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On Efficient Solutions for Semidefinite Linear Fractional Vector Optimization Problems

Moon Hee Kim
College of General Education, Tongmyong University, Busan 48520, Korea
mooni@tu.ac.kr

Gue Myung Lee*
Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea
gmlee@pknu.ac.kr

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We consider a semidefinite linear fractional vector optimization problem (FVP) and establish optimality theorem for efficient solutions for (FVP), which hold without any constraint qualification and which are expressed by sequences. Moreover, we discuss the relations between properly efficient solution of (FVP) and one of its related linear vector optimization problem (LVP). By using the relation, we obtain optimality theorem for properly efficient solutions for (FVP), which hold without any constraint qualification and which are expressed by sequences. Also, by using optimality theorem for efficient solutions of (FVP) and the relations, we obtain the semi-definite version of the Isermann's result for (FVP), which gives a sufficient condition that an efficient solution of (FVP) can be properly efficient for (FVP). We give examples for our results for properly efficient solutions for (FVP). Moreover, we formulate vector dual problem (VD) for (FVP) and establish duality theorems for (FVP) and (VD). Our approach for getting our main resuts is to use linear vector optimization problems related to (FVP).

Keywords: Semidefinite linear fractional vector optimization problem, efficient solutions, properly efficient solutions, optimality conditions, vector dual problem, weak duality theorem, strong duality theorem.

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1. Introduction and preliminaries

Semidefinite optimization problems have been intensively studied since many optimization problem can be changed into the problems which are very computationable [16]. Jeyakumar, Lee and Dinh [9] proved the optimality conditions for convex optimization problem, which held without any constraint qualification and which were expressed in terms of sequences. The optimality conditions have been studied for many kinds of convex optimization problems. In particular, Lee and Lee [13] studied optimality conditions for efficient solutions of an abstract convex vector optimization problems which held without any constraint qualification. Kim, Kim and Lee [11] investigated optimality conditions for a semidefinite linear opti-

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* Corresponding author.

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mization problems which held without any constraint qualification. Kim, Kim and Lee [12] studied optimality conditions for weakly efficient solutions of semidefinite linear fractional vector optimization problems which held without any constraint qualification. We can find important results for linear fractional vector optimization problems in [2, 3, 6].

In this paper, we consider a semidefinite linear fractional vector optimization problem (FVP) and establish optimality theorems for efficient solutions and properly efficient solutions for (FVP), which holds without any constraint qualification and which are expressed by sequences. We establish the semi-definite version of the Isermann's result [4, 7, 15] for (FVP), which gives a sufficient condition that an efficient solution of (FVP) can be properly efficient for (FVP). Moreover, we formulate vector dual problem (VD) for (FVP) and establish duality theorems for (FVP) and (VD). Our approach for getting our main results is to use linear vector optimization problems related to (FVP).

Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. For a subset $D \subset X$, the *closure* of D, induced by the norm topology on X, is denoted by $\mathrm{cl}D$.

Let C be a closed convex cone in X. Then the *positive dual cone* of C is defined by

$$C^* := \{z \in X \ : \ \langle x, z \rangle \geqq 0 \ \ \forall x \in C\}.$$

The *indicator function* $\delta_A : X \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_A := \begin{cases} 0 & \text{if } \ x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

Let $h : X \to \mathbb{R} \cup \{+\infty\}$ be a function. Then the *conjugate function* of h, denoted $h^* : X \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$h^*(v) := \sup\{\langle v, x \rangle - h(x) \mid x \in \mathrm{dom} h\}$$

where $\mathrm{dom} h := \{x \in X \mid h(x) < +\infty\}$.

The function h is said to be *proper* if h does not take on the valued $-\infty$ and $\mathrm{dom} h \neq \emptyset$. The *epigraph* of the function h is defined by

$$\mathrm{epi} h := \{(x, r) \in X \times \mathbb{R} \ : \ h(x) \leqq r\}.$$

Moreover if $\liminf_{y \to x} h(y) \geqq h(x)$ for all $x \in X$, we say that h is *lower semicontinuous.*

A function $h : X \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for all $\lambda \in [0, 1]$,

$$h(\lambda x + (1 - \lambda) y) \leqq \lambda h(x) + (1 - \lambda) h(y) \ \text{ for all } x, y \in X.$$

Lemma 1.1. [1] *Let $h_1, h_2 : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. Then epi$(h_1 + h_2)^* = cl($epi h_1^* + epi $h_2^*)$. If, in addition, one of h_1 and h_2 is continuous at some $x_0 \in$ dom $h_1 \cap$ dom h_2, then*

$$epi(h_1 + h_2)^* = epi\ h_1^* + epi\ h_2^*.$$

Lemma 1.2. [14] *Let I be an arbitrary index set and let $f_i : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. Suppose that there exists $x_0 \in X$ such that $sup_{i\in I} f_i(x_0) < \infty$. Then*

$$epi(sup_{i\in I} f_i)^* = cl\ (co \bigcup_{i\in I} epi f_i^*),$$

where $sup_{i\in I} f_i : X \to \mathbb{R} \cup \{+\infty\}$ is defined by $(sup_{i\in I} f_i)(x) = sup_{i\in I} f_i(x)$ for all $x \in X$.

TrA is the trace of $n \times n$ matrix A. For a symmetric $n \times n$ matrix A, $A \succeq 0$ means that A is positive semidefinite. Let S^n be the space of $n \times n$ symmetric matrices. Let $S^n_+ = \{X \in S^n \mid X \succeq 0\}$. Then $\mathrm{Tr}(\cdot\ \cdot)$ is the inner product on S^n and S^n is the finite-dimensional Hilbert space with $\mathrm{Tr}(\cdot\ \cdot)$, and $\|A\| = \sqrt{Tr(AA)}$ (where $A \in S^n$) is a norm on S^n, in fact, $\|A\|$ is the Frobenius norm defined on S^n ([10]).

2. Efficient solutions

In this paper, we consider the following semidefinite linear fractional vector optimization problem:

$$\begin{array}{lll} \text{(FVP)} & \text{Minimize} & \left(\dfrac{\mathrm{Tr}(C_1X)+\alpha_1}{\mathrm{Tr}(D_1X)+\beta_1}, \dots, \dfrac{\mathrm{Tr}(C_pX)+\alpha_p}{\mathrm{Tr}(D_pX)+\beta_p}\right) \\ & \text{subject to} & X \succeq 0,\ \mathrm{Tr}(A_jX) = b_j,\ j = 1, \dots, m, \end{array}$$

where $C_i \in S^n$, $\alpha_i \in \mathbb{R}$, $D_i \in S^n$, $\beta_i \in \mathbb{R}$, $i = 1, \dots, p$, $A_j \in S^n$, $b_j \in \mathbb{R}$, $j = 1, \dots, m$ are given real numbers.

Let $\triangle := \{X \in S^n | X \succeq 0,\ \mathrm{Tr}(A_iX) = b_i,\ i = 1, ..., m\}$. Assume that $\mathrm{Tr}(D_iX) + \beta_i > 0$ for any $X \in \triangle$, $i = 1, \cdots, p$. Let $q_i(X) = \frac{\mathrm{Tr}(C_iX)+\alpha_i}{\mathrm{Tr}(D_iX)+\beta_i}$ for any $X \in \triangle$.

Definition 2.1. (1) $\bar{X} \in \triangle$ is said to be an *efficient solution* for (FVP) if there exists no other feasible $X \in \triangle$ such that

$$\left(\frac{\mathrm{Tr}(C_1X)+\alpha_1}{\mathrm{Tr}(D_1X)+\beta_1}, \dots, \frac{\mathrm{Tr}(C_pX)+\alpha_p}{\mathrm{Tr}(D_pX)+\beta_p}\right) \leqq \left(\frac{\mathrm{Tr}(C_1\bar{X})+\alpha_1}{\mathrm{Tr}(D_1\bar{X})+\beta_1}, \dots, \frac{\mathrm{Tr}(C_p\bar{X})+\alpha_p}{\mathrm{Tr}(D_p\bar{X})+\beta_p}\right)$$

$$\text{and} \quad \left(\frac{\mathrm{Tr}(C_1X)+\alpha_1}{\mathrm{Tr}(D_1X)+\beta_1}, \dots, \frac{\mathrm{Tr}(C_pX)+\alpha_p}{\mathrm{Tr}(D_pX)+\beta_p}\right) \neq \left(\frac{\mathrm{Tr}(C_1\bar{X})+\alpha_1}{\mathrm{Tr}(D_1\bar{X})+\beta_1}, \dots, \frac{\mathrm{Tr}(C_p\bar{X})+\alpha_p}{\mathrm{Tr}(D_p\bar{X})+\beta_p}\right).$$

(2) $\bar{X} \in \triangle$ is said to be a *properly efficient solution* for (FVP) if $\bar{X}$ is an efficient solution of (FVP) and there exists a constant $M > 0$ such that for each $i = 1, \cdots, p$, we have

$$\frac{\dfrac{\mathrm{Tr}(C_i\bar{X})+\alpha_i}{\mathrm{Tr}(D_i\bar{X})+\beta_i} - \dfrac{\mathrm{Tr}(C_iX)+\alpha_i}{\mathrm{Tr}(D_iX)+\beta_i}}{\dfrac{\mathrm{Tr}(C_jX)+\alpha_j}{\mathrm{Tr}(D_jX)+\beta_j} - \dfrac{\mathrm{Tr}(C_j\bar{X})+\alpha_j}{\mathrm{Tr}(D_j\bar{X})+\beta_j}} \leqq M$$

for some j such that

$$\frac{\mathrm{Tr}(C_jX)+\alpha_j}{\mathrm{Tr}(D_jX)+\beta_j} > \frac{\mathrm{Tr}(C_j\bar{X})+\alpha_j}{\mathrm{Tr}(D_j\bar{X})+\beta_j} \quad \text{whenever } X \in \triangle \text{ and } \quad \frac{\mathrm{Tr}(C_iX)+\alpha_i}{\mathrm{Tr}(D_iX)+\beta_i} < \frac{\mathrm{Tr}(C_i\bar{X})+\alpha_i}{\mathrm{Tr}(D_i\bar{X})+\beta_i}.$$

For basic properties of efficient solutions the reader is referred to [5, 8, 15].

Now we give the following optimality theorems for an efficient solution and properly efficient solution of (FVP).

Theorem 2.2. *Let $\lambda \in \mathbb{R}^p$ be such that $\lambda_i > 0,\ i = 1, \ldots, p$. Let $\bar{X} \in \triangle$. Then the following are equivalent:*

(i) *$\bar{X}$ is an efficient solution of* (FVP)*;*

(ii) *there exist $\mu_j \in \mathbb{R}, j = 1, \cdots, m,\ y_j \geqq 0, j = 1, \cdots, p$ such that*

$$(0,0) \in \Big\{\sum_{i=1}^{p} \lambda_i(C_i - q_i(\bar{X})D_i, -\alpha_i + q_i(\bar{X})\beta_i)\Big\} + \{0\} \times \mathbb{R}^+$$
$$+ \,cl\Big(\bigcup_{\mu_j \in \mathbb{R}} \{\sum_{j=1}^{m} \mu_j(A_j, b_j)\} + \bigcup_{y_j \geqq 0} \{\sum_{j=1}^{p} y_j(C_j - q_j(\bar{X})D_j, -\alpha_j + q_j(\bar{X})\beta_j)\}$$
$$+ (-S^n_+) \times \mathbb{R}^+\Big).$$

(iii) *there exist $\mu^l_j \in \mathbb{R}, j = 1, \ldots, m,\ V^l \in S^n_+$ and $y^l_j \geqq 0,\ j = 1, \ldots, p$ such that*

$$\sum_{i=1}^{p} \lambda_i(C_i - q_i(\bar{X})D_i) + \lim_{l\to\infty}\Big[\sum_{j=1}^{m} \mu^l_j A_j + \sum_{j=1}^{p} y^l_j(C_j - q_j(\bar{X})D_j) - V^l\Big] = 0$$

and $\lim_{l\to\infty} \mathrm{Tr}(V^l\bar{X}) = 0$.

Proof. (i) ⇒ (ii) ⇒ (iii): Suppose that (i) holds, that is, $\bar{X}$ is an efficient solution of (FVP). Then $\bar{X}$ is an efficient solution of the following vector optimization problem:

$$\begin{aligned}\text{Minimize}_{X\in S^n}\quad & \Big(\mathrm{Tr}(C_1X) + \alpha_1 - q_1(\bar{X})(\mathrm{Tr}(D_1X) + \beta_1), \ldots,\\ & \qquad \mathrm{Tr}(C_pX) + \alpha_p - q_p(\bar{X})(\mathrm{Tr}(D_pX) + \beta_p)\Big)\\ \text{subject to}\quad & X \succeq 0,\ \mathrm{Tr}(A_jX) = b_j,\ j = 1, \cdots, m.\end{aligned}$$

Then we can check that $\bar{X}$ is an optimal solution of the following problem (P):

$$\begin{aligned}\text{(P)}\quad \text{Minimize}_{X\in S^n}\quad & \sum_{i=1}^{p} \lambda_i\Big[\mathrm{Tr}(C_iX) + \alpha_i - q_i(\bar{X})(\mathrm{Tr}(D_iX) + \beta_i)\Big]\\ \text{subject to}\quad & X \succeq 0,\ \mathrm{Tr}(A_jX) = b_j,\ j = 1, \ldots, m,\\ & \mathrm{Tr}(C_jX) + \alpha_j - q_j(\bar{X})(\mathrm{Tr}(D_jX) + \beta_j) \leqq 0,\quad j = 1, \ldots, p.\end{aligned}$$

Let $$A = \{X \in S^n \mid \mathrm{Tr}(A_jX) = b_j,\ j = 1, \cdots, m\}$$

and $$B = \{X \in S^n \mid \mathrm{Tr}(C_jX) + \alpha_j - q_j(\bar{X})(\mathrm{Tr}(D_jX) + \beta_j) \leqq 0,\ j = 1, \ldots, p\}$$

Let $\widetilde{\triangle} = S^n_+ \cap A \cap B$. Then $\delta_{\widetilde{\triangle}}(X) = \delta_{S^n_+}(X) + \delta_A(X) + \delta_B(X)$ for any $X \in S^n$. By Lemma 1.1, we obtain

$$\mathrm{epi}\delta^*_{\widetilde{\triangle}} = \mathrm{cl}(\mathrm{epi}\delta^*_{S^n_+} + \mathrm{epi}\delta^*_A + \mathrm{epi}\delta^*_B).$$

Let $$F(X) = \sum_{i=1}^{p} \lambda_i\Big[\mathrm{Tr}(C_iX) + \alpha_i - q_i(\bar{X})(\mathrm{Tr}(D_iX) + \beta_i)\Big].$$

Since $\bar{X}$ is an optimal solution of (P), $F(X) + \delta_{\widetilde{\triangle}}(X) \geqq F(\bar{X}) + \delta_{\widetilde{\triangle}}(\bar{X})$ for any $X \in S^n$. Thus $\mathrm{Tr}(0X) - [F(X) + \delta_{\widetilde{\triangle}}(X)] \leqq 0$ for any $X \in S^n$. Thus $(F + \delta_{\widetilde{\triangle}})^*(0) \leqq 0$. Hence

$$(0,0) \in \mathrm{epi}(F + \delta_{\widetilde{\triangle}})^* = \mathrm{epi}F^* + \mathrm{epi}\delta^*_{\widetilde{\triangle}}.$$

Using Lemma 1.2, we can check that

$$\begin{aligned}
&\mathrm{epi}F^* = \Big\{\sum_{i=1}^{p} \lambda_i (C_i - q_i(\bar{X}) D_i, -\alpha_i + q_i(\bar{X})\beta_i)\Big\} + \{0\} \times \mathbb{R}^+, \\
&\mathrm{epi}\delta^*_{S^n_+} = (-S^n_+) \times \mathbb{R}^+, \\
&\mathrm{epi}\delta^*_{A} = \mathrm{cl}\Big(\bigcup_{\mu_j \in \mathbb{R}} \{\sum_{j=1}^{m} \mu_j (A_j, b_j)\} + \{0\} \times \mathbb{R}^+\Big), \\
&\mathrm{epi}\delta^*_{B} = \mathrm{cl}\Big[\bigcup_{y_j \geqq 0} \{\sum_{j=1}^{p} y_j (C_j - q_j(\bar{X}) D_j, -\alpha_j + q_j(\bar{X})\beta_j)\} + \{0\} \times \mathbb{R}^+\Big].
\end{aligned}$$

Thus (ii) holds. Suppose that (ii) holds. Then there exist $\mu_j^l \in \mathbb{R}$, $V^l \in S^n_+$, $r \in \mathbb{R}^+$ and $r^l \in \mathbb{R}^+$ such that

$$\sum_{i=1}^{p} \lambda_i (C_i - q_i(\bar{X}) D_i) + \lim_{l \to \infty} \Big[\sum_{j=1}^{m} \mu_j^l A_j + \sum_{j=1}^{p} y_j^l (C_j - q_j(\bar{X}) D_j) - V^l\Big] = 0$$

and $$\sum_{i=1}^{p} \lambda_i (-\alpha_i + q_i(\bar{X})\beta_i) + r + \lim_{l \to \infty} \Big[\sum_{j=1}^{m} \mu_j^l b_j + \sum_{j=1}^{p} y_j^l (-\alpha_j + q_j(\bar{X})\beta_j) + r^l\Big] = 0.$$

Thus, we have,

$$\begin{aligned}
0 \;=\; & \sum_{i=1}^{p} \lambda_i \mathrm{Tr}(C_i - q_i(\bar{X}) D_i)\bar{X} \\
& + \lim_{l \to \infty} \mathrm{Tr}\Big[\sum_{j=1}^{m} \mu_j^l A_j + \sum_{j=1}^{p} y_j^l (C_j - q_j(\bar{X}) D_j) - V^l\Big]\bar{X} \\
& - \sum_{i=1}^{p} \lambda_i (-\alpha_i + q_i(\bar{X})\beta_i) - r - \lim_{l \to \infty} \Big[\sum_{j=1}^{m} \mu_j^l b_j + \sum_{j=1}^{p} y_j^l (-\alpha_j + q_j(\bar{X})\beta_j) + r^l\Big].
\end{aligned}$$

So, we obtain, $-\lim_{l \to \infty}\Big[r^l + \mathrm{Tr}(V^l \bar{X})\Big] - r = 0$. Since $r^l \geqq 0$ and $\mathrm{Tr}(V^l \bar{X}) \geqq 0$, then $r = 0$, $\lim_{l \to \infty} \mathrm{Tr}(V^l \bar{X}) = 0$. Thus (iii) holds.

(iii) ⇒ (i): Suppose that (iii) holds. Then for any $X \in \widetilde{\triangle}$,

$$\begin{aligned}
& \sum_{i=1}^{p} \lambda_i \Big[\mathrm{Tr}(C_i (X - \bar{X})) - q_i(\bar{X}) \mathrm{Tr}(D_i (X - \bar{X}))\Big] \\
& + \lim_{l \to \infty} \Big[\sum_{j=1}^{m} \mu_j^l \mathrm{Tr}(A_j (X - \bar{X})) + \sum_{j=1}^{p} y_j^l (\mathrm{Tr}(C_j (X - \bar{X})) - q_j(\bar{X}) \mathrm{Tr}(D_j (X - \bar{X}))) \\
& - \mathrm{Tr}(V^l (X - \bar{X}))\Big] = 0.
\end{aligned}$$

So, for any $X \in \widetilde{\triangle}$,

$$\sum_{i=1}^{p} \lambda_i \Big[\mathrm{Tr}(C_i(X-\bar{X})) - q_i(\bar{X})\mathrm{Tr}(D_i(X-\bar{X}))\Big] + \lim_{l\to\infty}\Big[\sum_{j=1}^{p} y_j^l(\mathrm{Tr}(C_j(X-\bar{X})) - q_j(\bar{X})\mathrm{Tr}(D_j(X-\bar{X}))) - \mathrm{Tr}(V^lX)\Big] = 0.$$

Since $\mathrm{Tr}(V^lX) \geqq 0,\ \ \mathrm{Tr}(C_j\bar{X}) + \alpha_j - q_j(\bar{X})(\mathrm{Tr}(D_j\bar{X}) + \beta_j) = 0$

and $\mathrm{Tr}(C_jX) + \alpha_j - q_j(\bar{X})(\mathrm{Tr}(D_jX) + \beta_j) \leqq 0$

for any $X \in \widetilde{\triangle}$, we have, for any $X \in \widetilde{\triangle}$,

$$\sum_{i=1}^{p} \lambda_i \Big[\mathrm{Tr}(C_iX) + \alpha_i - q_i(\bar{X})(\mathrm{Tr}(D_iX) + \beta_i)\Big] \geqq \sum_{i=1}^{p} \lambda_i \Big[\mathrm{Tr}(C_i\bar{X}) + \alpha_i - q_i(\bar{X})(\mathrm{Tr}(D_i\bar{X}) + \beta_i)\Big].$$

Thus $\bar{X}$ is an optimal solution of (P). We can check that $\bar{X}$ is an efficient solution of (FVP). Hence (i) holds. □

We give an assumption for (FVP):

Assumption (C): $\left\{\dfrac{\mathrm{Tr}(D_iX)+\beta_i}{\mathrm{Tr}(D_jX)+\beta_j} \mid i \neq j,\ X \in \triangle\right\}$ is bounded above.

Proposition 2.3. *The Assumption* (C) *is assumed. Let* $\bar{X} \in \triangle$. *Then the following are equivalent:*

(i) $\bar{X}$ *is a properly efficient solution of* (FVP)*;*

(ii) $\bar{X}$ *is a properly efficient solution of the following linear semidefinite vector optimization problem:*

$$\text{(LVP)}\qquad \textit{Minimize}\quad \Big(\mathrm{Tr}(C_1X) + \alpha_1 - q_1(\bar{X})(\mathrm{Tr}(D_1X) + \beta_1), \ldots, \mathrm{Tr}(C_pX) + \alpha_p - q_p(\bar{X})(\mathrm{Tr}(D_pX) + \beta_p)\Big)$$

$$\textit{subject to}\quad X \in \triangle.$$

Proof. Let $\bar{X} \in \triangle$. Then the following are equivalent:

(i) $\bar{X}$ is a properly efficient solution of (FVP).

(ii)(1) There does not exist $X \in \triangle$ such that $\frac{\mathrm{Tr}(C_iX)+\alpha_i}{\mathrm{Tr}(D_iX)+\beta_i} \leqq \frac{\mathrm{Tr}(C_i\bar{X})+\alpha_i}{\mathrm{Tr}(D_i\bar{X})+\beta_i}$ for all numbers $i = 1, \ldots, p$ and $\frac{\mathrm{Tr}(C_jX)+\alpha_j}{\mathrm{Tr}(D_jX)+\beta_j} < \frac{\mathrm{Tr}(C_j\bar{X})+\alpha_j}{\mathrm{Tr}(D_j\bar{X})+\beta_j}$ for some $j \in \{1, \ldots, p\}$;

(2) there exists $M > 0$ such that, for each $i = 1, \ldots, p$ and each $X \in \triangle$ satisfying $\frac{\mathrm{Tr}(C_iX)+\alpha_i}{\mathrm{Tr}(D_iX)+\beta_i} < \frac{\mathrm{Tr}(C_i\bar{X})+\alpha_i}{\mathrm{Tr}(D_i\bar{X})+\beta_i}$, there exists at least on number $j \in \{1, \ldots, p\}$ such that

$$\frac{\mathrm{Tr}(C_j\bar{X})+\alpha_j}{\mathrm{Tr}(D_j\bar{X})+\beta_j} < \frac{\mathrm{Tr}(C_jX)+\alpha_j}{\mathrm{Tr}(D_jX)+\beta_j} \quad\text{and}\quad \frac{\dfrac{\mathrm{Tr}(C_i\bar{X})+\alpha_i}{\mathrm{Tr}(D_i\bar{X})+\beta_i} - \dfrac{\mathrm{Tr}(C_iX)+\alpha_i}{\mathrm{Tr}(D_iX)+\beta_i}}{\dfrac{\mathrm{Tr}(C_jX)+\alpha_j}{\mathrm{Tr}(D_jX)+\beta_j} - \dfrac{\mathrm{Tr}(C_j\bar{X})+\alpha_j}{\mathrm{Tr}(D_j\bar{X})+\beta_j}} \leqq M$$

(iii) (1) there does not exist $X \in \triangle$ such that

$$\operatorname{Tr}(C_iX) + \alpha_i - q_i(\bar{X})(\operatorname{Tr}(D_iX) + \beta_i) \leqq 0 = \operatorname{Tr}(C_i\bar{X}) + \alpha_i - q_i(\bar{X})(\operatorname{Tr}(D_i\bar{X}) + \beta_i)$$

for all $i = 1, \dots, p$ and

$$\operatorname{Tr}(C_jX) + \alpha_j - q_j(\bar{X})(\operatorname{Tr}(D_jX) + \beta_j) < 0 = \operatorname{Tr}(C_j\bar{X}) + \alpha_j - q_j(\bar{X})(\operatorname{Tr}(D_j\bar{X}) + \beta_j),$$

for some $j \in \{1, \cdots, p\}$;

(2) there exists $M > 0$ such that for each $i = 1, \cdots, p$ and each $X \in \triangle$ satisfying

$$\operatorname{Tr}(C_iX) + \alpha_i - q_i(\bar{X})(\operatorname{Tr}(D_iX) + \beta_i) < 0 = \operatorname{Tr}(C_i\bar{X}) + \alpha_i - q_i(\bar{X})(\operatorname{Tr}(D_i\bar{X}) + \beta_i),$$

there are some $j \in \{1, \cdots, p\}$ such that

$$\operatorname{Tr}(C_j\bar{X}) + \alpha_j - q_j(\bar{X})(\operatorname{Tr}(D_j\bar{X}) + \beta_j) < \operatorname{Tr}(C_jX) + \alpha_j - q_j(\bar{X})(\operatorname{Tr}(D_jX) + \beta_j)$$

and

$$\frac{(\operatorname{Tr}(C_i\bar{X}) + \alpha_i - q_i(\bar{X})(\operatorname{Tr}(D_i\bar{X}) + \beta_i) - (\operatorname{Tr}(C_iX) + \alpha_i - q_i(\bar{X})(\operatorname{Tr}(D_iX) + \beta_i))}{(\operatorname{Tr}(C_jX) + \alpha_j - q_j(\bar{X})(\operatorname{Tr}(D_jX) + \beta_j)) - (\operatorname{Tr}(C_j\bar{X}) + \alpha_j - q_j(\bar{X})(\operatorname{Tr}(D_j\bar{X}) + \beta_j))}$$
$$\leqq \frac{\operatorname{Tr}(D_iX) + \beta_i}{\operatorname{Tr}(D_jX) + \beta_j} M.$$

Thus the result holds. □

Modifying Example 3.6 in [2], we give an example showing that if the Assumption C does not hold, then the result of Proposition 2.3 is not true.

Example 2.4. Consider the following semidefinite linear fractional vector optimization problem (FVP):

$$\text{Minimize} \quad \left(\frac{\operatorname{Tr}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} - 1}{\operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1}, \frac{\operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} - 1}{\operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1} \right)$$

$$\text{subject to} \quad \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in S^2_+, \quad \text{and} \quad \operatorname{Tr}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} = 0.$$

Let $\triangle$ be the feasible set for (FVP) and let $\bar{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $a = 1,\ b = 0$ and $c = 1.$ Then $\bar{X} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

Let M be arbitrarily positive number. We choose $\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in \triangle$ such that

$$\frac{\operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} - 1}{\operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1} < \min\left\{ -M, \ \frac{\operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\bar{X} - 1}{\operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\bar{X} + 1} \right\}.$$

Then
$$\frac{\dfrac{\operatorname{Tr}\begin{pmatrix}0&0\\0&-1\end{pmatrix}\bar{X}-1}{\operatorname{Tr}\begin{pmatrix}1&0\\0&0\end{pmatrix}\bar{X}+1}-\dfrac{\operatorname{Tr}\begin{pmatrix}0&0\\0&-1\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}-1}{\operatorname{Tr}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}+1}}{\dfrac{\operatorname{Tr}\begin{pmatrix}-1&0\\0&0\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}-1}{\operatorname{Tr}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}+1}-\dfrac{\operatorname{Tr}\begin{pmatrix}-1&0\\0&0\end{pmatrix}\bar{X}-1}{\operatorname{Tr}\begin{pmatrix}0&0\\0&1\end{pmatrix}\bar{X}+1}}$$
$$=\left(-\frac{c+1}{a+1}\right)\left(-\frac{x_3+1}{x_1+1}\right)>-\frac{c+1}{a+1}(-M)=M.$$

So, $\bar{X}$ is not a properly efficient solution of (FVP). Consider the following linear semidefinite vector optimization problem (LVP):

Minimize
$$\Bigg(\Big(\operatorname{Tr}\begin{pmatrix}-1&0\\0&0\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}-1\Big)-q_1(\bar{X})\Big(\operatorname{Tr}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}+1\Big),$$
$$\Big(\operatorname{Tr}\begin{pmatrix}0&0\\0&-1\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}-1\Big)-q_2(\bar{X})\Big(\operatorname{Tr}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}+1\Big)\Bigg)$$
subject to $\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}\in S^2_+$, and $\operatorname{Tr}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}=0$,

where $q_1(\bar{X})=\dfrac{\operatorname{Tr}\begin{pmatrix}-1&0\\0&0\end{pmatrix}\bar{X}-1}{\operatorname{Tr}\begin{pmatrix}0&0\\0&1\end{pmatrix}\bar{X}+1}=-1$ and $q_2(\bar{X})=\dfrac{\operatorname{Tr}\begin{pmatrix}0&0\\0&-1\end{pmatrix}\bar{X}-1}{\operatorname{Tr}\begin{pmatrix}1&0\\0&1\end{pmatrix}\bar{X}+1}=-1.$

(LVP) becomes: Minimize $(-x_1+x_3, x_1-x_3)$

subject to $\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}\in S^2_+$, and $x_2=0$.

Thus $\bar{X}$ is a properly efficient solution of (LVP). Notice that
$$\frac{\operatorname{Tr}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}+1}{\operatorname{Tr}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}+1}$$
is not bounded above over the feasible set of (FVP) and
$$\frac{\operatorname{Tr}\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}+1}{\operatorname{Tr}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_2&x_3\end{pmatrix}+1}$$
is not bounded above over the feasible set of (FVP). Thus, the result of Proposition 2.3 does not hold for this example. □

Following the proof of Theorem 2 in [5], and using Proposition 2.3, we can obtain the following proposition:

Proposition 2.5. *The Assumption* (C) *is assumed. Let $\bar{X} \in \triangle$. Then the following statements are equivalent:*

(i) *$\bar{X}$ is a properly efficient solution of (FVP);*

(ii) *there exist $\lambda_i > 0,\ i = 1, \cdots, p$ such that $\bar{X}$ is an optimal solution of the following linear optimization problem:*

$$\text{(LP)} \qquad \textit{Minimize} \qquad \sum_{i=1}^{p} \lambda_i \left(\mathrm{Tr}(C_i X) + \alpha_i - q_i(\bar{X})(\mathrm{Tr}(D_i X) + \beta_i)\right)$$

$$\textit{subject to} \qquad X \in \triangle.$$

Using Proposition 2.5 and following the proof of Theorem 2.2, we can obtain the following optimality theorem for properly efficient solution of (FVP):

Theorem 2.6. *The Assumption* (C) *is assumed. Let $\bar{X} \in \triangle$. Then the following statements are equivalent:*

(i) *$\bar{X}$ is a properly efficient solution of (FVP);*

(ii) *there exist $\lambda_i > 0, i = 1, \dots, p,\ \mu_j \in \mathbb{R},\ j = 1, \dots, m$ such that*

$$(0,0) \in \Big\{\sum_{i=1}^{p} \lambda_i (C_i - q_i(\bar{X}) D_i, -\alpha_i + q_i(\bar{X})\beta_i)\Big\} + \{0\} \times \mathbb{R}^+$$
$$+cl\Big(\bigcup_{\mu_j \in \mathbb{R}} \{\sum_{j=1}^{m} \mu_j (A_j, b_j)\} + (-S^n_+) \times \mathbb{R}^+\Big).$$

(iii) *there exist $\lambda_i > 0, i = 1, \cdots, p,\ \mu_j^l \in \mathbb{R},\ j = 1, \cdots, m,\ V^l \in S_n^+$ such that*

$$\sum_{i=1}^{p} \lambda_i (C_i - q_i(\bar{X}) D_i) + \lim_{l \to \infty} \Big[\sum_{j=1}^{m} \mu_j^l A_j - V^l\Big] = 0 \quad \textit{and} \quad \lim_{l \to \infty} \mathrm{Tr}(V^l \bar{X}) = 0.$$

The following theorem, which is semidefinite version of the Isermann's result [4, 7, 15], gives a sufficient condition that an efficient solution of (FVP) can be properly efficient for (FVP).

Theorem 2.7. *The Assumption* (C) *is assumed. Let $\bar{X} \in \triangle$. Assume that*

$$\bigcup_{\mu_j \in \mathbb{R}} \Big\{\sum_{j=1}^{m} \mu_j (A_j, b_j)\Big\} + \bigcup_{y_j \geqq 0} \Big\{\sum_{j=1}^{p} y_j (C_j - q_j(\bar{X}) D_j, -\alpha_j + q_j(\bar{X})\beta_j)\Big\} + (-S^n_+) \times \mathbb{R}_+$$

is closed. If $\bar{X}$ is an efficient solution of (FVP)*, then $\bar{X}$ is a properly efficient solution of* (FVP)*.*

Proof. Let $\lambda = (\lambda_1, \cdots, \lambda_p) \in \mathbb{R}^p$ with $\lambda_i > 0$ for all $i = 1, \cdots, p$. Let $\bar{X}$ is an efficient solution of (FVP). Then by Theorem 2.2,

$$\begin{aligned}(0,0) \in &\Big\{\sum_{i=1}^{p} \lambda_i (C_i - q_i(\bar{X}) D_i, -\alpha_i + q_i(\bar{X})\beta_i)\Big\} + \{0\} \times \mathbb{R}_+ \\ &+ \mathrm{cl}\Big(\bigcup_{\mu_j \in \mathbb{R}} \Big\{ \sum_{j=1}^{m} \mu_j (A_j, b_j) \Big\} + \bigcup_{y_j \geqq 0} \Big\{ \sum_{j=1}^{p} y_j (C_j - q_j(\bar{X}) D_j, -\alpha_j + q_j(\bar{X})\beta_j) \Big\} \\ &+ (-S^n_+) \times \mathbb{R}_+ \Big).\end{aligned}$$

By assumption,

$$\begin{aligned}(0,0) \in &\sum_{i=1}^{p} \lambda_i (C_i - q_i(\bar{X}) D_i, -\alpha_i + q_i(\bar{X})\beta_i) \\ &+ \bigcup_{\mu_j \in \mathbb{R}} \Big\{ \sum_{j=1}^{m} \mu_j (A_j, b_j) \Big\} + \bigcup_{y_j \geqq 0} \Big\{ \sum_{j=1}^{p} y_j (C_j - q_j(\bar{X}) D_j, -\alpha_j + q_j(\bar{X})\beta_j) \Big\} \\ &+ (-S^n_+) \times \mathbb{R}_+.\end{aligned}$$

So, there exist $\bar{\mu}_j \in \mathbb{R}$ and $\bar{y}_j \geqq 0,\ j = 1, \cdots, m,\ \bar{r} \geqq 0$ and $\bar{V} \in S^n_+$ such that

$$0 = \sum_{i=1}^{p} \lambda_i (C_i - q_i(\bar{X}) D_i) + \sum_{j=1}^{m} \bar{\mu}_j A_j + \sum_{j=1}^{p} \bar{y}_j (C_j - q_j(\bar{X}) D_j) - \bar{V}$$

and

$$0 = \sum_{i=1}^{p} (\lambda_i + \bar{y}_i)(-\alpha_i + q_i(\bar{X})\beta_i) + \sum_{j=1}^{m} \bar{\mu}_j b_j + \bar{r}.$$

So for any $X \in \triangle$,

$$0 = \sum_{i=1}^{p} \lambda_i \mathrm{Tr}(C_i - q_i(\bar{X}) D_i) X + \sum_{j=1}^{m} \bar{\mu}_j \mathrm{Tr}(A_j X) + \sum_{j=1}^{p} \bar{y}_j \mathrm{Tr}(C_j - q_j(\bar{X}) D_j) X - \mathrm{Tr}(\bar{V} X).$$

Hence, for any $X \in \triangle$,

$$\begin{aligned}&\mathrm{Tr}(\bar{V} X) + \bar{r} \\ &= \sum_{i=1}^{p} (\lambda_i + \bar{y}_i)(\mathrm{Tr}(C_i X) + \alpha_i - q_i(\bar{X})(\mathrm{Tr}(D_i X) + \beta_i)) + \sum_{j=1}^{m} \bar{\mu}_j (\mathrm{Tr}(A_j X) - b_j) \\ &= \sum_{i=1}^{p} (\lambda_i + \bar{y}_i)(\mathrm{Tr}(C_i X) + \alpha_i - q_i(\bar{X})(\mathrm{Tr}(D_i X) + \beta_i)).\end{aligned}$$

Since $\bar{r} \geqq 0$ and $\bar{V} \in S^n_+$, for any $X \in \triangle$,

$$\begin{aligned}&\sum_{i=1}^{p} (\lambda_i + \bar{y}_i)(\mathrm{Tr}(C_i X) + \alpha_i - q_i(\bar{X})(\mathrm{Tr}(D_i X) + \beta_i)) \\ &\geqq 0 = \sum_{i=1}^{p} (\lambda_i + \bar{y}_i)(\mathrm{Tr}(C_i \bar{X}) + \alpha_i - q_i(\bar{X})(\mathrm{Tr}(D_i \bar{X}) + \beta_i)).\end{aligned}$$

Since $\lambda_i + \bar{y}_i > 0$, by Proposition 2.5, $\bar{X}$ is a properly efficient solution of (FVP). $\square$

Now we give examples illustrating how to solve semidefinite linear fractional vector optimization problems by using Theorem 2.6.

Example 2.8. Consider the following semidefinite linear fractional vector optimization problem (FVP):

$$\begin{aligned} \text{Minimize} \quad & \left(\frac{\operatorname{Tr}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1}{\operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1}, \frac{\operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1}{\operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1} \right) \\ \text{subject to} \quad & \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in S^2_+, \ \operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} = 1 \ \text{ and} \\ & \operatorname{Tr}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} = 1. \end{aligned}$$

Let $\triangle$ be the feasible set for (FVP). Then

$$\triangle = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in S^2 \ \middle| \ 0 \leqq x_1 \leqq 1, \ x_3 = 1 - x_1, \ x_2 = 0 \right\}.$$

Since $\dfrac{\operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1}{\operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1} \leqq 2$ and $\dfrac{\operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1}{\operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1} \leqq 2$, the Assumption (C) in Theorem 2.6 is satisfied.

Let $C_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $C_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $D_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha_1 = 1$, $\beta_1 = 1$, $\alpha_2 = 1$, $\beta_2 = 1$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $b_1 = 1$ and $b_2 = 1$.

Let $\bar{X} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\bar{X} \in \triangle$.

Let $q_1(\bar{X}) = \dfrac{\operatorname{Tr}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}{\scriptstyle \bar{X}+1}}{\operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}{\scriptstyle \bar{X}+1}}$ and $q_2(\bar{X}) = \dfrac{\operatorname{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}{\scriptstyle \bar{X}+1}}{\operatorname{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}{\scriptstyle \bar{X}+1}}$.

Then $q_1(\bar{X}) = 1$ and $q_2(\bar{X}) = 2$,

$$\begin{aligned} & 1 \cdot (C_1 - q_1(\bar{X})D_1, -\alpha_1 + q_1(\bar{X})\beta_1) + \frac{1}{2}(C_2 - q_2(\bar{X})D_2, -\alpha_2 + q_2(\bar{X})\beta_2) \\ & \qquad + \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right) + \frac{1}{2}(A_1, b_1) + (-1)(A_2, b_2) + \left(-\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right) \\ & = 1 \cdot \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 0 \right) + \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 1 \right) + \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right) \\ & \qquad + \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) + (-1)\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, 1 \right) + \left(-\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right) \end{aligned}$$

$$= \left(\begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix}, \frac{1}{2}\right) + \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right) + \left(\begin{pmatrix} -\frac{1}{2} & -1 \\ -1 & -\frac{1}{2} \end{pmatrix}, -\frac{1}{2}\right) + \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right)$$

$$= \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right).$$

By Theorem 2.6, $\bar{X}$ is a properly efficient solution of (FVP). □

Example 2.9. Consider the following semidefinite linear fractional vector optimization problem (FVP):

$$\text{Minimize} \quad \left(\frac{\mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}}{\mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1}, \frac{\mathrm{Tr}\begin{pmatrix} -1 & \dfrac{1}{2} \\ \dfrac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}}{\mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1} \right)$$

$$\text{subject to} \quad \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in S^2_+, \ \text{ and } \ \mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} = 0.$$

Let $\triangle$ be the feasible set for (FVP). Then

$$\triangle = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in S^2 \ \middle|\ x_1 \geqq 0, x_2 = 0, x_3 = 0 \right\}$$

and the Assumption (C) in Theorem 2.6 is clearly satisfied.

Let $\bar{X} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\bar{X} \in \triangle$.

Let $q_1(\bar{X}) = \dfrac{\mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\bar{X}}{\mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\bar{X}+1}$ and $q_2(\bar{X}) = \dfrac{\mathrm{Tr}\begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}\bar{X}}{\mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\bar{X}+1}$.

Then $q_1(\bar{X}) = 1$ and $q_2(\bar{X}) = -1$.

Let $C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $D_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $C_2 = \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$, $D_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\text{Then} \quad 1 \cdot (C_1 - q_1(\bar{X})D_1) + 1 \cdot (C_2 - q_2(\bar{X})D_2) + \lim_{n\to\infty} \left[nA - \begin{pmatrix} \frac{1}{n} & \frac{1}{2} \\ \frac{1}{2} & n \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} + \lim_{n\to\infty} \begin{pmatrix} -\frac{1}{n} & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and $\lim_{n\to\infty} \begin{pmatrix} \frac{1}{n} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \bar{X} = 0$. In consequence of Theorem 2.6, $\bar{X}$ is a properly efficient solution of (FVP). □

Remark 2.10. In Example 2.9, there are no solutions $\lambda_1,\ \lambda_2,\ \mu$ for the following system:

$$\begin{aligned}&\lambda_1(C_1 - q_1(\bar{X})D_1) + \lambda_2(C_2 - q_2(\bar{X})D_2) + \mu A \in S^2_+,\\ &\lambda_1 > 0,\ \lambda_2 > 0,\ \mu \in \mathbb{R},\\ &\mathrm{Tr}[\lambda_1(C_1 - q_1(\bar{X})D_1) + \lambda_2(C_2 - q_2(\bar{X})D_2) + \mu A]\bar{X} = 0.\end{aligned}$$

So we should consider the limit in Example 2.9.

Applying Theorem 2.7, we will show that the matrix $\bar{X}$ in Example 2.8 is a properly efficient solution of the problem (FVP).

Example 2.11. Consider the problem (FVP) in Example 2.8. Let A_1, A_2, b_1, b_2, C_1, D_1, α_1, β_1, C_2, D_2, α_2, β_2 be same as in Example 2.8. Let $\bar{X} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then we can geometrically check that $\bar{X}$ is an efficient solution of (FVP). Let

$$\begin{aligned}\Lambda = &\bigcup_{\substack{u_1\in\mathbb{R}\\ u_2\in\mathbb{R}}} \{u_1(A_1, b_1) + u_2(A_2, b_2)\}\\ &+ \bigcup_{\substack{y_1\geqq 0\\ y_2\geqq 0}} \{y_1(C_1 - q_1(\bar{X})D_1, -\alpha_1 + q_1(\bar{X})\beta) + y_2(C_2 - q_2(\bar{X})D_2, -\alpha_2 + q_2(\bar{X})\beta_2)\}\\ &+ (-S^2_+) \times \mathbb{R}_+\end{aligned}$$

and
$$\begin{aligned}\widetilde{\Lambda} = &\{(u_1 + u_2, u_2, u_2, u_1 + u_2, u_1 + u_2) \mid u_1, u_2 \in \mathbb{R}\}\\ &+ \{(y_2, y_1, y_1, y_1 - y_2, y_2) \mid y_1 \geqq 0, y_2 \geqq 0\}\\ &+ \{(-a, -b, -b, -c, \widehat{r}) \mid a \geqq 0,\ c \geqq 0,\ ac - b^2 \geqq 0,\ \widehat{r} \geqq 0\}.\end{aligned}$$

Then Λ is closed if and only if $\widetilde{\Lambda}$ is closed.

We now prove that $\widetilde{\Lambda}$ is closed. For any $(r_1, r_2, r_3) \in \mathbb{R}^3$, there exist $y_1 \geqq 0$, $y_2 \geqq 0$ such that $r_1 - r_3 = -y_1 + 2y_2$. So, for any $(r_1, r_2, r_3) \in \mathbb{R}^3$,

$$\begin{aligned}&((r_1 - (r_2 - y_1) - y_2) + (r_2 - y_1) + y_2,\ (r_2 - y_1) + y_1,\\ &\quad (r_1 - (r_2 - y_1) - y_2) + (r_2 - y_1) + y_1 - y_2)\\ &= (r_1, r_2, r_1 + y_1 - 2y_2) = (r_1, r_2, r_1 - r_1 + r_3) = (r_1, r_2, r_3).\end{aligned}$$

In consequence,

$$\begin{aligned}&\{(u_1 + u_2, u_2, u_2, u_1 + u_2, u_1 + u_2) \mid u_1, u_2 \in \mathbb{R}\}\\ &\qquad + \{(y_2, y_1, y_1, y_1 - y_2, y_2) \mid y_1 \geqq 0, y_2 \geqq 0\}\\ &= \{(r_1, r_2, r_2, r_3, r_1) \mid r_1, r_2, r_3 \in \mathbb{R}\}.\end{aligned}$$

Hence

$$\begin{aligned}\widetilde{\Lambda} &= \{(r_1 - a, r_2 - b, r_2 - b, r_3 - c, r_1 + \widehat{r}) \mid a \geqq 0,\ c \geqq 0,\ ac - b^2 \geqq 0,\ \widehat{r} \geqq 0\}\\ &= \{(\alpha, \beta, \beta, \gamma, \delta) \mid \alpha, \beta, \gamma \in \mathbb{R},\ \delta \geqq \alpha\}.\end{aligned}$$

So, $\widetilde{\Lambda}$ is closed and hence Λ is closed. Hence by Theorem 2.7, $\bar{X}$ is a properly efficient solution of (FVP). □

3. Duality theorems

Now we formulate the dual problem (VD) for (FVP) as follows:

$$\text{(VD)} \qquad \text{Maximize } (v_1, \cdots, v_p)$$

$$\text{subject to } \sum_{i=1}^{p} \lambda_i (C_i - v_i D_i) + \lim_{l \to \infty} \Big[\sum_{j=1}^{m} \mu_j^l A_j - V^l \Big] = 0,$$

$$\limsup_{l \to \infty} \sum_{j=1}^{m} \mu_j^l b_j \leqq \sum_{i=1}^{p} \lambda_i (\alpha_i - v_i \beta_i), \ \lambda_i > 0, \ i = 1, \ldots, p,$$

$$\mu_j^l \in \mathbb{R}, \ j = 1, \ldots, m, \ V^l \in S_+^n, \ l = 1, 2, \ldots.$$

Theorem 3.1. (Weak duality) *Let* X *be feasible for* (FVP). *Let* $(v, \lambda, \{\mu_j^l\}, \{V^l\})$ *be feasible for* (VD). *Then the following does not hold:*

$$\Big(\frac{\mathrm{Tr}(C_1 X) + \alpha_1}{\mathrm{Tr}(D_1 X) + \beta_1}, \ldots, \frac{\mathrm{Tr}(C_p X) + \alpha_p}{\mathrm{Tr}(\mathrm{D_p X}) + \beta_{\mathrm{p}}} \Big) \leqq (v_1, \ldots, v_p)$$

and

$$\Big(\frac{\mathrm{Tr}(C_1 X) + \alpha_1}{\mathrm{Tr}(D_1 X) + \beta_1}, \ldots, \frac{\mathrm{Tr}(C_p X) + \alpha_p}{\mathrm{Tr}(D_p X) + \beta_p} \Big) \neq (v_1, \ldots, v_p).$$

Proof. Let X be feasible for (FVP) and let $(v, \lambda, \{\mu_j^l\}, \{V^l\})$ be feasible for (VD). Suppose to the contrary that

$$\Big(\frac{\mathrm{Tr}(C_1 X) + \alpha_1}{\mathrm{Tr}(D_1 X) + \beta_1}, \ldots, \frac{\mathrm{Tr}(C_p X) + \alpha_p}{\mathrm{Tr}(D_p X) + \beta_p} \Big) \leqq (v_1, \cdots, v_p)$$

and

$$\Big(\frac{\mathrm{Tr}(C_1 X) + \alpha_1}{\mathrm{Tr}(D_1 X) + \beta_1}, \ldots, \frac{\mathrm{Tr}(C_p X) + \alpha_p}{\mathrm{Tr}(D_p X) + \beta_p} \Big) \neq (v_1, \cdots, v_p).$$

Then for some i,

$$\frac{\mathrm{Tr}(C_i X) + \alpha_i}{\mathrm{Tr}(D_i X) + \beta_i} < v_i \ \text{ and for all } j, j \neq i, \ \ \frac{\mathrm{Tr}(C_j X) + \alpha_j}{\mathrm{Tr}(D_j X) + \beta_j} \leqq v_j.$$

Thus for some i, $\mathrm{Tr}(C_i X) + \alpha_i - v_i(\mathrm{Tr}(D_i X) + \beta_i) < 0$ and for all j, $j \neq i$,

$$\mathrm{Tr}(C_j X) + \alpha_j - v_j(\mathrm{Tr}(D_j X) + \beta_j) \leqq 0.$$

So for some i, $\alpha_i - v_i \beta_i + \mathrm{Tr}(C_i - v_i D_i) X < 0$ and for all j, $j \neq i$,

$$\alpha_j - v_j \beta_j + \mathrm{Tr}(C_j - v_j D_j) X \leqq 0.$$

Since $\lambda_i > 0, \ i = 1, \cdots, p$, we have

$$\sum_{i=1}^{p} \lambda_i (\mathrm{Tr}(C_i - v_i D_i) X + \alpha_i - v_i \beta_i) < 0$$

and hence

$$\sum_{i=1}^{p} \lambda_i (\alpha_i - v_i \beta_i) - \lim_{l \to \infty} \Big[\mathrm{Tr}(\sum_{j=1}^{m} \mu_j^l A_j - V^l) X \Big] < 0.$$

Hence, $$\sum_{i=1}^{p}\lambda_i(\alpha_i - v_i\beta_i) < \lim_{l\to\infty}\Big[\mathrm{Tr}(\sum_{j=1}^{m}\mu_j^l A_j X) - \mathrm{Tr}(V^l X)\Big]$$

$$\leqq \limsup_{l\to\infty}\mathrm{Tr}(\sum_{j=1}^{m}\mu_j^l A_j X) + \limsup_{l\to\infty}(-\mathrm{Tr}(V^l X))$$

$$\leqq \limsup_{l\to\infty}\sum_{j=1}^{m}\mu_j^l \mathrm{Tr}(A_j X) = \limsup_{l\to\infty}\sum_{j=1}^{m}\mu_j^l b_j.$$

This is a contradiction. Consequently we obtain

$$\Big(\frac{\mathrm{Tr}(C_1X)+\alpha_1}{\mathrm{Tr}(D_1X)+\beta_1},\dots,\frac{\mathrm{Tr}(C_pX)+\alpha_p}{\mathrm{Tr}(D_pX)+\beta_p}\Big) \nleq (v_1, v_2, \cdots, v_p). \qquad \square$$

Theorem 3.2. (Strong duality) *Let $\bar{X} \in \triangle$ be a properly efficient solution of* (FVP). *Suppose that Assumption* (C) *is assumed. Let $\bar{v}_i = \frac{\mathrm{Tr}(C_i\bar{X})+\alpha_i}{\mathrm{Tr}(D_i\bar{X})+\beta_i}$. Then there exists $(\lambda, \{\mu_j^l\}, \{V^l\})$ such that $(\bar{v}, \lambda, \{\mu_j^l\}, \{V^l\})$ is feasible for (VD) and moreover $(\bar{v}, \lambda, \{\mu_j^l\}, \{V^l\})$ is an efficient solution of (VD).*

Proof. Let $\bar{X} \in \triangle$ be a properly efficient solution of (FVP). By Theorem 2.6, there exist $\lambda_i > 0,\ i = 1, \cdots, p,\ \mu_j^l \in \mathbb{R},\ j = 1, \cdots, p,\ V^l \in S_+^n$ such that

$$\sum_{i=1}^{p}\lambda_i(C_i - q_i(\bar{X})D_i) + \lim_{l\to\infty}\Big[\sum_{j=1}^{m}\mu_j^l A_j - V^l\Big] = 0 \ \text{ and } \ \lim_{l\to\infty}\mathrm{Tr}(V^l\bar{X}) = 0.$$

Let $\bar{v}_i = q_i(\bar{X})$. Then $\sum_{i=1}^{p}\lambda_i(C_i - \bar{v}_i D_i) + \lim_{l\to\infty}\Big[\sum_{j=1}^{m}\mu_j^l A_j - V^l\Big] = 0.$

Hence $\sum_{i=1}^{p}\lambda_i \mathrm{Tr}(C_i - \bar{v}_i D_i)\bar{X} + \lim_{l\to\infty}\Big[\sum_{j=1}^{m}\mu_j^l \mathrm{Tr}(A_j\bar{X}) - \mathrm{Tr}(V^l\bar{X})\Big] = 0.$

So, $\sum_{i=1}^{p}\lambda_i \mathrm{Tr}(C_i - \bar{v}_i D_i)\bar{X} + \lim_{l\to\infty}\sum_{j=1}^{m}\mu_j^l \mathrm{Tr}(A_j\bar{X}) = 0.$

So, $\sum_{i=1}^{p}\lambda_i\Big[\mathrm{Tr}(C_i\bar{X}) - \bar{v}_i\mathrm{Tr}(D_i\bar{X})\Big] + \lim_{l\to\infty}\sum_{j=1}^{m}\mu_j^l b_j = 0.$

Since $\bar{v}_i = \frac{\mathrm{Tr}(C_i\bar{X})+\alpha_i}{\mathrm{Tr}(D_i\bar{X})+\beta_i}$, we have $\mathrm{Tr}(C_i\bar{X}) - \bar{v}_i\mathrm{Tr}(D_i\bar{X}) = -\alpha_i + \bar{v}_i\beta_i$, and so

$$\sum_{i=1}^{p}\lambda_i(-\alpha_i + \bar{v}_i\beta_i) + \lim_{l\to\infty}\sum_{j=1}^{m}\mu_j^l b_j = 0.$$

Thus $(\bar{v}, \lambda, \{\mu_j^l\}, \{V^l\})$ is feasible for (VD). By Theorem 3.1, $(\bar{v}, \lambda, \{\mu_j^l\}, \{V^l\})$ is an efficient solution of (VD). □

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