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# A One-Step Tikhonov Regularization Iterative Scheme for Solving Split Feasibility and Fixed Point Problems

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We study split feasibility and fixed point problems for Lipschitzian pseudocontractive and nonexpansive mappings in real Hilbert spaces. Using Tikhonov's regularization technique, we first propose an Ishikawa-type gradient-projection iterative scheme for approximating solutions to such problems and then carry out its convergence analysis. A weak convergence theorem is established, applications are derived, and several numerical examples are presented.

*Keywords:* Fixed point problem, Hilbert space, minimization problem, pseudocontractive mapping.

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## 1. Introduction

Let $C$ and $Q$ be nonempty, closed and convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and let its adjoint be denoted by $A^*$. The split feasibility problem (SFP) is to find

$$x \in C \text{ such that } Ax \in Q. \tag{1}$$

The SFP was introduced by Censor and Elfving [7] in Euclidean spaces. It models inverse problems which arise from phase retrieval, medical image reconstruction and intensity modulated radiation therapy [8]. In the case where (1) has a solution, one can easily verify that a point $x^*$ is a solution of (1) if and only if $x^*$ solves the following fixed point equation:
Find $x^* \in C$ such that

$$x^* = P_C(x^* - \gamma A^*(I - P_Q)Ax^*),$$

where $\gamma$ is a positive constant, $I : H_2 \to H_2$ is the identity operator, and $P_C$ and $P_Q$ are the metric projections of $H_1$ onto $C$ and of $H_2$ onto $Q$, respectively. Byrne [5] introduced an algorithm, simply known as the $CQ$-algorithm, for solving the SFP in (finite-dimensional) Euclidean spaces as follows:

$$\begin{cases} x_1 \in \mathbb{R}^n, \\ x_{n+1} = P_C(x_n - \gamma A^T(I - P_Q)Ax_n), \ n \in \mathbb{N}, \end{cases}$$

where $A^T$ is the transpose of $A$.

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Later Xu [26] proposed the following algorithm for solving the SFP in infinite-dimensional Hilbert spaces: Let $x_1 \in H_1$ and compute

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \ n \in \mathbb{N}. \tag{2}$$

The sequences generated by the $CQ$-algorithm (2) converge weakly to a solution of the SFP, whenever a solution exists. When the SFP has no solution, these sequences converge to a minimizer of $\|P_QAx - Ax\|$ over all $x \in C$, whenever such a minimizer exists (see [28]).

Now consider the following minimization problem:

$$\min_{x \in C} f(x) := \frac{1}{2}\|Ax - P_QAx\|^2. \tag{3}$$

The objective function $f$ is continuously differentiable and its gradient $\nabla f$ is given by

$$\nabla f(x) = A^*(I - P_Q)Ax.$$

Thus one can rewrite (2) as follows:

$$x_{n+1} = P_C(x_n - \gamma \nabla f(x_n)), \ n \in \mathbb{N}.$$

However, the minimization problem (3) is, in general, ill posed. Therefore regularization is needed. Xu [26] considered the following Tikhonov regularization:

$$\min_{x \in C} f^\tau(x) := \frac{1}{2}\|Ax - P_QAx\|^2 + \frac{1}{2}\tau\|x\|^2,$$

where $\tau > 0$ is the regularization parameter. In this case, the gradient of the objective function $f^\tau$ is given by

$$\nabla f^\tau(x) = A^*(I - P_Q)Ax + \tau x.$$

Taking into account the above facts, Xu [26] proved the following theorem:

**Theorem 1.1.** *Assume that the SFP* (1) *is consistent. Let* $\{x_n\}$ *be the sequence defined by* $x_1 \in H_1$ *and*

$$x_{n+1} = P_C(x_n - \gamma \nabla f^{\tau_n}(x_n)), \ n \in \mathbb{N}, \quad \text{where } \gamma \in \left(0, \frac{2}{\|A\|^2}\right) \ \text{and} \ \sum_{n=1}^{\infty} \tau_n < \infty.$$

*Then* $\{x_n\}$ *converges weakly to a solution of the SFP* (1).

For more information on the SFP and its generalizations, see [19] and references therein.

Next, let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be two mappings with nonempty fixed point sets $Fix(T)$ and $Fix(S)$, respectively, where $Fix(T) := \{x \in H_1 : Tx = x\}$ and $Fix(S) := \{y \in H_2 : Sy = y\}$. The split common fixed point problem (SCFPP) is to find a point

$$x \in Fix(T) \text{ such that } Ax \in Fix(S). \tag{4}$$

If in (4) $T$ and $S$ are the metric projections onto $C$ and $Q$, respectively, then (4) reduces to (1). Thus the SCFPP generalizes the SFP. In the framework of Hilbert spaces, the SCFPP was first studied by Censor and Segal [9] for directed operators.

They proposed the following algorithm for approximating its solutions:

$$\begin{cases} x_1 \in H_1, \\ x_{n+1} = T[x_n - \gamma A^*(I - S)Ax_n], n \in \mathbb{N}, \end{cases} \tag{5}$$

where $\gamma \in (0, \frac{2}{\lambda})$ and $\lambda$ is the spectral radius of the operator $A^*A$. Assuming some suitable conditions, they established a weak convergence theorem for it. Moudafi [15] studied the SCFPP in infinite dimensional Hilbert spaces in the case where $T$ and $S$ are quasi-nonexpansive mappings, and proposed an iterative scheme that generates sequences which converge weakly to a solution of the problem. For recent studies of the SCFPP and of algorithms for solving it, see [13, 20, 23] and references therein.

We now consider the following composite problem:

$$\text{Find } x^* \in C \cap Fix(T) \text{ such that } Ax^* \in Q \cap Fix(S). \tag{6}$$

The composite problem (6), which is simply called the split feasibility and fixed point problem (SFFPP), consists of an SFP and an SCFPP. We denote the solution set of the SFFPP by $\Gamma$.

In the case where $S$ is the identity mapping on $H_1$ and $T$ is nonexpansive, Ceng et al. [6] proposed the following extragradient iterative algorithm for solving the SFFPP (6):

$$\begin{cases} x_1 \in C, \\ z_n = P_C(I - \gamma_n \nabla f^{\tau_n})x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)TP_C(x_n - \gamma_n \nabla f^{\tau_n}(z_n)), \quad n \in \mathbb{N}. \end{cases}$$

They proved that under some suitable conditions, the sequences $\{z_n\}$ and $\{x_n\}$ converge weakly to a solution of the problem.

Also recently, Chen et al. [10] studied the SFFPP in the case where $T$ is an $L$-Lipschitzian pseudocontractive mapping and $S$ is a nonexpansive mapping. They proposed the following Ishikawa-type extragradient iterative algorithm for approximating a solution:

$$\begin{cases} x_1 \in C, \\ q_n = P_C(I - \gamma_n \nabla f^S)x_n, \\ w_n = P_C(x_n - \gamma_n \nabla f^S(q_n)), \\ z_n = (1 - \beta_n)w_n + \beta_n T w_n, \\ x_{n+1} = (1 - \alpha_n)w_n + \alpha_n T z_n, n \in \mathbb{N}, \end{cases} \tag{7}$$

where
$$\nabla f^S := A^*(I - SP_Q)A,$$

and proved a weak convergence theorem for it.

Motivated by Ceng et al. [6], Chen et al. [10], and Phuengrattana and Suantai [18], Wongsasinchai [25] has recently studied the SFFPP in the case where $T$ is an $L$-Lipschitzian pseudocontractive mapping and $S$ is a nonexpansive mapping, and has proposed the following SP-type extragradient algorithm:

$$\begin{cases} x_1 \in C, \\ q_n = P_C(I - \gamma_n \nabla f^{S\tau_n})x_n, \\ w_n = P_C(x_n - \gamma_n \nabla f^{S\tau_n} q_n), \\ s_n = (1-\delta_n)w_n + \delta_n T w_n, \\ z_n = (1-\beta_n)s_n + \beta_n T s_n, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n T z_n, n \in \mathbb{N}, \end{cases} \tag{8}$$

where $$\nabla f^{S\tau_n} := A^*(I - SP_Q)A + \tau_n I,$$

$$\{\gamma_n\} \subset [a,b] \text{ for some } a, b \in (0, \frac{1}{\tau_n + 2\|A\|^2}),$$

$$\{\tau_n\} \subset (0,\infty), \quad \sum_{n=1}^{\infty} \tau_n < \infty$$

and $$0 < a < \alpha_n < b < \beta_n < c < \delta_n < d < \frac{1}{\sqrt{L^2+1}+1+L^2}.$$

It has been established that the sequences generated by (8) converge weakly to a point in the solution set $\Gamma$.

It is important to note that the aforementioned extragradient algorithms for solving the SFFPP involve the computation of four metric projections per iteration. Computing the metric projection amounts to solving a minimization problem. Note that even in the case where the projection onto the feasible sets $C$ and $Q$ can be computed easily, the computation of four metric projections per iteration does increase the computational burden of the algorithm. Thus a better approach is to reduce the number of projections per iteration. Motivated by these facts and following the regularization technique of Xu [26], we propose in the present paper a new gradient-projection iterative scheme for approximating solutions of the SFFPP (6) for the case where $T$ is an $L$-Lipschitzian pseudocontractive mapping and $S$ is a nonexpansive mapping, and carry out its convergence analysis.

The organization of our paper is as follows: In Section 2, we recall some useful definitions and preliminary results, which are essential for the analysis of our algorithm. In Section 3, we present the algorithm and its convergence analysis. In Section 4, we provide some applications of our main result. In Section 5, we give numerical examples to illustrate our algorithm and compare it with some existing algorithms in the literature. Finally, we present a few conclusions in Section 6.

## 2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and let $K$ be a nonempty, closed and convex subset of $H$. We denote by '$x_n \rightharpoonup x$' the weak convergence of $\{x_n\}$ to a point $x \in H$.

We begin by recalling the following well-known identity:

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y \rangle + \|y\|^2 \quad \forall x, y \in H. \tag{9}$$

**Definition 2.1.** A mapping $T : H \to H$ is said to be:

(i) *$L$-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \ \ \forall x, y \in H.$$

If $L \in [0, 1)$ then $T$ is said to be a *contraction*. If $L = 1$ then $T$ is said to be *nonexpansive*;

(ii) *Quasi-nonexpansive* if $Fix(T) \neq \emptyset$ and

$$\|Tu - v\| \leq \|u - v\| \quad \forall u \in H, v \in Fix(T);$$

(iii) *$k$-strictly pseudocontractive* if there exists $k \in [0,1)$ such that

$$\|Tu - Tv\|^2 \leq \|u - v\|^2 + k\|(I-T)u - (I-T)v\|^2 \quad \forall u, v \in H;$$

(iv) *Pseudocontractive* if $\langle Tu - Tv, u - v\rangle \leq \|u - v\|^2 \quad \forall u, v \in H$
or, equivalently,

$$\|Tu - Tv\|^2 \leq \|u - v\|^2 + \|(I-T)u - (I-T)v\|^2 \quad \forall u, v \in H;$$

(v) *Quasi-pseudocontractive* if $Fix(T) \neq \emptyset$ and

$$\langle Tu - v, u - v\rangle \leq \|u - v\|^2 \quad \forall u \in H, v \in Fix(T)$$

or, equivalently,

$$\|Tu - v\|^2 \leq \|u - v\|^2 + \|(I-T)u\|^2 \quad \forall u \in H, v \in Fix(T).$$

The class of pseudocontractive mappings contains the classes of nonexpansive and strictly pseudocontractive mappings. Also, every pseudocontractive mapping with a nonempty fixed point set is quasi-pseudocontractive. The class of pseudocontractive mappings is an important class of mappings mainly because of its connection with the class of monotone operators. Recall that a mapping $T : H \to H$ is said to be monotone [4] if

$$\langle Tu - Tv, u - v\rangle \geq 0 \; \forall u, v \in H.$$

Thus $T$ is monotone if and only if $I - T$ is pseudocontractive. Moreover, the solutions of the operator equation $Tu = 0$ coincide with the fixed points of $I - T$. We denote the solution set of the operator equation $Tu = 0$ by zer $T$.

The following lemma reveals that the fixed point set of a pseudocontractive mapping is closed and convex.

**Lemma 2.2.** *If $T : H \to H$ is a pseudocontractive mapping, then its fixed point set $Fix(T)$ is closed and convex.*

**Proof.** Since $T$ is pseudocontractive, the operator $(I - T)$ is monotone. It is known that for a monotone operator $M$, zer $M$ is a closed and convex set (see, for example, [2] Prop. 23.39). Since $Fix(T) = \text{zer}\,(I - T)$, it follows that $Fix(T)$ is indeed closed and convex, as asserted. □

The metric projection of $H$ onto $K$, denoted by $P_K$, is the mapping that maps each point $x \in H$ to its unique nearest point in $K$, that is,

$$\|x - P_K x\| \leq \|x - y\| \; \forall y \in K.$$

The metric projection is characterized by $P_K x \in K$ and

$$\|x - P_K x\|^2 + \|y - P_K x\|^2 \leq \|x - y\|^2 \quad \forall y \in K \tag{10}$$

or, equivalently, $\langle x - P_K x, y - P_K x\rangle \leq 0 \quad \forall y \in K$.

Moreover, $P_K$ is nonexpansive and $Fix(P_K) = K$.

**Definition 2.3.** Let $T : H \to H$ be a mapping. The mapping $I - T$ is said to be demiclosed at 0 if for any sequence $\{x_n\}$ in $H$, the assumptions $x_n \rightharpoonup x^*$ and $(I - T)x_n \to 0$ imply that $Tx^* = x^*$.

**Lemma 2.4.** [29] *Let $H$ be a real Hilbert space and $K$ be a closed and convex subset of $H$. Let $T : K \to K$ be a continuous pseudocontractive mapping. Then $I - T$ is demiclosed at zero.*

**Lemma 2.5.** [10] *Let $K$ be a nonempty, closed and convex subset of a real Hilbert space $H$ and let $S : K \to K$ be a nonexpansive mapping. Set $\nabla f^S := A^*(I - SP_K)A$.*

*Then*
$$\langle x - y, \nabla f^S(x) - \nabla f^S(y)\rangle \geq \frac{1}{2\|A\|^2}\|\nabla f^S(x) - \nabla f^S(y)\|^2. \tag{11}$$

**Lemma 2.6.** [24] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \leq a_n + b_n$ for all $n \in \mathbb{N}$.*
*If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

**Lemma 2.7.** [22, 28] *Let $H$ be a real Hilbert space and $T : H \to H$ be an $L$-Lipschitzian mapping with $L \geq 1$. Set $T_\alpha = (1 - \alpha)I + \alpha T((1 - \kappa)I + \kappa T)$.*
*If $0 < \alpha < \kappa < \frac{1}{1+\sqrt{1+L^2}}$ and $T$ is quasi-pseudocontractive, then $T_\alpha$ is quasi-nonexpansive and*
$$\|T_\alpha x - x^*\|^2 \leq \|x - x^*\|^2 - \alpha(\kappa - \alpha)(1 - \kappa L)^2\|Tx - x\|^2 \ \forall x \in H, x^* \in Fix(T).$$

**Lemma 2.8.** [26] *Let $K$ be a nonempty, closed and convex subset of a Hilbert space $H$. Let $\{x_n\}$ be a bounded sequence which satisfies the following two properties:*

- *every weak limit point of $\{x_n\}$ lies in $K$; and*
- *$\lim_{n\to\infty} \|x_n - x\|$ exists for every $x \in K$.*

*Then $\{x_n\}$ converges weakly to a point in $K$.*

## 3. Main results

In this section we present our algorithm and establish a weak convergence theorem for it.

**Algorithm 3.1.** *Let $C$ and $Q$ be nonempty, closed and convex subsets of the real Hilbert spaces $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and let $A^* : H_2 \to H_1$ be the adjoint of $A$. Let $S : H_2 \to H_2$ be a nonexpansive mapping and $T : H_1 \to H_1$ be an $L$-Lipschitzian pseudocontractive mapping with $L \geq 1$. Suppose that $\Gamma := \{x^* \in H_1 : x^* \in C \cap Fix(T),\ Ax^* \in Q \cap Fix(S)\} \neq \emptyset$ and let $\{x_n\}$ be a sequence generated as follows:*
$$\begin{cases} x_1 \in C, \\ u_n = P_C(x_n - \gamma_n \nabla f^{S\tau_n}(x_n)), \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n T((1 - \beta_n)u_n + \beta_n T u_n), \ \ n \in \mathbb{N}, \end{cases} \tag{12}$$

*where*
$$\nabla f^{S\tau_n}(x_n) = \nabla f^S(x_n) + \tau_n x_n,\ \nabla f^S(x_n) = A^*(I - SP_Q)Ax_n,$$

*and the following conditions are satisfied:*

(i) $\tau_n \in [0,1)$ with $\sum_{n=1}^{\infty} \tau_n < \infty$;

(ii) $0 < a \leq \gamma_n \leq b < \frac{1}{\tau_n + \|A\|^2}$;

(iii) $0 < c < \alpha_n < d < \beta_n < e < \frac{1}{1+\sqrt{1+L^2}}$.

The following lemmata are used in the proof of our main theorem.

**Lemma 3.2.** *Given the data in Algorithm* 3.1, *the map* $P_C(I - \gamma_n \nabla f^{S\tau_n}) : C \to C$ *is a strict contraction with constant* $(1 - \gamma_n \tau_n)$.

**Proof.** Let the points $x$ and $y$ belong to $C$. We first note that

$$\begin{aligned} &\langle x - y, A^*(I - SP_Q)Ax - A^*(I - SP_Q)Ay\rangle \\ &\quad = \langle Ax - Ay, (I - SP_Q)Ax - (I - SP_Q)Ay\rangle \\ &\quad = \|Ax - Ay\|^2 - \langle Ax - Ay, SP_QAx - SP_QAy\rangle \\ &\quad \geq \|Ax - Ay\|^2 - \|Ax - Ay\|\|SP_QAx - SP_QAy\| \geq 0. \end{aligned} \tag{13}$$

Using now (9), (11), the fact that $P_C$ is nonexpansive and (13), we get

$$\begin{aligned} &\|P_C(x - \gamma_n \nabla f^{S\tau_n} x) - P_C(y - \gamma_n \nabla f^{S\tau_n} y)\|^2 \\ &\quad \leq \|(x - \gamma_n \nabla f^{S\tau_n} x) - (y - \gamma_n \nabla f^{S\tau_n} y)\|^2 \\ &\quad = \|x - y\|^2 - 2\gamma_n \langle x - y, \nabla f^{S\tau_n} x - \nabla f^{S\tau_n} y\rangle + \gamma_n^2 \|\nabla f^{S\tau_n} x - \nabla f^{S\tau_n} y\|^2 \\ &\quad = \|x - y\|^2 - 2\gamma_n \tau_n \|x - y\|^2 \\ &\qquad - 2\gamma_n \langle x - y, \nabla f^S x - \nabla f^S y\rangle + \gamma_n^2 \|\nabla f^{S\tau_n} x - \nabla f^{S\tau_n} y\|^2 \\ &\quad = \|x - y\|^2 - 2\gamma_n \tau_n \|x - y\|^2 - 2\gamma_n \langle x - y, \nabla f^S x - \nabla f^S y\rangle \\ &\qquad + \gamma_n^2 (\|\nabla f^S x - \nabla f^S y\|^2 + 2\tau_n \langle x - y, \nabla f^S x - \nabla f^S y\rangle + \tau_n^2 \|x - y\|^2) \\ &\quad \leq (1 - 2\gamma_n \tau_n + \gamma_n^2 \tau_n^2)\|x - y\|^2 - (2\gamma_n - 2\gamma_n^2 \tau_n - 2\gamma_n^2 \|A\|^2)\langle x - y, \nabla f^S x - \nabla f^S y\rangle \\ &\quad \leq (1 - 2\gamma_n \tau_n + \gamma_n^2 \tau_n^2)\|x - y\|^2 = (1 - \gamma_n \tau_n)^2 \|x - y\|^2 \leq \|x - y\|^2, \end{aligned} \tag{14}$$

as asserted. □

**Lemma 3.3.** *The sequence* $\{x_n\}$ *generated by* (12) *is bounded.*

**Proof.** Let $p \in \Gamma$. Then $\nabla f^S p = 0$. It follows from (12) and (14) that

$$\begin{aligned} \|u_n - p\| &= \|P_C(x_n - \gamma_n \nabla f^{S\tau_n}(x_n)) - P_C(p - \gamma_n \nabla f^S p)\| \\ &= \|P_C(x_n - \gamma_n \nabla f^{S\tau_n}(x_n)) - P_C(p - \gamma_n \nabla f^{S\tau_n} p)\| \\ &\quad + \|P_C(p - \gamma_n \nabla f^{S\tau_n} p) - P_C(p - \gamma_n \nabla f^S p)\| \\ &\leq \|x_n - p\| + \|(p - \gamma_n \nabla f^{S\tau_n} p) - (p - \gamma_n \nabla f^S p)\| \\ &\leq \|x_n - p\| + \gamma_n \tau_n \|p\|. \end{aligned} \tag{15}$$

Using Lemma 2.7 and (15), we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n)u_n + \alpha_n T((1 - \beta_n)u_n + \beta_n T u_n) - p\| \\ &\leq \|u_n - p\| \leq \|x_n - p\| + \gamma_n \tau_n \|p\|. \end{aligned}$$

Note that $\gamma_n < \frac{1}{\|A\|^2}$. Therefore $\sum_{n=1}^{\infty} \gamma_n \tau_n \|p\| < \infty$. Using Lemma 2.6, we now see that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \Gamma$. This implies that $\{x_n\}$ is bounded, as asserted. □

**Theorem 3.4.** *The sequence $\{x_n\}$ generated by Algorithm* 3.1 *converges weakly to a point in* $\Gamma$.

**Proof.** Let $x \in \Gamma$ and set $t_n := x_n - \gamma_n \nabla f^{S\tau_n}(x_n)$. It follows from Lemma 2.7 that

$$\begin{aligned} \|x_{n+1} - x\|^2 &= \|(1-\alpha_n)u_n + \alpha_n T((1-\beta_n)u_n + \beta_n T u_n) - x\|^2 \\ &\le \|u_n - x\|^2 - \alpha_n(\beta_n - \alpha_n)(1-\beta_n L)^2 \|T u_n - u_n\|^2. \end{aligned} \tag{16}$$

Note that from (10) it follows that

$$\|u_n - x\|^2 = \|P_C t_n - x\|^2 \le \|t_n - x\|^2 - \|t_n - P_C t_n\|^2. \tag{17}$$

Since by (9) and (10),

$$\begin{aligned} -\langle x_n - x, \nabla f^S x_n\rangle &= -\langle A x_n - Ax, A x_n - S P_Q A x_n\rangle \\ &= -\langle S P_Q A x_n - Ax, A x_n - S P_Q A x_n\rangle - \|A x_n - S P_Q A x_n\|^2 \\ &= \frac{1}{2}(\|S P_Q A x_n - S P_Q A x\|^2 + \|A x_n - S P_Q A x_n\|^2 - \|A x_n - Ax\|^2) \\ &\quad - \|A x_n - S P_Q A x_n\|^2 \\ &\le \frac{1}{2}(\|P_Q A x_n - P_Q A x\|^2 + \|A x_n - S P_Q A x_n\|^2 - \|A x_n - Ax\|^2) \\ &\quad - \|A x_n - S P_Q A x_n\|^2 \\ &\le \frac{1}{2}(\|A x_n - Ax\|^2 - \|P_Q A x_n - A x_n\|^2 + \|A x_n - S P_Q A x_n\|^2 - \|A x_n - Ax\|^2) \\ &\quad - \|A x_n - S P_Q A x_n\|^2 \\ &= -\frac{1}{2}(\|A x_n - S P_Q A x_n\|^2 + \|P_Q A x_n - A x_n\|^2), \end{aligned}$$

we have

$$\begin{aligned} \|t_n - x\|^2 &= \|x_n - \gamma_n \nabla f^{S\tau_n} x_n - x\|^2 = \|(x_n - x) - \gamma_n \nabla f^S x_n - \gamma_n \tau_n x_n\|^2 \\ &= \|(x_n - x) - \gamma_n \nabla f^S x_n\|^2 + \gamma_n^2 \tau_n^2 \|x_n\|^2 - 2\gamma_n \tau_n \langle x_n - x - \gamma_n \nabla f^S x_n, x_n\rangle \\ &= \|x_n - x\|^2 - 2\gamma_n \langle x_n - x, \nabla f^S x_n\rangle + \gamma_n^2 \|\nabla f^S x_n\|^2 + \gamma_n^2 \tau_n^2 \|x_n\|^2 \\ &\quad - 2\gamma_n \tau_n \langle x_n - x - \gamma_n \nabla f^S x_n, x_n\rangle \\ &\le \|x_n - x\|^2 - \gamma_n(1 - \gamma_n \|A\|^2)\|A x_n - S P_Q A x_n\|^2 - \gamma_n \tau_n \|x_n\|^2 (2 - \gamma_n \tau_n) \\ &\quad - \gamma_n \|P_Q A x_n - A x_n\|^2 + 2\gamma_n \tau_n \langle x + \gamma_n \nabla f^S x_n, x_n\rangle. \end{aligned} \tag{18}$$

Substituting (18) in (17), we obtain

$$\begin{aligned} \|u_n - x\|^2 &\le \|x_n - x\|^2 - \gamma_n(1 - \gamma_n \|A\|^2)\|A x_n - S P_Q A x_n\|^2 - \gamma_n \tau_n \|x_n\|^2 (2 - \gamma_n \tau_n) \\ &\quad - \gamma_n \|P_Q A x_n - A x_n\|^2 + 2\gamma_n \tau_n \langle x + \gamma_n \nabla f^S x_n, x_n\rangle - \|t_n - P_C t_n\|^2. \end{aligned} \tag{19}$$

Next, substituting (19) in (16), we have

$$\begin{aligned}\|x_{n+1}-x\|^2 \le \|x_n-x\|^2 &- \gamma_n(1-\gamma_n\|A\|^2)\|Ax_n-SP_QAx_n\|^2 - \gamma_n\tau_n\|x_n\|^2(2-\gamma_n\tau_n)\\ &- \gamma_n\|P_QAx_n - Ax_n\|^2 + 2\gamma_n\tau_n\langle x+\gamma_n\nabla f^S x_n, x_n\rangle\\ &- \|t_n - P_Ct_n\|^2 - \alpha_n(\beta_n-\alpha_n)(1-\beta_nL)^2\|Tu_n-u_n\|^2. \qquad (20)\end{aligned}$$

Since the sequence $\{x_n\}$ is bounded, there exists a number $M > 0$ such that we have $\langle x + \gamma_n\nabla f^S x_n, x_n\rangle \le M$. Therefore it follows from (20) that

$$\begin{aligned}&\gamma_n(1-\gamma_n\|A\|^2)\|Ax_n - SP_QAx_n\|^2 + \gamma_n\|P_QAx_n - Ax_n\|^2\\ &\qquad + \|t_n - P_Ct_n\|^2 + \alpha_n(\beta_n-\alpha_n)(1-\beta_nL)^2\|Tu_n-u_n\|^2\\ &\le \|x_n-x\|^2 - \|x_{n+1}-x\|^2 - \gamma_n\tau_n\|x_n\|^2(2-\gamma_n\tau_n) + 2\gamma_n\tau_nM. \qquad (21)\end{aligned}$$

Taking the limit of (21) as $n\to\infty$, we see that

$$\|Ax_n - SP_QAx_n\| \to 0 \text{ as } n\to\infty, \qquad (22)$$

$$\|P_QAx_n - Ax_n\| \to 0 \text{ as } n\to\infty, \qquad (23)$$

$$\|t_n - P_Ct_n\| \to 0 \text{ as } n\to\infty \qquad (24)$$

and

$$\|Tu_n - u_n\| \to 0 \text{ as } n\to\infty. \qquad (25)$$

It follows from (22) that

$$\begin{aligned}\|t_n - x_n\| &= \|x_n - \gamma_n\nabla f^{S\tau_n}(x_n) - x_n\|\\ &\le \gamma_n\|A^*\|\|Ax_n - SP_QAx_n\| + \gamma_n\tau_n\|x_n\| \to 0 \quad \text{as } n\to\infty. \qquad (26)\end{aligned}$$

Using (25) and the fact that $T$ is $L$-Lipschitzian, we find that

$$\begin{aligned}\|x_{n+1}-u_n\| &= \alpha_n\|T((1-\beta_n)u_n+\beta_nTu_n) - u_n\|\\ &\le \alpha_n\big(L\|(1-\beta_n)u_n + \beta_nTu_n - u_n\| + \|Tu_n-u_n\|\big)\\ &= \alpha_n(L\beta_n+1)\|Tu_n-u_n\| \to 0 \text{ as } n\to\infty. \qquad (27)\end{aligned}$$

Furthermore, from (22) and (23) it follows that

$$\|P_QAx_n - SP_QAx_n\| \to 0 \text{ as } n\to\infty.$$

Using (24), (26) and (27), we obtain that

$$\|x_{n+1}-x_n\| \to 0 \text{ as } n\to\infty.$$

Since the sequence $\{x_n\}$ is bounded, it has weakly convergent subsequences. Let $\{x_{n_i}\}$ of $\{x_n\}$ be such a subsequence and let $p$ be its weak limit. Then we have $u_{n_i} \rightharpoonup p$, $t_{n_i} \rightharpoonup p$ and $Ax_{n_i} \rightharpoonup Ap$ as $i\to\infty$. Since $I-T$ and $I-S$ are demiclosed at zero by Lemma 2.4, it follows that $p\in Fix(T)$ and $Ap \in Fix(S)$. Furthermore, since $C$ and $Q$ are weakly closed, it also follows that $p\in C$ and $Ap\in Q$. Thus we have shown that $p\in\Gamma$. Note that $\Gamma$ is nonempty, closed and convex by its definition and Lemma 2.2. Since $\lim_{n\to\infty}\|x_n-q\|$ exists for each $q\in\Gamma$ according to the proof of Lemma 3.3, it follows from Lemma 2.8 that $\{x_n\}$ converges weakly to a point in $\Gamma$, as asserted. □

We next present three consequences of our main result.

**Corollary 3.5.** *Let $C$ and $Q$ be nonempty, closed and convex subsets of the real Hilbert spaces $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and let $A^* : H_2 \to H_1$ be the adjoint of $A$. Let $T : H_1 \to H_1$ be a nonexpansive mapping. Suppose that $\Gamma_1 := \{x^* \in H_1 : x^* \in C \cap Fix(T), Ax^* \in Q\} \neq \emptyset$, let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_1 \in C, \\ u_n = P_C(x_n - \gamma_n \nabla f^{\tau_n}(x_n)), \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n T((1-\beta_n)u_n + \beta_n T u_n), \quad n \in \mathbb{N}, \end{cases} \tag{28}$$

*where* $\qquad \nabla f^{\tau_n}(x_n) = A^*(I - P_Q)Ax_n + \tau_n x_n,$

*and let the following conditions be satisfied:*

(i) $\tau_n \in [0,1)$ *with* $\sum_{n=1}^{\infty} \tau_n < \infty$;

(ii) $0 < a \leq \gamma_n \leq b < \frac{1}{\tau_n + \|A\|^2}$;

(iii) $0 < c < \alpha_n < d < \beta_n < e < \frac{1}{1+\sqrt{2}}$.

*Then the sequence $\{x_n\}$ converges weakly to a point in $\Gamma_1$.*

**Proof.** Let $S$ in Algorithm 3.1 be the identity mapping. Then (12) becomes (28). Since a nonexpansive mapping is, by definition, 1-Lipschitzian, this result is seen to immediately follow from the proof of Theorem 3.4. □

**Corollary 3.6.** *Let $C$ and $Q$ be nonempty, closed and convex subsets of the real Hilbert spaces $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and let $A^* : H_2 \to H_1$ be the adjoint of $A$. Let $S : H_2 \to H_2$ be a nonexpansive mapping and $T : H_1 \to H_1$ be an $L$-Lipschitzian pseudocontractive mapping with $L \geq 1$. Suppose that $\Gamma_2 := \{x^* \in H_1 : x^* \in C \cap Fix(T),\ Ax^* \in Q \cap Fix(S)\} \neq \emptyset$, let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_1 \in C, \\ u_n = P_C(x_n - \gamma_n \nabla f^S(x_n)), \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n T((1-\beta_n)u_n + \beta_n T u_n), \quad n \in \mathbb{N}, \end{cases}$$

*where* $\qquad \nabla f^S(x_n) = A^*(I - SP_Q)Ax_n,$

*and let the following conditions be satisfied:*

(i) $0 < a \leq \gamma_n \leq b < \frac{1}{\|A\|^2}$;

(ii) $0 < c < \alpha_n < d < \beta_n < e < \frac{1}{1+\sqrt{1+L^2}}$.

*Then the sequence $\{x_n\}$ converges weakly to a point in $\Gamma_2$.*

**Proof.** This result follows from Theorem 3.4 by letting $\tau_n = 0$ in Algorithm 3.1. □

**Corollary 3.7.** *Let $H_1$, $H_2$ be real Hilbert spaces, and let $C$ and $Q$ be nonempty, closed and convex subsets of $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of $A$. Suppose that $\Gamma_3 := \{x^* \in H_1 : x^* \in C, Ax^* \in Q\} \neq \emptyset$, let a sequence $\{x_n\}$ be generated by*

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C(x_n - \gamma_n \nabla f^{S\tau_n}(x_n)), \quad n \in \mathbb{N}, \end{cases} \tag{29}$$

*where* $\qquad \nabla f^{S\tau_n}(x_n) = A^*(I - SP_Q)Ax_n + \tau_n x_n,$

*and let the following conditions be satisfied:*

(i) $\tau_n \in [0,1)$ *with* $\sum_{n=1}^{\infty} \tau_n < \infty$;

(ii) $0 < a \leq \gamma_n \leq b < \frac{1}{\tau_n + \|A\|^2}$.

*Then the sequence $\{x_n\}$ converges weakly to a point in $\Gamma_3$.*

**Proof.** Let $S = T = I$ in Algorithm 3.1, where $I$ is the identity mapping. Then (12) becomes (29). Thus the result follows from Theorem 3.4. □

## 4. An application

### 4.1. Split feasibility and convex minimization problem

Let $C$ and $Q$ be nonempty, closed and convex subsets of the real Hilbert spaces $H_1$ and $H_2$, respectively. Let $g : H_1 \to \mathbb{R}$ be a convex and differentiable function with Lipschitz continuous gradient $\nabla g$ and let $A : H_1 \to H_2$ be a bounded linear operator. We consider the following split feasibility and convex minimization problem (SFCMP):

$$\text{Find } x^* \in H_1 \text{ such that } \quad x^* \in C \cap \arg\min_{x \in H_1} g(x) \text{ and } Ax^* \in Q. \tag{30}$$

The SFCMP and its generalizations have recently been studied by many authors; see, for instance, [1, 21] and references therein. The SFCMP serves as a model for some applied problems in image processing and signal recovery such as finding the minimum energy for bandlimited signals [3, 14] and constrained denoising problems [12].

It is known that the mapping $\nabla g : H_1 \to H_1$ satisfies the following inequality (see [17]):

$$\langle \nabla g(x) - \nabla g(y), x - y \rangle \geq \frac{1}{L}\|\nabla g(x) - \nabla g(y)\|^2, \tag{31}$$

where $L$ is the Lipschitz constant of $\nabla g$. Thus it follows from (31) that $\nabla g$ is monotone and consequently, $I - \nabla g$ is a Lipschitzian pseudocontractive mapping with Lipschitz constant $(1 + L)$.

The SFCMP (30) can be recast as follows:

$$\text{Find } x^* \in H_1 \text{ such that } \quad x^* \in C \cap \text{zer } (\nabla g) \text{ and } Ax^* \in Q.$$

We therefore obtain the following theorem regarding the approximation of solutions to the SFCMP (30).

**Theorem 4.1.** *Let $H_1$, $H_2$ be real Hilbert spaces, and let $C$ and $Q$ be nonempty, closed and convex subsets of $H_1$ and $H_2$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and let $A^* : H_2 \to H_1$ be the adjoint of $A$. Let $g : H_1 \to \mathbb{R}$ be a convex and differentiable function with Lipschitz continuous gradient $\nabla g$, and let $L$ be the Lipschitz constant of $\nabla g$. Suppose that*

$$\Gamma_4 := \{x^* \in H_1 : x^* \in C \cap \arg\min_{x \in H_1} g(x), Ax^* \in Q\} \neq \emptyset$$

*and let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in C, \\ u_n = P_C(x_n - \gamma_n \nabla f^{\tau_n}(x_n)), \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n T((1-\beta_n)u_n + \beta_n T u_n), n \in \mathbb{N}, \end{cases} \tag{32}$$

*where $T = I - \nabla g$, $\nabla f^{\tau_n}(x_n) = A^*(I - P_Q)Ax_n + \tau_n x_n$, and the following conditions are satisfied:*

(i) $\tau_n \in [0,1)$ *with* $\sum_{n=1}^{\infty} \tau_n < \infty$;

(ii) $0 < a \leq \gamma_n \leq b < \frac{1}{\tau_n + \|A\|^2}$;

(iii) $0 < c < \alpha_n < d < \beta_n < e < \frac{1}{1+\sqrt{1+(1+L)^2}}$.

*Then the sequence $\{x_n\}$ converges weakly to a point in $\Gamma_4$.*

**Proof.** Note that $T = I - \nabla g$ is a Lipschitzian pseudocontractive mapping with Lipschitz constant $(1+L)$, and that $Fix(T) = \text{zer}\ (\nabla g)$. Let $S = I$ in Algorithm 3.1. Then (12) reduces to (32). Therefore the result follows from Theorem 3.4. □

## 5. Numerical examples

In this section we present two numerical examples to illustrate the convergence of our algorithm and compare its performance with that of other algorithms having similar features. Our numerical experiments have been performed in Windows 10 using Matlab R2021b, and have run on an HP Laptop with Intel(R) Core(TM) i5 CPU and 4GB RAM.

**Example 5.1.** Let $H_1 = H_2$ be the real Hilbert space $\mathbb{R}^2$ equipped with the usual Euclidean inner product and let the induced norm be denoted by $\|\cdot\|$.
For $x = (a,b) \in H_1$, define $x^\perp$ to be $(b, -a)$. Let $C = \{x \in H_1 : \|x\| \leq 1\}$ and $Q = \{x \in H_2 : \|x\| \leq 2\}$. Define $T : C \to C$ by

$$Tx := \begin{cases} x + x^\perp & \text{if } \|x\| \leq \frac{1}{2}, \\ \frac{x}{\|x\|} - x + x^\perp & \text{if } \frac{1}{2} \leq \|x\| \leq 1, \end{cases}$$

and $S : H_2 \to H_2$ by $Sx := \frac{2}{3}x \quad \forall x \in H_2$. Then $T$ is an $L$-Lipschitzian pseudocontractive mapping with $L = 5$ (see [11]) and $S$ is nonexpansive. We define $A : H_1 \to H_2$ by $Ax := \frac{1}{3}x$ for all $x \in \mathbb{R}^2$, and $\tau_n = \frac{1}{n^2+1}$, $\gamma_n = \frac{2(n^2+1)}{n^2+10}$, $\delta_n = 0.05$, $\beta_n = 0.035$ and $\alpha_n = 0.01$ for all $n \geq 1$.

We make a few different choices of the initial value $x_1$ as follows:
*Case Ia:* $x_1 = (-0.1217, 0.5694)$;
*Case Ib:* $x_1 = (0.5501, -0.0234)$;
*Case Ic:* $x_1 = (0.2365, 0.4427)$;
*Case Id:* $x_1 = (-0.2921, -0.0276)$.

We compare the performance of our Algorithm 3.1 (Ade) with algorithm (8) (Won). The stopping criterion used for our computations is error $= \|x_{n+1} - x_n\| < 10^{-10}$. We plot the graphs of errors against the number of iterations in each case. The figures and the numerical results we obtained are shown in Figure 5.1 and Table 5.1, respectively.

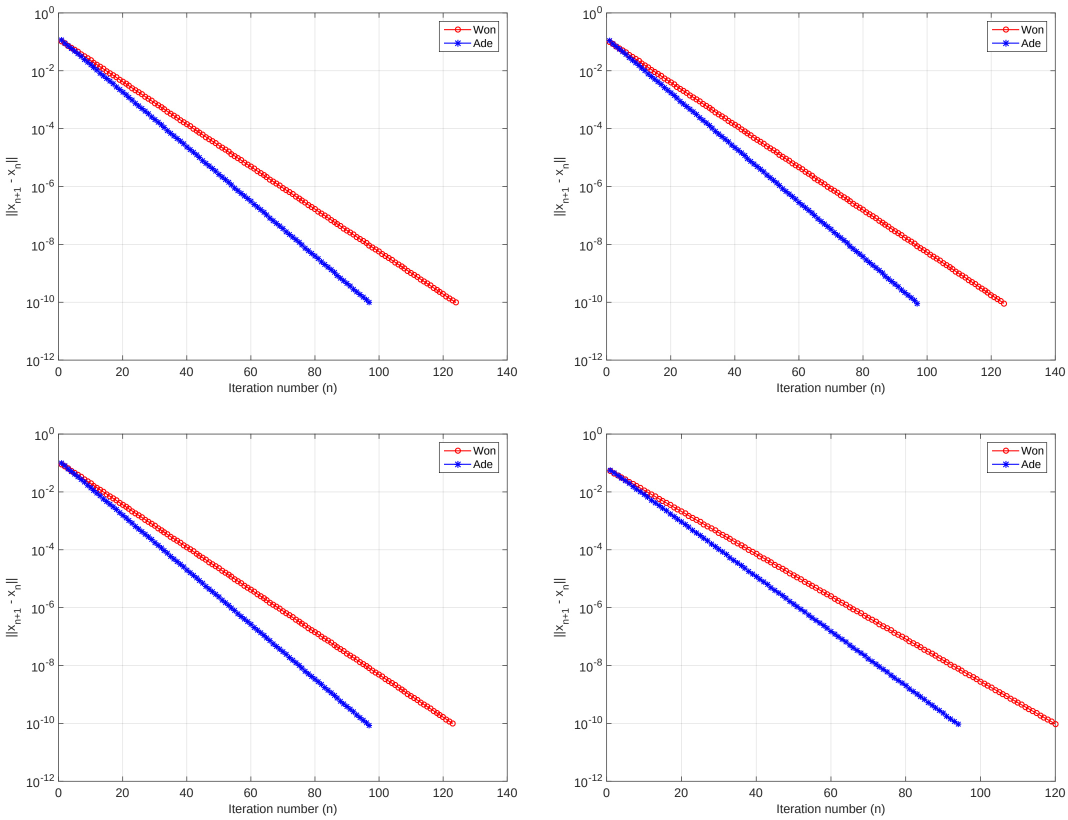


Figure 5.1: Top left: Case Ia; Top right: Case Ib;
Bottom left: Case Ic; Bottom right: Case Id.

**Example 5.2.** Let $H_1 = \mathbb{R}$ and $H_2 = \mathbb{R} \times \mathbb{R}$ equipped with the Euclidean inner products and the induced norms denoted by $|\cdot|$ and $\|\cdot\|$, respectively. Let $C = [0, \infty)$ and $Q = \{x \in H_2 : 1 \leq \|x\| \leq 3\}$. Define the mappings $T : H_1 \to H_1$ by

$$Tx := \begin{cases} x - 1 + \frac{4}{x+1} & \text{if } x \in [0, \infty), \\ 3 & \text{otherwise,} \end{cases}$$

and $S : H_2 \to H_2$ by

$$Sx := (\frac{-x_1}{2} + \frac{3}{2}, \frac{x_2}{3} + \frac{1}{3}) \quad \forall x = (x_1, x_2) \in H_2.$$

Table 5.1: Numerical results

| | | Ade | Won |
|---|---|---|---|
| Case Ia | CPU time (sec)<br>No of Iter. | 0.0155<br>97 | 0.0330<br>124 |
| Case Ib | CPU time (sec)<br>No. of Iter. | 0.0249<br>97 | 0.1233<br>124 |
| Case Ic | CPU time (sec)<br>No of Iter. | 0.0254<br>97 | 0.0332<br>123 |
| Case Id | CPU time (sec)<br>No of Iter. | 0.0088<br>94 | 0.0210<br>120 |

It is known (see [27]) that if $x, y \in [0, \infty)$, then

$$\langle Tx - Ty, x - y\rangle \leq |x - y|^2 \quad \text{and} \quad |Tx - Ty| \leq 5|x - y|.$$

Now take $x \in [0, \infty)$ and $y \in (-\infty, 0)$, and note that in this case $(x - y) > 0$ and $y - \frac{4x}{x+1} < 0$. Thus we have

$$\begin{aligned}\langle Tx - Ty, x - y\rangle &= \langle x - 1 + \frac{4}{x+1} - 3, x - y\rangle = \langle x - \frac{4x}{x+1}, x - y\rangle \\ &= |x - y|^2 + \langle y - \frac{4x}{x+1}, x - y\rangle < |x - y|^2.\end{aligned}$$

Also, $$|Tx - Ty| = |x - 1 + \frac{4}{x+1} - 3| = x|1 - \frac{4}{x+1}| < 5|x - y|.$$

The above inequalities obviously hold if $x, y \in (-\infty, 0)$. Therefore, it follows that $T$ is an $L$-Lipschitzian pseudocontractive mapping with $L = 5$. The mapping $S$ is nonexpansive. Let $A : H_1 \to H_2$ be defined by $Ax = (\frac{x}{3}, \frac{x}{6})$ for all $x \in H_1$ with $A^* : H_2 \to H_1$ defined by $A^*(y_1, y_2) = \frac{y_1}{3} + \frac{y_2}{6}$ for all $(y_1, y_2) \in H_2$.

In this example $\Gamma = \{3\}$. We choose

$$\tau_n = \tfrac{1}{n^2+1}, \ \gamma_n = \tfrac{2n}{n+5}, \ \delta_n = \tfrac{3n}{100n+1}, \ \beta_n = \tfrac{7n}{100n+1}, \ \alpha_n = \tfrac{13n}{100n+1} \text{ for all } n \geq 1.$$

Again we make a few different choices of the initial value $x_1$ as follows:

*Case IIa:* $x_1 = 6$;

*Case IIb:* $x_1 = 25.65$;

*Case IIc:* $x_1 = 98.22$;

*Case IId:* $x_1 = 0.222$.

We compare the performance of our Algorithm 3.1 (Ade) with that of algorithms (8) (Won) and (7) (Che). The stopping criterion used for our computations is error $= |x_{n+1} - x_n| < 10^{-7}$. In each case we plot the graphs of errors against the number of iterations. The figures and the numerical results we have obtained are shown in Figure 5.2 and Table 5.2, respectively.

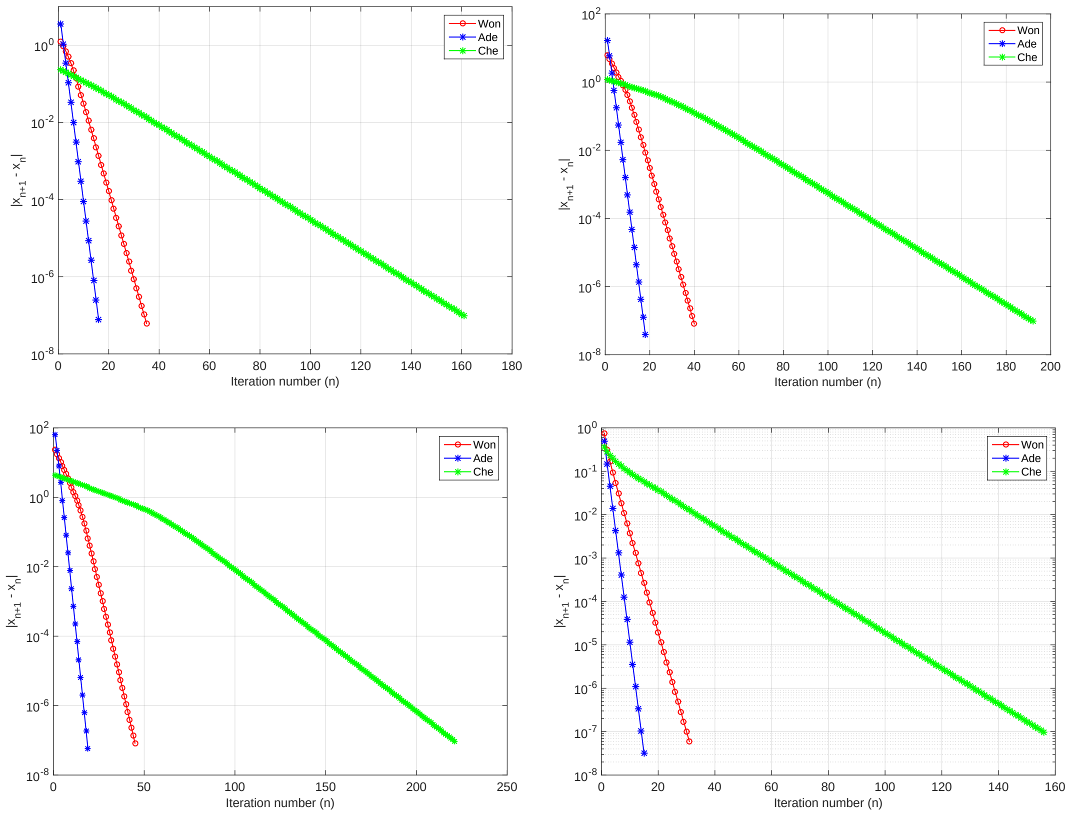


Figure 5.2: Top left: Case IIa; Top right: Case IIb;
Bottom left: Case IIc; Bottom right: Case IId.

Table 5.2: Numerical results

| | | Ade | Won | Che |
|---|---|---|---|---|
| Case IIa | CPU time (sec)<br>No of Iter. | 0.0011<br>16 | 0.0233<br>35 | 0.0044<br>161 |
| Case IIb | CPU time (sec)<br>No. of Iter. | 0.0012<br>18 | 0.0023<br>40 | 0.0030<br>192 |
| Case IIc | CPU time (sec)<br>No of Iter. | 0.0022<br>19 | 0.0238<br>45 | 0.0028<br>221 |
| Case IId | CPU time (sec)<br>No of Iter. | 0.0015<br>15 | 0.0025<br>31 | 0.0086<br>156 |

## 6. Conclusions

We have studied the split feasibility and fixed point problem for Lipschitzian pseudocontractive and nonexpansive mappings in real Hilbert spaces. By combining the gradient-projection method with Ishikawa iterations, we have proposed a new iterative scheme that involves the computation of just two metric projections per iteration. We have established a weak convergence theorem and have given an application of our main result. The numerical results we have obtained confirm that our algorithm performs better than the existing algorithms for solving the problem in terms of both the number of iterations and the computation time. Our result improves and complements many existing results in the literature.

As part of our future research in this direction, we intend to extend our study by proposing a new algorithm with a more general form of the Tikhonov regularization (see, for example, [16]). We also intend to examine the convergence rate of our algorithm and the influence of the decay of the regularization parameter on its performance. Moreover, we intend to perform numerical experiments in higher and infinite dimensional spaces, and investigate the efficiency of our algorithm when it is applied to problems which are derived by discretizing problems that are posed in such spaces.

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