l(F, S) \subset \text{swMin}_l(F, S)$  is clear from the definitions. Let us prove the reverse inclusion  $\text{swMin}_l(F, S) \subset \text{sMin}_l(F, S)$ . Indeed, let  $x_0 \in \text{swMin}_l(F, S)$  and assume that there exists  $x_1 \in S \setminus \{x_0\}$  such that  $F(x_1) \leq_K^l F(x_0)$ . Then

$$F(x_0) \subset F(x_1) + K. \quad (10)$$

As  $F$  is strictly natural quasi  $\leq_K^l$ -convex on  $S$ , for all  $t \in (0, 1)$ , there exists  $\lambda \in [0, 1]$  such that

$$\lambda F(x_0) + (1 - \lambda)F(x_1) \subset F(x_t) + \text{int } K, \quad (11)$$

where  $x_t := tx_0 + (1 - t)x_1$ . From (10) and since  $K$  is a cone we have

$$(1 - \lambda)F(x_0) \subset (1 - \lambda)F(x_1) + K,$$

and so, using (11) it follows

$$\begin{aligned} F(x_0) &\subset \lambda F(x_0) + (1 - \lambda)F(x_0) \subset \lambda F(x_0) + (1 - \lambda)F(x_1) + K \\ &\subset F(x_t) + \text{int } K + K = F(x_t) + \text{int } K. \end{aligned}$$

Therefore,  $F(x_t) \ll_K^l F(x_0)$ , which contradicts the fact that  $x_0 \in \text{swMin}_l(F, S)$ .

Finally,  $\text{sMin}_l(F, S) \subset \text{Min}_l(F, S)$  is obvious, and by [15, Proposition 2.7(i)] we know that  $\text{Min}_l(F, S) \subset \text{wMin}_l(F, S)$ . In consequence, the proof is finished.  $\square$

**Lemma 3.4.** *Assume that  $S$  is a convex set, and  $F$  is strictly natural quasi  $\leq_K^u$ -convex and  $(-K)$ -compact and  $(-K)$ -convex valued on  $S$ . Then*

$$\text{swMin}_u(F, S) = \text{wMin}_u(F, S) = \text{Min}_u(F, S) = \text{sMin}_u(F, S).$$

**Proof.** It is similar to Lemma 3.3 observing the following:

- (i) For the equality  $\text{swMin}_u(F, S) = \text{wMin}_u(F, S)$  we apply Theorem 5.15 in [20].
- (ii) To prove  $\text{swMin}_u(F, S) \subset \text{sMin}_u(F, S)$ , assume  $x_0 \in \text{swMin}_u(F, S) \setminus \text{sMin}_u(F, S)$ ; then there exists  $x_1 \in S \setminus \{x_0\}$  such that

$$F(x_1) \leq_K^u F(x_0). \tag{12}$$

As  $F$  is strictly natural quasi  $\leq_K^u$ -convex on  $S$ , for all  $t \in (0, 1)$ , there exists  $\lambda \in [0, 1]$  such that

$$F(x_t) \ll_K^u \lambda F(x_0) + (1 - \lambda)F(x_1), \tag{13}$$

where  $x_t := tx_0 + (1 - t)x_1$ . From (12),  $(1 - \lambda)F(x_1) \leq_K^u (1 - \lambda)F(x_0)$ . Using this, (13) and Lemma 2.7(iii) since  $F$  is  $(-K)$ -convex valued on  $S$ , we derive

$$F(x_t) \ll_K^u \lambda F(x_0) + (1 - \lambda)F(x_1) \leq_K^u \lambda F(x_0) + (1 - \lambda)F(x_0) \leq_K^u F(x_0),$$

and so  $F(x_t) \ll_K^u F(x_0)$ , which contradicts that  $x_0 \in \text{swMin}_u(F, S)$ . □

Let us observe that  $F$  is  $(-K)$ -convex valued in Lemma 3.4, in contrast to Lemma 3.3, where the corresponding requirement,  $F$  is  $K$ -convex valued, it is not necessary.

**Proposition 3.5.** *Let  $S$  be a convex subset of  $X$ . If  $F : X \rightrightarrows Y$  is natural quasi  $\leq_K^l$ -convex on  $S$ , then  $\phi(\cdot, y)$  is natural quasi convex on  $S$  for all  $y \in X$ .*

**Proof.** Let  $y \in X$ ,  $x_1, x_2 \in S$ ,  $B = F(y)$  and  $t \in (0, 1)$ . As  $F$  is natural quasi  $\leq_K^l$ -convex on  $S$ , there exists  $\lambda \in [0, 1]$  such that

$$F(tx_1 + (1 - t)x_2) \leq_K^l \lambda F(x_1) + (1 - \lambda)F(x_2).$$

Then, on the one hand, by Theorem 3.22(iii-b) in Jiménez et al. [21], one has that  $\mathbb{D}^{si}(\cdot, B)$  is  $\leq_K^l$ -increasing on  $\mathcal{P}_0(Y)$ , and on the other hand,  $\mathbb{D}^{si}(\cdot, B)$  is convex on  $\mathcal{P}_0(Y)$  by [17, Theorem 2(ii)]. Therefore by applying both properties we obtain

$$\begin{aligned} \mathbb{D}^{si}(F(tx_1 + (1 - t)x_2), B) &\leq \mathbb{D}^{si}(\lambda F(x_1) + (1 - \lambda)F(x_2), B) \\ &\leq \lambda \mathbb{D}^{si}(F(x_1), B) + (1 - \lambda) \mathbb{D}^{si}(F(x_2), B), \end{aligned}$$

which proves that  $\phi(\cdot, y)$  is natural quasi convex. □

**Proposition 3.6.** *Let  $S$  be a convex subset of  $X$ . If  $F : X \rightrightarrows Y$  is natural quasi  $\leq_K^u$ -convex on  $S$  with  $(-K)$ -convex values, then  $\hat{\phi}(\cdot, y)$  is natural quasi convex on  $S$  for all  $y \in X$ .*

**Proof.** The proof is similar as Proposition 3.5. Now we use [21, Theorem 3.22(v-b)] which states that  $\widehat{\mathbb{D}}^{si}(\cdot, B)$  is  $\leq_K^u$ -increasing on  $\mathcal{P}_0(Y)$ , and [17, Theorem 2(ii)] which shows that  $\widehat{\mathbb{D}}^{si}(\cdot, B)$  is convex on  $\mathcal{P}_0(Y)$  if  $B$  is  $(-K)$ -convex. For this reason, in this proposition we need the additional assumption  $F$  is  $(-K)$ -convex valued. □

**Lemma 3.7.** *Assume that  $F$  is  $K$ -compact valued. Then*

$$\bigcup_{y \in S} \text{Sol}(\phi(\cdot, y), S) = \text{swMin}_l(F, S) = \text{wMin}_l(F, S).$$

**Proof.** Let  $x_0 \in \bigcup_{y \in S} \text{Sol}(\phi(\cdot, y), S)$ . Then there exists  $y_0 \in S$  such that

$$\mathbb{D}^{si}(F(x), F(y_0)) \geq \mathbb{D}^{si}(F(x_0), F(y_0)), \quad \forall x \in S. \tag{14}$$

Suppose that  $x_0 \notin \text{swMin}_l(F, S)$ ; then there exists  $x_1 \in S$  such that  $F(x_1) \ll_K^l F(x_0)$ . By using that  $\mathbb{D}^{si}(\cdot, F(y_0))$  is strictly  $\leq_K^l$ -increasing on  $K$ -compact subsets of  $Y$  since  $F(y_0)$  is  $K$ -compact (see Lemma 2.14), we get  $\mathbb{D}^{si}(F(x_1), F(y_0)) < \mathbb{D}^{si}(F(x_0), F(y_0))$ , which contradicts (14).

The equality  $\text{swMin}_l(F, S) = \text{wMin}_l(F, S)$  follows from Lemma 2.15. Now, assume  $x_0 \in \text{wMin}_l(F, S)$ . Then by Lemma 2.15 we get  $\mathbb{D}^{si}(F(x), F(x_0)) \geq \mathbb{D}^{si}(F(x_0), F(x_0))$ , for all  $x \in S$ , which implies  $x_0 \in \text{Sol}(\phi(\cdot, x_0), S)$ , and so  $x_0 \in \bigcup_{y \in S} \text{Sol}(\phi(\cdot, y), S)$ .  $\square$

**Lemma 3.8.** *Assume that  $F$  is  $(-K)$ -compact valued. Then*

$$\bigcup_{y \in S} \text{Sol}(\widehat{\phi}(\cdot, y), S) = \text{swMin}_u(F, S) = \text{wMin}_u(F, S).$$

The proof is similar to the one of Lemma 3.7, now we use [20, Theorem 4.17(vi)], which shows that  $\widehat{\mathbb{D}}^{si}(\cdot, B)$  is strictly  $\leq_K^u$ -increasing on  $(-K)$ -compact sets if  $B$  is  $(-K)$ -compact, and [20, Theorem 5.15], which proves that

$$x_0 \in \text{wMin}_u(F, S) \Leftrightarrow x_0 \in \text{swMin}_u(F, S) \Leftrightarrow \widehat{\mathbb{D}}^{si}(F(x), F(x_0)) \geq 0, \quad \forall x \in S.$$

**Theorem 3.9.** *Assume that  $S$  is a compact and convex subset of  $X$ ,  $F : X \rightrightarrows Y$  is  $K$ -continuous on  $S$ , natural quasi  $\leq_K^l$ -convex on  $S$  and  $K$ -compact valued on  $S$ . Then  $\text{wMin}_l(F, S)$  is connected.*

**Proof. Step 1.**  $\text{Sol}(\phi(\cdot, y), S) \neq \emptyset$  for all  $y \in S$ . Indeed,  $\phi(x, y) = \mathbb{D}^{si}(F(x), F(y))$  is continuous on  $S \times S$  by [17, Corollary 1(i)]. As  $S$  is compact,  $\phi(\cdot, y)$  attains a minimum on  $S$  at some point  $x_0 \in S$ , and so  $x_0 \in \text{Sol}(\phi(\cdot, y), S)$ .

**Step 2.**  $\text{Sol}(\phi(\cdot, y), S)$  is a convex set for all  $y \in S$ . Indeed, as  $F$  is natural  $\leq_K^l$ -convex on  $S$ , by Proposition 3.5 we have that the function  $\phi(\cdot, y)$  is natural quasi convex on  $S$ . Now, by Proposition 2.11 we obtain that  $\text{Sol}(\phi(\cdot, y), S)$  is convex.

**Step 3.** The set-valued map  $M : S \rightrightarrows S$  defined by  $M(y) = \text{Sol}(\phi(\cdot, y), S)$  is u.c. on  $S$ . Indeed, by contradiction, suppose that  $M$  is not u.c. at  $y_0 \in S$ . Then there exists an open set  $V$  such that  $M(y_0) \subset V$  and for all neighbourhood  $U$  of  $y_0$  one has  $M(U) \not\subset V$ . This implies that there exists a sequence  $(y_n) \subset S$ , with  $y_n \rightarrow y_0$  and  $M(y_n) \not\subset V$ . So, for each  $n \in \mathbb{N}$  there is  $x_n \in M(y_n)$  satisfying

$$x_n \notin V, \quad \forall n \in \mathbb{N}. \tag{15}$$

As  $S$  is compact, without lost of generality we may assume that  $x_n \rightarrow x_0 \in S$ . Since  $x_n \in M(y_n)$  one has

$$\mathbb{D}^{si}(F(x), F(y_n)) \geq \mathbb{D}^{si}(F(x_n), F(y_n)), \quad \forall x \in S, n \in \mathbb{N}. \tag{16}$$

As  $\phi$  is continuous on  $S \times S$  by [17, Corollary 1(i)] because  $F$  is  $K$ -continuous and  $K$ -compact valued on  $S$  by assumption, and as  $(x_n, y_n) \rightarrow (x_0, y_0)$ , we get from (16)

$$\mathbb{D}^{si}(F(x), F(y_0)) \geq \mathbb{D}^{si}(F(x_0), F(y_0)), \quad \forall x \in S,$$

which means that  $x_0 \in M(y_0)$ . As  $x_n \rightarrow x_0 \in M(y_0) \subset V$ , we derive that  $x_n \in V$  for all  $n$  large enough, which contradicts (15).

**Step 4.**  $w\text{Min}_l(F, S)$  is connected. Indeed, by Lemma 3.7 one has  $w\text{Min}_l(F, S) = \cup_{y \in S} M(y)$ . As  $S$  is convex,  $M(y) \neq \emptyset$  (step 1) and  $M$  is u.c. on  $S$  (step 3) with connected values since  $M(y)$  is convex for every  $y \in S$  (step 2), by using Lemma 2.16 it follows that  $w\text{Min}_l(F, S) = M(S)$  is connected. The proof is finished.  $\square$

We remark that the proofs of Lemma 3.7 and Theorem 3.9 are similar to those of Theorem 3.1 and Theorem 4.1(i), respectively, in [11].

**Theorem 3.10.** *Assume that  $S$  is a compact and convex subset of  $X$ ,  $F : X \rightrightarrows Y$  is  $K$ -continuous on  $S$ , strictly natural quasi  $\leq_K^l$ -convex on  $S$  and  $K$ -compact valued on  $S$ . Then  $\text{Min}_l(F, S)$  is connected.*

**Proof.** By Lemma 3.3 we have  $\text{Min}_l(F, S) = w\text{Min}_l(F, S)$ . Now the theorem follows from Theorem 3.9 taking into account that a strictly natural quasi  $\leq_K^l$ -convex set-valued map is also natural quasi  $\leq_K^l$ -convex.  $\square$

This theorem is in the line of Theorem 3.1 in [13], where it is proved that  $\text{Min}_l(F, S)$  is arcwise connected (a stronger conclusion) but it is assumed that  $F$  is continuous with compact values (stronger assumptions).

**Theorem 3.11.** *Assume that  $S$  is a compact and convex subset of  $X$ ,  $F : X \rightrightarrows Y$  is  $(-K)$ -continuous on  $S$ , natural quasi  $\leq_K^u$ -convex on  $S$  and  $(-K)$ -compact and  $(-K)$ -convex valued on  $S$ . Then  $w\text{Min}_u(F, S)$  is connected.*

The proof is similar to the one of Theorem 3.9 and for this reason is omitted, we only have to take into account the following: In the proof of Theorem 3.11 are used the function  $\widehat{\phi}$ , [17, Corollary 1(i)] with  $F$   $(-K)$ -continuous and  $(-K)$ -compact valued, Proposition 3.6 obtaining that  $\widehat{\phi}(\cdot, y)$  is natural quasi convex on  $S$  and Lemma 3.8 instead of the function  $\phi$ , [17, Corollary 1(i)] with  $F$   $K$ -continuous and  $K$ -compact valued, Proposition 3.5 and Lemma 3.7, respectively, which are used in the proof of Theorem 3.9.

**Theorem 3.12.** *Assume that  $S$  is a compact and convex subset of  $X$ ,  $F : X \rightrightarrows Y$  is  $(-K)$ -continuous on  $S$ , strictly natural quasi  $\leq_K^u$ -convex on  $S$  and  $(-K)$ -compact and  $(-K)$ -convex valued on  $S$ . Then  $\text{Min}_u(F, S)$  is connected.*

This theorem follows directly from Theorem 3.11 if we take into account that we have  $\text{Min}_u(F, S) = w\text{Min}_u(F, S)$  by Lemma 3.4.

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