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Generalized Characterization Theorems for Set Relations and an Application to Multi-Valued Optimization

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This paper proposes relaxed characterization for set relations by scalarization functions and states a generalized Gordan's theorem of the alternative for set-valued maps. As an application, we introduce some robustness of multi-valued optimization problems together with an algorithm by which its criteria can be calculated.

Keywords: Set optimization, set relation, scalarization, theorem of the alternative, robustness.

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1. Introduction

A set optimization problem is a further applied mathematical optimization model, in which we find a minimal (optimal) set of a given set-valued map in various ways. Basic concepts of set order relations have been studied by Nishnianidze [19] and Young [24], and nowadays Kuroiwa-Tanaka-Ha's set relations [14] are used in many concerned papers. Set relations usually focus on infinitely many elements in given sets, although it is difficult to check all the elements in a practical period of time. Here, we have a reason to consider set functions called scalarization functions.

To quantify the set relations, [2, 15] introduced set scalarization functions by using Tammer's function [5, 6]. These functions have a lot of practical properties in set-valued analysis (e.g., [4, 15, 16]). By utilizing the functions, characterization theorems for set relations have been proposed [1, 20, 21, 22]. As parallel approaches in a normed space, oriented distance functions characterize the set relations and similar properties have been found [3, 11, 12, 13].

In this paper, we investigate characterization theorems for $(\mathrm{cl}\, C)$-type set relations to propose a Gordan's type theorem of the alternative. Gordan's theorem [8] contrasts solutions of linear simultaneous inequalities and those of the dual equations. Jeyakumar [10] proved a Gordan-type theorem of the alternative addressing a subconvexlike multi-valued function. Li [17] and Yang et. al. [23] suggest applied versions of Gordan's theorem for convexlike set-valued maps. Nishizawa et. al. [20] and Ogata et. al. [22] propose different types of Gordan-type theorems with the set scalarization functions, which features removal of any convexity. This paper gives more generalized conditions to prove the theorems of the alternative than ones proposed in [21].

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As application, we discuss robustness of multi-valued optimization problems and its calculation. Speaking of calculation of the scalarization functions, Yu et al. [25, 26] established six kinds of algorithm in a finite dimensional space when given sets and an ordering cone are polyhedral. Each algorithm consists of the solutions of finitely many linear programming problems and the values of the scalarization functions come as the minimum or maximum of them. We show some calculation algorithm based on [26] to estimate for the problem how feasible it is under slight perturbation.

This paper is organized as follows. Section 2 contains basic notation and properties of set relations and scalarization functions. Section 3 is devoted to our main results. We prove relaxed characterization theorems and state a generalized Gordan's type theorem of the alternative in the last part of the section. In Section 4, our results are applied to robustness of multi-valued optimization problems and its calculation procedures are briefly shown.

2. Preliminaries

Throughout this paper, we let X be a topological vector space, $\mathcal{P}(X)$ the family of all nonempty subsets in X, and $C \subset X$ denotes a convex solid (i.e., $\operatorname{int} C \neq \emptyset$) cone. Note that X is partially ordered by the convex cone C with $\leq_C$ (i.e. it holds that $x \leq_C y$ if $y - x \in C$ for $x, y \in X$). For $S_1, S_2 \in \mathcal{P}(X)$ and $t \in \mathbb{R}$, the algebraic sum $S_1 + S_2$ and the scalar multiplication tS_1 are defined by

$$\begin{aligned} S_1 + S_2 &= \{s_1 + s_2 \mid s_1 \in S_1, s_2 \in S_2\}, \\ tS_1 &= \{ts_1 \mid s_1 \in S_1\}. \end{aligned}$$

Let us introduce convex conical properties ([18]). $S \in \mathcal{R}(X)$ is C-bounded if for all neighborhood U of zero, there exists $t > 0$ such that $S \subset tU + C$. S is C-closed if $S + \operatorname{cl} C$ is closed where $\operatorname{cl} C$ denotes the closure of C. S is C-compact if any cover of S in the form of $\{U_\lambda + C \mid U_\lambda \text{ are open}\}$ admits a finite subcover. We easily confirm that C-compactness leads to C-closedness, which implies C-boundedness.

In this paper, we introduce the following set relations proposed by Kuroiwa et al. In set optimization, various preference relations are suggested based on these six relations.

Definition 2.1. (Set relations [14]) For $S_1, S_2 \in \mathcal{P}(X)$,

$$\begin{aligned}
S_1 \preceq_C^{(1)} S_2 &\overset{\text{def}}{\Longleftrightarrow} S_1 \subset \bigcap_{s \in S_2} (s - C); \\
S_1 \preceq_C^{(2)} S_2 &\overset{\text{def}}{\Longleftrightarrow} S_1 \cap \bigcap_{s \in S_2} (s - C) \neq \emptyset; \\
S_1 \preceq_C^{(3)} S_2 &\overset{\text{def}}{\Longleftrightarrow} S_2 \subset S_1 + C; \\
S_1 \preceq_C^{(4)} S_2 &\overset{\text{def}}{\Longleftrightarrow} S_2 \cap \bigcap_{s \in S_1} (s + C) \neq \emptyset; \\
S_1 \preceq_C^{(5)} S_2 &\overset{\text{def}}{\Longleftrightarrow} S_1 \subset S_2 - C; \\
S_1 \preceq_C^{(6)} S_2 &\overset{\text{def}}{\Longleftrightarrow} S_2 \cap (S_1 + C) \neq \emptyset.
\end{aligned}$$

Unless otherwise described, the relation $\preceq_C^{(i)}$ is denoted by "type" (i) or just (i) in context. We note that the relations in Definition 2.1 can be expressed by the pointwise ordering $\leq_C$ as follows.

Lemma 2.2. *For $S_1, S_2 \in \mathcal{P}(X)$, it holds that*

$$\begin{aligned}
S_1 \preceq_C^{(1)} S_2 &\iff \forall s_1 \in S_1,\ \forall s_2 \in S_2,\ s_1 \leq_C s_2;\\
S_1 \preceq_C^{(2)} S_2 &\iff \exists s_1 \in S_1,\ \forall s_2 \in S_2,\ s_1 \leq_C s_2;\\
S_1 \preceq_C^{(3)} S_2 &\iff \forall s_2 \in S_2,\ \exists s_1 \in S_1,\ s_1 \leq_C s_2;\\
S_1 \preceq_C^{(4)} S_2 &\iff \exists s_2 \in S_2,\ \forall s_1 \in S_1,\ s_1 \leq_C s_2;\\
S_1 \preceq_C^{(5)} S_2 &\iff \forall s_1 \in S_1,\ \exists s_2 \in S_2,\ s_1 \leq_C s_2;\\
S_1 \preceq_C^{(6)} S_2 &\iff \exists s_1 \in S_1,\ \exists s_2 \in S_2,\ s_1 \leq_C s_2.
\end{aligned}$$

Lemma 2.3. *For $S_1, S_2 \in \mathcal{P}(X)$, it holds that*

$$\begin{aligned}
S_1 \preceq_C^{(1)} S_2 \Longrightarrow S_1 \preceq_C^{(2)} S_2 \Longrightarrow S_1 \preceq_C^{(3)} S_2 \Longrightarrow S_1 \preceq_C^{(6)} S_2,\\
S_1 \preceq_C^{(1)} S_2 \Longrightarrow S_1 \preceq_C^{(4)} S_2 \Longrightarrow S_1 \preceq_C^{(5)} S_2 \Longrightarrow S_1 \preceq_C^{(6)} S_2.
\end{aligned}$$

We remark that types (3) and (5) are preorders when $\mathbf{0} \in C$, that is, they are both reflexive and transitive. These relations have been especially called "set less order relations" and commonly used in the literature (see [15, 21] and the references cited therein).

Proposition 2.4. *Let $S \in \mathcal{P}(X)$. Then it holds that $W^{\mathrm{c}} = W^{\mathrm{c}} + C$ (i.e. W is free-disposal with respect to C) where*

$$W := \bigcap_{s \in S} (s - \operatorname{cl} C) \neq X.$$

Proof. $(\subset)$ Let $x \in W^{\mathrm{c}}$. There exists a neighborhood U_x of zero with $x + U_x \subset W^{\mathrm{c}}$. Then, we have $x + d \in W^{\mathrm{c}}$ for $d \in U_x \cap (-C)$. Therefore, $x \in W^{\mathrm{c}} - d \subset W^{\mathrm{c}} + C$.

$(\supset)$ Let $x \in W^{\mathrm{c}} + C$. Then, $x = w + d$ for some $w \in W^{\mathrm{c}}$ and some $d \in C$. There exists a neighborhood U_w of zero such that $w + U_w \subset W^{\mathrm{c}}$. If $x \notin W$, it holds that

$$(w - d + U_w) \cap W \neq \emptyset.$$

Since $W - d = W$, then we have the contradiction such as

$$(w + d + U_w) \cap W = (w + U_w) \cap (W - d) = (w + U_w) \cap W \neq \emptyset.$$

Thus, it is true that $x \in W^{\mathrm{c}}$. $\square$

Similarly to Proposition 2.4, the next lemma follows.

Lemma 2.5. *Let $S \subset X$. Then it holds that $W^{\mathrm{c}} = W^{\mathrm{c}} - C$ (i.e. W is free-disposal with respect to $(-C)$) where*

$$W := \bigcap_{s \in S} (s + \operatorname{cl} C) \neq X.$$

Definition 2.6. (Scalarization functions [15]) Let $S_1, S_2 \in \mathcal{P}(X)$ be nonempty sets and $k \in \operatorname{int} C$. The scalarization functions $Z^{(i)}_{C,k} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$, where $i = 1, \ldots, 6$, are defined by

$$Z^{(i)}_{C,k}(S_1, S_2) = \inf\{t \in \mathbb{R} \mid S_1 \preceq^{(i)}_C S_2 + tk\}.$$

It holds that $Z^{(i)}_{C,k}(S_1, S_2) \in \mathbb{R}$ under several conditions for S_1, S_2 [16]. If S_1, S_2, C are convex polyhedral, these values $Z^{(i)}_{C,k}(S_1, S_2)$ are calculable by finitely many steps [25, 26]. These functions are based on a Minkowski-type nonlinear functional

$$\phi_{C,k}(x) = \inf\{t \in \mathbb{R} \mid x \in tk - C\}$$

given in [5, 7, 6] as the fact that $Z^{(i)}_{C,k}(\{x\}, \{\mathbf{0}\})$ is equivalent to $\phi_{C,k}(x)$ for all $i = 1, \ldots, 6$. The function $\phi_{C,k}$ has sublinear contour lines and coincides with linear functional when C is a half space.

Next, we recall characterization theorems for the set relations of Definition 2.1. In the literature, these kinds of equivalence relations often called "alternative theorems" have been mainly studied for optimality conditions ([20, 22, 1]) in set optimization. For detail and examples see [21].

Proposition 2.7. [21] *Let $S_1, S_2 \in \mathcal{P}(X)$. Then,*

$$S_1 \preceq^{(1)}_{\operatorname{cl} C} S_2 \Longleftrightarrow Z^{(1)}_{C,k}(S_1, S_2) \leq 0.$$

Proposition 2.8. [21] *Let $S_1, S_2 \in \mathcal{P}(X)$. If S_1 is compact, then*

$$S_1 \preceq^{(2)}_{\operatorname{cl} C} S_2 \Longleftrightarrow Z^{(2)}_{C,k}(S_1, S_2) \leq 0,$$

$$S_1 \preceq^{(3)}_{\operatorname{cl} C} S_2 \Longleftrightarrow Z^{(3)}_{C,k}(S_1, S_2) \leq 0.$$

Proposition 2.9. [21] *Let $S_1, S_2 \in \mathcal{P}(X)$. If S_2 is compact, then*

$$S_1 \preceq^{(4)}_{\operatorname{cl} C} S_2 \Longleftrightarrow Z^{(4)}_{C,k}(S_1, S_2) \leq 0,$$

$$S_1 \preceq^{(5)}_{\operatorname{cl} C} S_2 \Longleftrightarrow Z^{(5)}_{C,k}(S_1, S_2) \leq 0.$$

Proposition 2.10. [21] *Let $S_1, S_2 \in \mathcal{P}(X)$. If S_2 and S_2 are compact, then*

$$S_1 \preceq^{(6)}_{\operatorname{cl} C} S_2 \Longleftrightarrow Z^{(6)}_{C,k}(S_1, S_2) \leq 0.$$

3. Main results

As a main part of the research, we would like to introduce a Gordan-type theorem of the alternative for set-valued maps. Gordan's theorem of the alternative was proposed in 1873 ([8]), which has been generalized for multi-valued functions ([10]) and set-valued maps ([17, 23]). In 2005, Nishizawa et al. introduced some nonlinear scalarization functions to omit convexity assumptions ([20]). Ogata et al. studied six Gordan-type theorems ([22, 21]) by using the scalarization function given in the previous section. This paper suggests these six theorems with relaxed conditions.

In this part, let V be a nonempty set, $S \in \mathcal{P}(X)$, $F : V \to \mathcal{P}(X)$ a set-valued map.

Proposition 3.1. [21] *It holds for all* $i = 1, \ldots, 6$ *and* $v \in V$ *that*

$$F(v) \preceq^{(i)}_{\operatorname{cl} C} S \implies Z^{(i)}_{C,k}(F(v), S) \leq 0, \ \forall k \in \operatorname{int} C.$$

Examples for Proposition 3.1 are shown in Figure 1. The left side shows $F(v) \preceq^{(1)}_{C} S$ and the right one does the other case $Z^{(1)}_{C,k}(F(v), S) > 0$. This paper proves the converse implication being true with relaxed conditions.

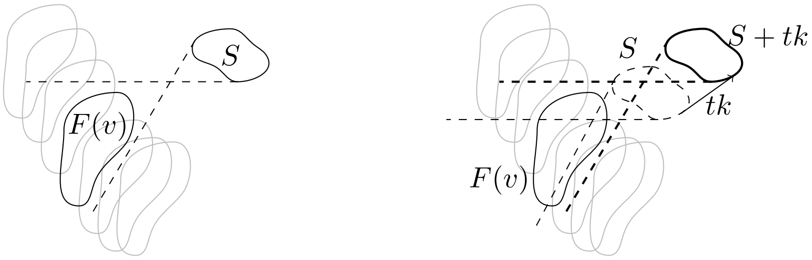

Figure 1: Examples of type (1) for Proposition 3.1

Proposition 3.2. *Let* $k \in \operatorname{int} C$ *and* $v \in V$. *Then,*

$$F(v) \not\preceq^{(1)}_{\operatorname{cl} C} S \implies Z^{(1)}_{C,k}(F(v), S) > 0.$$

Proposition 3.2 is proved in Theorem 4.2 of [21].

Theorem 3.3. *Let* $k \in \operatorname{int} C$ *and* $v \in V$. *If* F *is* C*-compact valued on* V,

$$F(v) \not\preceq^{(2)}_{\operatorname{cl} C} S \implies Z^{(2)}_{C,k}(F(v), S) > 0.$$

Proof. By the assumption, We have $F(v) \subset W^{\mathrm{c}}$ where

$$W := \bigcap_{s \in S} (s - \operatorname{cl} C).$$

For all $y \in F(v)$, there exists $t_y > 0$ such that $y - t_y k \in W^{\mathrm{c}}$. In consequence $\{t_y k + W^{\mathrm{c}} + C\}_{y \in F(v)}$ is an open cover of $F(v)$ since we have for all $y \in F(v)$ that

$$y \in t_y k + W^{\mathrm{c}} = t_y k + W^{\mathrm{c}} + C$$

by Proposition 2.4. Thus, we obtain

$$F(v) \subset \bigcup_{i=1}^{n} (t_{y_i} k + W^{\mathrm{c}} + C) = \bigcup_{i=1}^{n} (t_{y_i} k + W^{\mathrm{c}})$$

for some $y_1, \ldots, y_n \in F(v)$ since $F(v)$ is C-compact. This implies

$$Z^{(2)}_{C,k}(F(v), S) \geq \bar{t} > 0$$

for $\bar{t} := \min\{t_{y_1}, \ldots, t_{y_n}\}$. □

Proposition 3.4. *Let* $k \in \operatorname{int} C$ *and* $v \in V$. *If* F *is* C*-closed valued on* V,

$$F(v) \not\preceq^{(3)}_{\operatorname{cl} C} S \implies Z^{(3)}_{C,k}(F(v), S) > 0.$$

Proof. We assume $Z^{(3)}_{C,k}(F(v),S) \leq 0$. Then we obtain $tk + S \subset F(v) + \operatorname{cl} C$ for all $t > 0$. Therefore,

$$S \subset \bigcap_{t>0}(-tk + F(v) + \operatorname{cl} C) = \operatorname{cl}(F(v) + \operatorname{cl} C) = F(v) + \operatorname{cl} C$$

since $F(v)$ is C-closed. □

Proposition 3.5. *Let $k \in \operatorname{int} C$ and $v \in V$. If S is $(-C)$-closed then,*

$$F(v) \not\preceq^{(5)}_{\operatorname{cl} C} S \implies Z^{(5)}_{C,k}(F(v),S) > 0.$$

Proposition 3.5 directly follows from Proposition 3.4 by replacing S_1,S_2, and C with S_2,S_1, and $-C$, respectively. Propositions 3.4, 3.5 have been also studied in different ways or forms (e.g., [6, 9]). The above proof is a relatively (and expectedly) simpler one.

Theorem 3.6. *Let $k \in \operatorname{int} C$ and $v \in V$. If S is $(-C)$-compact then,*

$$F(v) \not\preceq^{(4)}_{\operatorname{cl} C} S \implies Z^{(4)}_{C,k}(F(v),S) > 0.$$

Proof. By the assumption, we have $S \subset W^{\mathrm{c}}$ where

$$W := \bigcap_{y \in F(v)} (y + \operatorname{cl} C).$$

For all $s \in S$, there exists $t_s > 0$ such that $s + t_s k \in W^{\mathrm{c}}$. Thus, $\{-t_s k + W^{\mathrm{c}} - C\}_{s \in S}$ is an open cover of S since it holds for all $s \in S$ that

$$s \in -t_s k + W^{\mathrm{c}} = -t_s k + W^{\mathrm{c}} - C$$

by Lemma 2.5. Thus, we obtain

$$S \subset \bigcup_{i=1}^{n}(-t_{s_i} k + W^{\mathrm{c}} - C) = \bigcup_{i=1}^{n}(-t_{s_i} k + W^{\mathrm{c}})$$

for some $s_1, \ldots, s_n \in S$ since S is $(-C)$-compact. This implies that

$$Z^{(4)}_{C,k}(F(v),S) \geq \bar{t} > 0$$

for $\bar{t} := \min\{t_{s_1}, \ldots, t_{s_n}\}$. □

Theorem 3.7. *Let $k \in \operatorname{int} C$ and $v \in V$. If F is C-closed valued and S is $(-C)$-compact, or F is C-compact valued and S is $(-C)$-closed, then*

$$F(v) \not\preceq^{(6)}_{\operatorname{cl} C} S \implies Z^{(6)}_{C,k}(F(v),S) > 0.$$

Proof. Here, we assume that F is C-closed valued and S is $(-C)$-compact. Then, we have $S \subset (F(v) + \operatorname{cl} C)^{\mathrm{c}}$. Since $F(v)$ is C-closed, it holds that $(F(v) + \operatorname{cl} C)^{\mathrm{c}}$ is open.

Hence, for all $s \in S$, there exists $t_s > 0$ such that $s + t_s k \in (F(v) + \operatorname{cl} C)^{\mathrm{c}}$. Thus,

$$S \subset \bigcup_{s\in S} ((F(v) + \operatorname{cl} C)^{\mathrm{c}} - t_s k) = \bigcup_{s\in S} ((F(v) + \operatorname{cl} C)^{\mathrm{c}} - t_s k - C)$$

since $(F(v) + \operatorname{cl} C)^{\mathrm{c}} - t_s k$ is free disposal with respect to $(-C)$ for all $s \in S$. This implies $\{(F(v) + \operatorname{cl} C)^{\mathrm{c}} - t_s k - C\}_{s\in S}$ is a subcover of S. Therefore,

$$S \subset \bigcup_{i=1}^{n} ((F(v) + \operatorname{cl} C)^{\mathrm{c}} - t_{s_i} k - C)$$

for some $s_1, \dots, s_n \in S$ since S is $(-C)$-compact. By taking $\bar{t} := \min\{t_{s_1}, \dots, t_{s_n}\}$, it holds that $S \subset (F(v) + \operatorname{cl} C)^{\mathrm{c}} - \bar{t}k$, that is,

$$(S + \bar{t}k) \cap (F(v) + \operatorname{cl} C) = \emptyset$$

for all $t \leq \bar{t}$. Thus, we conclude that $Z^{(6)}_{C,k}(F(v), S) > \bar{t} > 0$. □

By Propositions 3.2, 3.4, 3.5 and Theorems 3.3, 3.6, 3.7, we obtain the following generalized Gordan's theorem of the alternative, which is a relaxation of Theorem 3.1 in [22].

Lemma 3.8. (Generalized Gordan's theorem of the alternative) *Let V be a nonempty set, $F : V \to \mathcal{P}(X)$. If*

- *F is C-compact valued on V when $i = 2$;*
- *F is C-closed valued on V when $i = 3$;*
- *S is $(-C)$-compact when $i = 4$;*
- *S is $(-C)$-closed when $i = 5$;*
- *F is C-closed valued on V and V is $(-C)$-compact, or F is C-compact valued on V and S is $(-C)$-closed when $i = 6$,*

then exactly one of the following two statements is true:

(i) $\exists v \in V$ *s.t.* $F(v) \preceq^{(i)}_{\operatorname{cl} C} S$;

(ii) $\exists k \in \operatorname{int} C$ *s.t.* $Z^{(i)}_{C,k}(F(v), S) > 0, \quad \forall v \in V$.

4. Application

In this section, we show calculation algorithm in computing values of the scalarization functions in a finite dimensional space and discuss some robustness in multi-valued optimization.

For $S \subset \mathbb{R}^n$, $\operatorname{co} S$ and $\operatorname{cone} S$ denote the convex hull and the conical hull of S, respectively. The *recession cone* of S is defined to be

$$\{d \in \mathbb{R}^n \mid s + td \in S,\ \forall s \in S, \forall t \geq 0\}$$

and is denoted by $\operatorname{rec}(S)$. $M(\alpha, \beta)$ is the set of all $\alpha \times \beta$ matrices. S is said to be *polyhedral* if $S = \{x \in \mathbb{R} \mid Px \geq q\}$ for some $P \in M(\alpha, n)$ and some $q \in \mathbb{R}^\alpha$. S is *finitely generated* if $S = \operatorname{co} S_1 + \operatorname{co}\operatorname{cone} S_2$ for some finite sets $S_1, S_2 \subset \mathbb{R}^n$. $s \in S$ is an *extreme point* of S if $s \neq ts_1 + (1-t)s_2$ for all $s_1, s_2 \in S$ and all $t \in [0, 1]$.

The set of all extreme points of S is denoted by $\operatorname{ext}(S)$. Moreover if $S \in \mathbb{R}^n$ is a convex set, we have $S = \operatorname{co}\operatorname{ext}(S) + \operatorname{rec}(S)$ (e.g. Proposition 1.12 in [27]). Also, polyhedrality and finitely generatedness coincide in a finite dimensional space (see Theorem 1.2 in [27]).

Proposition 4.1. *Let $C \neq \mathbb{R}^n$ be a convex solid cone and $S \subset \mathbb{R}^n$ finitely generated. Then, S is C-bounded if and only if $\operatorname{rec}(S) \subset \operatorname{cl} C$.*

Proof. ($\Rightarrow$) Assume that we have $k \in \operatorname{rec}(S) \cap (\operatorname{cl} C)^{\mathrm{c}}$. Then, there exists a neighborhood U_k of zero such that $k - U_k \subset (\operatorname{cl} C)^{\mathrm{c}}$. Thus, $k \notin U_k + \operatorname{cl} C$, that is, $tk \notin t(U_k + C) = tU_k + C$ for all $t > 0$. Since $tk \in S$, S is not C-bounded.

($\Leftarrow$) For all neighborhood U of zero, there exists $t_U > 0$ such that $\operatorname{co}\operatorname{ext}(S) \subset t_U U$ since $\operatorname{co}\operatorname{ext}(S)$ is bounded. Thus, we have

$$S = \operatorname{co}\operatorname{ext}(S) + \operatorname{rec}(S) \subset t_U U + \operatorname{cl} C = t_U U + C. \qquad \square$$

By Theorem 1.12 in [27], $\operatorname{rec}(S) = \operatorname{co}\operatorname{cone} S_0$ for some finite set S_0 for a finitely generated set S. If C is also polyhedral (that is, $C = \{x \mid Px \leq 0\}$ for some $P \in M(\alpha, n)$ and $q \in \mathbb{R}^\alpha$), Proposition 4.1 simply describes the fact that

$$S \text{ is } C\text{-bounded} \iff Ps \leq 0,\ \forall s \in S_0.$$

Proposition 4.2. *Let $C \neq \mathbb{R}^n$ be a convex solid cone and $S \subset \mathbb{R}^n$ be finitely generated. Then, S is C-bounded if and only if S is C-compact.*

Proof. It is sufficient that we prove C-boundedness leads to C-compactness. Let $\{U_\lambda + C\}_{\lambda \in \Lambda}$ be a cover of S for some open sets U_λ. Since $\operatorname{co}\operatorname{ext}(S)$ is C-compact, there exists $\lambda_1, \ldots, \lambda_m \in \Lambda$ such that

$$\operatorname{co}\operatorname{ext}(S) \subset \bigcup_{i=1,\ldots,m} (U_{\lambda_i} + C).$$

By the assumption, it holds that $\operatorname{rec}(S) \subset \operatorname{cl} C$, which implies

$$S = \operatorname{co}\operatorname{ext}(S) + \operatorname{rec}(S) \subset \bigcup_{i=1,\ldots,m} (U_{\lambda_i} + C) + \operatorname{cl} C \subset \bigcup_{i=1,\ldots,m} (U_{\lambda_i} + C).$$

Therefore, S admits a finite subcover. $\square$

Proposition 4.3. *Let $C \neq \mathbb{R}^n$ be a convex solid cone and $S \subset \mathbb{R}^n$. If S and C are both finitely generated, then S is C-closed.*

Proof. We assume that $\operatorname{rec}(S) = \operatorname{co}\operatorname{cone} S_0$ and $C = \operatorname{co}\operatorname{cone} C_0$ for some finite sets $S_0, C_0 \subset \mathbb{R}^n$. Then,

$$S + \operatorname{cl} C = \operatorname{ext}(S) + \operatorname{co}\operatorname{cone} S_0 + \operatorname{co}\operatorname{cone} C_0 = \operatorname{ext}(S) + \operatorname{co}\operatorname{cone}(S_0 + C_0).$$

Since $\operatorname{ext}(S)$ is compact and $\operatorname{co}\operatorname{cone}(S_0 + C_0)$ is closed, $S + \operatorname{cl} C$ is proved to be closed. $\square$

Optimization problems may solve real issues mathematically, although their ideal models can't be exactly same as real ones due to perturbed factors and small errors. In perturbation theory, we give small deviation to an optimization problem to figure out how its feasible set and solutions are robust. In this part, we discuss computable criteria for robustness of the feasibility of a multi-valued optimization problem.

Let $S \subset \mathbb{R}^\gamma$ be nonempty, $\phi : S \to \mathbb{R}$, $f : S \to \mathbb{R}^n$, $r \in \mathbb{R}^n$. Consider the following multi-valued optimization problem:

$$\text{(P)} \qquad \text{Minimize } \phi(s) \text{ subject to } f(s) \leq_C r$$

under a convex cone C defined by

$$C := \{x \in \mathbb{R}^n \mid \langle p_j, x\rangle \leq 0, \ \forall j = 1, \ldots, m\}$$

for some $p_1, \ldots, p_m \in \mathbb{R}^n$, or simply, $C := \{x \in \mathbb{R}^n \mid P_C x \leq 0\}$ for an $m \times n$ matrix P_C where each row is p_j^{T}.

We assume in (P) that f and r are perturbed in the sets F and R respectively where

$$\begin{aligned} F(s) &: = \{f(s) \mid f \in F\} = \{x \in \mathbb{R}^n \mid P_F(s)x \leq q_F(s)\}, \\ R &: = \{x \in \mathbb{R}^n \mid P_R x \leq q_R\} \end{aligned}$$

for $P_F : S \to M(\alpha, n)$, $q_F : S \to \mathbb{R}^\alpha$, $P_R \in M(\beta, n)$, and $q_R \in \mathbb{R}^\beta$. Then, it holds that for all $s \in S$,

$$\begin{aligned} F(s) &= \operatorname{co} F_1(s) + \operatorname{co}\operatorname{cone} F_2(s) \\ R &= \operatorname{co} R_1 + \operatorname{co}\operatorname{cone} R_2 \end{aligned}$$

for some finite set-valued maps $F_1(s), F_2(s)$ and some finite set R_1, R_2 by processing Fourier-Mozkin Elimination (see Section 1.2 in [27]). Then, by Theorems 3.3 and 3.4 in [26], it clearly holds that

$$\begin{aligned} Z^{(1)}_{C,k}(F(s), R) &= \min_{j=1,\ldots,m} \left\{ \mathrm{Val}\left(\mathrm{LP}^{(1)}(j, s)\right)\right\}; \\ Z^{(2)}_{C,k}(F(s), R) &= \mathrm{Val}\left(\mathrm{LP}^{(2)}(s)\right); \\ Z^{(3)}_{C,k}(F(s), R) &= \sup_{r \in R} \left\{ \mathrm{Val}\left(\mathrm{LP}^{(3)}(r, s)\right)\right\}; \\ Z^{(4)}_{C,k}(F(s), R) &= \mathrm{Val}\left(\mathrm{LP}^{(4)}(s)\right); \\ Z^{(5)}_{C,k}(F(s), R) &= \sup_{x \in F(s)} \left\{ \mathrm{Val}\left(\mathrm{LP}^{(5)}(x)\right)\right\}; \\ Z^{(6)}_{C,k}(F(s), R) &= \mathrm{Val}\left(\mathrm{LP}^{(6)}(s)\right); \end{aligned}$$

where $\mathrm{Val}(\cdot)$ denotes the optimal values of the following problems:

$$\mathrm{LP}^{(1)}(j, s) \qquad \begin{cases} \text{Maximize} & \dfrac{\langle p_j, x - r\rangle}{\langle p_j, k\rangle} \\ \text{subject to} & P_F(s)x \leq q_F(s) \text{ and } P_R r \leq q_R; \end{cases}$$

$$\mathrm{LP}^{(2)}(s)\qquad \begin{cases} \text{Minimize} & t\in\mathbb{R} \\ \text{subject to} & \dfrac{\langle p_j, x\rangle}{\langle p_j, k\rangle} + \sup\limits_{r\in R}\dfrac{\langle p_j, -r\rangle}{\langle p_j, k\rangle} \le t \ \text{ for all } \ j=1,\dots,m, \\ & P_F(s)x \le q_F(s); \end{cases}$$

$$\mathrm{LP}^{(3)}(r,s)\qquad \begin{cases} \text{Minimize} & t\in\mathbb{R} \\ \text{subject to} & \dfrac{\langle p_j, x-r\rangle}{\langle p_j, k\rangle} \le t \ \text{ for all } \ j=1,\dots,m, \\ & P_F(s)x \le q_F(s); \end{cases}$$

$$\mathrm{LP}^{(4)}(s)\qquad \begin{cases} \text{Minimize} & t\in\mathbb{R} \\ \text{subject to} & \sup\limits_{x\in F(s)}\dfrac{\langle p_j, x\rangle}{\langle p_j, k\rangle} + \dfrac{\langle p_j, -r\rangle}{\langle p_j, k\rangle} \le t \ \text{ for all } \ j=1,\dots,m, \\ & P_R r \le q_R; \end{cases}$$

$$\mathrm{LP}^{(5)}(x)\qquad \begin{cases} \text{Minimize} & t\in\mathbb{R} \\ \text{subject to} & \dfrac{\langle p_j, x-r\rangle}{\langle p_j, k\rangle} \le t \ \text{ for all } \ j=1,\dots,m, \\ P_R r \le q_R; & \end{cases}$$

$$\mathrm{LP}^{(6)}(s)\qquad \begin{cases} \text{Minimize} & t\in\mathbb{R} \\ \text{subject to} & \dfrac{\langle p_j, x-r\rangle}{\langle p_j, k\rangle} \le t \ \text{ for all } \ j=1,\dots,m, \\ P_F(s)x \le q_F(s) \ \text{ and } \ P_R r \le q_R. & \end{cases}$$

By Lemma 3.8 and Proposition 4.1, the feasibility of (P) is confirmed by calculating values of $Z^{(i)}_{C,k}(F(s),R)$ as follows:

- (P) is feasible for all $f(s)\in F(s)$ and $r\in R$ when

$$\exists s\in S \text{ s.t. } \min_{j=1,\dots,m}\left\{\mathrm{Val}\left(\mathrm{LP}^{(1)}(j,s)\right)\right\}\le 0;$$

- there exists $f\in F$ such that (P) is feasible for all $r\in R$ when there exists $s\in S$ such that

$$P_C\tilde{s}\le 0\ \ (\forall\tilde{s}\in F_2(s)),\qquad \mathrm{Val}\left(\mathrm{LP}^{(2)}(s)\right)\le 0;$$

- for all $r\in R$, there exists $f\in F$ such that (P) is feasible when there exists $s\in S$ such that

$$P_C\tilde{s}\le 0\ \ (\forall\tilde{s}\in F_2(s)),\qquad \sup_{r\in R}\left\{\mathrm{Val}\left(\mathrm{LP}^{(3)}(r,s)\right)\right\}\le 0;$$

- there exists $r\in R$ such that (P) is feasible for all $f\in F$ when

$$P_C\tilde{r}\le 0\ (\forall\tilde{r}\in R_2),\qquad \exists s\in S \text{ s.t. } \mathrm{Val}\left(\mathrm{LP}^{(4)}(s)\right)\le 0;$$

- for all $f\in F$, there exists $r\in R$ such that (P) is feasible when

$$P_C\tilde{r}\le 0\ (\forall\tilde{r}\in R_2),\qquad \exists s\in S \text{ s.t. } \sup_{x\in F(s)}\left\{\mathrm{Val}\left(\mathrm{LP}^{(5)}(x)\right)\right\}\le 0;$$

- there exist $f \in G$ and $r \in R$ such that (P) is feasible when

$$P_C\tilde{r} \leq 0 \ (\forall \tilde{r} \in R_2), \ \ \exists s \in S \text{ s.t. } (P_C\tilde{s} \leq 0, \ \forall \tilde{s} \in F_2(s)) \wedge \left(\mathrm{Val}\left(\mathrm{LP}^{(6)}(s)\right) \leq 0\right).$$

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