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Existence of Bounded Solutions for some Quasilinear Degenerate Elliptic Systems

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We prove the existence of a bounded solution to a quasilinear system of degenerate equations. The main assumption asks the off-diagonal coefficients to have a “butterfly” support.

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1. Introduction

We consider the boundary value problem

$$\begin{cases} -\operatorname{div}(a(x,u(x))Du(x)) = f(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, $f,u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^{N^2 n^2}$ is matrix valued with components $a_{i,j}^{\alpha,\beta}(x,u)$ where $i,j \in \{1,...,n\}$ and $\alpha,\beta \in \{1,\dots,N\}$. Note that the first line in (1) is a system of N equations of the form

$$-\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} \sum_{\beta=1}^{N} a_{i,j}^{\alpha,\beta}(x,u) D_j u^\beta \right) = f^\alpha \qquad \alpha = 1,\dots,N. \tag{2}$$

We want to show the existence of a bounded weak solution $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ for the problem (1) under suitable assumptions on the coefficients $a_{i,j}^{\alpha,\beta}(x,u)$ and on the datum f. The main feature in this study is a kind of degenerate ellipticity, that is, ellipticity that vanishes when $|u|$ is large. This topic has been studied when (2) is a single equation, that is, $N = 1$: a priori estimates are contained in [2], existence results can be found in [9, 10, 14, 34, 40], uniqueness is studied in [38], lower order terms appear in [8, 11, 12, 15, 16, 18], the nonlinear case has been dealt with in

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[1, 11], regularity for the gradient is contained in [24], the anisotropic case can be found in [5, 22]. Let us also mention [23], where the right hand side is in divergence form, that is $f = -\mathrm{div} F$, and [4], where the biharmonic operator is considered.

In the vectorial case $N \geq 2$, existence of bounded solutions for elliptic systems (2) is a delicate matter: De Giorgi counterexample says that, in general, it cannot happen, see section 6. In order to keep away such a counterexample, we need some additional assumptions on coefficients $a_{i,j}^{\alpha,\beta}(x,u)$: to this aim, we require that the support of $u \to a_{i,j}^{\alpha,\beta}(x,u)$ is confined in a "butterfly" set, see assumption $(\mathcal{A}_3)$ and Figure 2.1 in the sequel. Precise assumptions and our result are in section 2; in order to prove it, we use an approximation argument: we consider auxiliary non degenerate problems and we derive estimates in section 3; then, in section 4, we pass to the limit and we get the result for the degenerate system. An example is given in section 5; such an example shows that the aforementioned condition on the support helps us to deal with systems that cannot be treated using the structure conditions employed in [7, 19, 20, 28, 42]. Section 6 is devoted to De Giorgi counterexample and its application to the present situation.

2. Assumptions and a first result

For all $i, j \in \{1, \dots, n\}$ and all $\alpha, \beta \in \{1, \dots, N\}$ we assume that $a_{i,j}^{\alpha,\beta} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:

$(\mathcal{A}_0)$ $x \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ is measurable and $y \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ is continuous;

$(\mathcal{A}_1)$ (boundedness of all the coefficients) there exists $c > 0$ such that $|a_{i,j}^{\alpha,\beta}(x,y)| \leq c$ for almost every $x \in \Omega$ and for all $y \in \mathbb{R}^N$;

$(\mathcal{A}_2)$ (degenerate ellipticity of all the coefficients) there exist constants $\nu > 0$ and $\theta \in (0,1)$ such that

$$\sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,y) \xi_i^\alpha \xi_j^\beta \geq \nu \sum_{\alpha=1}^{N} \frac{|\xi^\alpha|^2}{(1+|y^\alpha|)^\theta}$$

for almost every $x \in \Omega$, for all $y \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{N \times n}$;

$(\mathcal{A}_3)$ (support of off-diagonal coefficients) there exists $L_0 > 0$ such that for all $L \geq L_0$ we have

$$(a_{i,j}^{\alpha,\beta}(x,y) \neq 0, \quad |y^\alpha| > L) \quad \Longrightarrow \quad |y^\beta| > L.$$

Please, note that $(\mathcal{A}_3)$ is always fulfilled when $\alpha = \beta$; on the contrary, when $\alpha \neq \beta$, $(\mathcal{A}_3)$ forces the support of $a_{i,j}^{\alpha,\beta}(x,y)$ to be contained in the grey region of Figure 2.1.

The assumptions on the support of off-diagonal coefficients are sometimes used when dealing with elliptic systems: in [37] off-diagonal coefficients $a_{i,j}^{\alpha,\beta}(x,y)$ are required to vanish when y^α is large; in [41] the system is assumed to be tridiagonal, that is, $a_{i,j}^{\alpha,\beta} = 0$ for $\beta > \alpha$; [32] and [33] require that the support of $a_{i,j}^{\alpha,\beta}(x,y)$ is confined in squares along the $y^\alpha = \pm y^\beta$ diagonals; in [30] various conditions on the support are given; [29] requires a stronger condition than the one in the present work.

As far as f is concerned, we assume that $f \in L^m(\Omega, \mathbb{R}^N)$ with $m > \frac{n2}{n+2}$; this implies that $v \mapsto \int_\Omega fv$ is a linear and continuous functional on $W_0^{1,2}(\Omega, \mathbb{R}^N)$. Note that, when $n \geq 3$, $\frac{n2}{n+2} = (2^*)'$, where $2^* = \frac{n2}{n-2}$ is the Sobolev exponent and $p' = \frac{p}{p-1}$ is the Hölder conjugate of p.

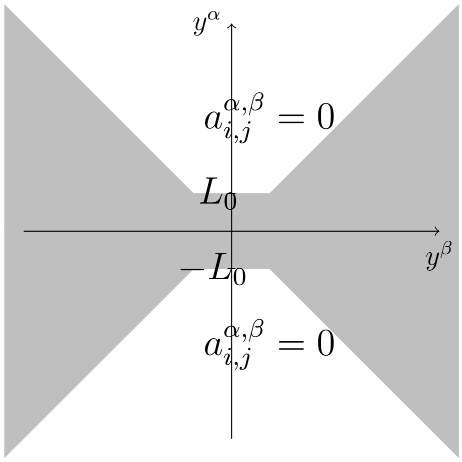

Figure 2.1: Assumption $(\mathcal{A}_3)$: off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ vanish on the white part of the picture; they might be non zero only on the grey part.

Theorem 2.1. *Under the assumptions* $(\mathcal{A}_0)$, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, *and* $(\mathcal{A}_3)$, *with* $n \geq 3$, $f \in L^m(\Omega, \mathbb{R}^N)$ *and* $\frac{n}{2} < m \leq n$, *there exists a bounded weak solution* u *of the problem* (1), *that is, there exists* $u \in W_0^{1,2}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ *such that*

$$\int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x, u(x)) D_j u^\beta(x) D_i \varphi^\alpha(x) dx = \int_\Omega \sum_{\alpha=1}^{N} f^\alpha(x) \varphi^\alpha(x) dx$$

for all $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$.

Remark 1. If we drop assumption $(\mathcal{A}_3)$, then Theorem 2.1 is false, in general, see section 6.

Remark 2. Assumption $(\mathcal{A}_3)$ is used at the beginning of the proof in order to get equality (7).

Remark 3. Note that our result is valid for $\theta < 1$. Indeed, in the present vectorial situation we cannot argue as in [9] and [1], where the scalar case and $\theta \leq 1$ is dealt with. We take advantage of the technique in [14] that requires $\theta < 1$.

Remark 4. An example for our Theorem 2.1 is given in section 5.

Remark 5. Assumption $(\mathcal{A}_2)$ says that degeneracy in the α equation of system (2) occurs when u^α is large; on the contrary, in [13] degeneracy in the α equation occurs when u^β is large, with $\beta \neq \alpha$ and $\alpha, \beta \in \{1, 2\}$.

Remark 6. Our Theorem 2.1 deals with right hand side $f \in L^m$ when integrability exponent m is bigger than $n/2$. It would be nice to deal with case $m \leq n/2$: we leave it to some future work.

3. Approximation and estimates

We set $\tilde{a}^{\alpha,\beta}_{i,j,k}(x,y) = a^{\alpha,\beta}_{i,j}(x,y) + \frac{1}{k}\delta_{\alpha,\beta}\delta_{i,j}$ with

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

We consider the following family of approximate problems

$$(\tilde{\mathcal{P}}_k) \qquad \begin{cases} -\sum_{i=1}^n D_i \left(\sum_{j=1}^n \sum_{\beta=1}^N \tilde{a}^{\alpha,\beta}_{i,j,k}(x,u_k) D_j u_k^\beta \right) = f^\alpha, & x \in \Omega \\ u_k = 0, & x \in \partial\Omega. \end{cases}$$

We want to show the existence of a weak solution for each problem $(\tilde{\mathcal{P}}_k)$, that is a function $u_k \in W^{1,2}_0(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \tilde{a}^{\alpha,\beta}_{i,j,k}(x, u_k(x)) D_j u_k^\beta(x) D_i \varphi^\alpha(x) dx = \int_\Omega \sum_{\alpha=1}^N f^\alpha(x)\varphi^\alpha(x) dx \tag{3}$$

for all $\varphi \in W^{1,2}_0(\Omega, \mathbb{R}^N)$. Let us first give some properties of the coefficients $\tilde{a}^{\alpha,\beta}_{i,j,k}$. From assumption $(\mathcal{A}_1)$ it follows that

$$(\tilde{\mathcal{A}}_1) \qquad |\tilde{a}^{\alpha,\beta}_{i,j,k}(x,y)| \leq c + 1.$$

Using assumption $(\mathcal{A}_2)$ we have the following non degenerate ellipticity condition

$$(\tilde{\mathcal{A}}_2) \qquad \begin{aligned} &\sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \tilde{a}^{\alpha,\beta}_{i,j,k}(x,y)\xi_i^\alpha \xi_j^\beta \\ &= \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a^{\alpha,\beta}_{i,j}(x,y)\xi_i^\alpha \xi_j^\beta + \frac{1}{k} \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \delta_{\alpha,\beta}\delta_{i,j}\xi_i^\alpha \xi_j^\beta \\ &\geq \nu \sum_{\alpha=1}^N \frac{|\xi^\alpha|^2}{(1+|y^\alpha|)^\theta} + \frac{1}{k}|\xi|^2. \end{aligned}$$

In the last line, $\frac{1}{k}|\xi|^2$ makes $\tilde{a}^{\alpha,\beta}_{i,j,k}$ non degenerate, so we can apply the surjectivity result of Leray-Lions, see [35]. This gives the existence of a weak solution u_k for the problem $\tilde{\mathcal{P}}_k$, that is, there exists $u_k \in W^{1,2}_0(\Omega, \mathbb{R}^N)$ such that (3) holds true for every $\varphi \in W^{1,2}_0(\Omega, \mathbb{R}^N)$. In Lemma 3.1, arguing as in [14], we prove that weak solutions u_k to problem $\tilde{\mathcal{P}}_k$ are bounded. We first recall the following elementary inequalities that will be used in the proof of Lemma 3.1. We have

$$\sum_{\alpha=1}^M a_\alpha^p \leq M \left(\sum_{\alpha=1}^M a_\alpha \right)^p, \tag{4}$$

and

$$\left(\sum_{\alpha=1}^M a_\alpha \right)^p \leq M^p \sum_{\alpha=1}^M (a_\alpha)^p, \tag{5}$$

provided $a_\alpha \geq 0$ for all $\alpha \in \{1, \dots, M\}$ and $p > 0$.

Lemma 3.1. *Assume that $f \in L^m(\Omega, \mathbb{R}^N)$ with $\frac{n}{2} < m \leq n$ and let u_k be a weak solution of $\tilde{\mathcal{P}}_k$. Then the norms of u_k in $L^\infty(\Omega, \mathbb{R}^N)$ and in $W_0^{1,2}(\Omega, \mathbb{R}^N)$ are bounded by a constant which depends only on L_0, θ, m, n, N, ν, $|\Omega|$ and the norm of f in $L^m(\Omega, \mathbb{R}^N)$.*

Proof. For every $L \geq \bar{L} := \max(1, L_0)$, let us define the function

$$G_L(s) = \begin{cases} 0 & \text{if } |s| \leq L \\ s - L\frac{s}{|s|} & \text{if } |s| > L, \end{cases}$$

and let us consider as test function in (3) the function $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ defined as

$$\varphi = (\varphi^1, \dots, \varphi^N) = (G_L(u_k^1), \dots, G_L(u_k^N)); \tag{6}$$

let us remark that, using the notation $A_{k,L}^\alpha := \{x \in \Omega : |u_k^\alpha(x)| > L\}$, we have

$$D_i\varphi^\alpha = D_i G_L(u_k^\alpha) = D_i u_k^\alpha 1_{A_{k,L}^\alpha},$$

where $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ otherwise. Using $(\mathcal{A}_3)$ we obtain

$$a_{i,j}^{\alpha,\beta}(x, u_k(x))1_{A_{k,L}^\alpha}(x) = a_{i,j}^{\alpha,\beta}(x, u_k(x))1_{A_{k,L}^\alpha}(x)1_{A_{k,L}^\beta}(x). \tag{7}$$

Now, let us also remark that by definition of the coefficients $\tilde{a}_{i,j,k}^{\alpha,\beta}$ and in view of (7) and $(\mathcal{A}_2)$, we have

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \tilde{a}_{i,j,k}^{\alpha,\beta}(x, u_k(x))D_j u_k^\beta(x) D_i\varphi^\alpha(x) = \\
&= \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n (a_{i,j}^{\alpha,\beta}(x, u_k(x)) + \frac{1}{k}\delta_{\alpha,\beta}\delta_{i,j}) D_j u_k^\beta(x) D_i u_k^\alpha(x) 1_{A_{k,L}^\alpha}(x) = \\
&= \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, u_k(x)) D_j u_k^\beta(x) 1_{A_{k,L}^\beta}(x) D_i u_k^\alpha(x) 1_{A_{k,L}^\alpha}(x) + \\
&\qquad + \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \frac{1}{k}\delta_{\alpha,\beta}\delta_{i,j} D_j u_k^\beta(x) D_i u_k^\alpha(x) 1_{A_{k,L}^\alpha}(x) \geq \\
&\geq \nu \sum_{\alpha=1}^N \frac{|Du_k^\alpha(x) 1_{A_{k,L}^\alpha}(x)|^2}{(1 + |u_k^\alpha(x)|)^\theta} + \sum_{\alpha=1}^N \sum_{i=1}^n \frac{1}{k}|D_i u_k^\alpha(x)|^2 1_{A_{k,L}^\alpha}(x) \geq \\
&\geq \nu \sum_{\alpha=1}^N \frac{|Du_k^\alpha(x)|^2 1_{A_{k,L}^\alpha}(x)}{(1 + |u_k^\alpha(x)|)^\theta}.
\end{aligned}$$

Therefore, if we substitute the test function (6) in (3) we get

$$\int_\Omega \sum_{\alpha=1}^N f^\alpha(x)\varphi^\alpha(x)dx =$$

$$= \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \tilde{a}_{i,j,k}^{\alpha,\beta}(x,u_k(x)) D_j u_k^\beta(x) D_i\varphi^\alpha(x)dx \geq$$

$$\geq \int_\Omega \nu \sum_{\alpha=1}^N \frac{|Du_k^\alpha|^2 1_{A_{k,L}^\alpha}}{(1+|u_k^\alpha|)^\theta} = \nu \sum_{\alpha=1}^N \int_{A_{k,L}^\alpha} \frac{|Du_k^\alpha|^2}{(1+|u_k^\alpha|)^\theta}.$$

From the previous inequality and using Hölder inequality we have

$$\nu \sum_{\alpha=1}^N \int_{A_{k,L}^\alpha} \frac{|Du_k^\alpha|^2}{(1+|u_k^\alpha|)^\theta} \leq \int_\Omega \sum_{\alpha=1}^N f^\alpha\varphi^\alpha = \sum_{\alpha=1}^N \int_{A_{k,L}^\alpha} f^\alpha G_L(u_k^\alpha) \leq$$

$$\leq \sum_{\alpha=1}^N \|f^\alpha\|_{L^m(A_{k,L}^\alpha)} \left(\int_{A_{k,L}^\alpha} |G_L(u_k^\alpha)|^{m'}\right)^{\frac{1}{m'}} \leq C_f \sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} |G_L(u_k^\alpha)|^{m'}\right)^{\frac{1}{m'}},$$

where $m' = \frac{m}{m-1}$ and $C_f := \|f\|_{L^m(\Omega)}$ (here and sometimes in the sequel, we drop the target space $\mathbb{R}^N$); then, we have in particular

$$\sum_{\alpha=1}^N \int_{A_{k,L}^\alpha} \frac{|Du_k^\alpha|^2}{(1+|u_k^\alpha|)^\theta} \leq \frac{C_f}{\nu} \sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} |G_L(u_k^\alpha)|^{m'}\right)^{\frac{1}{m'}}. \tag{8}$$

Now, for any $1 \leq \sigma < 2$, using Hölder's inequality with exponents $\frac{2}{\sigma}$ and $\frac{2}{2-\sigma}$, (4) and (8), we get

$$\sum_{\alpha=1}^N \int_{A_{k,L}^\alpha} |Du_k^\alpha|^\sigma = \sum_{\alpha=1}^N \int_{A_{k,L}^\alpha} \frac{|Du_k^\alpha|^\sigma}{(1+|u_k^\alpha|)^{\frac{\theta\sigma}{2}}} (1+|u_k^\alpha|)^{\frac{\theta\sigma}{2}} \leq$$

$$\leq \sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} \frac{|Du_k^\alpha|^2}{(1+|u_k^\alpha|)^\theta}\right)^{\frac{\sigma}{2}} \left(\int_{A_{k,L}^\alpha} (1+|u_k^\alpha|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{\frac{2-\sigma}{2}} \leq$$

$$\leq \sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} \frac{|Du_k^\alpha|^2}{(1+|u_k^\alpha|)^\theta}\right)^{\frac{\sigma}{2}} \cdot \sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} (1+|u_k^\alpha|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{\frac{2-\sigma}{2}} \leq$$

$$\leq N \left(\sum_{\alpha=1}^N \int_{A_{k,L}^\alpha} \frac{|Du_k^\alpha|^2}{(1+|u_k^\alpha|)^\theta}\right)^{\frac{\sigma}{2}} \cdot \sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} (1+|u_k^\alpha|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{\frac{2-\sigma}{2}} \leq$$

$$\leq N \left[\frac{C_f}{\nu} \sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} |G_L(u_k^\alpha)|^{m'}\right)^{\frac{1}{m'}}\right]^{\frac{\sigma}{2}} \cdot \sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} (1+|u_k^\alpha|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{\frac{2-\sigma}{2}}.$$

Since $\frac{n}{2} < m \leq n$, we can choose $\sigma = \dfrac{n\,m}{n\,m + m - n}$; it results that $1 \leq \sigma < 2$ and the exponent $\sigma^* = \dfrac{n\sigma}{n-\sigma}$ is equal to m'.

A simple use of the Sobolev embedding theorem gives

$$\|G_L(u_k^\alpha)\|_{L^{m'}(A_{k,L}^\alpha)} = \|G_L(u_k^\alpha)\|_{L^{\sigma^*}(A_{k,L}^\alpha)} = \|G_L(u_k^\alpha)\|_{L^{\sigma^*}(\Omega)} \tag{9}$$

$$\leq C_S\|DG_L(u_k^\alpha)\|_{L^\sigma(\Omega)} = C_S\|DG_L(u_k^\alpha)\|_{L^\sigma(A_{k,L}^\alpha)} = C_S\|Du_k^\alpha\|_{L^\sigma(A_{k,L}^\alpha)},$$

where $C_S := (n-1)\sigma/(n-\sigma)$. Then, using (5) and the last two inequalities, we get

$$\left(\sum_{\alpha=1}^N \|Du_k^\alpha\|_{L^\sigma(A_{k,L}^\alpha)}\right)^\sigma \leq N^\sigma \sum_{\alpha=1}^N \|Du_k^\alpha\|_{L^\sigma(A_{k,L}^\alpha)}^\sigma = N^\sigma \sum_{\alpha=1}^N \int_{A_{k,L}^\alpha} |Du_k^\alpha|^\sigma \leq$$

$$\leq N^{\sigma+1}\left(\frac{C_f C_S}{\nu}\right)^{\frac{\sigma}{2}} \left(\sum_{\alpha=1}^N \|Du_k^\alpha\|_{L^\sigma(A_{k,L}^\alpha)}\right)^{\frac{\sigma}{2}} \cdot \sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} (1+|u_k^\alpha|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{\frac{2-\sigma}{2}};$$

then

$$\left(\sum_{\alpha=1}^N \|Du_k^\alpha\|_{L^\sigma(A_{k,L}^\alpha)}\right)^{\frac{\sigma}{2}} \leq N^{\sigma+1}\left(\frac{C_f C_S}{\nu}\right)^{\frac{\sigma}{2}} \sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} (1+|u_k^\alpha|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{\frac{2-\sigma}{2}}$$

therefore

$$\left(\sum_{\alpha=1}^N \|Du_k^\alpha\|_{L^\sigma(A_{k,L}^\alpha)}\right)^\sigma \leq N^{2\sigma+2}\left(\frac{C_f C_S}{\nu}\right)^\sigma \left[\sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} (1+|u_k^\alpha|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{\frac{2-\sigma}{2}}\right]^2.$$

Then, using (4) and (5) we have

$$\sum_{\alpha=1}^N \|Du_k^\alpha\|_{L^\sigma(A_{k,L}^\alpha)}^\sigma \leq N\left(\sum_{\alpha=1}^N \|Du_k^\alpha\|_{L^\sigma(A_{k,L}^\alpha)}\right)^\sigma \leq$$

$$\leq N^{2\sigma+3}\left(\frac{C_f C_S}{\nu}\right)^\sigma \left[\sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} (1+|u_k^\alpha|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{\frac{2-\sigma}{2}}\right]^2 \leq \tag{10}$$

$$\leq N^{2\sigma+5}\left(\frac{C_f C_S}{\nu}\right)^\sigma \sum_{\alpha=1}^N \left(\int_{A_{k,L}^\alpha} (1+|u_k^\alpha|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{2-\sigma}.$$

Now, for any $\alpha = 1, \ldots, N$ we want to give an estimate from above of the following integral

$$\left(\int_{A_{k,L}^\alpha} (1+|u_k^\alpha|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{2-\sigma} \tag{11}$$

depending on $\|Du_k^\alpha\|_{L^\sigma(A_{k,L}^\alpha)}$ and on the measure of $A_{k,L}^\alpha$. First, let us remark that, since $L \geq \max(1, L_0) \geq 1$, it results that

$$1+|u_k^\alpha(x)| \leq 2(L+|G_L(u_k^\alpha(x))|), \qquad \text{for all } x \in A_{k,L}^\alpha. \tag{12}$$

Then, using (5) we have

$$\begin{aligned}
\left(\int_{A^\alpha_{k,L}} (1+|u^\alpha_k|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{2-\sigma} &\le \left(\int_{A^\alpha_{k,L}} 2^{\frac{\theta\sigma}{2-\sigma}}(L+|G_L(u^\alpha_k)|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{2-\sigma}\\
&\le \left(\int_{A^\alpha_{k,L}} 2^{\frac{\theta\sigma}{2-\sigma}}2^{\frac{\theta\sigma}{2-\sigma}}(L^{\frac{\theta\sigma}{2-\sigma}}+|G_L(u^\alpha_k)|^{\frac{\theta\sigma}{2-\sigma}})\right)^{2-\sigma} \qquad (13)\\
&= 2^{2\theta\sigma}\left(L^{\frac{\theta\sigma}{2-\sigma}}|A^\alpha_{k,L}|+\int_{A^\alpha_{k,L}}|G_L(u^\alpha_k)|^{\frac{\theta\sigma}{2-\sigma}}\right)^{2-\sigma}\\
&\le 2^{2\theta\sigma}\left[L^{\theta\sigma}|A^\alpha_{k,L}|^{2-\sigma}+\left(\int_{A^\alpha_{k,L}}|G_L(u^\alpha_k)|^{\frac{\theta\sigma}{2-\sigma}}\right)^{2-\sigma}\right].
\end{aligned}$$

We recall that $\theta<1$; so, it results $\frac{\theta\sigma}{2-\sigma}<\frac{\sigma}{2-\sigma}$ and it is easy to check that $\frac{\sigma}{2-\sigma}<\sigma^*$ since $m>\frac{n}{2}$. Then, in the following relation we apply first Hölder inequality with exponents $\frac{\sigma^*}{\frac{\theta\sigma}{2-\sigma}}>1$ and its conjugate $\frac{\sigma^*(2-\sigma)}{\sigma^*(2-\sigma)-\theta\sigma}$, then Sobolev inequality as in (9), finally Young inequality with exponents $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$:

$$\begin{aligned}
&\left(\int_{A^\alpha_{k,L}}|G_L(u^\alpha_k)|^{\frac{\theta\sigma}{2-\sigma}}\right)^{2-\sigma} \le \left(\int_{A^\alpha_{k,L}}|G_L(u^\alpha_k)|^{\sigma^*}\right)^{\frac{\theta\sigma}{\sigma^*}}|A^\alpha_{k,L}|^{\frac{\sigma^*(2-\sigma)-\theta\sigma}{\sigma^*}}\\
&=\|G_L(u^\alpha_k)\|^{\theta\sigma}_{L^{\sigma^*}(A^\alpha_{k,L})}|A^\alpha_{k,L}|^{\frac{\sigma^*(2-\sigma)-\theta\sigma}{\sigma^*}} \le C_S^{\theta\sigma}\|Du^\alpha_k\|^{\theta\sigma}_{L^\sigma(A^\alpha_{k,L})}|A^\alpha_{k,L}|^{\frac{\sigma^*(2-\sigma)-\theta\sigma}{\sigma^*}} \qquad (14)\\
&\le C_S^{\theta\sigma}\left(\delta\|Du^\alpha_k\|^\sigma_{L^\sigma(A^\alpha_{k,L})}+C_\delta|A^\alpha_{k,L}|^{\frac{\sigma^*(2-\sigma)-\theta\sigma}{\sigma^*(1-\theta)}}\right)
\end{aligned}$$

where $\delta>0$ can be arbitrarily chosen and $C_\delta=\delta^{-\frac{\theta}{1-\theta}}$. Since $m>\frac{n}{2}$, we have $\sigma<\frac{n}{n-1}$ and

$$2-\sigma<\frac{(2-\sigma)\sigma^*-\theta\sigma}{\sigma^*(1-\theta)};$$

moreover
$$\frac{(2-\sigma)\sigma^*-\theta\sigma}{\sigma^*(1-\theta)}=2-\sigma+\frac{\theta}{1-\theta}(1-\sigma+\frac{\sigma}{n})$$

so that $|A^\alpha_{k,L}|^{\frac{\sigma^*(2-\sigma)-\theta\sigma}{\sigma^*(1-\theta)}}=|A^\alpha_{k,L}|^{2-\sigma}|A^\alpha_{k,L}|^{\frac{\theta}{1-\theta}(1-\sigma+\frac{\sigma}{n})}\le|A^\alpha_{k,L}|^{2-\sigma}|\Omega|^{\frac{\theta}{1-\theta}(1-\sigma+\frac{\sigma}{n})}$.

Then, putting together (13), (14) and the last inequality, we have the following estimate of (11):

$$\begin{aligned}
&\left(\int_{A^\alpha_{k,L}}(1+|u^\alpha_k|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{2-\sigma} \le 2^{2\theta\sigma}\left[L^{\theta\sigma}|A^\alpha_{k,L}|^{2-\sigma}+\left(\int_{A^\alpha_{k,L}}|G_L(u^\alpha_k)|^{\frac{\theta\sigma}{2-\sigma}}\right)^{2-\sigma}\right]\\
&\le 2^{2\theta\sigma}L^{\theta\sigma}|A^\alpha_{k,L}|^{2-\sigma}+2^{2\theta\sigma}C_S^{\theta\sigma}\left(\delta\|Du^\alpha_k\|^\sigma_{L^\sigma(A^\alpha_{k,L})}+C_\delta|A^\alpha_{k,L}|^{\frac{\sigma^*(2-\sigma)-\theta\sigma}{\sigma^*(1-\theta)}}\right)\\
&\le 2^{2\theta\sigma}L^{\theta\sigma}|A^\alpha_{k,L}|^{2-\sigma}+2^{2\theta\sigma}C_S^{\theta\sigma}\delta\|Du^\alpha_k\|^\sigma_{L^\sigma(A^\alpha_{k,L})}+2^{2\theta\sigma}C_S^{\theta\sigma}C_\delta|\Omega|^{\frac{\theta}{1-\theta}(1-\sigma+\frac{\sigma}{n})}|A^\alpha_{k,L}|^{2-\sigma}\\
&=2^{2\theta\sigma}(L^{\theta\sigma}+C_S^{\theta\sigma}C_\delta|\Omega|^{\frac{\theta}{1-\theta}(1-\sigma+\frac{\sigma}{n})})|A^\alpha_{k,L}|^{2-\sigma}+2^{2\theta\sigma}C_S^{\theta\sigma}\delta\|Du^\alpha_k\|^\sigma_{L^\sigma(A^\alpha_{k,L})}.
\end{aligned}$$

Now, from (10) and the last inequality we have

$$\sum_{\alpha=1}^{N} \|Du_k^\alpha\|^\sigma_{L^\sigma(A^\alpha_{k,L})} \le N^{2\sigma+5}\left(\frac{C_f C_S}{\nu}\right)^\sigma \sum_{\alpha=1}^{N}\left(\int_{A^\alpha_{k,L}} (1+|u_k^\alpha|)^{\frac{\theta\sigma}{2-\sigma}}\right)^{2-\sigma}$$

$$\le N^{2\sigma+5}\left(\frac{C_f C_S}{\nu}\right)^\sigma \sum_{\alpha=1}^{N} 2^{2\theta\sigma}(L^{\theta\sigma} + C_S^{\theta\sigma} C_\delta |\Omega|^{\frac{\theta}{1-\theta}(1-\sigma+\frac{\sigma}{n})})|A^\alpha_{k,L}|^{2-\sigma}$$

$$+ N^{2\sigma+5}\left(\frac{C_f C_S}{\nu}\right)^\sigma \sum_{\alpha=1}^{N} 2^{2\theta\sigma} C_S^{\theta\sigma}\delta \|Du_k^\alpha\|^\sigma_{L^\sigma(A^\alpha_{k,L})}.$$

Choosing δ sufficiently small (for example $\delta \le \frac{1}{2}\frac{1}{N^{2\sigma+5}}\left(\frac{\nu}{1+C_f C_S}\right)^\sigma \frac{1}{2^{2\theta\sigma}C_S^{\theta\sigma}}$), we have

$$\sum_{\alpha=1}^{N} \|Du_k^\alpha\|^\sigma_{L^\sigma(A^\alpha_{k,L})} \le 2N^{2\sigma+5}\left(\frac{C_f C_S}{\nu}\right)^\sigma \sum_{\alpha=1}^{N} 2^{2\theta\sigma}(L^{\theta\sigma} + C_S^{\theta\sigma} C_\delta |\Omega|^{\frac{\theta}{1-\theta}(1-\sigma+\frac{\sigma}{n})})|A^\alpha_{k,L}|^{2-\sigma}.$$

To simplify the notation we set

$$C_1 := 2N^{2\sigma+5}\left(\frac{C_f C_S}{\nu}\right)^\sigma 2^{2\theta\sigma}, \quad C_2 := 2N^{2\sigma+5}\left(\frac{C_f C_S}{\nu}\right)^\sigma 2^{2\theta\sigma} C_S^{\theta\sigma} C_\delta |\Omega|^{\frac{\theta}{1-\theta}(1-\sigma+\frac{\sigma}{n})},$$

so that the last inequality becomes

$$\sum_{\alpha=1}^{N} \|Du_k^\alpha\|^\sigma_{L^\sigma(A^\alpha_{k,L})} \le \sum_{\alpha=1}^{N} (C_1 L^{\theta\sigma} + C_2)|A^\alpha_{k,L}|^{2-\sigma}.$$

By Sobolev's embedding as in (9) and the last inequality we obtain

$$\sum_{\alpha=1}^{N} \|G_L(u_k^\alpha)\|^\sigma_{L^{\sigma^*}(A^\alpha_{k,L})} \le C_S^\sigma \sum_{\alpha=1}^{N} \|DG_L(u_k^\alpha)\|^\sigma_{L^\sigma(A^\alpha_{k,L})} =$$

$$= C_S^\sigma \sum_{\alpha=1}^{N} \|Du_k^\alpha\|^\sigma_{L^\sigma(A^\alpha_{k,L})} \le C_S^\sigma (C_1 L^{\theta\sigma} + C_2)\sum_{\alpha=1}^{N} |A^\alpha_{k,L}|^{2-\sigma}.$$

By our choice of $L \ge 1$, we have $C_2 \le C_2 L^{\theta\sigma}$; then, setting $C := C_S^\sigma(C_1 + C_2)$, we can write

$$\sum_{\alpha=1}^{N} \|G_L(u_k^\alpha)\|^\sigma_{L^{\sigma^*}(A^\alpha_{k,L})} \le C L^{\theta\sigma}\sum_{\alpha=1}^{N} |A^\alpha_{k,L}|^{2-\sigma}.$$

Since $m > \frac{n}{2}$, then $2-\sigma = \frac{\sigma}{\sigma^*} + \varepsilon$ for some $\varepsilon > 0$; then we can write

$$\sum_{\alpha=1}^{N}\left(\int_{A^\alpha_{k,L}} |G_L(u_k^\alpha)|^{\sigma^*}\right)^{\frac{\sigma}{\sigma^*}} \le C L^{\theta\sigma}\sum_{\alpha=1}^{N} |A^\alpha_{k,L}|^{\frac{\sigma}{\sigma^*}+\varepsilon}. \tag{15}$$

Let us remark that for all $h > L \ge 1$ it results

$$\int_{A^\alpha_{k,L}} |G_L(u_k^\alpha)|^{\sigma^*} \ge \int_{A^\alpha_{k,h}} |G_L(u_k^\alpha)|^{\sigma^*} \ge (h-L)^{\sigma^*}|A^\alpha_{k,h}|;$$

then $$\left(\int_{A^\alpha_{k,L}} |G_L(u^\alpha_k)|^{\sigma^*}\right)^{\frac{\sigma}{\sigma^*}} \geq (h-L)^\sigma |A^\alpha_{k,h}|^{\frac{\sigma}{\sigma^*}}.$$

Putting together (15) and the last inequality, we get

$$(h-L)^\sigma \sum_{\alpha=1}^N |A^\alpha_{k,h}|^{\frac{\sigma}{\sigma^*}} \leq \sum_{\alpha=1}^N \left(\int_{A^\alpha_{k,L}} |G_L(u^\alpha_k)|^{\sigma^*}\right)^{\frac{\sigma}{\sigma^*}} \leq CL^{\theta\sigma} \sum_{\alpha=1}^N |A^\alpha_{k,L}|^{\frac{\sigma}{\sigma^*}+\varepsilon}$$

then $$\sum_{\alpha=1}^N |A^\alpha_{k,h}|^{\frac{\sigma}{\sigma^*}} \leq \frac{CL^{\theta\sigma}}{(h-L)^\sigma} \sum_{\alpha=1}^N |A^\alpha_{k,L}|^{\frac{\sigma}{\sigma^*}+\varepsilon}.$$

Now, using (4) and (5), we have

$$\left(\sum_{\alpha=1}^N |A^\alpha_{k,h}|\right)^{\frac{\sigma}{\sigma^*}} \leq N^{\frac{\sigma}{\sigma^*}} \sum_{\alpha=1}^N |A^\alpha_{k,h}|^{\frac{\sigma}{\sigma^*}} \leq \frac{N^{\frac{\sigma}{\sigma^*}} CL^{\theta\sigma}}{(h-L)^\sigma} \sum_{\alpha=1}^N |A^\alpha_{k,L}|^{\frac{\sigma}{\sigma^*}+\varepsilon} \leq$$
$$\leq \frac{N^{\frac{\sigma}{\sigma^*}} CL^{\theta\sigma} N}{(h-L)^\sigma} \left(\sum_{\alpha=1}^N |A^\alpha_{k,L}|\right)^{\frac{\sigma}{\sigma^*}+\varepsilon}$$

that is $$\sum_{\alpha=1}^N |A^\alpha_{k,h}| \leq \frac{\bar{C}}{(h-L)^{\sigma^*}} L^{\theta\sigma^*} \left(\sum_{\alpha=1}^N |A^\alpha_{k,L}|\right)^{1+\varepsilon\frac{\sigma^*}{\sigma}} \tag{16}$$

where $\bar{C} := N(CN)^{\frac{\sigma^*}{\sigma}}$. Now, we define the function $\psi : [\bar{L}, +\infty) \to [0, +\infty)$ by

$$\psi(h) = \sum_{\alpha=1}^N |A^\alpha_{k,h}|;$$

let us remark that the function ψ is decreasing and, in view of (16), we can say that for all $h > L \geq \bar{L}$ it results

$$\psi(h) \leq \frac{\bar{C}}{(h-L)^{\sigma^*}} L^{\theta\sigma^*} (\psi(L))^{1+\varepsilon\frac{\sigma^*}{\sigma}}.$$

Then, we can use a variant of Lemma 4.1 at page 93 of [39] that we write for the convenience of the reader, see Lemma A.2 in [14], Lemma 2 in [27], Lemma 2.3 in [22] and Lemma 2.3 in [21].

Lemma 3.2. *Let $c_*, \alpha, \beta, \gamma$ be positive constants. Let $\psi : [\bar{L}, +\infty) \to [0, +\infty)$ be decreasing and such that*

$$\psi(h) \leq \frac{c_* L^{\gamma\alpha}}{(h-L)^\alpha} [\psi(L)]^\beta \tag{17}$$

for every h, L with $h > L \geq \bar{L} \geq 0$. If $\beta > 1$ and $\gamma < 1$, then we have $\psi(\bar{L}+d) = 0$,

where $$d = \left\{\Lambda c_* [\psi(\bar{L})]^{\beta-1} 2^{\alpha\beta/(\beta-1)}\right\}^{1/\alpha}$$

and $$\Lambda = \max\left\{(2\bar{L})^{\gamma\alpha}; \left[2\left(c_*[\psi(\bar{L})]^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}\right)^{\frac{1}{\alpha}}\right]^{\frac{\gamma\alpha}{1-\gamma}}\right\}.$$

In view of the previous lemma, we deduce that

$$\psi(\bar{L}+d)=\sum_{\alpha=1}^{N}|A^{\alpha}_{k,\bar{L}+d}|=0.$$

Let us note that $\psi(\bar{L})\leq N|\Omega|$; then we have

$$|u^{\alpha}_k(x)|\leq \bar{L}+\tilde{d},\qquad \text{for all } \alpha=1,\dots,N,\qquad \text{a. e. in } \Omega,$$

that is, the norms of functions u^{α}_k are bounded in $L^{\infty}(\Omega)$ by a constant $C^*:=\bar{L}+\tilde{d}$ depending only on L_0, $\theta, m, n, N, \nu, |\Omega|$ and on $\|f\|_{L^m(\Omega)}$.

Now, it remains to show that also the norms in $W^{1,2}_0(\Omega)$ are bounded. Taking u_k as test function in (3) and using assumption $(\mathcal{A}_2)$, we have

$$\int_{\Omega}\sum_{\alpha=1}^{N}f^{\alpha}(x)u^{\alpha}_k(x)dx=\int_{\Omega}\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}\tilde{a}^{\alpha,\beta}_{i,j,k}(x,u_k(x))D_ju^{\beta}_k(x)D_iu^{\alpha}_k(x)dx=$$

$$=\int_{\Omega}\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}a^{\alpha,\beta}_{i,j}(x,u_k(x))D_ju^{\beta}_k(x)D_iu^{\alpha}_k(x)dx+$$

$$+\int_{\Omega}\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}\frac{1}{k}\delta_{\alpha,\beta}\delta_{i,j}D_ju^{\beta}_k(x)D_iu^{\alpha}_k(x)dx\geq$$

$$\geq\nu\int_{\Omega}\sum_{\alpha=1}^{N}\frac{|Du^{\alpha}_k|^2}{(1+|u^{\alpha}_k|)^{\theta}}\geq\frac{\nu}{(1+C^*)^{\theta}}\int_{\Omega}\sum_{\alpha=1}^{N}|Du^{\alpha}_k|^2$$

since $\|u^{\alpha}_k\|_{L^{\infty}(\Omega)}\leq C^*$ for all k; we have

$$\int_{\Omega}\sum_{\alpha=1}^{N}|Du^{\alpha}_k|^2\leq\frac{(1+C^*)^{\theta}}{\nu}\int_{\Omega}\sum_{\alpha=1}^{N}f^{\alpha}(x)u^{\alpha}_k(x)dx\leq\frac{(1+C^*)^{\theta}}{\nu}C^*N\|f\|_{L^1(\Omega)};$$

since $f\in L^m(\Omega,\mathbb{R}^N)$ with $m>\frac{n}{2}$, it follows also $f\in L^1(\Omega,\mathbb{R}^N)$ and the boundedness of u_k in $W^{1,2}_0(\Omega,\mathbb{R}^N)$ is proved. □

4. Passing to the limit: Proof of Theorem 2.1

Proof of Theorem 2.1 In Lemma 3.1 the boundedness in $L^{\infty}(\Omega)$ and in $W^{1,2}_0(\Omega)$ is proved for the sequence of solutions u_k to (3). Then we can say that there exists a constant K such that $\|u_k\|_{L^{\infty}(\Omega)}\leq K$ and $\|u_k\|_{W^{1,2}_0(\Omega)}\leq K$ for all $k\in\mathbb{N}$. Boundedness of u_k in $W^{1,2}_0(\Omega)$ implies the existence of a subsequence u_{k_l} weakly converging in $W^{1,2}_0(\Omega)$ to a function $u\in W^{1,2}_0(\Omega)$, that is $u_{k_l}\rightharpoonup u$. Moreover, by Rellich-Kondrachov embedding Theorem, Sobolev space $W^{1,2}_0(\Omega)$ is compactly embedded in $L^2(\Omega)$; then, there exists a subsequence $u_{k_{l_p}}$ strongly converging to u in L^2. From L^2 convergence we get pointwise convergence almost everywhere, up to a further subsequence.

To simplify the notation, we can say that there exists a subsequence u_{k_λ} such that

$$\begin{aligned}
&u_{k_\lambda} \rightharpoonup u \quad \text{in } W_0^{1,2}(\Omega),\\
&u_{k_\lambda} \to u \quad \text{in } L^2(\Omega),\\
&u_{k_\lambda}(x) \to u(x) \quad \text{a. e. in } \Omega,\\
&\|u_{k_\lambda}\|_{L^\infty(\Omega)} \le K, \quad \|u_{k_\lambda}\|_{W_0^{1,2}(\Omega)} \le K.
\end{aligned} \tag{18}$$

Pointwise convergence and L^∞ bound show that $u \in L^\infty(\Omega)$. In (3), written for the elements of subsequence u_{k_λ}, we pass to the limit as $\lambda \to +\infty$ and we want to prove

$$\lim_{\lambda\to+\infty} \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \tilde{a}_{i,j,k_\lambda}^{\alpha,\beta}(x, u_{k_\lambda}(x)) D_j u_{k_\lambda}^\beta(x) D_i \varphi^\alpha(x) dx =$$
$$= \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, u(x)) D_j u^\beta(x) D_i \varphi^\alpha(x) dx.$$

Indeed, we have

$$\begin{aligned}
&\left| \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \tilde{a}_{i,j,k_\lambda}^{\alpha,\beta}(x, u_{k_\lambda}(x)) D_j u_{k_\lambda}^\beta(x) D_i \varphi^\alpha(x) dx + \right.\\
&\qquad \left. - \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, u(x)) D_j u^\beta(x) D_i \varphi^\alpha(x) dx \right| =\\
&= \left| \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) D_j u_{k_\lambda}^\beta(x) D_i \varphi^\alpha(x) dx + \right.\\
&\qquad + \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \frac{1}{k_\lambda} \delta_{\alpha,\beta} \delta_{i,j} D_j u_{k_\lambda}^\beta(x) D_i \varphi^\alpha(x) dx +\\
&\qquad \left. - \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, u(x)) D_j u^\beta(x) D_i \varphi^\alpha(x) dx \right| \le\\
&\le \left| \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n [a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))] D_j u_{k_\lambda}^\beta(x) D_i \varphi^\alpha(x) dx \right| +\\
&\qquad + \left| \int_\Omega \sum_{\alpha=1}^N \sum_{i=1}^n \frac{1}{k_\lambda} D_i u_{k_\lambda}^\alpha(x) D_i \varphi^\alpha(x) dx \right| +\\
&\qquad + \left| \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, u(x)) [D_j u_{k_\lambda}^\beta(x) - D_j u^\beta(x)] D_i \varphi^\alpha(x) dx \right| =\\
&= I_k + II_k + III_k.
\end{aligned}$$

Let us estimate the three items in the last sum.

Using Hölder's inequality and boundedness of the sequence u_{k_λ} in $W_0^{1,2}(\Omega)$ we get

$$\begin{aligned} I_k &= \left| \int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} [a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))] D_j u_{k_\lambda}^\beta(x) D_i \varphi^\alpha(x) dx \right| \\ &\leq \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} \left(\int_\Omega |a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))|^2 |D_i \varphi^\alpha(x)|^2 dx \right)^{\frac{1}{2}} \|D_j u_{k_\lambda}^\beta\|_{L^2} \\ &\leq K \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} \left(\int_\Omega |a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))|^2 |D_i \varphi^\alpha(x)|^2 dx \right)^{\frac{1}{2}} . \end{aligned} \tag{19}$$

For any $i, j = 1, \ldots, n$ and for any $\alpha, \beta = 1, \ldots, N$, using pointwise convergence in (18) and continuity of functions $y \to a_{i,j}^{\alpha,\beta}(x, y)$ we have that

$$|a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))|^2 |D_i \varphi^\alpha(x)|^2 \to 0 \quad \text{as } \lambda \to +\infty;$$

moreover from $(\mathcal{A}_1)$ we have

$$\begin{aligned} &|a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))|^2 |D_i \varphi^\alpha(x)|^2 \leq \\ &\leq (|a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x))| + |a_{i,j}^{\alpha,\beta}(x, u(x))|)^2 |D_i \varphi^\alpha(x)|^2 \leq \\ &\leq (c + c)^2 |D_i \varphi^\alpha(x)|^2 \in L^1(\Omega); \end{aligned}$$

therefore by dominated convergence theorem we have that

$$\left(\int_\Omega |a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))|^2 |D_i \varphi^\alpha(x)|^2 dx \right)^{\frac{1}{2}} \to 0 \quad \text{as } \lambda \to +\infty;$$

this and (19) imply that I_k tends to zero as $\lambda \to +\infty$. In view of the boundedness of sequence u_{k_λ} and using Hölder's inequality we have

$$\begin{aligned} II_k &= \left| \int_\Omega \sum_{\alpha=1}^{N} \sum_{i=1}^{n} \frac{1}{k_\lambda} D_i u_{k_\lambda}^\alpha(x) D_i \varphi^\alpha(x) dx \right| \leq \frac{1}{k_\lambda} \sum_{\alpha=1}^{N} \sum_{i=1}^{n} \|D_i u_{k_\lambda}^\alpha\|_{L^2} \|D_i \varphi^\alpha\|_{L^2} \leq \\ &\leq \frac{1}{k_\lambda} nNK \|\varphi\|_{W_0^{1,2}(\Omega, R^N)}, \end{aligned}$$

so that II_k tends to zero as $\lambda \to +\infty$. Finally, since $u_{k_\lambda}^\beta \rightharpoonup u^\beta$ in $W_0^{1,2}(\Omega)$, it follows that for any $i, j = 1, \ldots, n$ and for any $\alpha, \beta = 1, \ldots, N$ we have

$$\int_\Omega a_{i,j}^{\alpha,\beta}(x, u(x)) [D_j u_{k_\lambda}^\beta(x) - D_j u^\beta(x)] D_i \varphi^\alpha(x) dx \to 0 \quad \text{as } \lambda \to +\infty;$$

then also III_k tends to zero as $\lambda \to +\infty$. This ends the proof of Theorem 2.1.

5. Example

Let us consider $N = 2$ and set $0 < \theta < 1$; let us take

$$a_{i,j}^{\alpha,\beta}(x,y) = \delta_{i,j}\,\psi^{\alpha,\beta}(y)\,\frac{1}{(1+|y^\alpha|)^{\theta/2}}\,\frac{1}{(1+|y^\beta|)^{\theta/2}} \tag{20}$$

where $\delta_{i,j}$ is the Kronecker symbol defined at the beginning of Section 3, the map $\psi^{\alpha,\beta}: \mathbb{R}^2 \to \mathbb{R}$ is continuous with

$$\psi^{\alpha,\alpha}(y) = 2 \tag{21}$$

and
$$\alpha \neq \beta \implies 0 \leq \psi^{\alpha,\beta}(y) \leq 1; \tag{22}$$

moreover
$$\alpha \neq \beta \implies \text{support of } \psi^{\alpha,\beta}(y) \subset \{y \in \mathbb{R}^2 : |y^\alpha| \leq |y^\beta|\} \tag{23}$$

as it is shown in Figure 5.1.

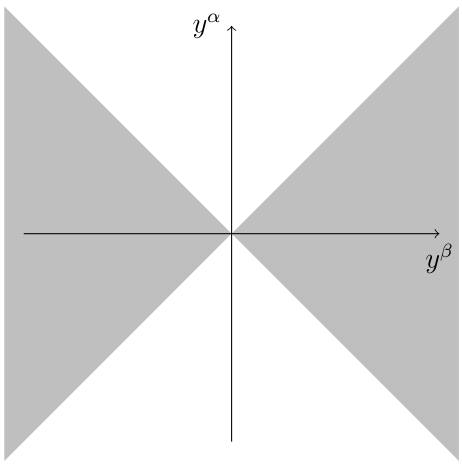

Figure 5.1: Assumption (23): when $\alpha \neq \beta$, the support of $\psi^{\alpha,\beta}$ is contained in the grey part.

The last condition (23) implies $(\mathcal{A}_3)$ for every $L_0 > 0$. Furthermore, $(\mathcal{A}_0)$ and $(\mathcal{A}_1)$ are satisfied with $c = 2$. Let us check $(\mathcal{A}_2)$: we have

$$\sum_{\alpha,\beta=1}^{2}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,y)\xi_i^\alpha\xi_j^\beta = \sum_{\alpha,\beta=1}^{2}\sum_{i,j=1}^{n}\delta_{i,j}\,\psi^{\alpha,\beta}(y)\,\frac{\xi_i^\alpha}{(1+|y^\alpha|)^{\theta/2}}\,\frac{\xi_j^\beta}{(1+|y^\beta|)^{\theta/2}}$$

$$= \sum_{\alpha,\beta=1}^{2}\sum_{i=1}^{n}\psi^{\alpha,\beta}(y)\,\frac{\xi_i^\alpha}{(1+|y^\alpha|)^{\theta/2}}\,\frac{\xi_i^\beta}{(1+|y^\beta|)^{\theta/2}}$$

$$= \sum_{i=1}^{n}\psi^{1,1}(y)\,\frac{\xi_i^1}{(1+|y^1|)^{\theta/2}}\,\frac{\xi_i^1}{(1+|y^1|)^{\theta/2}} + \sum_{i=1}^{n}\psi^{1,2}(y)\,\frac{\xi_i^1}{(1+|y^1|)^{\theta/2}}\,\frac{\xi_i^2}{(1+|y^2|)^{\theta/2}}$$

$$+ \sum_{i=1}^{n}\psi^{2,1}(y)\,\frac{\xi_i^2}{(1+|y^2|)^{\theta/2}}\,\frac{\xi_i^1}{(1+|y^1|)^{\theta/2}} + \sum_{i=1}^{n}\psi^{2,2}(y)\,\frac{\xi_i^2}{(1+|y^2|)^{\theta/2}}\,\frac{\xi_i^2}{(1+|y^2|)^{\theta/2}}$$

$$= 2\frac{|\xi^1|^2}{(1+|y^1|)^{\theta}} + [\psi^{1,2}(y)+\psi^{2,1}(y)]\sum_{i=1}^{n}\frac{\xi_i^1}{(1+|y^1|)^{\theta/2}}\,\frac{\xi_i^2}{(1+|y^2|)^{\theta/2}} + 2\frac{|\xi^2|^2}{(1+|y^2|)^{\theta}};$$

we note that $0 \leq \psi^{1,2}(y) + \psi^{2,1}(y) \leq 2$ and

$$\begin{aligned}
&\left| \sum_{i=1}^{n} \frac{\xi_i^1}{(1+|y^1|)^{\theta/2}} \frac{\xi_i^2}{(1+|y^2|)^{\theta/2}} \right| \\
&\leq \left[\sum_{i=1}^{n} \left(\frac{\xi_i^1}{(1+|y^1|)^{\theta/2}} \right)^2 \right]^{1/2} \left[\sum_{i=1}^{n} \left(\frac{\xi_i^2}{(1+|y^2|)^{\theta/2}} \right)^2 \right]^{1/2} \\
&\leq \frac{1}{2} \sum_{i=1}^{n} \left(\frac{\xi_i^1}{(1+|y^1|)^{\theta/2}} \right)^2 + \frac{1}{2} \sum_{i=1}^{n} \left(\frac{\xi_i^2}{(1+|y^2|)^{\theta/2}} \right)^2 \\
&= \frac{1}{2} \frac{|\xi^1|^2}{(1+|y^1|)^{\theta}} + \frac{1}{2} \frac{|\xi^2|^2}{(1+|y^2|)^{\theta}}.
\end{aligned}$$

Then,

$$[\psi^{1,2}(y) + \psi^{2,1}(y)] \sum_{i=1}^{n} \frac{\xi_i^1}{(1+|y^1|)^{\theta/2}} \frac{\xi_i^2}{(1+|y^2|)^{\theta/2}} \geq -\frac{|\xi^1|^2}{(1+|y^1|)^{\theta}} - \frac{|\xi^2|^2}{(1+|y^2|)^{\theta}}.$$

Now we are ready to check $(\mathcal{A}_2)$: we have

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^{2} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x,y) \xi_i^\alpha \xi_j^\beta \\
&= 2\frac{|\xi^1|^2}{(1+|y^1|)^{\theta}} + [\psi^{1,2}(y) + \psi^{2,1}(y)] \sum_{i=1}^{n} \frac{\xi_i^1}{(1+|y^1|)^{\theta/2}} \frac{\xi_i^2}{(1+|y^2|)^{\theta/2}} + 2\frac{|\xi^2|^2}{(1+|y^2|)^{\theta}} \\
&\geq 2\frac{|\xi^1|^2}{(1+|y^1|)^{\theta}} - \frac{|\xi^1|^2}{(1+|y^1|)^{\theta}} - \frac{|\xi^2|^2}{(1+|y^2|)^{\theta}} + 2\frac{|\xi^2|^2}{(1+|y^2|)^{\theta}} \\
&= \frac{|\xi^1|^2}{(1+|y^1|)^{\theta}} + \frac{|\xi^2|^2}{(1+|y^2|)^{\theta}}.
\end{aligned}$$

Thus, $(\mathcal{A}_2)$ holds true with $\nu = 1$. Then, our Theorem 2.1 can be applied to the present example. On the contrary, this example cannot be handled by means of [31]: in order to show that, we further require that

$$\psi^{1,2}(y_M) = 1 \tag{24}$$

for every $M \in \{1, 2, 3, ...\}$, where

$$y_M = (M, M+1). \tag{25}$$

Note that, in the quasilinear case, [31] requires that there exist constants $c_1 \in (0, +\infty)$ and $c_2 \in [0, +\infty)$ such that

$$\frac{c_1 |\xi^\gamma|^2}{(1+|y^\gamma|)^{\theta}} - c_2 \leq \sum_{\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\gamma,\beta}(x,y) \xi_j^\beta \xi_i^\gamma \tag{26}$$

for every $\gamma = 1, ..., N$, for every y and ξ, for almost every x.

We now show that this cannot hold true in our example. Indeed, in our example $N = 2$ and we can take $\gamma = 1$, $y = y_M$, so that (26) becomes

$$\frac{c_1|\xi^1|^2}{(1+M)^\theta} - c_2 \le \sum_{i,j=1}^n a_{i,j}^{1,1}(x, y_M)\xi_j^1\xi_i^1 + \sum_{i,j=1}^n a_{i,j}^{1,2}(x, y_M)\xi_j^2\xi_i^1$$
$$= \frac{2|\xi^1|^2}{(1+M)^\theta} + \psi^{1,2}(y_M)\sum_{i=1}^n \frac{\xi_i^2}{(1+M+1)^{\theta/2}}\frac{\xi_i^1}{(1+M)^{\theta/2}};$$

we can take $\xi^1 = (1, 0, ..., 0)$ and $\xi^2 = (t, 0, ..., 0)$ so that (26) now reads as follows

$$\frac{c_1}{(1+M)^\theta} - c_2 \le \frac{2}{(1+M)^\theta} + \frac{t}{(1+M+1)^{\theta/2}(1+M)^{\theta/2}} : \tag{27}$$

but it is false when t goes to $-\infty$; this shows that (26) cannot hold true and [31] cannot be used to deal with the present example. Now we recall assumption (H3)' in [42]: in the quasilinear case it requires that

$$0 \le \sum_{\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\gamma,\beta}(x, y)\xi_j^\beta\xi_i^\gamma \tag{28}$$

for every $\gamma = 1, ..., N$, for every y and ξ, for almost every x. This cannot hold true in our example. Indeed, in our example $N = 2$ and we can take $\gamma = 1$, $y = y_M$, so that, considering ξ^1 and ξ^2 as before, (28) reads as follows

$$0 \le \frac{2}{(1+M)^\theta} + \frac{t}{(1+M+1)^{\theta/2}(1+M)^{\theta/2}} : \tag{29}$$

but it is false when t goes to $-\infty$; this shows that (28) cannot hold true in our example. Let us now recall assumption (A_5) in [28]: in the quasilinear case it requires that

$$0 \le \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, y)\xi_j^\beta \left[(\mathrm{Id} - b\otimes b)\,\xi\right]_i^\alpha \tag{30}$$

for every $y \in \mathbb{R}^N$, for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^{N\times n}$, for every $b \in \mathbb{R}^N$ with $|b| = 1$, see also [20], [7]. In [19], inequality (30) is required for $|b| \le 1$. Note that, when $N = 2$, (30) implies (28): take first $b = (0, 1)$, then $b = (1, 0)$. Since we are in case $N = 2$, failure of (28) implies failure of (30); so, our example does not verify (30).

6. Remarks on De Giorgi's counterexample

Concerning De Giorgi's counterexample we refer to [17], page 54 in [25], page 183 in [26], Section 3 in [36], page 86 in [6] and page 99 in [3]. In the sequel we adapt it to the case in which the boundary datum is zero. De Giorgi counterexample says that the unbounded map $u(x) = \frac{x}{|x|^\gamma}$ is a weak solution of the system

$$-\sum_{i=1}^n D_i\left(\sum_{\beta,j=1}^n a_{i,j}^{\alpha,\beta}(x)D_ju^\beta(x)\right) = 0, \qquad \alpha = 1, ..., n$$

when $x \in B(0,1) \subset \mathbb{R}^n$; here, $B(0,1)$ is the open ball around 0, with radius 1; moreover, $n \geq 3$,

$$\gamma = \frac{n}{2}\left\{1 - [(2n-2)^2+1]^{-1/2}\right\} > 1$$

and
$$a_{i,j}^{\alpha,\beta}(x) = \delta_{\alpha,\beta}\,\delta_{i,j} + b_{\alpha,i}(x)\,b_{\beta,j}(x)$$

with
$$b_{\alpha,i}(x) = \left[(n-2)\delta_{\alpha,i} + n\frac{x_i x_\alpha}{|x|^2}\right].$$

This means that $u \in W^{1,2}(B(0,1),\mathbb{R}^n)$ and

$$\int_{B(0,1)} \sum_{\alpha,\beta,i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x) D_j u^\beta(x) D_i \phi^\alpha(x)dx = 0$$

for every $\phi \in W_0^{1,2}(B(0,1),\mathbb{R}^n)$. Note that $b_{\alpha,i} \in L^\infty(B(0,1))$ and

$$\sum_{\alpha,\beta,i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x)\xi_j^\beta \xi_i^\alpha = |\xi|^2 + \langle b(x),\xi\rangle^2 \geq |\xi|^2. \tag{31}$$

Let us remark that $u(x) = x$ on $\partial B(0,1)$: in order to have zero boundary value, we need to consider the difference $v(x) = u(x) - x$; it turns out that $v \in W_0^{1,2}(B(0,1),\mathbb{R}^n)$ and

$$\int_{B(0,1)} \sum_{\alpha,\beta,i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x) D_j v^\beta(x) D_i \phi^\alpha(x)dx = -\int_{B(0,1)} \sum_{\alpha,\beta,i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x) D_j x^\beta D_i \phi^\alpha(x)dx$$

for every $\phi \in W_0^{1,2}(B(0,1),\mathbb{R}^n)$. Let us set $F_i^\alpha(x) = -\sum_{\beta,j=1}^{n} a_{i,j}^{\alpha,\beta}(x) D_j x^\beta$; then

$$\int_{B(0,1)} \sum_{\alpha,\beta,i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x) D_j v^\beta(x) D_i \phi^\alpha(x)dx = \int_{B(0,1)} \sum_{\alpha,i=1}^{n} F_i^\alpha(x) D_i \phi^\alpha(x)dx$$

for every $\phi \in W_0^{1,2}(B(0,1),\mathbb{R}^n)$. Note that $F_i^\alpha \in W^{1,p}(B(0,1)) \cap L^\infty(B(0,1))$ for every $p < n$; since $3 \leq n$, we can take $p \geq 2$ and, integrating by parts in the right hand side, we get

$$\int_{B(0,1)} \sum_{\alpha,\beta,i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x) D_j v^\beta(x) D_i \phi^\alpha(x)dx = \int_{B(0,1)} \sum_{\alpha,i=1}^{n} -D_i F_i^\alpha(x) \phi^\alpha(x)dx;$$

if we set $f^\alpha(x) = \sum_{i=1}^{n} -D_i F_i^\alpha(x)$, then

$$\int_{B(0,1)} \sum_{\alpha,\beta,i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x) D_j v^\beta(x) D_i \phi^\alpha(x)dx = \int_{B(0,1)} \sum_{\alpha=1}^{n} f^\alpha(x) \phi^\alpha(x)dx \tag{32}$$

for every $\phi \in W_0^{1,2}(B(0,1),\mathbb{R}^n)$.

Note that $f^\alpha \in L^p(B(0,1))$ for every $p < n$; this means that p can be chosen larger than $\frac{n}{2}$; since (31) holds true, v is the unique solution to (32); moreover, v is unbounded; this means that, in general, for systems, we cannot hope for existence of bounded solutions $v \in W_0^{1,2}$, unless additional conditions on coefficients $a_{i,j}^{\alpha,\beta}$ are assumed. This was done in this paper by requiring that off-diagonal coefficients $a_{i,j}^{\alpha,\beta}$ have a "butterfly" support. When dealing with this kind of systems, we have to do a different approximation: compare our choice of approximate coefficients, in the second line of Section 3, with the one made in (2.3) of [14], when dealing with a single equation.

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