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Alternated Inertial Fixed Point Algorithms

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We study a weakly convergent fixed point algorithm with alternated inertial step to approximate a common fixed point of a sequence of mappings of nonexpansive type in a real Hilbert space. We present convergence analysis for the proposed algorithm under mild assumptions. Then, we apply the results to iterative algorithms of type alternating projection, forward-backward and primal-dual splitting and derive some convergence results. To demonstrate the effectiveness of our proposed algorithm, we present numerical comparisons of the algorithm with the existing ones.

Keywords: Nonexpansive mapping, fixed point, alternated inertial term, weak convergence, primal-dual splitting algorithm, Hilbert space.

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1. Introduction

In various areas of nonlinear analysis and optimization to unify the convergence proofs of algorithms, Fejér monotonicity has been widely exploited see, e.g., [5, 6, 7, 13]. In particular, this notion has proved to be an efficient tool to simplify and unify the convergence analysis of the Krasnoselskiĭ-Mann algorithm arising in fixed point problems (see [7, Chapter 5]). It is noteworthy that the Krasnoselskiĭ-Mann algorithm is that the proximal point algorithm [27, 32], the forward-backward algorithm [24, 31] and the Douglas-Rachford splitting algorithm [20, 24] may well be interpreted as its special cases.

The inertial versions of the Krasnoselskiĭ-Mann algorithm in [10, 19, 26] are existing algorithms that do solve the fixed point problems, and possible to prove the generated sequence weak convergence to a solution. Such inertial variants are of interest in practice because they have the potential to improve the performance of the algorithm. Indeed, when we apply this algorithm to minimize the sum of a proper, lower semicontinuous and convex function, and a differentiable and convex function whose gradient is Lipschitz continuous, the theoretical convergence rates of the algorithm can be improved under a suitable choice of inertial parameters [8, Theorem 4.4], [4, Theorem 1], [3, Theorem 16]. However, the sequence generated by the inertial

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algorithm is not Fejér monotone and this may make the sequence generated by the algorithm to move or swing back and forth around the set of solutions [29, Examples 1 and 2].

Recently, Iutzeler and Hendrickx [22] introduced a fixed point algorithm with alternated inertial step to approximate a fixed point of an averaged mapping in finite dimensional spaces. Their algorithm is based on the algorithm considered in [29]. We emphasize that alternating inertia allows to keep the Fejér monotonicity property and has been shown to exhibit attractive performances in practice (see [22, 21]).

Motivated and inspired by algorithms considered in [21, 22, 29], we investigate a fixed point algorithm with alternated inertial step to approximate a common fixed point of mappings of nonexpansive type in real Hilbert spaces. We show that the sequence generated by our proposed algorithm is Fejér monotone and converges weakly to a common fixed point under mild assumptions on inertial parameters.

Then, the proposed algorithm can be directly applied to solve the convex feasibility and the (structured) monotone inclusion problems. Indeed, by employing some elaborated techniques (see [2, 15, 23, 34]), we know that solutions of the convex feasibility and the (structured) monotone inclusion problems can be viewed as a common fixed point of mappings of nonexpansive type. Relying on this, we apply the results to iterative algorithms of type alternating projection, forward-backward and primal-dual splitting and are able to guarantee convergence of these algorithms. Our main contributions of the present paper are as follows:

- We present the fixed point algorithm (Algorithm 3.1) designed for finding a common fixed point of a sequence of mappings of nonexpansive type. Under suitable conditions (Assumption 3.4), it is guaranteed that the sequence generated by the algorithm is Fejér monotone when the iteration counter is even. Moreover, the generated entire sequence converges weakly to the common fixed point (Theorem 3.5). Our results extend and improve the corresponding results of [29, Theorem 1] and [22, Lemma 3.3].
- By making use of the proposed algorithm and elaborated techniques (see [2, 15, 23, 34]), we then solve the convex feasibility and the (structured) monotone inclusion problems. We present the weakly convergent alternating projection, forward-backward and primal-dual splitting algorithms. Under suitable conditions on the iterative parameters, it is guaranteed that the sequences generated by the algorithm converges weakly to the solution.
- The proposed algorithm is applicable to solve optimization problems involving compositions with linear continuous operators. In particular, we consider an ℓ^1 optimization problem that minimizes the sum of three convex functions, which appears in the context of control theory [30]. We provide numerical experiments to confirm practical usefulness of our proposed algorithm.

For related inertial algorithms, we refer to [1, 10, 19, 25, 26, 28]. Note that different to the existing inertial algorithms, the assumptions on the inertial parameters of the proposed algorithm are mild and the sequence generated by the algorithm is Fejér monotone when the iteration counter is even.

The rest of this paper is organized as follows. Some preliminaries are presented in Section 2. Then we show the convergence of the proposed algorithm in Section 3.

Concrete problems and special cases of our algorithm are investigated in Sections 4, 5 and 6. The numerical experiments in Section 7 show the effectiveness of the proposed algorithm by comparing with other existing methods, and finally, Section 8 concludes the paper.

2. Preliminaries

The following notations will be used in this paper: $\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}_{++}$ denote the set of strictly positive real numbers; $\overline{\mathbb{R}}$ denotes the extended real line, i.e., $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$; $\mathbb{N} = \{1, 2, \dots\}$ denotes the set of positive integers; $\mathcal{H}$ denotes a real Hilbert space; for any $x, y \in \mathcal{H}$, $\langle x, y\rangle$ denotes the inner product of x and y; for any $z \in \mathcal{H}$, $\|z\|$ denotes the norm of z, i.e., $\|z\| = \sqrt{\langle z, z\rangle}$; for any $\{x_k\} \subset \mathcal{H}$ and $x \in \mathcal{H}$, $x_k \to x$ and $x_k \rightharpoonup x$ denote the strong and weak convergences of $\{x_k\}$ to x, respectively; I denotes the identity mapping on $\mathcal{H}$.

For an arbitrary set-valued operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$, $\mathrm{dom}(A)$ denotes the *domain* of A, that is, $\mathrm{dom}(A) = \{x \in \mathcal{H} : A(x) \neq \emptyset\}$; $\mathrm{ran}(A)$ denotes the *range* of A, i.e., $\mathrm{ran}(A) = \bigcup\{A(x) : x \in \mathrm{dom}(A)\}$, $\mathrm{gr}(A)$ denotes the *graph* of A, that is, $\mathrm{gr}(A) = \{(x, x^*) : x^* \in A(x)\}$. The set of zero points of A is denoted by $A^{-1}(0)$, i.e., $A^{-1}(0) = \{z \in \mathrm{dom}(A) : 0 \in A(z)\}$. For a function $f : \mathcal{H} \to \overline{R}$, $\mathrm{dom}(f)$ denotes the domain of f, i.e., $\mathrm{dom}(f) = \{x \in \mathcal{H} : f(x) < \infty\}$. $\Gamma(\mathcal{H})$ denotes the family of proper, convex and lower semicontinuous extended real-valued functions.

Let $x, y \in \mathcal{H}$ and let $\alpha \in \mathbb{R}$. Then, the following identity will be used in the paper:

$$\|(1-\alpha)x + \alpha y\|^2 = (1-\alpha)\|x\|^2 + \alpha\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{1}$$

([7, Corollary 2.15]). Let C be a nonempty subset of $\mathcal{H}$ and let $\{x_k\}$ be a sequence in $\mathcal{H}$. $\{x_k\}$ is said to be *Fejér monotone* with respect to C if

$$\|x_{k+1} - u\| \leq \|x_k - u\|$$

for any $u \in C$ and for all $k \in \mathbb{N}$.

A *fixed point* of a mapping $T \colon \mathcal{H} \to \mathcal{H}$ is a point $x \in \mathcal{H}$ satisfying $T(x) = x$. The set

$$\mathrm{Fix}(T) := \{x \in \mathcal{H} : T(x) = x\}$$

is called a *fixed point set* of T. $I - T$ is said to be *demiclosed at zero* if $p \in \mathrm{Fix}(T)$ whenever $\{x_k\}$ is a sequence in H such that $x_k \rightharpoonup p$ and $\lim_{k\to} \|x_k - T(x_k)\| = 0$. Let $\alpha \in (0, 1)$ and $\gamma > 0$. T is said to be

(i) *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ $(\forall x, y \in \mathcal{H})$;

(ii) *firmly nonexpansive* if
$$\|T(x) - T(y)\|^2 \leq \langle x - y, T(x) - T(y)\rangle \ (\forall x, y \in \mathcal{H});$$

(iii) *α-averaged* if there exists a nonexpansive mapping $R : \mathcal{H} \to \mathcal{H}$ such that $T = (1-\alpha)I + \alpha R$;

(iv) *strongly nonexpansive* if T is nonexpansive and $x_k - y_k - (T(x_k) - T(y_k)) \to 0$ $(k \to \infty)$, whenever $\{x_k\}$ and $\{y_k\}$ are sequences in $\mathcal{H}$ such that $\{x_k - y_k\}$ is bounded and $\|x_k - y_k\| - \|T(x_k) - T(y_k)\| \to 0$ $(k \to \infty)$;

(v) *γ-cocoercive* if $\langle x - y, T(x) - T(y)\rangle \geq \gamma\|T(x) - T(y)\|^2$ $(\forall x, y \in \mathcal{H})$.

For properties and insights into these mappings we refer to [13, 7, 33]. It is known that if T is nonexpansive, then $I - T$ is demiclosed at zero and the fixed point set of T is closed and convex [33, 7]. When T is α-averaged, the following inequality holds (see [7, Proposition 4.35]):

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{1-\alpha}{\alpha}\|(I - T)x - (I - T)y\|^2 \ (\forall x, y \in \mathcal{H}) \tag{2}$$

Let $\mathcal{H}$ and $\mathcal{G}$ be real Hilbert spaces and let $L : \mathcal{H} \to \mathcal{G}$ be a nonzero bounded linear operator with induced norm $\|L\| = \sup\{\|Lx\| : x \in \mathcal{H} \text{ with } \|x\| \leq 1\}$. The *adjoint operator* $L^* : \mathcal{G} \to \mathcal{H}$ of L is defined by $\langle Lx, y\rangle_{\mathcal{G}} = \langle x, L^* y\rangle_{\mathcal{H}}$ for all $x \in \mathcal{H}$ and all $y \in \mathcal{G}$.

A set-valued operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is said to be

(i) *monotone* if, for all $(x, x^*), (y, y^*) \in \mathrm{gr}(A)$ we have $\langle x - y, x^* - y^*\rangle \geq 0$;

(ii) *ν-strongly monotone* for some $\nu > 0$, if, for all (x, x^*), $(y, y^*) \in \mathrm{gr}(A)$,

$$\langle x - y, x^* - y^*\rangle \geq \nu\|x - y\|^2;$$

(iii) *maximally monotone* if A is monotone and $A = B$ whenever $B : \mathcal{H} \rightrightarrows \mathcal{H}$ is a monotone mapping such that $\mathrm{gr}(A) \subset \mathrm{gr}(B)$.

For set-valued operator $A : \mathcal{H} \rightrightarrows \mathcal{H}$, and for $\gamma \in \mathbb{R}_{++}$, the *resolvent* $J_{\gamma A} : \mathcal{H} \rightrightarrows \mathcal{H}$ of A is defined by $J_{\gamma A} = (I + \gamma A)^{-1}$. Moreover, if A is maximally monotone, then $J_{\gamma A}$ is single-valued and $\mathrm{dom}(J_A) = \mathcal{H}$. The resolvent of the inverse operator of maximally monotone operator A can be computed as follows [7, Proposition 23.20]:

$$I = J_{\gamma A} + \gamma J_{\gamma^{-1} A^{-1}} \circ \gamma^{-1} I.$$

For set-valued operators $A, B : \mathcal{H} \rightrightarrows \mathcal{H}$, the *parallel sum* is defined as

$$A \,\square\, B := (A^{-1} + B^{-1})^{-1}.$$

For a function $f \in \Gamma(\mathcal{H})$, the *subdifferential* $\partial f : \mathcal{H} \rightrightarrows \mathcal{H}$ of f at $x \in \mathcal{H}$ is defined by

$$\partial f(x) = \{x^* \in \mathcal{H} : f(y) \geq f(x) + \langle y - x, x^*\rangle \ (\forall y \in \mathcal{H})\}. \tag{3}$$

We know that the subdifferential ∂f is maximally monotone (see [33, Theorem 4.6.6], [7, Theorem 20.40]) and its resolvent is given by $J_{\gamma\partial f} = \mathrm{prox}_{\gamma f}$ (see [7]), where $\mathrm{prox}_{\gamma f}(x) = \mathrm{argmin}_{y\in\mathcal{H}}\{f(y) + \frac{1}{2\gamma}\|y - x\|^2\}$ denotes the *proximal mapping* of f.

We say that f is *ν-strongly convex* for some $\nu > 0$ if $f - \nu\|\cdot\|^2/2$ is convex. The *conjugate* of f is $f^* : \mathcal{H} \to f^*(p) = \sup\{\langle p, x\rangle - f(x) : x \in \mathcal{H}\}$ for all $p \in \mathcal{H}$.

Moreover, if $f \in \Gamma(\mathcal{H})$, then $f^* \in \Gamma(\mathcal{H})$, as well, and $(\partial f)^{-1} = \partial f^*$.

Finally, for $g \in \Gamma(\mathcal{H})$, the *infimal convolution* $f \,\square\, g : \mathcal{H} \to \overline{R}$ of f and g is defined by $f \,\square\, g(x) = \inf_{y\in\mathcal{H}}\{f(y) + g(x - y)\}$ for all $x \in \mathcal{H}$.

3. Fixed point algorithm and convergence results

We provide a weakly convergent fixed point algorithm together with convergence results. We consider the following iterative algorithm.

Algorithm 3.1.
$$\begin{cases} \overline{x}_k = \begin{cases} x_k & (\textit{if } k \textit{ is even}) \\ x_k + \gamma_k(x_k - x_{k-1}) & (\textit{if } k \textit{ is odd}) \end{cases} \\ x_{k+1} = T_k(\overline{x}_k), \end{cases}$$

where $k \geq 1$, $x_0, x_1 \in \mathcal{H}$, $\{\gamma_k\} \subset [0, \infty)$, and $\{T_k\}$ is a sequence of mappings of $\mathcal{H}$ into itself that satisfies $\cap_{k=1}^{\infty} Fix(T_k) \neq \emptyset$ and the following condition:

$$\begin{cases} \textit{if } \{x_{k_j}\} \subset \mathcal{H} \textit{ and } \{T_{k_j}\} \subset \{T_k\} \textit{ such that} \\ x_{k_j} \rightharpoonup x \in \mathcal{H} \textit{ and } x_{k_j} - T_{k_j}(x_{k_j}) \to 0, \textit{ then } x \in \cap_{k=1}^{\infty} Fix(T_k) \end{cases} \tag{4}$$

(see [23, p. 1564]*). Note that condition* (4) *can be considered as a generalization of the demiclosedness property for one mapping.*

Remark 3.2.

- In the case when $T_k = J_{\gamma A}$ for every $k \in \mathbb{N}$, where $A : \mathcal{H} \rightrightarrows \mathcal{H}$ is a maximally monotone operator and $\gamma > 0$, Algorithm 3.1 becomes the algorithm considered in [29]. Moreover, in the case when $T_k = T$ for every $k \in \mathbb{N}$, where $\alpha \in (0, 1)$ and $T : \mathcal{H} \to \mathcal{H}$ is an α-averaged mapping, Algorithm 3.1 becomes the algorithm considered in [22].
- Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximally monotone operator such that $A^{-1}(0) \neq \emptyset$, and let $\{\lambda_k\} \subset (0, \infty)$ such that $\inf_k \lambda_k > 0$. Then $\{J_{\lambda_k}\}$ is a sequence of mappings of firmly nonexpansive and satisfies (4) [23, Lemma 5.1].
- Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximally monotone operator, $B : \mathcal{H} \to \mathcal{H}$ be β-cocoercive, for $\beta > 0$. Suppose that $(A + B)^{-1}(0) \neq \emptyset$ and that $\lambda_k \in (0, 2\beta)$ and $\inf_k \lambda_k > 0$, and set $\alpha_k := 2\beta/(4\beta - \lambda_k)$ and
$$R_{\lambda_k} := J_{\lambda_k} \circ (I - \lambda_k B)$$
for every $k \in \mathbb{N}$. Then $\{R_{\lambda_k}\}$ is a sequence of mappings of α_k-averaged [7, Proposition 26.1(iv)(d)] and satisfies (4) [11, 18].

3.1. Convergence analysis

To establish convergence of the sequence generated by Algorithm 3.1, we consider the case when $\{T_k\}$ is a sequence of averaged mappings. In this case, T_k can be written as
$$T_k = (1 - \alpha_k)I + \alpha_k S_k, \tag{5}$$
where $\{\alpha_k\} \subset (0, 1)$ and $\{S_k\}$ is a sequence of nonexpansive mappings of $\mathcal{H}$ into itself.

Remark 3.3. From the definition of T_k we have
$$\cap_{k=1}^{\infty} \mathrm{Fix}(T_k) = \cap_{k=1}^{\infty} \mathrm{Fix}(S_k). \tag{6}$$

To establish the convergence, we need the following assumption:

Assumption 3.4. Assume that the following conditions:

(A1) $\{\alpha_k\} \subset (0, c]$ for some $c \in (0, 1)$.

(A2) $\limsup_k \gamma_k < (1 - c)/c$.

Theorem 3.5. *Assume that Assumption* 3.4 *holds for* $\{\alpha_k\}$ *and* $\{\gamma_k\}$, *and define* $C := \cap_{k=1}^{\infty} Fix(T_k)$. *Then the following hold:*

(i) $\{x_{2k}\}$ *generated by Algorithm* 3.1 *is Fejér monotone with respect to* C.

(ii) $\{x_k\}$ *converges weakly to a point in* C.

(iii) *Suppose that* $0 < d \leq \alpha_k$ *for some* $d > 0$, *and* $S_k = S$ *for every* $k \in \mathbb{N}$, *where* $S : \mathcal{H} \to \mathcal{H}$ *is nonexpansive. Then*

$$\lim_{k\to\infty} \|x_k - T_k(x_k)\| = \lim_{k\to\infty} \|x_k - S(x_k)\| = 0.$$

Proof. Let $u \in C$. From (A2), one can assume without loss of generality that $\gamma_k \leq b < (1-c)/c$ for some $b \in (0,1)$ for every $k \in \mathbb{N}$. We set $y_k := x_{2k}$ for every $k \in \mathbb{N}$. It follows from (1) and (2) that

$$\begin{aligned}
&\|y_{k+1} - u\|^2 \\
&= \|T_{2k+1}(T_{2k}(y_k) + \gamma_{2k+1}(T_{2k}(y_k) - y_k)) - u\|^2 \\
&\leq \|T_{2k}(y_k) + \gamma_{2k+1}(T_{2k}(y_k) - y_k) - u\|^2 \\
&\quad - \frac{1-\alpha_{2k}}{\alpha_{2k}} \|T_{2k}(y_k) + \gamma_{2k-1}(T_{2k}(y_k) - y_k) - y_{k+1}\|^2 \\
&\leq (1+\gamma_{2k+1})\|T_{2k}(y_k) - u\|^2 - \gamma_{2k+1}\|y_k - u\|^2 + \gamma_{2k+1}(1+\gamma_{2k+1})\|y_k - T_{2k}(y_k)\|^2 \\
&\leq (1+\gamma_{2k+1})\|y_k - u\|^2 - (1+\gamma_{2k+1})\frac{1-\alpha_{2k}}{\alpha_{2k}}\|y_k - T_{2k}(y_k)\| \\
&\quad - \gamma_{2k+1}\|y_k - u\|^2 + \gamma_{2k+1}(1+\gamma_{2k+1})\|y_k - T_{2k}(y_k)\|^2 \\
&= \|y_k - u\|^2 - (1+\gamma_{2k+1})\left(\frac{1-\alpha_{2k}}{\alpha_{2k}} - \gamma_{2k+1}\right)\|y_k - T_{2k}(y_k)\|^2. \qquad (7)
\end{aligned}$$

Using (A1), we obtain $(1-\alpha_k)/\alpha_k \geq (1-c)/c$ for all $k \in \mathbb{N}$. It follows from (A2) and (7) that $\{x_{2k}\}$ is Fejér monotone with respect to C and hence (i) holds.

Next, we consider (ii). From (7), we have

$$\begin{aligned}
((1-c)/c - b)\|y_k - T_{2k}(y_k)\|^2 &\leq ((1-c)/c - \gamma_{2k+1})\|y_k - T_{2k}(y_k)\|^2 \\
&\leq \|y_k - u\|^2 - \|y_{k+1} - u\|^2.
\end{aligned}$$

and hence
$$\|y_k - T_{2k}(y_k)\| \to 0 \ (k \to \infty) \qquad (8)$$

because $\{\|y_k - u\|\}$ converges. From (4) and (8), it is guaranteed that every weak sequential cluster point of $\{y_k\}$ belongs to C. According to [7, Theorem 5.5], $\{x_{2k}\}$ converges weakly to some u in C. On the other hand, it follows from the definition of $\{x_k\}$ and (8) that

$$\|x_{2k} - x_{2k+1}\| = \|x_{2k} - T_{2k}(x_{2k})\| \to 0 \ (k \to \infty). \qquad (9)$$

Hence, $\{x_{2+1}\}$ converges weakly to u, and so does the entire sequence $\{x_k\}$.

Finally, we consider (iii). It follows from (9) and $0 < d \leq \alpha_k$ that

$$d\|x_{2k} - S(x_{2k})\| \leq \alpha_{2k}\|x_{2k} - S(x_{2k})\| = \|x_{2k} - T_{2k}(x_{2k})\| \to 0 \ (k \to \infty). \qquad (10)$$

Moreover, we obtain from (9), (10) and the nonexpansiveness of S that

$$\begin{aligned}\|x_{2k+1} - S(x_{2k+1})\| &= \|x_{2k+1} - x_{2k} + x_{2k} - S(x_{2k}) + S(x_{2k}) - S(x_{2k+1})\| \\ &\leq 2\|x_{2k} - x_{2k+1}\| + \|x_{2k} - S(x_{2k})\| \to 0 \ (k \to \infty),\end{aligned}$$

and hence

$$\begin{aligned}d\|x_{2k+1} - T_{2k+1}(x_{2k+1})\| &\leq \alpha_{2k+1}\|x_{2k+1} - S(x_{2k+1})\| \\ &\leq c\|x_{2k+1} - S(x_{2k+1})\| \to 0 \ (k \to \infty).\end{aligned}$$

This completes the proof. □

Remark 3.6.

- In the Hilbert space setting, Theorem 3.5 lead us to Fejér monotonicity of $\{x_{2k}\}$ and weak convergence of $\{x_k\}$ to the common fixed point. We note that Theorem 3.5 is a generalization of Lemma 3.3 in [22].
- Note that different to other existing algorithms [1, 4, 8, 9, 17, 18, 19, 25, 26], the inertial parameters $\{\gamma_k\}$ can be chosen bigger than 1. For example, set $c := 1/3$. Then (A2) of Assumption 3.4 can be written as $\limsup_k \gamma_k < 2$.

4. Projection algorithm and convergence result

This section presents convergence results of projection algorithm, which generate sequences of iterates that converge weakly to solutions. To this end, we consider the convex feasibility problem.

Problem 4.1. *Let $\{C_i\}$ be a sequence of closed convex subsets of $\mathcal{H}$ such that $\cap_{i=1}^{\infty} C_i \neq \emptyset$. We consider the following convex feasibility problem*

$$\text{find } x \in \mathcal{H} \text{ such that } x \in \bigcap_{i=1}^{\infty} C_i. \tag{11}$$

Let us introduce some technical results which are needed in the following.

Proposition 4.2. [23, Proposition 4.1] *Let $\{R_k\}$ be a sequence of mappings of firmly nonexpasive on $\mathcal{H}$ such that $\cap_{k=1}^{\infty} Fix(R_k) \neq \emptyset$ and $\{\beta_k^j\}$ be a sequence of nonnegative real numbers with indices $j, k \in \mathbb{N}$ such that $j \leq k$. Consider the following conditions:*

(i) $\sum_{j=1}^{k} \beta_k^j = 1$ $(\forall k \in \mathbb{N})$

(ii) $\lim_{k\to\infty} \beta_k^j > 0$ $(\forall j \in \mathbb{N})$

(iii) $\sum_{k=1}^{\infty} \sum_{j=1}^{k} |\beta_{k+1}^j - \beta_k^j| < \infty$.

Set $S_k = \sum_{j=1}^{k} \beta_j^k R_k$ for every $k \in \mathbb{N}$. Then the following hold:

(1) *If $\{\beta_k^j\}$ satisfies* (i) *and* (ii)*, then $\{S_k\}$ is a sequence of strongly nonexpasive mappings and $\cap_{j=1}^{\infty} Fix(R_j) = \cap_{k=1}^{\infty} Fix(S_k)$.*

(2) *If $\{\beta_k^j\}$ satisfies (i) and (ii), and for every $j \in \mathbb{N}$, R_j is demiclosed, then $\{S_k\}$ satisfies* (4)*.*

(3) *If $\{\beta_k^j\}$ satisfies* (i), (ii) *and* (iii)*, then $\{S_k\}$ satisfies* (4)*.*

Remark 4.3. Conditions (i), (ii) and (iii) have been considered in [2, Section 4]. An example of $\{\beta_k^j\}$ satisfying these conditions is (see [2, p. 2358])

$$\beta_k^j = \begin{cases} 2^{-j} & (j < k) \\ 2^{1-j} & (j = k). \end{cases}$$

Lemma 4.4. [23, Lemma 5.2] *Let $\{S_k\}$ be a sequence of mappings on $\mathcal{H}$ and $\{\delta_k\}$ is a sequence in $[0,1]$ such that $0 < \liminf_k \delta_k$. Set*

$$T_k := (1-\delta_k)I + \delta_k S_k$$

for every $k \in \mathbb{N}$. Then the following hold:

(1) *If $\{S_k\}$ is a sequence of strongly nonexpasive mappings, then $\{T_k\}$ is a sequence of strongly nonexpasive mappings.*

(2) *If $\{S_k\}$ satisfies (4), then $\{T_k\}$ satisfies (4).*

4.1. Alternated inertial projection algorithm

We will formulate in this subsection weakly convergent alternated inertial projection algorithm. Let $\{C_i\}$ be a sequence of simple[1] closed convex subsets of $\mathcal{H}$ such that $\cap_{i=1}^m C_i \neq \emptyset$. The intersection of simple sets is not necessarily simple. Using techniques developed in Section 3, we show the following result.

Corollary 4.5. *Let $\{x_k\}$ be a sequence generated by*

$$\begin{cases} \overline{x}_k = \begin{cases} x_k & (\text{if } k \text{ is even}) \\ x_k + \gamma_k(x_k - x_{k-1}) & (\text{if } k \text{ is odd}) \end{cases} \\ x_{k+1} = \left((1-\delta_k)I + \delta_k \left(\left(\sum_{i=1}^k \lambda_i\right)^{-1} \sum_{j=1}^k \lambda_j P_{C_j}\right)\right)(\overline{x}_k) \end{cases} \tag{12}$$

where $k \geq 1$, $x_0, x_1 \in \mathcal{H}$, $\{\lambda_k\} \subset (0,\infty)$ such that $\sum_{k=1}^\infty \lambda_k < \infty$, $c \in (0,1)$, $\{\delta_k\} \subset [0,c]$ such that $0 < \liminf_k \delta_k$, $\{\gamma_k\} \subset [0,\infty)$ such that $\limsup_k \gamma_k < (1-c)/c$, and P_{C_j} is the metric projection onto C_j for every $j \in \mathbb{N}$. Then the following hold:

(i) *$\{x_{2k}\}$ is Fejér monotone with respect to $\cap_{i=1}^\infty C_i$.*

(ii) *$\{x_k\}$ converges weakly to a point in $\cap_{i=1}^\infty C_i$.*

Proof. Set $S_k = \sum_{j=1}^k \beta_k^j P_{C_j}$, where

$$\beta_k^j = \begin{cases} \frac{\lambda_j}{\sum_{j=1}^k \lambda_j} & (j \leq k); \\ 0 & (j > k). \end{cases} \tag{13}$$

Then (12) has the structure of Algorithm 3.1 and $\{\beta_k^j\}$ satisfies (i), (ii) and (iii) of Proposition 4.2 (see [23, Theorem 4.3]). According to Proposition 4.2, S_k is strongly nonexpansive with $\cap_{j=1}^\infty \mathrm{Fix}(S_k) = \cap_{j=1}^\infty C_j$, and satisfies (4) because P_{C_j} is firmly nonexpansive with $\mathrm{Fix}(P_{C_j}) = C_j$. We also obtain that $(1-\delta_k)I + \delta_k S_k$ satisfies (4) using Lemma 4.4.

According to Theorem 3.5, the sequence $\{x_{2k}\}$ is Fejér monotone with respect $\cap_{j=1}^\infty C_j$. Moreover, $\{x_k\}$ converges weakly to a point in $\cap_{j=1}^\infty C_j$. □

[1]A set in $\mathcal{H}$ is said to be *simple* if its associated projection mapping has a closed form expression.

Remark 4.6. For other related projection algorithms, we refer to [5, 6, 7, 13, 23]. We mention that inertial versions of the projection algorithm have been proposed in [17, Algorithm 2]. Note that different to [17], the sequence generated by our algorithm is Fejér monotone with respect to the set of solutions and the assumptions on the iterative parameters are mild.

5. Alternated inertial forward-backward algorithms and convergence results

This section presents convergence results of alternated inertial forward-backward algorithms, which generate sequences of iterates that converge weakly to solutions. To this end, we consider the following monotone inclusion problem.

Problem 5.1. *Let $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ be a maximally monotone operator and $B\colon \mathcal{H} \to \mathcal{H}$ a β-cocoercive mapping for some $\beta > 0$ such that $(A+B)^{-1}(0) \neq \emptyset$. The problem is to solve the inclusion*

$$\text{find } z \in \mathcal{H} \text{ such that } 0 \in (A+B)(z). \tag{14}$$

5.1. Alternated inertial forward-backward algorithm

Using techniques developed in Section 3, we show the following result.

Corollary 5.2. *Let $\{x_k\}$ be a sequence generated by*

$$\begin{cases} \overline{x}_k = \begin{cases} x_k & (\text{if } k \text{ is even}) \\ x_k + \gamma_k(x_k - x_{k-1}) & (\text{if } k \text{ is odd}) \end{cases} \\ x_{k+1} = J_{\delta_k A} \circ (I - \delta_k B)(\overline{x}_k) \end{cases} \tag{15}$$

where $k \geq 1$, $x_0, x_1 \in \mathcal{H}$, $\{\delta_k\} \subset [a, b]$ for some $a, b \in (0, 2\beta)$, $\alpha_k := (2\beta)/(4\beta - \delta_k)$, $c := (2\beta)/(4\beta - b)$, and $\{\gamma_k\} \subset [0, \infty)$ such that $\limsup_k \gamma_k < (1-c)/c$. Then the following hold:

(i) *$\{x_{2k}\}$ is Fejér monotone with respect to $(A+B)^{-1}(0)$.*

(ii) *$\{x_k\}$ converges weakly to a point in $(A+B)^{-1}(0)$.*

Proof. Set $R_k := J_{\delta_k A} \circ (I - \delta_k B)$ for every $k \in \mathbb{N}$. Then $\mathrm{Fix}(R_k) = (A+B)^{-1}(0)$ and R_k is α_k-averaged for every $k \in \mathbb{N}$ [7, Proposition 26.1 (iv)]. Moreover, $\{R_k\}$ satisfies (4) [11, Proposition 3.1].

According to Theorem 3.5, the sequence $\{x_{2k}\}$ is Fejér monotone with respect to $(A+B)^{-1}(0)$. Moreover, $\{x_k\}$ converges weakly to a point in $(A+B)^{-1}(0)$. □

Remark 5.3. In case $Bx = 0$ for all $x \in \mathcal{H}$ (15) becomes

$$\begin{cases} \overline{x}_k = \begin{cases} x_k & (\text{if } k \text{ is even}) \\ x_k + \gamma_k(x_k - x_{k-1}) & (\text{if } k \text{ is odd}) \end{cases} \\ x_{k+1} = J_{\delta_k A}(\overline{x}_k) \end{cases} \tag{16}$$

and is to the best of our knowledge new in the context of solving the monotone inclusion problem $0 \in Az$. Assuming that $\inf \delta_k > 0$, and $\limsup_k \gamma_k < 1$, the conclusion of Corollary 5.2 remains valid.

For other related algorithms, we refer to [1, 18].

6. Primal-dual splitting algorithms and convergence results

Based on the general algorithm in Section 3, we will formulate in this section weakly convergent primal-dual splitting algorithms. To this end, we first consider the following structured monotone inclusion problem.

Problem 6.1. *Let m be a strictly positive integer and let $I := \{1, 2, \ldots, m\}$. We consider the following primal inclusion problem [14, 34]*

$$\text{find } x \in \mathcal{H} \text{ such that } z \in Ax + \sum_{i=1}^{m} L_i^* \left((B_i \square D_i) (L_i x - r_i) \right) + Cx, \tag{17}$$

and its dual inclusion problem

$$\begin{aligned} &\text{find } v_1 \in \mathcal{G}_1, \ldots, v_m \in \mathcal{G}_m \text{ such that} \\ &(\exists\, x \in \mathcal{H}) \begin{cases} z - \sum_{i=1}^m L_i^* v_i \in Ax + Cx \\ v_i \in (B_i \,\square\, D_i)(L_i x - r_i), \;\; i = 1, 2, \ldots, m, \end{cases} \end{aligned} \tag{18}$$

where

- *$\mathcal{H}, \mathcal{G}_1, \ldots, \mathcal{G}_m$ are real Hilbert spaces.*
- *$z \in \mathcal{H}$ and $(r_1, \ldots, r_m) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$.*
- *$A : \mathcal{H} \rightrightarrows \mathcal{H}$ and $B_i : \mathcal{G}_i \rightrightarrows \mathcal{G}_i$ $(i \in I)$ are maximally monotone operators.*
- *$C : \mathcal{H} \to \mathcal{H}$ is cocoercive for some $\mu > 0$.*
- *$D_i : \mathcal{G}_i \rightrightarrows \mathcal{G}_i$ $(i \in I)$ is maximally monotone.*
- *$L_i : \mathcal{H} \to \mathcal{G}$ $(i \in I)$ is a nonzero bounded linear operator with adjoint L_i^*.*

We say that $(\overline{x}, \overline{v}_1, \ldots, \overline{v}_m) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ is a primal-dual solution to Problem 6.1 if

$$z - \sum_{j=1}^{m} L_j^* \overline{v}_j \in A\overline{x} + C\overline{x} \text{ and } \overline{v}_i \in (B_i \,\square\, D_i)(L_i \overline{x} - r_i) \; i = 1, 2, \ldots, m. \tag{19}$$

If $\overline{x}$ is a solution to (17), then there exists $(\overline{v}_1, \ldots, \overline{v}_m) \in \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ such that $(\overline{x}, \overline{v}_1, \ldots, \overline{v}_m)$ is a primal-dual solution to Problem 6.1, and if $(\overline{v}_1, \ldots, \overline{v}_m)$ is a solution to (18), then there exists $\overline{x} \in \mathcal{H}$ such that $(\overline{x}, \overline{v}_1, \ldots, \overline{v}_m)$ is a primal-dual solution to Problem 6.1. If $(\overline{x}, \overline{v}_1, \ldots, \overline{v}_m)$ is a primal-dual solution to Problem 6.1, then $\overline{x}$ is a solution to (17) and $(\overline{v}_1, \ldots, \overline{v}_m)$ is a solution to (18).

Example 6.2. Problem 6.1 is very useful in applications to many practical problems. Indeed, consider the convex optimization problems of the form

$$\min_{x \in \mathcal{H}} \left\{ f(x) + \sum_{i=1}^{m} g_i(L_i x) + h(x) \right\}, \tag{20}$$

where $f \in \Gamma(\mathcal{H})$, $h : \mathcal{H} \to \mathbb{R}$ is differentiable with Lipschitz continuous gradient, and for every $i \in I$, $g_i \in \Gamma(\mathcal{G}_i)$ and $L_i : \mathcal{H} \to \mathcal{G}_i$ is a bounded linear operator. Under mild assumptions (see [15, Proposition 4.3]) the equivalent monotone inclusion problem takes the form

$$\text{find } u \in \mathcal{H} \text{ such that } 0 \in \partial f(x) + \sum_{i=1}^{m} L_i^* \partial g_i(L_i x) + \nabla h(x), \tag{21}$$

which is a special instance of (17).

In order to show the results, we need the following notation. We consider the Hilbert space $\boldsymbol{\mathcal{G}} := \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ endowed with the inner product and associated norm defined for $\boldsymbol{u} = (u_1, \dots, u_m), \boldsymbol{v} = (v_1, \dots, v_m) \in \boldsymbol{\mathcal{G}}$ as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{G}}} := \sum_{i=1}^{m} \langle u_i, v_i \rangle_{\mathcal{G}_i} \text{ and } \|\boldsymbol{u}\|_{\boldsymbol{\mathcal{G}}} := \sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle_{\boldsymbol{\mathcal{G}}}},$$

respectively. Furthermore, we let $\boldsymbol{\mathcal{K}} = \mathcal{H} \times \boldsymbol{\mathcal{G}}$ be the Hilbert space endowed with inner product and associated norm defined for every $(x, \boldsymbol{u}), (y, \boldsymbol{v}) \in \boldsymbol{\mathcal{K}}$ as

$$\langle (x, \boldsymbol{u}), (y, \boldsymbol{v}) \rangle_{\boldsymbol{\mathcal{K}}} := \langle x, y \rangle_{\mathcal{H}} + \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{\mathcal{G}}} \text{ and } \|(x, \boldsymbol{u})\|_{\boldsymbol{\mathcal{K}}} := \sqrt{\langle (x, \boldsymbol{u}), (x, \boldsymbol{u}) \rangle_{\boldsymbol{\mathcal{K}}}}, \tag{22}$$

respectively. Furthermore, we consider the set-valued operator

$$\boldsymbol{M} : \boldsymbol{\mathcal{K}} \rightrightarrows \boldsymbol{\mathcal{K}} : (x, v_1, \dots, v_m) \mapsto (-z + Ax, r_1 + B_1^{-1}(v_1), \dots, r_m + B_m^{-1}(v_m)), \tag{23}$$

We next consider the linear continuous operator of

$$\boldsymbol{S} : \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} : (x, v_1, \dots, v_m) \mapsto \left(\sum_{i=1}^{m} L_i^* v_i, -L_1 x, \dots, -L_m x \right). \tag{24}$$

We consider the single-valued operator

$$\boldsymbol{Q} : \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} : (x, v_1, \dots, v_m) \mapsto \left(Cx, D_1^{-1} v_1, \dots, D_m^{-1} v_m \right). \tag{25}$$

According to [34, page 672], $\boldsymbol{Q}$ is β-cocoercive with

$$\beta = \min\{\mu, \nu_1, \dots, \nu_m\}.$$

We next introduce the bounded linear operator as

$$\begin{aligned} &\boldsymbol{V} : \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \\ &\qquad (x, v_1, ..., v_m) \mapsto \left(\frac{1}{\tau} x - \sum_{i=1}^{m} L_i^* v_i, -L_1 x + \frac{1}{\sigma_1} v_1, ..., -L_m x + \frac{1}{\sigma_m} v_m \right). \end{aligned} \tag{26}$$

Then $\boldsymbol{V}$ is self-adjoint, with ρ-strongly positive for

$$\rho := \min\left\{ \tau^{-1}, \sigma_1^{-1}, \dots, \sigma_m^{-1} \right\} \left(1 - \sqrt{\tau \sum_{i=1}^{m} \sigma_i \|L_i\|^2} \right) > 0$$

when (28) holds (see [34]).

6.1. Alternated inertial primal-dual splitting algorithm

This subsection presents the following algorithm for solving Problem 6.1 in the case when for every $i \in I$,

$$D_i : \mathcal{G}_i \rightrightarrows \mathcal{G}_i \ (i \in I) \text{ is } \nu_i\text{-strongly monotone for some } \nu_i \in (0,\infty). \tag{27}$$

Algorithm 6.3. (Alternated inertial primal-dual splitting algorithm)

$$\begin{cases} \overline{x}_k = \begin{cases} x_k & (\text{if } k \text{ is even}) \\ x_k + \gamma_k(x_k - x_{k-1}) & (\text{if } k \text{ is odd}) \end{cases} \\ \overline{v}_{i,k} = \begin{cases} v_{i,k} & (\text{if } k \text{ is even}) \\ v_{i,k} + \gamma_k(v_{i,k} - v_{i,k-1}) & (\text{if } k \text{ is odd}) \end{cases} \quad (\forall i \in I) \\ x_{k+1} = J_{\tau A}\left(\overline{x}_k - \tau\left(\sum_{i=1}^m L_i^* \overline{v}_{i,k} + C\overline{x}_k - z\right)\right) \\ v_{i,k+1} = J_{\sigma_i B_i^{-1}}\left(\overline{v}_{i,k} + \sigma_i\left(L_i(2x_{k+1} - \overline{x}_k) - D_i^{-1}\overline{v}_{i,k} - r_i\right)\right) \ (\forall i \in I) \end{cases}$$

where $(x_0, v_{1,0}, \ldots, v_{m,0}) \in \mathcal{H} \times \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$, $\{\gamma_k\} \subset (0,1)$, *and* $\tau, \sigma_1, \ldots, \sigma_m > 0$ *such that*

$$2 \cdot \min\{\tau^{-1}, \sigma_1^{-1}, \ldots, \sigma_m^{-1}\} \cdot \min\{\mu, \nu_1, \ldots, \nu_m\} \cdot \left(1 - \sqrt{\tau \sum_{i=1}^m \sigma_i \|L_i\|^2}\right) > 1. \tag{28}$$

Remark 6.4. In case $\gamma_k = 0$ for every $k \in \mathbb{N}$, Algorithm 6.3 can be written in the following way:

$$\begin{cases} x_{k+1} = J_{\tau A}\left(x_k - \tau\left(\sum_{i=1}^m L_i^* v_{i,k} + Cx_k - z\right)\right), \\ v_{1,k+1} = J_{\sigma_i B_i^{-1}}(v_{i,k} + \sigma_i L_i(2x_{k+1} - x_k) - D_i^{-1}v_{i,k} - r_i) \ \ (i \in I). \end{cases} \tag{29}$$

Algorithm (29) is the error-free case of the primal-dual splitting algorithm and its fundamental convergence properties have been investigated in [34, Theorem 3.1]. The case $m = 1$ has been addressed in [16]. Moreover, the case $C \equiv 0$, Algorithm (29) exactly reverts to the primal-dual algorithms of Chambolle and Pock [14].

By making use of primal-dual techniques (see [15, 34]) and Theorem 3.5, we derive weak convergence of Algorithm 6.3.

Theorem 6.5. *In Problem* 6.1, *suppose that the operators* $\{D_i\}_{i\in I}$ *are as in* (27), *and*

$$z \in ran\left(A + \sum_{i=1}^m L_i^*\left((B_i \Box D_i)(L_i \cdot - r_i)\right) + C\right). \tag{30}$$

Let $\{(x_k, v_{1,k}, \ldots, v_{m,k})\}$ *be a sequence generated by Algorithm* 6.3. *Set*

$$\beta := \min\{\mu, \nu_1, \ldots, \nu_m\}, \quad \textit{and}$$

$$\rho := \min\left\{\tau^{-1}, \sigma_1^{-1}, \ldots, \sigma_m^{-1}\right\}\left(1 - \sqrt{\tau \sum_{i=1}^m \sigma_i \|L_i\|^2}\right),$$

and $c := 2\beta\rho/(4\beta\rho - 1)$ *and suppose that* (A2) *of Assumption* 3.4 *holds for* $\{\gamma_k\}$.

Then the following hold:

(1) $\{(x_{2k}, v_{1,2k}, \ldots, v_{m,2k})\}$ *is Fejér monotone with respect to the set of primal-dual solutions of Problem* 6.1.

(2) *There exists a primal-dual solution* $\overline{\boldsymbol{v}} = (\overline{x}, \overline{v}_1, \ldots, \overline{v}_m)$ *to Problem* 6.1 *such that* $\{(x_k, v_{1,k}, \ldots, v_{m,k})\}$ *converges weakly to* $\overline{\boldsymbol{v}}$.

(3) *Suppose that C is uniformly monotone*[2]*. Then* $\{x_k\}$ *converges strongly to* $\overline{x}$.

(4) *Suppose that* D_i^{-1} *is uniformly monotone for some* $i \in I$*. Then* $\{v_{i,k}\}$ *converges strongly to* $\overline{v}_i$.

Proof. By using an argument similar to that in [34, Theorem 3.1], Algorithm 6.3 becomes

$$\boldsymbol{v}_{k+1} = J_{\boldsymbol{A}} \circ (\boldsymbol{I} - \boldsymbol{B})(\overline{\boldsymbol{v}}_k), \tag{31}$$

where $\overline{\boldsymbol{v}}_k := (\overline{x}_k, \overline{v}_{1,k}, \ldots, \overline{v}_{m,k})$, $\boldsymbol{A} := \boldsymbol{V}^{-1}(\boldsymbol{M}+\boldsymbol{S})$ and $\boldsymbol{B} := \boldsymbol{V}^{-1}\boldsymbol{Q}$. Consequently, it has the structure of (15) when $T_k := J_{\boldsymbol{A}} \circ (\boldsymbol{I} - \boldsymbol{B})$ for every $k \in \mathbb{N}$. Hence, it is sufficient to check the convergence conditions of (15) to show our claim.

Consider the Hilbert space $\boldsymbol{\mathcal{K}_V}$ endowed with inner product and norm defined for $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\mathcal{K}}$ as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_{\boldsymbol{\mathcal{K}_V}} := \langle \boldsymbol{x}, \boldsymbol{V}\boldsymbol{y} \rangle_{\boldsymbol{K}} \text{ and } \|\boldsymbol{x}\|_{\boldsymbol{\mathcal{K}_V}} := \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle_{\boldsymbol{\mathcal{K}_V}}} \tag{32}$$

respectively. Because the set-valued operator $\boldsymbol{M} + \boldsymbol{S}$ and $\boldsymbol{Q}$ are maximally monotone on $\boldsymbol{\mathcal{K}}$, the operators $\boldsymbol{A}$ and $\boldsymbol{B}$ are maximally monotone on $\boldsymbol{\mathcal{K}_V}$ (see [34]). Furthermore, $\boldsymbol{B}$ is $\beta\rho$-cocoercive on $\boldsymbol{\mathcal{K}_V}$. In addition, we have

$$(\boldsymbol{A} + \boldsymbol{B})^{-1}(\boldsymbol{0}) = (\boldsymbol{M} + \boldsymbol{S} + \boldsymbol{Q})^{-1}(\boldsymbol{0}).$$

From (28), we have $2\beta\rho > 1$. Set $\alpha := c$. Then it follows from [7, Proposition 26.1] that $T_k = J_{\boldsymbol{A}} \circ (\boldsymbol{I} - \boldsymbol{B})$ is α-averaged and

$$\cap_{k=1}^{\infty} \mathrm{Fix}(T_k) = \mathrm{Fix}(J_{\boldsymbol{A}} \circ (\boldsymbol{I} - \boldsymbol{B})) = (\boldsymbol{A} + \boldsymbol{B})^{-1}(\boldsymbol{0}) = (\boldsymbol{M} + \boldsymbol{S} + \boldsymbol{Q})^{-1}(\boldsymbol{0}).$$

Moreover, because $\boldsymbol{V}$ is self-adjoint and ρ-strongly positive, weak and strong convergence in $\boldsymbol{\mathcal{K}_V}$ are equivalent with weak and strong convergence in $\boldsymbol{\mathcal{K}}$, respectively.

We consider (1) and (2). According to Corollary 5.2, the sequence $\{\boldsymbol{v}_{2k}\}$ is Fejér monotone with respect to $\cap_{k=1}^{\infty} \mathrm{Fix}(T_k) = (\boldsymbol{M} + \boldsymbol{S} + \boldsymbol{Q})^{-1}(\boldsymbol{0})$. Moreover, $\{\boldsymbol{v}_k\}$ converges weakly to $\overline{\boldsymbol{v}}$ in $\boldsymbol{\mathcal{K}_V}$ and, consequently, in $\boldsymbol{\mathcal{K}}$ to $\overline{\boldsymbol{v}} \in (\boldsymbol{M} + \boldsymbol{S} + \boldsymbol{Q})^{-1}(\boldsymbol{0})$. Therefore, the claim follows from Corollary 5.2.

We consider (3) and (4). We prove the statement in case C is uniformly monotone, the situation when D_i^{-1} $(i \in I)$ fulfills this condition being similar. By taking into consideration (iii) of Theorem 3.5 and (31), we have

$$\boldsymbol{v}_k - J_{\boldsymbol{A}} \circ (\boldsymbol{I} - \boldsymbol{B})(\boldsymbol{v}_k) \to 0 \ (k \to \infty). \tag{33}$$

It follows from (33) that

$$\boldsymbol{B}(\boldsymbol{v}_k) - \boldsymbol{B}(\overline{\boldsymbol{v}}) \to 0 \ (k \to \infty). \tag{34}$$

[2] $B : \mathcal{H} \rightrightarrows \mathcal{H}$ is *uniformly monotone* if there exists an increasing function $\phi_B \colon [0, \infty) \to [0, \infty]$ that vanishes only at 0, and $\langle x - y, u - v \rangle \geq \phi_B(\|x - y\|)$ for all $(x, u), (y, v) \in \mathrm{gr}(B)$.

The proof of (34) follows the lines of the proof of [7, Theorem 26.14 (ii)], thus is omitted. On the other hand, (34) and the strongly positivity of $\boldsymbol{V}$ yield

$$\boldsymbol{V}^{-1}(\boldsymbol{Q}(\boldsymbol{v}_k) - \boldsymbol{Q}(\overline{\boldsymbol{v}})) = \boldsymbol{B}(\boldsymbol{v}_k) - \boldsymbol{B}(\overline{\boldsymbol{v}}) \to 0 \quad (k \to \infty),$$

which implies that $\boldsymbol{Q}(\boldsymbol{v}_k) - \boldsymbol{Q}(\overline{\boldsymbol{v}}) \to 0$ $(k \to \infty)$. Hence,

$$Cx_k \to C\overline{x} \text{ and } D_i^{-1} v_{i,k} \to D_i^{-1}\overline{v}_i \ (\forall i \in I). \tag{35}$$

There exists an increasing function $\phi_C \colon [0,\infty) \to [0,\infty]$ that vanishes only at 0 such that

$$\phi_C(\|x_k - \overline{x}\|) \leq \langle x_k - \overline{x}, Cx_k - C\overline{x}\rangle \leq \|x_k - \overline{x}\|\|Cx_k - C\overline{x}\|. \tag{36}$$

Because $\{x_k - \overline{x}\}$ is bounded, it follows from (36) that $x_k \to \overline{x}$ $(k \to \infty)$. □

Remark 6.6. In the case when

$$C \equiv 0 \text{ and, for every } i \in \mathbb{N} \ D_i(v) = \begin{cases} \mathcal{G}_i & (v = 0), \\ \emptyset & (v \neq 0), \end{cases}$$

(6.3) becomes

$$\begin{cases} \overline{x}_k = \begin{cases} x_k & \text{(if } k \text{ is even)} \\ x_k + \gamma_k(x_k - x_{k-1}) & \text{(if } k \text{ is odd)} \end{cases} \\ \overline{v}_{i,k} = \begin{cases} v_{i,k} & \text{(if } k \text{ is even)} \\ v_{i,k} + \gamma_k(v_{i,k} - v_{i,k-1}) & \text{(if } k \text{ is odd)} \end{cases} \quad (\forall i \in I) \\ x_{k+1} = J_{\tau A}\left(\overline{x}_k - \tau\left(\sum_{i=1}^m L_i^* \overline{v}_{i,k} - z\right)\right) \\ v_{i,k+1} = J_{\sigma_i B_i^{-1}}\left(\overline{v}_{i,k} + \sigma_i\left(L_i(2x_{k+1} - \overline{x}_k) - r_i\right)\right) \ (\forall i \in I) \end{cases}$$

and is to the best of our knowledge new. In this case, the conclusion of Theorem 6.5 remains valid with conditions (28) and (A2) of Assumption 3.4 replaced by

$$\tau \sum_{i=1}^m \sigma_i \|L_i\|^2 < 1 \tag{37}$$

(see [7, Remark 3.3]) and $\limsup_k \gamma_k < 1$, respectively.

6.2. Convex minimization problems

In this section, we provide a concrete problem that reduce to Problem 6.1. We introduce to the following optimization problem [15, Problem 4.1].

Problem 6.7. *Let $f \in \Gamma(\mathcal{H})$ and $h : \mathcal{H} \to \mathbb{R}$ be a convex and differentiable function with a μ^{-1}-Lipschitz continuous gradient, for some $\mu > 0$. For every $i \in I$, let $\mathcal{G}_i$ be a real Hilbert space, let $r_i \in \mathcal{G}_i$, let $g_i, l_i \in \Gamma(\mathcal{G}_i)$ such that l_i is ν_i-strongly convex, for some $\nu_i \geq 0$ and $L_i : \mathcal{H} \to \mathcal{G}_i$ a nonzero bounded linear operator. Consider the primal problem*

$$\min_{x \in \mathcal{H}} \left\{ f(x) + \sum_{i=1}^m (g_i \,\square\, l_i)(L_i x - r_i) + h(x) - \langle x, z\rangle_{\mathcal{H}} \right\} \tag{38}$$

and the dual problem

$$\min_{v_1\in\mathcal{G}_1,\dots,v_m\in\mathcal{G}_m} \left\{ (f^* \,\square\, h^*)\left(z-\sum_{i=1}^m L_i^* v_i\right) + \sum_{i=1}^m (g_i^*(v_i)+l_i^*(v_i)+\langle v_i, r_i\rangle_{\mathcal{G}_i})\right\}. \quad (39)$$

Remark 6.8. In Problem 6.7, if $z=0$, and each l_i and r_i are the indicator function of $\{0\}$ and $r_i=0$, respectively, then (38) reduces to (20).

We apply the proposed algorithm to Problem 6.7.

Corollary 6.9. *In Problem* 6.7*, suppose that*

$$z \in ran\Bigg(\partial f + \sum_{i=1}^m L_i^*\left((\partial g_i \,\square\, \partial l_i)\left(L_i\cdot - r_i\right)\right) + \nabla h\Bigg). \quad (40)$$

Let $\{(x_k, v_{1,k},\dots,v_{m,k})\}$ *be a sequence generated by*

$$\begin{cases} \overline{x}_k = \begin{cases} x_k & (\textit{if } k \textit{ is even}) \\ x_k+\gamma_k(x_k-x_{k-1}) & (\textit{if } k \textit{ is odd}) \end{cases} \\ \overline{v}_{i,k} = \begin{cases} v_{i,k} & (\textit{if } k \textit{ is even}) \\ v_{i,k}+\gamma_k(v_{i,k}-v_{i,k-1}) & (\textit{if } k \textit{ is odd}) \end{cases} \quad (\forall i\in I) \\ x_{k+1} = \mathrm{prox}_{\tau f}\left(\overline{x}_k - \tau\left(\sum_{i=1}^m L_i^*\overline{v}_{i,k} + \nabla h(\overline{x}_k) - z\right)\right) \\ v_{i,k+1} = \mathrm{prox}_{\sigma_i g_i^*}\left(\overline{v}_{i,k} + \sigma_i\left(L_i(2x_{k+1}-\overline{x}_k) - l_i^*(\overline{v}_{i,k}) - r_i\right)\right) \ (\forall i\in I), \end{cases}$$

where $(x_0, v_{1,0},\dots,v_{m,0}) \in \mathcal{H}\times\mathcal{G}_1\times\cdots\times\mathcal{G}_m$*, and* $\tau,\sigma_1,\dots,\sigma_m>0$ *such that* (28) *holds. Let* β*,* ρ *and* c *be defined as in Theorem* 6.5 *and suppose that* (A2) *of Assumption* 3.4 *holds for* $\{\gamma_k\}$*. In the case when* $A:=\partial f, C:=\nabla h$*, and for every* $i\in\mathbb{N}$*,* $B_i=\partial g_i$ *and* $D_i:=\partial l_i$*, the following hold:*

(1) $\{(x_{2k}, v_{1,2k},\dots,v_{m,2k})\}$ *is Fejér monotone with respect to the set of primal-dual solutions of Problem* 6.1*.*

(2) *There exists* $\overline{\boldsymbol{v}}=(\overline{x},\overline{v}_1,\dots,\overline{v}_m)\in\boldsymbol{\mathcal{K}}$ *such that* $\{(x_k,v_{1,k},\dots,v_{m,k})\}$ *converges weakly to* $\overline{\boldsymbol{v}}$ *and* $\overline{x}$ *is an optimal solution of the problem (38) and* $(\overline{v}_1,\dots,\overline{v}_m)$ *is an optimal solution of* (39)*.*

(3) *Suppose that* h *is uniformly convex*[3]*. Then* $\{x_k\}$ *converges strongly* $\overline{x}$*.*

(4) *Suppose that* l_i^* *is uniformly convex for some* $i\in I$*. Then* $\{v_{i,k}\}$ *converges strongly to* $\overline{v}_i$*.*

Proof. It follows from [7, Theorem 20.40] that the operators A and B_i $(i\in I)$ are maximally monotone. Moreover, the Baillon-Haddad Theorem (see [7, Corollary

[3] $f\colon\mathcal{H}\to\overline{\mathbb{R}}$ is said to be *uniformly convex*, if there exists an increasing function $\phi\colon[0,\infty)\to(0,\infty]$ which vanishes only at 0 and such that for all $x,y\in\mathrm{dom} f$ and for all $t\in(0,1)$:

$$f((1-t)x+ty)+t(1-t)\phi(\|x-y\|)\le(1-t)f(x)+tf(y).$$

18.16]) ensures that C is μ-cocoercive. Because l_i is ν_i-strongly convex, D_i is ν_i-strongly monotone, for every $i \in I$. On the other hand, for every $i \in I$, it follows from the ν_i-strong convexity of l_i and [7, Corollary 13.38 and Theorem 18.15] that l_i^* is Fréchet differentiable on $\mathcal{G}_i$ with a $1/\nu_i$-Lipschitz continuous gradient, and from [7, Corollary 16.30] that $D_i^{-1} = \nabla l_i^*$. The strong convexity of the functions l_i guarantees that $g_i \,\square\, l_i \in \Gamma(\mathcal{G}_i)$ (see [7, Corollary 11.17, Proposition 12.14]) and $\partial(g_i \,\square\, l_i) = \partial g_i \,\square\, \partial l_i$ $(i \in I)$ (see [7, Proposition 15.7]). According to Theorem 6.5 (1) that the sequence $\{(x_{2k}, v_{1,2k}, \ldots, v_{m,2k})\}$ is Fejér monotone with respect to the set of primal-dual solutions of Problem 6.1. This proves (1).

By Theorem 6.5 (2), $\{(x_{2k}, v_{1,2k}, \ldots, v_{m,2k})\}$ converges weakly to a point in the set of primal-dual solutions of Problem 6.1. In particular, $\{x_k\}$ converges weakly to some $\overline{x} \in \mathcal{H}$ such that

$$z \in \partial f(\overline{x}) + \sum_{i=1}^{m} L_i^*((\partial g_i \,\square\, \partial l_i))(L_i \overline{x} - r_i) + \nabla h(\overline{x}),$$

and the sequence $\{(v_{1,k}, \ldots, v_{m,k})\}$ converges weakly to some $(\overline{v}_1, \ldots, \overline{v}_m)$ such that

$$(\exists x \in \mathcal{H}) \begin{cases} z - \sum_{i=1}^{m} L_i^* \overline{v}_i \in Ax + Cx \\ \overline{v}_i \in (B_i \,\square\, D_i)(L_i x - r_i), \ i = 1, 2, \ldots, m. \end{cases}$$

Then $\overline{x}$ is a solution of the problem (38) and $(\overline{v}_1, \ldots, \overline{v}_m)$ is a solution of the problem (39) (see [15, Theorem 4.2]). This proves (2).

(3) follows from Theorem 6.5 (3) because ∇h is uniformly monotone when h is uniformly convex. Similarly, (4) follows form Theorem 6.5 (4). $\square$

Remark 6.10. In Problem 6.7, if $z = 0$, $h \equiv 0$, and each l_i and r_i are the indicator function of $\{0\}$ and $r_i = 0$, respectively, problems (38) and (39) become

$$\min_{x \in \mathcal{H}} \left\{ f(x) + \sum_{i=1}^{m} g_i(L_i x - r_i) \right\}$$

and

$$\min_{v_1 \in \mathcal{G}_1, \ldots, v_m \in \mathcal{G}_m} \left\{ f^*\left(z - \sum_{i=1}^{m} L_i^* v_i \right) + \sum_{i=1}^{m} g_i^*(v_i) \right\}.$$

In this case, the conclusion of Corollary 6.9 remains valid with conditions (28) and (A2) of Assumption 3.4 replaced by (37) and $\limsup_k \gamma_k < 1$, respectively.

7. Numerical experiments

In this section, we provide numerical experiments and compare our proposed algorithms with some existing algorithms in the literature. All codes were written in MATLAB R2022b and performed on a PC laptop Intel(R) Core(TM) i7-8650U CPU @ 1.90GHz 2.11 GHz, RAM 16.00 GB.

We consider the following optimization problem:

Problem 7.1. (ℓ^1 optimization problem)

$$\min_{\boldsymbol{u}\in\mathbb{R}^n} \{\|\boldsymbol{u}\|_1 + i_{\mathcal{C}_1}(\boldsymbol{u}) + i_{\mathcal{C}_2}(\Phi\boldsymbol{u})\}, \tag{41}$$

where $\Phi \in \mathbb{R}^{d\times n}$, $\boldsymbol{\zeta} \in \mathbb{R}^d$, $\mathcal{C}_1 = \{\boldsymbol{v} \in \mathbb{R}^n : \|\boldsymbol{v}\|_\infty \le 1\}$ and $\mathcal{C}_2 = \{\boldsymbol{\zeta}\}$; here $\|\cdot\|_1$ is the ℓ^1 norm and $\|\cdot\|_\infty$ is the ℓ^∞ norm, respectively.

Remark 7.2. Problem 7.1 was originally introduced in [30] for solving optimal control problems in the context of control theory. We consider applying the proposed algorithm to solve (41).

In the experiment, we set $d = 2$ and $n = 1,000$, and used Φ and $\boldsymbol{\zeta}$ given in [30, Chapter 9]. In this case, (41) can be stated in the framework of Problem 6.7 by taking $f = \|\cdot\|_1$, $m = 2$, $g_1 = i_{\mathcal{C}_1}$, $g_2 = i_{\mathcal{C}_2}$, $L_1 = I$, $L_2 = \Phi$, $l_1 = l_2 = i_{\{\mathbf{0}\}}$, $r_1 = r_2 = 0$, $z = 0$ and $h = 0$.

The specific implementation of Algorithm 6.3 for Problem 7.1 reads

$$\begin{cases} \overline{x}_k = \begin{cases} x_k & \text{(if } k \text{ is even)} \\ x_k + \gamma_k(x_k - x_{k-1}) & \text{(if } k \text{ is odd)} \end{cases} \\ \overline{v}_{i,k} = \begin{cases} v_{i,k} & \text{(if } k \text{ is even)} \\ v_{i,k} + \gamma_k(v_{i,k} - v_{i,k-1}) & \text{(if } k \text{ is odd)} \end{cases} \quad (\forall i \in \{1,2\}) \\ x_{k+1} = \operatorname{prox}_{\tau f}\left(\overline{x}_k - \tau\left(\sum_{i=1}^{2} L_i^*\overline{v}_{i,k}\right)\right) \\ v_{i,k+1} = \sigma_i(I - P_{\mathcal{C}_i})\left((1/\sigma_i)\overline{v}_{i,k} + (L_i(2x_{k+1} - \overline{x}_k))\right) \ (\forall i \in \{1,2\}), \end{cases} \tag{42}$$

where $\operatorname{prox}_{\tau f}$ is the soft thresholder on $\{0\}$ (see [7, Example 24.20]) and $P_{\mathcal{C}_i}$ is given by

$$P_{\mathcal{C}_1}(\boldsymbol{u}) = \begin{pmatrix} \operatorname{sgn}(u_1)\min\{|u_1|, 1\} \\ \operatorname{sgn}(u_2)\min\{|u_2|, 1\} \\ \vdots \\ \operatorname{sgn}(u_n)\min\{|u_n|, 1\} \end{pmatrix} \quad \text{and} \quad P_{\mathcal{C}_2}(\boldsymbol{v}) = \boldsymbol{\zeta},$$

(see [30]). Thus, the implementation of (42) is easy.

We give numerical comparisons of (42) and other algorithms, namely the algorithms of the primal-dual forward-backward-type (PD) from [16, 34] and the inertial primal-dual forward-backward-type (IPD) from [18, 10, 19]. Furthermore, we discuss the result for the case in which (42) corresponds to different choices of $\{\gamma_k\}$. We initialize all algorithms at the origin. As choices for τ, σ_1 and σ_2 occuring in algorithms, we let $\tau = 1.5$, $\sigma_1 = 0.3$ and $\sigma_2 = 0.5$, which satisfy (37). We chose 10 random initial points $(x_0^{(i)}, v_{1,0}^{(i)}, v_{2,0}^{(i)}) = (x_1^{(i)}, v_{1,1}^{(i)}, v_{2,1}^{(i)}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ $(i = 1, 2, \ldots, 10)$ and every entry of the initial point is uniformly generated from $[-100, 100]$.

The computational results are reported in Figure 7.1. In particular, we demonstrate there the distance $D_k := (1/10)\left(\sum_{i=1}^{10} \|(x_k^{(i)},\ v_{1,k}^{(i)},\ v_{2,k}^{(i)}) - (x^*, v_1^*, v_2^*)\|_{\boldsymbol{\mathcal{K}_V}}\right)$, where

(x^*, v_1^*, v_2^*) denotes the approximate solution and $\|\cdot\|_{\boldsymbol{\mathcal{K}_V}}$ is defined in (32). Note that (x^*, v_1^*, v_2^*) was obtained using MATLAB CVX package (http://cvxr.com/cvx/).

The computation results of (42), PD and IPD are presented in the left part of Figure 7.1. The inertial parameter for IPD is chosen as $\gamma_k \equiv 0.2$ according to the theoretic results provided in [10, 18, 19]. We also consider (42) with $\gamma_k \equiv 0.9$. One can observe that the proposed algorithm is faster than both the PD and the IPD.

The right part of Figure 7.1 shows the computational results of (42). We chose the associated parameters as $\gamma_k \equiv 0.1$, $\gamma_k \equiv 0.5$ and $\gamma_k \equiv 0.9$, respectively. As we can see, the increase of γ_k implies a faster approach of the solution. We see also from Figure 7.1 that both sequences generated by our algorithm are convergent, which conforms with our theory.

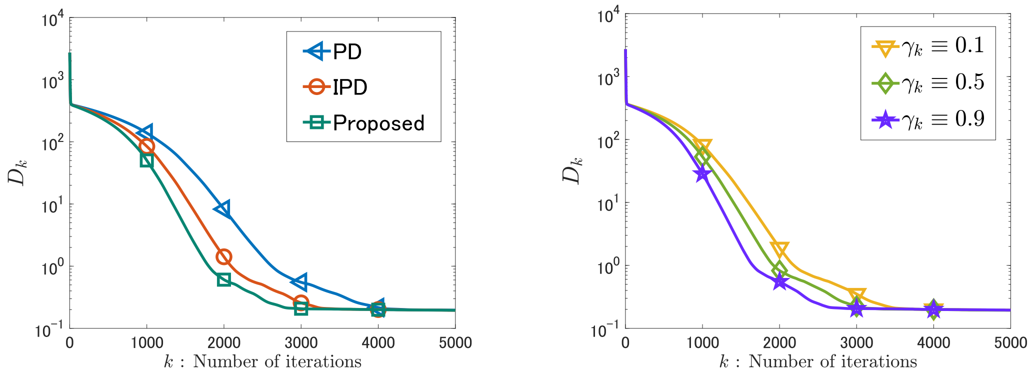

Figure 7.1: The computation results of the relation between the distance to a solution and iteration number. The left figure shows the numerical comparison of (42) with the PD and the IPD. The right figure shows the result of (42) for the case in which correspond to different choices of $\{\gamma_k\}$.

8. Conclusions

In this paper, we have proposed an alternated inertial fixed point algorithm and studied its convergence properties. The proposed algorithm can be applied for finding a common fixed point of a sequence of mappings of nonexpansive type in a real Hilbert space. In contrast to the existing inertial-type algorithms, the sequence generated by the algorithm is Fejér monotone when the iteration counter is even, and the assumptions on the iterative parameters are mild. Moreover, we applied the results to iterative algorithms of type alternating projection, forward-backward and the primal-dual splitting and derived some convergence results. To demonstrate the algorithm's effectiveness, performance, and convergence, we conducted numerical comparisons with the existing algorithms. Numerical results showed that the proposed algorithm converges faster than the existing algorithms.

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