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# On Weak Solutions to First-Order Discount Mean Field Games

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We establish the existence and uniqueness of weak solutions to first-order discount mean field games and a stability result to give the existence for the ergodic problem. We show an example to illustrate the multiplicity of weak solutions to the ergodic problem. With this motivation, we address a selection condition, which is a necessary condition that any limit of solutions under subsequence satisfies. As an application, we show a nontrivial example to get the convergence of weak solutions.

*Keywords:* Mean field games, ergodic problem, vanishing discount approximation.

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## 1. Introduction

In this paper, we consider the stationary first-order discount mean field game systems with a local coupling

$$\begin{cases} \varepsilon u^\varepsilon + H(x, Du^\varepsilon) = f(x, m^\varepsilon) & \text{in } \mathbb{T}^d, \\ \varepsilon m^\varepsilon - \operatorname{div}(m^\varepsilon D_p H(x, Du^\varepsilon)) = \varepsilon & \text{in } \mathbb{T}^d, \end{cases} \tag{1}$$

where $\mathbb{T}^d$ is the $d$-dimensional flat torus identified with $[0,1]^d$, and $\varepsilon$ is a given positive number. The functions $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ and $f : \mathbb{T}^d \times [0, +\infty) \to \mathbb{R}$ are given continuous functions. Here, $u^\varepsilon : \mathbb{T}^d \to \mathbb{R}$ and $m^\varepsilon : \mathbb{T}^d \to [0, \infty)$ are the unknowns.

Mean field game (MFG) systems have been introduced simultaneously by Lasry and Lions [18] and by Huang, Caines and Malhamé [15] to describe the optimal strategies of a population of agents and their macroscopic distribution. These games are often determined by a system of a Hamilton-Jacobi equation coupled with a transport or Fokker-Planck equation. In this paper, we focus on deterministic games, and therefore we consider the first order Hamilton-Jacobi equation coupled with a transport equation. From a perspective of mathematical analysis, these systems are not monotone systems, and not uniformly elliptic. For first order MFG systems in which the coupling $f$ is of nonlocal nature, the existence and uniqueness of solutions are well-understood (see [19]). On the other hand, in the case of local couplings,

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in general, we cannot expect the solvability in the classical sense, and therefore it is reasonable to introduce the notion of weak solutions. Recently the framework of weak solutions has been developed. See [1, 4, 5, 8, 13, 24].

Our main interest in this paper is to establish the well-posedness of weak solutions to (1), and to study the asymptotic behavior of the solution $(u^\varepsilon, m^\varepsilon)$ as the discount factor vanishes $\varepsilon \to 0$. We call this asymptotic problem the *vanishing discount problem*. As an analogy with the study of the vanishing discount problem for Hamilton-Jacobi equations (see [7, 17, 22]) and also for discount MFG systems (see [11]) we can naturally expect that the limit problem of (1), which is called the *ergodic problem*, is described by

$$\begin{cases} H(x, Du) = f(x, m) + \lambda & \text{in } \mathbb{T}^d, \\ -\mathrm{div}(m D_p H(x, Du)) = 0 & \text{in } \mathbb{T}^d, \end{cases} \tag{2}$$

where $u : \mathbb{T}^d \to \mathbb{R}$, $m : \mathbb{T}^d \to [0, \infty)$ and $\lambda \in \mathbb{R}$ are the unknowns. We call $\lambda$ an *ergodic constant*. The ergodic problem (2) appears in many contexts of the asymptotic problem of MFG systems. For instance, we refer to [3, 5] for the study of the long time average of solutions.

The discount problem naturally arises in optimal control theory and differential game theory, where $\varepsilon$ is a discount factor. In recent years, there has been much interest and progress on the vanishing discount problem for Hamilton-Jacobi equations. The ergodic problem for Hamilton-Jacobi equations is given by

$$H(x, Dv) = c \quad \text{in } \mathbb{T}^d, \tag{3}$$

where $v : \mathbb{T}^d \to \mathbb{R}$ and $c \in \mathbb{R}$ are the unknowns. One of the standard ways to establish the existence of viscosity solutions to (3) is to consider the solution $v^\varepsilon \in \mathrm{Lip}(\mathbb{T}^d)$ to the discount problem

$$\varepsilon v^\varepsilon + H(x, Dv^\varepsilon) = 0 \quad \text{in } \mathbb{T}^d,$$

and pass to the limit as $\varepsilon \to 0$. Under a coercivity assumption on the Hamiltonians, we can easily get an a priori estimate on $\|Dv^\varepsilon\|_{L^\infty(\mathbb{T}^d)}$, and by the Arzelá-Ascoli theorem, we can prove that there exists a subsequence $\{\varepsilon_j\}_{j \in \mathbb{N}}$ with $\varepsilon_j \to 0$ as $j \to \infty$ such that for some $(v, c) \in \mathrm{Lip}\,(\mathbb{T}^d) \times \mathbb{R}$,

$$\varepsilon_j v^{\varepsilon_j} \to -c, \quad v^{\varepsilon_j} - \min_{\mathbb{T}^d} v^{\varepsilon_j} \to v \quad \text{in } C(\mathbb{T}^d) \quad \text{as } j \to \infty.$$

Here, it is worth emphasizing that due to the lack of uniqueness of viscosity solutions to (3), it is nontrivial whether the whole convergence of $v^\varepsilon - \min_{\mathbb{T}^d} v^\varepsilon$ holds. Recently, the convergence to a unique limit and its characterization has been established in [7, 17, 22] using weak KAM theory, the nonlinear adjoint method, and the duality method, respectively. We also refer [20, 28] and the references therein for further development.

For second-order MFG systems which are uniformly elliptic systems, the vanishing discount problem is studied in [6, 21]. For first-order MFG systems, in [11], the authors study the existence of classical solutions under the specific Hamiltonian $H(x, p) = \frac{1}{2}|p|^2 + V(x)$ with a small oscillatory potential.

In this setting, since the ergodic problem has uniqueness of solutions up to constants, the convergence of the whole sequence is rather easily proved. We also point out that in the argument in [11], the specific form of Hamiltonian is crucial. However, if one considers weak solutions, then the multiplicity of weak solutions to (2) rather naturally appears. In [11, Section 2.2], the authors consider a weak solution which is introduced in [8], and show an example to illustrate the non-uniqueness issue (see also [12]). This multiplicity of weak solutions makes the asymptotic problem harder and more interesting.

The main feature of this paper, compared particularly to [11], is the study of the vanishing discount problem in a different framework of weak solutions which are introduced in [4, 5, 13]. In [8], the authors construct weak solutions based on variational inequality techniques and Minty's method, and on the other hand, in [4, 5, 13], the authors introduce weak solutions by using variational structures and the Fenchel type of duality theorem. For a comparison between these two notions of weak solutions, we refer the reader to [8].

The main contributions of this paper are firstly a proof of existence of a unique weak solution in the sense of [4, 5, 13] to the discount MFG system (1). Next, we get a stability result for $(u^\varepsilon, m^\varepsilon)$ as $\varepsilon \to 0$, which is a new way to prove the existence of weak solutions to the ergodic problem (2). Moreover, we prove that the limit function satisfies a viscosity supersolution property, which will be clearly explained in Section 3. We also show an example to illustrate the multiplicity of weak solutions to the ergodic problem. With this motivation, we address a selection condition, which shows a necessary condition that any limit of solutions under subsequence satisfies. By using this condition, we show a nontrivial example to get the convergence result. In connection with the non-uniqueness issue on weak solutions to the ergodic problem (2), we give several uniqueness results. We explain the main results in the next sections in more details.

### 1.1. Assumptions

Throughout the paper we assume the following conditions:

(H1) The coupling term $f : \mathbb{T}^d \times [0,\infty) \to \mathbb{R}$ is continuous in both variables, strictly increasing with respect to the second variable, and there exists $q > 1$ and $C > 0$ such that

$$\frac{1}{C}|m|^{q-1} - C \le f(x,m) \le C|m|^{q-1} + C \quad \text{for all } (x,m) \in \mathbb{T}^d \times [0,\infty).$$

By replacing $H$ and $f$ by

$$\tilde{H}(x,p) := H(x,p) - \max_{\mathbb{T}^d} f(\cdot,0) \quad \text{and} \quad \tilde{f}(x,m) := f(x,m) - \max_{\mathbb{T}^d} f(\cdot,0)$$

if necessary, we can always assume $f(x,0) \le 0$ for all $x \in \mathbb{T}^d$ without loss of generality.

(H2) The Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous in both variables, strictly convex and differentiable on the second variable, with $D_pH$ continuous in both variables. Moreover, there exists $r > 1$ and $C > 0$ such that

$$\frac{1}{C}|p|^r - C \le H(x,p) \le C|p|^r + C \quad \text{for all } (x,p) \in \mathbb{T}^d \times \mathbb{R}^d. \tag{4}$$

For later purposes, we define $F : \mathbb{T}^d \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$F(x,m) := \begin{cases} \int_0^m f(x,s)\,ds & \text{if } m \geq 0 \\ +\infty & \text{if } m < 0. \end{cases}$$

It follows that $F$ is continuous on $\mathbb{T}^d \times (0,\infty)$, differentiable and strictly convex in $m$ and satisfies, for some $C > 0$,

$$\frac{1}{C}|m|^q - C \leq F(x,m) \leq C|m|^q + C \quad \text{for all } (x,m) \in \mathbb{T}^d \times [0,\infty).$$

Let $F^*$ be the convex conjugate of $F$ with respect to the second variable, i.e.,

$$F^*(x,a) = \sup_{m\in\mathbb{R}}\{am - F(x,m)\} = \sup_{m\geq 0}\{am - F(x,m)\}.$$

Then, $F^*$ satisfies that, for some $C > 0$,

$$\frac{1}{C}|a|^p - C \leq F^*(x,a) \leq C|a|^p + C \quad \text{for all } (x,a) \in \mathbb{T}^d \times [0,\infty), \tag{5}$$

where $p$ is the conjugate exponent of $q$, that is, $\frac{1}{p} + \frac{1}{q} = 1$.

We denote by $H^*$ the convex conjugate of $H$ with respect to the second variable. Note that $H^*$ satisfies

$$\frac{1}{C}|p|^{r'} - C \leq H^*(x,p) \leq C|p|^{r'} + C \quad \text{for all } (x,p) \in \mathbb{T}^d \times \mathbb{R}^d, \tag{6}$$

where $r'$ is the conjugate exponent of $r$. Moreover, $H^*(x,\cdot)$ is strictly convex, because $H(x,\cdot)$ is convex, superlinear, and in $C^1(\mathbb{R}^d)$ (see [2, Theorem A.2.4, p. 283] for instance) for each $x \in \mathbb{T}^d$.

We give the definition of weak solutions to (1) following the works in [4, 5, 13].

**Definition 1.1.** We call a pair $(u^\varepsilon, m^\varepsilon) \in W^{1,pr}(\mathbb{T}^d) \times L^q(\mathbb{T}^d)$ *weak solution* to (1) if

(i) $m^\varepsilon \geq 0$ a.e. in $\mathbb{T}^d$, $\int_{\mathbb{T}^d} m^\varepsilon\,dx = 1$ and $m^\varepsilon D_pH(\cdot, Du^\varepsilon) \in L^1(\mathbb{T}^d;\mathbb{R}^d)$,

(ii) the first equation of (1) holds in the following sense:

$$\varepsilon u^\varepsilon + H(x,Du^\varepsilon) = f(x,m^\varepsilon) \quad \text{a.e. in } \{m^\varepsilon > 0\}, \tag{7}$$

and

$$\varepsilon u^\varepsilon + H(x,Du^\varepsilon) \leq f(x,m^\varepsilon) \quad \text{a.e. in } \mathbb{T}^d, \tag{8}$$

(iii) the second equation of (1)

$$\varepsilon m^\varepsilon - \operatorname{div}(m^\varepsilon D_pH(x,Du^\varepsilon)) = \varepsilon \quad \text{in } \mathbb{T}^d, \tag{9}$$

holds in the sense of distributions.

We notice here that, if $pr > d$, then $u^\varepsilon \in C^{0,\gamma}(\mathbb{T}^d)$ with $\gamma = 1 - \frac{d}{pr}$, and $u^\varepsilon$ is differentiable almost everywhere.

### 1.2. Main results

Here, we present the main results of the paper.

**Theorem 1.2.** (Well-posedness) *The discount mean field game system* (1) *has a unique weak solution* $(u^\varepsilon, m^\varepsilon) \in W^{1,pr}(\mathbb{T}^d) \times L^q(\mathbb{T}^d)$. *Moreover, if* $pr > d$, *then it holds that*

$$\varepsilon u^\varepsilon + H(x, Du^\varepsilon) \geq f(x,0) \quad \text{in } \mathbb{T}^d \quad \textit{in the sense of viscosity solutions.} \tag{10}$$

Next, we obtain the weak compactness of $(u^\varepsilon, m^\varepsilon)$ and a stability result. Here we set

$$\langle f \rangle := f(x) - \int_{\mathbb{T}^d} f dx \quad \text{for any } f \in L^1(\mathbb{T}^d).$$

**Theorem 1.3.** (Stability) *Assume that either* $q \geq d$, *or* $r' \leq \frac{qd}{d-q}$ *if* $q < d$. *Let* $(u^\varepsilon, m^\varepsilon) \in W^{1,pr}(\mathbb{T}^d) \times L^q(\mathbb{T}^d)$ *be the weak solution to* (1). *There exists a subsequence* $(u^{\varepsilon_n}, m^{\varepsilon_n})$ *such that*

$$\langle u^{\varepsilon_n} \rangle \rightharpoonup u \quad \textit{weakly in } W^{1,pr}(\mathbb{T}^d), \quad m^{\varepsilon_n} \rightharpoonup m \quad \textit{weakly in } L^q(\mathbb{T}^d),$$

$$\varepsilon_n \int u^{\varepsilon_n} \, dx \to -\lambda \quad \textit{as} \ \ \varepsilon_n \to 0$$

*for some* $(u, m, \lambda) \in W^{1,pr}(\mathbb{T}^d) \times L^q(\mathbb{T}^d) \times \mathbb{R}$, *which is a weak solution to* (2) *defined by Definition* 3.1. *Moreover, if* $pr > d$, *we have*

$$H(x, Du) \geq f(x,0) + \lambda \quad \text{in } \mathbb{T}^d \quad \textit{in the sense of viscosity solutions.} \tag{11}$$

We emphasize here that as in Proposition 4.1 we have the uniqueness of $(m, \lambda)$, where $(u, m, \lambda) \in W^{1,pr}(\mathbb{T}^d) \times L^q(\mathbb{T}^d) \times \mathbb{R}$ is a weak solution to (2). However, we have multiplicity of $u$ in general. We show an example to illustrate the multiplicity of weak solutions to (2) in Section 4.2. Therefore, it is not clear whether $u^\varepsilon$ converges or not. In the next main theorem, we give a condition which any limit of $\langle u^\varepsilon \rangle$ satisfies.

**Theorem 1.4.** (Necessary condition) *Assume that either* $q \geq d$, *or* $r' \leq \frac{qd}{d-q}$ *if* $q < d$. *Let* $(u^\varepsilon, m^\varepsilon)$ *be the weak solution to* (1). *Assume that* $\langle u^\varepsilon \rangle \rightharpoonup \bar{u}$ *weakly in* $W^{1,pr}(\mathbb{T}^d)$, $m^\varepsilon \rightharpoonup m$ *weakly in* $L^q(\mathbb{T}^d)$ *and* $\int_{\mathbb{T}^d} \varepsilon u^\varepsilon \, dx \to -\lambda$ *as* $\varepsilon \to 0$. *Then,* $\bar{u}$ *is a minimizer of*

$$\inf_{u \in \mathcal{E}} \int_{\mathbb{T}^d} \langle u \rangle m \, dx, \tag{12}$$

*where we set* $\mathcal{E} := \{u \in W^{1,pr}(\mathbb{T}^d) \mid (u, m, \lambda)$ *is a weak solution to* $(2)\}$.

As an application of Theorem 1.4, we present a nontrivial example where the whole sequence $u^\varepsilon$ converges.

We also study the uniqueness issue of the ergodic problem (2). We define $\mathcal{Z} \subset \mathbb{T}^d$ by

$$\mathcal{Z} := \{x \in \mathbb{T}^d \mid H(x,0) - \lambda - f(x,0) \geq 0\} \cup \overline{\{x \in \mathbb{T}^d \mid m(x) > 0\}}. \tag{13}$$

We give a comparison principle on $\mathcal{Z}$ for the ergodic problem (2). We denote by $USC(\mathbb{T}^d)$ and $LSC(\mathbb{T}^d)$ the set of all upper and lower, respectively, semi-continuous functions on $\mathbb{T}^d$.

**Theorem 1.5.** (Comparison principle) *Let*

$$(u, m, \lambda) \in (W^{1,pr}(\mathbb{T}^d) \cap USC(\mathbb{T}^d)) \times L^q(\mathbb{T}^d) \times \mathbb{R}$$

*be a weak solution to* (2). *Let* $v \in W^{1,pr}(\mathbb{T}^d) \cap LSC(\mathbb{T}^d)$ *satisfy* (11). *If* $u \le v$ *on* $\mathcal{Z}$, *then* $u \le v$ *on* $\mathbb{T}^d$.

Furthermore, we consider the case where $\quad m \in C(\mathbb{T}^d). \qquad (14)$

Under assumption (14), we have an equivalence between weak solutions and viscosity solutions to (2) in the following sense.

**Theorem 1.6.** (Equivalence) *Let* $(u, m, \lambda) \in W^{1,pr}(\mathbb{T}^d) \times L^q(\mathbb{T}^d) \times \mathbb{R}$ *be a weak solution to* (2). *Assume that* (14) *holds.*

(i) *Let* $v \in C(\mathbb{T}^d)$ *be any viscosity solution to*

$$H(x, Dv) = f(x, m) + \lambda \quad \text{in } \mathbb{T}^d. \tag{15}$$

*Then,* $(v, m, \lambda)$ *is a weak solution to* (2) *and satisfies* (11).

(ii) *Conversely, if* $u$ *satisfies* (11), *then* $u$ *is a viscosity solution to* (15).

We give sufficient conditions to have (14) in Section 3.2.

This paper is organized as follows. In Section 2, we prove Theorem 1.2, that is, we establish the existence and uniqueness of weak solutions to (1). In Section 3, we investigate weak compactness of $(u^\varepsilon, m^\varepsilon)$ and construct a weak solution to the ergodic problem (2) as a limit, which implies Theorem 1.3. Then, we observe lack of uniqueness of weak solutions for ergodic problem (2) in Section 4. We prove Theorem 1.4 and show a nontrivial example to get a convergence of $u^\varepsilon$ in Section 5. In Section 6, we prove Theorems 1.5, 1.6.

## 2. Well-posedness of the discount problem

In this section, we establish the well-posedness result for (1) by following the arguments in [4, 13, 5] with a careful modification to the discount problem.

### 2.1. Optimization problems

We consider two optimization problems corresponding to (1). Define

$$\mathcal{A}^\varepsilon : W^{1,pr}(\mathbb{T}^d) \to \mathbb{R} \text{ by } \mathcal{A}^\varepsilon(\phi) := \int_{\mathbb{T}^d} F^*(x, \varepsilon\phi + H(x, D\phi)) - \varepsilon\phi \, dx. \tag{16}$$

We first notice that $\phi \mapsto \mathcal{A}^\varepsilon(\phi)$ is convex due to the convexity of $F^*(x, \cdot)$ and $H(x, \cdot)$.

Let $K_\varepsilon$ be the set of pairs $(m, w) \in L^q(\mathbb{T}^d) \times (L^{pr})' = L^q(\mathbb{T}^d) \times L^{\frac{r'q}{r'+q-1}}(\mathbb{T}^d; \mathbb{R}^d)$ such that $m \ge 0$ a.e. in $\mathbb{T}^d$, and satisfying

$$\varepsilon m + \operatorname{div}(w) = \varepsilon \quad \text{in } \mathbb{T}^d, \quad \text{in the sense of distributions,}$$

that is,
$$\int_{\mathbb{T}^d} D\psi \cdot w \, dx = \int_{\mathbb{T}^d} \varepsilon(m-1)\psi \, dx \quad \text{for all } \psi \in C^1(\mathbb{T}^d). \tag{17}$$

Note that $r' > 1$ and $q > 1$ implies $\frac{r'q}{r'+q-1} > 1$. We remark that (17) yields $\int_{\mathbb{T}^d} m \, dx = 1$.

We next define $\mathcal{B} : K_\varepsilon \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{B}(m, w) := \int_{\mathbb{T}^d} mH^*(x, -\frac{w}{m}) + F(x, m)\, dx, \tag{18}$$

where if $m(x) = 0$, then we set

$$mH^*(x, -\frac{w}{m}) := \begin{cases} 0 & \text{if } w(x) = 0 \\ +\infty & \text{if } w(x) \neq 0. \end{cases}$$

It is easy to see that $(m, w) \mapsto mH^*(x, -\frac{w}{m})$ is convex. Take $(m_1, w_1), (m_2, w_2) \in K_\varepsilon$ and $t \in (0, 1)$ and set $(m^t, w^t) := (tm_1 + (1-t)m_2, tw_1 + (1-t)w_2)$. Then

$$\begin{aligned} m^t H^*(x, -\frac{w^t}{m^t}) &= \sup_{p\in\mathbb{R}^d}\{-w^t \cdot p - m^t H(x, p)\} \\ &= \sup_{p\in\mathbb{R}^d}\{t(-w_1 \cdot p - m_1 H(x, p)) + (1-t)(-w_2 \cdot p - m_2 H(x, p))\} \\ &\leq tm_1 H^*(x, -\frac{w_1}{m_1}) + (1-t)m_2 H^*(x, -\frac{w_2}{m_2}). \end{aligned}$$

Moreover, noting that $m \mapsto F(x, m)$ and $w \mapsto H^*(x, -\frac{w}{m})$ are strictly convex since $f(x, \cdot)$ is strictly increasing, we see that $(m, w) \mapsto \mathcal{B}(m, w)$ is strictly convex.

First, we check that functionals $\mathcal{A}^\varepsilon$ and $\mathcal{B}$ are weakly lower semi-continuous.

**Lemma 2.1.** *Let $\mathcal{A}^\varepsilon$ and $\mathcal{B}$ be the functionals defined by* (16) *and* (18)*, respectively. Then the functionals $\mathcal{A}^\varepsilon$ and $\mathcal{B}$ are weakly lower semi-continuous in $W^{1,pr}(\mathbb{T}^d)$ and $L^q(\mathbb{T}^d) \times L^{\frac{r'q}{r'+q-1}}(\mathbb{T}^d; \mathbb{R}^d)$, respectively.*

**Proof.** Since $\phi \mapsto \mathcal{A}^\varepsilon(\phi)$ is convex, it suffices to show that $\mathcal{A}^\varepsilon$ is lower semi-continuous. Suppose that there exists $\{\phi_n\}_{n\in\mathbb{N}} \subset W^{1,pr}(\mathbb{T}^d)$ such that $\phi_n \to \phi$ in $W^{1,pr}(\mathbb{T}^d)$ and $\liminf_{n\to\infty} \mathcal{A}^\varepsilon(\phi_n) < \mathcal{A}^\varepsilon(\phi)$. Then, taking a subsequence if necessary, we have $\phi_n(x) \to \phi(x)$ and $D\phi_n(x) \to D\phi(x)$ for almost all $x \in \mathbb{T}^d$. As a preliminary, we show that there exists a constant $M > 0$ such that

$$F^*(x, \varepsilon\phi_n(x) + H(x, D\phi_n(x))) - \varepsilon\phi_n(x) > -M \tag{19}$$

for almost all $x \in \mathbb{T}^d$ and any $n \in \mathbb{N}$. Indeed, because (4) holds and $F^*(x, \cdot)$ is nondecreasing, we have

$$\begin{aligned} F^*(x, \varepsilon\phi_n(x) + H(x, D\phi_n(x))) - \varepsilon\phi_n(x) &\geq F^*\big(x, \varepsilon\phi_n(x) + \frac{1}{C}|D\phi_n(x)|^r - C\big) - \varepsilon\phi_n(x) \\ &\geq F^*(x, \varepsilon\phi_n(x) - C) - \varepsilon\phi_n(x). \end{aligned}$$

If $\varepsilon\phi_n(x) \leq C$ holds, noting that

$$F^*(x, a) = 0 \quad \text{for all } x \in \mathbb{T}^d \text{ and } a \leq f(x, 0) \leq 0,$$

we have $\quad F^*(x, \varepsilon\phi_n(x) + H(x, D\phi_n(x))) - \varepsilon\phi_n(x) \geq -\varepsilon\phi_n(x) \geq -C.$

On the other hand, if there exists a subsequence satisfying $\varepsilon\phi_n(x) > C$, it holds

$$F^*(x, \varepsilon\phi_n(x) + H(x, D\phi_n(x))) - \varepsilon\phi_n(x) \geq \frac{1}{\hat{C}}|\varepsilon\phi_n(x)|^p - \varepsilon\phi_n(x) - \hat{C},$$

where $\hat{C}$ is a sufficiently large constant. Because $p > 1$, we obtain (19).

It follows from Fatou's lemma that

$$\liminf_{n\to\infty} \mathcal{A}^{\varepsilon}(\phi_n) + M = \liminf_{n\to\infty} \int_{\mathbb{T}^d} F^*(x, \varepsilon\phi_n + H(x, D\phi_n)) - \varepsilon\phi_n + M \ dx$$
$$\geq \int_{\mathbb{T}^d} F^*(x, \varepsilon\phi + H(x, D\phi)) - \varepsilon\phi + M \ dx = \mathcal{A}^{\varepsilon}(\phi) + M,$$

which is a contradiction. Hence, (16) is lower semi-continuous.

The weakly lower semi-continuity of $\mathcal{B}$ can be proved similarly, provided the lower semi-continuity of its integrand is verified. Then it suffices to check that

$$\liminf_{n\to\infty} \left( a_n H^*(x, -\frac{b_n}{a_n}) + F(x, a_n) \right) \geq a_0 H^*(x, -\frac{b_0}{a_0}) + F(x, a_0) \tag{20}$$

holds for any $x \in \mathbb{T}^d$, if $(a_n, b_n) \to (a_0, b_0)$ on $\mathbb{R} \times \mathbb{R}^d$. In the case $a_0 < 0$, (20) holds because $F(x, a_0) = F(x, a_n) = +\infty$ for any sufficiently large $n \in \mathbb{N}$. On the other hand, in the case $a_0 > 0$, we have $a_n > 0$ for any sufficiently large $n \in \mathbb{N}$. Thus, it is clear that (20) holds. In what follows, we consider the case $a_0 = 0$. Note that, taking a sufficiently large constant $C > 0$, we have

$$a_n H^*(x, -\frac{b_n}{a_n}) \geq \begin{cases} 0 & \text{if } a_n = 0 \text{ and } b_n = 0 \\ +\infty & \text{if } a_n = 0 \text{ and } b_n \neq 0 \\ \dfrac{a_n}{C} \left| \dfrac{b_n}{a_n} \right|^{r'} - a_n C & \text{otherwise.} \end{cases} \tag{21}$$

In case (21) if $b_0 = 0$, (20) holds because

$$\liminf_{n\to\infty} \left( \frac{a_n}{C} \left| \frac{b_n}{a_n} \right|^{r'} - a_n C \right) \geq \liminf_{n\to\infty} -a_n C = 0.$$

If $b_0 \neq 0$, there exists $\tilde{c} > 0$ such that $|b_n| > \tilde{c}$ for sufficiently large $n \in \mathbb{N}$. Hence,

$$\liminf_{n\to\infty} a_n H^*(x, -\frac{b_n}{a_n}) \geq \liminf_{n\to\infty} \left( \frac{a_n}{C} \left| \frac{\tilde{c}}{a_n} \right|^{r'} - a_n C \right) = +\infty,$$

which implies (20). □

Here, we consider two optimization problems. At first, we study the one corresponding to the first equation of (1).

**Proposition 2.2.** *For all $\varepsilon > 0$, the optimization problem*

$$\inf_{\phi \in W^{1,pr}(\mathbb{T}^d)} \mathcal{A}^{\varepsilon}(\phi) \tag{22}$$

*has a minimizer $\phi \in W^{1,pr}(\mathbb{T}^d)$. Moreover, if $pr > d$, there exists a minimizer $\bar{\phi} \in W^{1,pr}(\mathbb{T}^d)$ satisfying*

$$\varepsilon\bar{\phi} + H(x, D\bar{\phi}) \geq f(x, 0) \quad \textit{in } \mathbb{T}^d \quad \textit{in the sense of viscosity solutions.} \tag{23}$$

**Proof.** Note that $\inf_{\phi\in W^{1,pr}}\mathcal{A}^\varepsilon(\phi)=\inf_{\phi\in C^1}\mathcal{A}^\varepsilon(\phi)$. Take a sequence $\{\phi_n\}_{n\in\mathbb{N}}\subset C^1(\mathbb{T}^d)$ satisfying $\mathcal{A}^\varepsilon(\phi_n)\to\inf_{\phi\in W^{1,pr}}\mathcal{A}^\varepsilon(\phi)$ as $n\to\infty$. Here, we prove that $\{\phi_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,pr}(\mathbb{T}^d)$. As a preliminary, we check that $\int_{\mathbb{T}^d}\phi_n dx$ is bounded. Since $F^*(x,\cdot)$ is nondecreasing, we have

$$\mathcal{A}^\varepsilon(\phi_n)\geq\int_{\mathbb{T}^d}F^*\left(x,-C_0+\int_{\mathbb{T}^d}\varepsilon\phi_n\right)-\varepsilon\phi_n\,dx \tag{24}$$

for some $C_0>0$. Noting that $F^*(x,a)=0$ for any $a\in\mathbb{T}^d$ and $a\leq 0$, by (5), we get

$$\mathcal{A}^\varepsilon(\phi_n)\geq\int_{\{\varepsilon\phi_n>C_0\}}\frac{1}{C}|\varepsilon\phi_n|^p-C\,dx-\int_{\mathbb{T}^d}\varepsilon\phi_n\,dx. \tag{25}$$

Here, we divide the argumentation into two cases.

**Case 1:** $\varepsilon\phi_n\leq C_0$ a.e. in $\mathbb{T}^d$.

**Case 2:** $\mathcal{L}^d(\{\varepsilon\phi_n>C_0\})\neq 0$, where $\mathcal{L}^d(A)$ denotes the Lebesgue measure of $A\subset\mathbb{T}^d$.

In Case 1, by (24), we have $C_0\geq\int_{\mathbb{T}^d}\varepsilon\phi_n\,dx\geq-\mathcal{A}^\varepsilon(\phi_n)\geq -C$ for some constant $C>0$. In Case 2, it follows from (25) and Jensen's inequality that

$$\begin{aligned}\mathcal{A}^\varepsilon(\phi_n)&\geq\frac{1}{C}\int_{\{\varepsilon\phi_n>C_0\}}|\varepsilon\phi_n|^p\,dx-\int_{\{\varepsilon\phi_n>C_0\}}\varepsilon\phi_n\,dx-\int_{\{\varepsilon\phi_n\leq C_0\}}\varepsilon\phi_n\,dx-C.\\&\geq\frac{1}{C}\left|\int_{\{\varepsilon\phi_n>C_0\}}\varepsilon\phi_n\,dx\right|^p-\int_{\{\varepsilon\phi_n>C_0\}}\varepsilon\phi_n\,dx-C,\end{aligned} \tag{26}$$

which implies $\int_{\{\varepsilon\phi_n>C_0\}}\varepsilon\phi_n\,dx$ is bounded. Then, (26) yields

$$\int_{\{\varepsilon\phi_n\leq C_0\}}\varepsilon\phi_n\,dx\geq -C.$$

In conclusion, we obtain the boundness of $\int_{\mathbb{T}^d}\varepsilon\phi_n\,dx$.

Next, we prove that $D\phi_n$ is bounded in $L^{pr}(\mathbb{T}^d)$. We have

$$\begin{aligned}&\|D\phi_n\|^{pr}_{L^{pr}(\mathbb{T}^d)}=\int_{\mathbb{T}^d}||D\phi_n|^r|^p\,dx\leq C\int_{\mathbb{T}^d}|H(x,D\phi_n)+C|^p\,dx\\&\leq C\int_{\mathbb{T}^d}|\varepsilon\phi_n+H(x,D\phi_n)|^p+|C-\varepsilon\phi_n|^p\,dx\\&\leq C\int_{\mathbb{T}^d}F^*(x,\varepsilon\phi_n+H(x,D\phi_n))+C+|C-\varepsilon\phi_n|^p\,dx\\&\leq C\left(\mathcal{A}^\varepsilon(\phi_n)+\int_{\mathbb{T}^d}\varepsilon\phi_n\,dx+\|\varepsilon\phi_n\|^p_{L^p}+1\right)\\&\leq C\left(\inf\mathcal{A}^\varepsilon+1+\varepsilon^p\|\langle\phi_n\rangle\|^p_{L^p(\mathbb{T}^d)}+C\left|\varepsilon\int_{\mathbb{T}^d}\phi_n\,dx\right|^p+\left|\varepsilon\int_{\mathbb{T}^d}\phi_n\,dx\right|+1\right)\\&\leq C\varepsilon^p\|D\phi_n\|^p_{L^p(\mathbb{T}^d)}+C\leq\delta C\varepsilon^p\|D\phi_n\|^{pr}_{L^{pr}(\mathbb{T}^d)}+C_\delta+C,\end{aligned} \tag{27}$$

by using the Poincaré-Wirtinger inequality and the Young inequality with arbitrary $\delta > 0$ in the last inequality. By taking a small $\delta > 0$, we obtain $\|D\phi_n\|_{L^{pr}(\mathbb{T}^d)} \leq C$.

It follows from the Poincaré-Wirtinger inequality that $\phi_n$ is bounded in $W^{1,pr}(\mathbb{T}^d)$. Thus, we can choose a subsequence such that $\phi_n$ converges to some $\phi$ weakly in $W^{1,pr}(\mathbb{T}^d)$ as $n \to \infty$. In light of Lemma 2.1, $\phi$ is a minimizer of (22).

On the other hand, set $\alpha_n(x) := \varepsilon\phi_n(x) + H(x, D\phi_n(x))$.

Let $\bar{\phi}_n$ be the viscosity solution to

$$\varepsilon\bar{\phi}_n + H(x, D\bar{\phi}_n(x)) = \max\{\alpha_n(x), f(x,0)\} \quad \text{in } \mathbb{T}^d.$$

By comparison, it holds $\bar{\phi}_n \geq \phi_n$. Noting that

$$F^*(x,a) = 0 \quad \text{for all } x \in \mathbb{T}^d \text{ and } a \leq f(x,0) \leq 0, \tag{28}$$

we have

$$\begin{aligned} F^*(x, \varepsilon\bar{\phi}_n + H(x, D\bar{\phi}_n(x))) &= F^*(x, \max\{\alpha_n(x), f(x,0)\}) \\ &= \begin{cases} F^*(x, \alpha_n(x)) & \text{if } \alpha_n(x) \geq f(x,0) \\ 0 & \text{if } \alpha_n(x) \leq f(x,0), \end{cases} \end{aligned}$$

which implies that

$$\begin{aligned} \int_{\mathbb{T}^d} F^*(x, \varepsilon\bar{\phi}_n + H(x, D\bar{\phi}_n(x)))\, dx &= \int_{\{\alpha_n \geq f(\cdot,0)\}} F^*(x, \varepsilon\bar{\phi}_n + H(x, D\bar{\phi}_n(x)))\, dx \\ &= \int_{\mathbb{T}^d} F^*(x, \alpha_n(x))\, dx. \end{aligned}$$

Thus, we have

$$\mathcal{A}^\varepsilon(\bar{\phi}_n) = \int_{\mathbb{T}^d} F^*(x, \alpha_n) - \varepsilon\bar{\phi}_n \, dx \leq \int_{\mathbb{T}^d} F^*(x, \alpha_n) - \varepsilon\phi_n \, dx = \mathcal{A}^\varepsilon(\phi_n).$$

Hence, we obtain $\mathcal{A}^\varepsilon(\bar{\phi}_n) \to \inf_{\phi \in W^{1,pr}} \mathcal{A}^\varepsilon(\phi)$ as $n \to \infty$. Similarly, $\bar{\phi}_n$ is bounded in $W^{1,pr}(\mathbb{T}^d)$. Then, we can choose a subsequence such that $\bar{\phi}_n$ converges to some $\bar{\phi}$ weakly in $W^{1,pr}(\mathbb{T}^d)$. In light of Lemma 2.1, $\bar{\phi}$ is a minimizer of (22). If $pr > d$, by the Rellich-Kondrachov compact embedding theorem, it holds that $\bar{\phi}_n \to \bar{\phi}$ uniformly on $\mathbb{T}^d$ as $n \to \infty$. Because

$$\varepsilon\bar{\phi}_n + H(x, D\bar{\phi}_n(x)) \geq f(x,0) \quad \text{in } \mathbb{T}^d$$

holds in the viscosity sense, due to the stability of viscosity solutions, $\bar{\phi}$ satisfies (23). □

Next, we study the dual optimization problems, which corresponds to the second equation of (1).

**Proposition 2.3.** *The optimization problem*

$$\inf_{(m,w) \in K_\varepsilon} \mathcal{B}(m,w) \tag{29}$$

*has the unique minimizer* $(m,w) \in K_\varepsilon$.

**Proof.** Take a sequence $(m_n, w_n) \in K_\varepsilon$ such that $\mathcal{B}(m_n, w_n) \to \inf_{(m,w)\in K_\varepsilon} \mathcal{B}(m,w)$ as $n \to \infty$. Then, noting that $(1,0) \in K_\varepsilon$, for sufficiently large $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathcal{B}(1,0) + 1 \geq \mathcal{B}(m_n, w_n) &= \int_{\mathbb{T}^d} m_n H^*(x, -\frac{w_n}{m_n}) + F(x, m_n)\, dx \\ &\geq \int_{\mathbb{T}^d} \frac{1}{C}|m_n|^q + \frac{m_n}{C}\left|\frac{w_n}{m_n}\right|^{r'} - C\, dx. \qquad (30)\end{aligned}$$

In particular, $\|m_n\|_{L^q}$ is bounded. Note that $w_n = 0$ *a.e.* in $\{m_n = 0\}$ because $\mathcal{B}(m_n, w_n) < +\infty$. By the Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{T}^d} |w_n|^{\frac{r'q}{r'+q-1}}\, dx &= \int_{\{m_n>0\}} |w_n|^{\frac{r'q}{r'+q-1}}\, dx \\ &\leq \|m_n\|_{L^q(\mathbb{T}^d)}^{\frac{r'-1}{r'+q-1}} \left(\int_{\{m_n>0\}} \frac{|w_n|^{r'}}{m_n^{r'-1}}\, dx\right)^{\frac{q}{r'+q-1}} \leq C. \qquad (31)\end{aligned}$$

Hence, we can choose a subsequence such that $(m_n, w_n) \rightharpoonup (m, w)$ weakly as $n \to \infty$ in $L^q(\mathbb{T}^d) \times L^{\frac{r'q}{r'+q-1}}(\mathbb{T}^d; \mathbb{R}^d)$. In light of Lemma 2.1, $(m, w)$ is a minimizer of (29). Since $(m, w) \mapsto \mathcal{B}(m, w)$ is strictly convex, $m$ is unique and so is $\frac{w}{m}$ in $\{m > 0\}$. As $w = 0$ in $\{m = 0\}$, uniqueness of $w$ follows as well, which implies the uniqueness of the minimizer of (29). □

Finally, we prove that the two optimization problems are in duality.

**Proposition 2.4.** *It holds that*

$$\min_{\phi\in W^{1,pr}(\mathbb{T}^d)} \mathcal{A}^\varepsilon(\phi) = -\min_{(m,w)\in K_\varepsilon} \mathcal{B}(m,w).$$

**Proof.** Let $X := L^q(\mathbb{T}^d) \times L^{\frac{r'q}{r'+q-1}}(\mathbb{T}^d; \mathbb{R}^d)$. We can rewrite (29) as

$$\min_{(m,w)\in K_\varepsilon} \mathcal{B}(m,w) = \min_{(m,w)\in X} \sup_{\phi\in C^1} \int_{\mathbb{T}^d} mH^*(x, -\frac{w}{m}) + F(x,m) + w\cdot D\phi + \varepsilon(1-m)\phi\, dx.$$

Indeed, because $F(x, m) = +\infty$ for $m < 0$, it suffices to pay attention to the case in which $(m, w) \in X$ does not satisfy (17). Then, there exists $\hat{\phi} \in C^1(\mathbb{T}^d)$ satisfying

$$\int_{\mathbb{T}^d} w\cdot D\hat{\phi} + \varepsilon(1-m)\hat{\phi}\, dx \neq 0.$$

Setting $\hat{\phi}_n := n\hat{\phi}$ and changing the signature if necessary, to yield

$$\int_{\mathbb{T}^d} mH^*(x, -\frac{w}{m}) + F(x,m) + w\cdot D\hat{\phi}_n + \varepsilon(1-m)\hat{\phi}_n\, dx \to +\infty \quad (n\to\infty),$$

which can not be the infimum. Hence, this infimum is attained at $(m, w) \in K_\varepsilon$ obtained by Proposition 2.3.

By Sion's min-max theorem, it holds that

$$\min_{(m,w)\in X}\sup_{\phi\in C^1}\int_{\mathbb{T}^d} mH^*(x,-\frac{w}{m})+F(x,m)+w\cdot D\phi+\varepsilon(1-m)\phi\,dx$$
$$=\sup_{\phi\in C^1}\min_{(m,w)\in X}\int_{\mathbb{T}^d} mH^*(x,-\frac{w}{m})+F(x,m)+w\cdot D\phi+\varepsilon(1-m)\phi\,dx.$$

By using the interchange of minimization and integration (see [27, Theorem 14.60, p. 677]), we obtain

$$\sup_{\phi\in C^1}\inf_{(m,w)\in X}\int_{\mathbb{T}^d} mH^*(x,-\frac{w}{m})+F(x,m)+w\cdot D\phi+\varepsilon(1-m)\phi\,dx$$
$$=\sup_{\phi\in C^1}\int_{\mathbb{T}^d}\inf_{(a,b)\in\mathbb{R}\times\mathbb{R}^d} aH^*(x,-\frac{b}{a})+F(x,a)+b\cdot D\phi+\varepsilon(1-a)\phi\,dx.$$

An easy computation shows that

$$F^*(x,\varepsilon\phi+H(x,D\phi))=-\inf_{(a,b)\in\mathbb{R}\times\mathbb{R}^d} aH^*(x,-\frac{b}{a})+F(x,a)+b\cdot D\phi-\varepsilon a\phi,$$

so that $\displaystyle\min_{(m,w)\in K_\varepsilon}\mathcal{B}(m,w)=\sup_{\phi\in C^1(\mathbb{T}^d)}\int_{\mathbb{T}^d}-F^*(x,\varepsilon\phi+H(x,D\phi))+\varepsilon\phi\,dx.$ □

### 2.2. Existence and uniqueness of weak solutions to the discount problem

In this section, we prove that a pair of minimizers of the optimization problems (22) and (29) is a weak solution to (1).

**Proposition 2.5.** *Let $\phi\in W^{1,pr}(\mathbb{T}^d)$ and $(m,w)\in K_\varepsilon$ be minimizers of* (22) *and* (29)*, respectively. Then, $(\phi,m)$ is a weak solution to* (1)*.*
*Conversely, if $(\phi,m)$ is a weak solution to* (1)*, then $\phi$ and $(m,w)$ are minimizers of* (22) *and* (29)*, respectively, where we set $w:=-mD_pH(x,D\phi)$.*

**Proof.** Let $\phi$ and $(m,w)$ be minimizers of (22) and (29), respectively. In view of the duality, Proposition 2.4, we have

$$\int_{\mathbb{T}^d} F^*(x,\varepsilon\phi+H(x,D\phi))-\varepsilon\phi\,dx=-\int_{\mathbb{T}^d} mH^*(x,-\frac{w}{m})+F(x,m)\,dx.$$

By the convexity of $F(x,\cdot)$ and $H(x,\cdot)$, we get

$$0=\int_{\mathbb{T}^d} mH^*(x,-\frac{w}{m})+F(x,m)+F^*(x,\varepsilon\phi+H(x,D\phi))-\varepsilon\phi\,dx$$
$$\geq\int_{\mathbb{T}^d} mH^*(x,-\frac{w}{m})+F(x,m)+m(\varepsilon\phi+H(x,D\phi))-F(x,m)-\varepsilon\phi\,dx$$
$$\geq\int_{\mathbb{T}^d} -w\cdot D\phi+\varepsilon\phi(m-1)\,dx. \tag{32}$$

We now use the fact that $(m,w)$ satisfies (17), $D\phi\in L^{pr}(\mathbb{T}^d)$ and that we have $w\in L^{\frac{r'q}{r'+q-1}}(\mathbb{T}^d;\mathbb{R}^d)=(L^{pr}(\mathbb{T}^d;\mathbb{R}^d))'$, where we denote by $X'$ the dual space of a Banach space $X$.

The conditions imply $\displaystyle\int_{\mathbb{T}^d} -w \cdot D\phi + \varepsilon\phi(m-1)\, dx = 0.$

Hence, equalities hold in estimate (32). In particular, combined with dualities of convexity of $F(x,\cdot)$ and $H(x,\cdot)$, we have

$$F(x,m) + F^*(x, \varepsilon\phi + H(x,D\phi)) = m(\varepsilon\phi + H(x,D\phi)) \quad \text{a.e. in } \mathbb{T}^d,$$

and
$$m\left\{H^*(x, -\frac{w}{m}) + H(x,D\phi)\right\} = -w \cdot D\phi \quad \text{a.e. in } \mathbb{T}^d. \tag{33}$$

Since $F(x,\cdot)$ is strictly convex and smooth on $(0,+\infty)$, we get

$$\varepsilon\phi + H(x,D\phi) = \frac{\partial F}{\partial m} = f(x,m) \quad \text{a.e. in } \{m > 0\}.$$

Moreover, if $x \in \{m = 0\}$, we have

$$\varepsilon\phi(x) + H(x, D\phi(x)) \in D_m^- F(x,0),$$

where we write $D_m^- F(x,0)$ for the subdifferential of $F(x,0)$ in $m$. Noting that $D_m^- F(x,0) = (-\infty, f(x,0)]$, we obtain (8). Also, (33) implies

$$w(x) = -m(x) D_p H(x, D\phi) \quad \text{a.e. in } \mathbb{T}^d.$$

By the duality of $H(x,\cdot)$, combined with $(m,w) \in K_\varepsilon$, we obtain

$$\varepsilon \int_{\mathbb{T}^d} (m-1)\psi\, dx = \int_{\mathbb{T}^d} w \cdot D\psi\, dx = -\int_{\mathbb{T}^d} m D_p H(x, D\phi) \cdot D\psi\, dx,$$

for $\psi \in C^1(\mathbb{T}^d)$, which yields (9).

Next, let $(\phi, m)$ be a weak solution to (1) and define $w = -m D_p H(x, D\phi)$. Then, $(m,w)$ belongs to $K_\varepsilon$. Moreover,

$$H^*(x, -\frac{w}{m}) = D\phi \cdot D_p H(x, D\phi) - H(x, D\phi). \tag{34}$$

We prove that $(m,w)$ is a minimizer for (29). Let $(m', w') \in K_\varepsilon$. By convexity of $F(x,\cdot)$ and $H^*(x,\cdot)$, using (34), we obtain

$$\begin{aligned}
\mathcal{B}(m',w') &\geq \int_{\mathbb{T}^d} m' H^*(x, -\frac{w'}{m'}) + F(x,m) + f(x,m)(m'-m)\, dx \\
&\geq \int_{\mathbb{T}^d} m' H^*(x, -\frac{w'}{m'}) + F(x,m) + \{\varepsilon\phi + H(x,D\phi)\}(m'-m)\, dx \\
&= \int_{\mathbb{T}^d} m'\{H^*(x, -\frac{w'}{m'}) + H(x,D\phi)\} + F(x,m) - mH(x,D\phi) + \varepsilon\phi(m'-m)\, dx \\
&\geq \int_{\mathbb{T}^d} -w' \cdot D\phi + F(x,m) - mH(x,D\phi) + \varepsilon\phi(m'-m)\, dx \\
&= \int_{\mathbb{T}^d} -\varepsilon\phi(m-1) + F(x,m) - mH(x,D\phi)\, dx \\
&= \int_{\mathbb{T}^d} -\varepsilon\phi(m-1) + F(x,m) + w \cdot D\phi + mH^*(x, -\frac{w}{m})\, dx = \mathcal{B}(m,w).
\end{aligned}$$

Hence, $(m,w)$ is a minimizer for (29).

Finally, we prove that $\phi$ is a minimizer for (22). Let $\phi' \in W^{1,pr}(\mathbb{T}^d)$. Note that

$$F^*(x, f(x,m)) = \sup_{s\in\mathbb{R}}\{sf(x,m) - F(x,s)\} = mf(x,m) - F(x,m).$$

If $m(x) > 0$, from (7) and the above equality, we get

$$F^*(x, \varepsilon\phi + H(x, D\phi)) + F(x,m) = m\{\varepsilon\phi + H(x, D\phi)\}. \tag{35}$$

On the other hand, if $m(x) = 0$, from (8), for all $a \in \mathbb{R}$, we have

$$\begin{aligned} F^*(x,a) - F^*(x, \varepsilon\phi + H(xD\phi)) &\geq F^*(x,a) - F^*(x, f(x,0)) = F^*(x,a) \\ &\geq 0 = m(x)\{a - (\varepsilon\phi + H(x, D\phi))\}. \end{aligned} \tag{36}$$

Thus, (35) and (36) implies $m \in D_a^- F^*(x, \varepsilon\phi + H(x, D\phi))$. Then, we obtain

$$\begin{aligned} \mathcal{A}^\varepsilon(\phi') &\geq \int_{\mathbb{T}^d} F^*(x, \varepsilon\phi + H(x, D\phi)) + m\{\varepsilon\phi' - \varepsilon\phi + H(x, D\phi') - H(x, D\phi)\} - \varepsilon\phi' \, dx \\ &\geq \int_{\mathbb{T}^d} F^*(x, \varepsilon\phi + H(x, D\phi)) + m\,\{\varepsilon\phi' - \varepsilon\phi + D_pH(x, D\phi) \cdot D(\phi' - \phi)\} - \varepsilon\phi' \, dx \\ &= \int_{\mathbb{T}^d} F^*(x, \varepsilon\phi + H(x, D\phi)) + m(\varepsilon\phi' - \varepsilon\phi) - w \cdot D(\phi' - \phi) - \varepsilon\phi' \, dx \\ &= \int_{\mathbb{T}^d} F^*(x, \varepsilon\phi + H(x, D\phi)) - \varepsilon\phi \, dx = \mathcal{A}^\varepsilon(\phi), \end{aligned} \tag{37}$$

since $(m, w) \in K_\varepsilon$. Hence, $\phi$ is a minimizer for (22). $\square$

Here, we give a proof of Theorem 1.2.

**Proof of Theorem 1.2.** The existence of weak solutions is given by Proposition 2.5. Here, we only prove the uniqueness. Let $(u_1^\varepsilon, m_1^\varepsilon)$ and $(u_2^\varepsilon, m_2^\varepsilon)$ be weak solutions to (1). It follows from the uniqueness of minimizers of (29) that $m^\varepsilon := m_1^\varepsilon = m_2^\varepsilon$ a.e. in $\mathbb{T}^d$. Next, suppose that

$$\mathcal{L}^d\left(\{x \in \mathbb{T}^d \mid m^\varepsilon(x) > 0 \text{ and } Du_1^\varepsilon(x) \neq Du_2^\varepsilon(x)\}\right) > 0,$$

where $\mathcal{L}^d(A)$ denotes the Lebesgue measure of $A \subset \mathbb{T}^d$. Because $H(x, \cdot)$ is strictly convex, as in (37), we have

$$\begin{aligned} \mathcal{A}^\varepsilon(u_2^\varepsilon) &\geq \int_{\mathbb{T}^d} F^*(x, \varepsilon u_1^\varepsilon + H(x, Du_1^\varepsilon)) + m\{\varepsilon u_2^\varepsilon - \varepsilon u_1^\varepsilon + H(x, Du_2^\varepsilon) - H(x, Du_1^\varepsilon)\} - \varepsilon u_2^\varepsilon \, dx \\ &> \int_{\mathbb{T}^d} F^*(x, \varepsilon u_1^\varepsilon + H(x, Du_1^\varepsilon)) + m\{\varepsilon u_2^\varepsilon - \varepsilon u_1^\varepsilon + D_pH(x, Du_1^\varepsilon) \cdot D(u_2^\varepsilon - u_1^\varepsilon)\} - \varepsilon u_2^\varepsilon \, dx \\ &\geq \int_{\mathbb{T}^d} F^*(x, \varepsilon u_1^\varepsilon + H(x, Du_1^\varepsilon)) - \varepsilon u_1^\varepsilon \, dx = \mathcal{A}^\varepsilon(u_1^\varepsilon). \end{aligned}$$

Hence, $u_2^\varepsilon$ is not a minimizer of (22), which contradicts the result of Proposition 2.5. Thus, we conclude that $Du_1^\varepsilon = Du_2^\varepsilon$ a.e. in $\{x \in \mathbb{T}^d \mid m^\varepsilon(x) > 0\}$. In particular, by (7), $u_1^\varepsilon = u_2^\varepsilon$ a.e. in $\{x \in \mathbb{T}^d \mid m^\varepsilon(x) > 0\}$.

Next, we prove the uniqueness of $u^\varepsilon$. We define

$$\bar{u}^\varepsilon(x) := \max\{u_1^\varepsilon(x), u_2^\varepsilon(x)\} = \frac{1}{2}(u_1^\varepsilon + u_2^\varepsilon + |u_1^\varepsilon - u_2^\varepsilon|).$$

As in [9, Lemma 7.6], it follows that $\bar{u}^\varepsilon \in W^{1,pr}(\mathbb{T}^d)$ with

$$D\bar{u}^\varepsilon = \chi_{\{u_1^\varepsilon > u_2^\varepsilon\}} Du_1^\varepsilon + \chi_{\{u_1^\varepsilon < u_2^\varepsilon\}} Du_2^\varepsilon + \frac{1}{2}\chi_{\{u_1^\varepsilon = u_2^\varepsilon\}}(Du_1^\varepsilon + Du_2^\varepsilon) \quad \text{a.e. in } \mathbb{T}^d,$$

where $\chi_A$ is the characteristic function for $A \subset \mathbb{T}^d$, that is, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. We claim that $(\bar{u}^\varepsilon, m^\varepsilon)$ is a weak solution to (1). Indeed, because $D\bar{u}^\varepsilon = Du_1^\varepsilon = Du_2^\varepsilon$ a.e. in $\{x \in \mathbb{T}^d \mid m^\varepsilon(x) > 0\}$, it is clear that (7) and (9) hold. Moreover, we can see that

$$\varepsilon\bar{u}^\varepsilon + H(x, D\bar{u}^\varepsilon) \le f(x, m^\varepsilon) \quad \text{a.e. in } \{u_1^\varepsilon \neq u_2^\varepsilon\}.$$

For almost all $x \in \{u_1^\varepsilon = u_2^\varepsilon\}$, by the convexity of the Hamiltonian, we get

$$\varepsilon\bar{u}^\varepsilon + H(x, D\bar{u}^\varepsilon) \le \frac{1}{2}(\varepsilon u_1^\varepsilon + H(x, Du_1^\varepsilon)) + \frac{1}{2}(\varepsilon u_2^\varepsilon + H(x, Du_2^\varepsilon)) \le f(x, m^\varepsilon),$$

which implies that $(\bar{u}^\varepsilon, m^\varepsilon)$ is a weak solution to (1).

Note that $x \in \{m^\varepsilon = 0\}$, by (8) and (28), we have

$$\int_{\{m^\varepsilon = 0\}} F^*(x, \varepsilon u_1^\varepsilon + H(x, Du_1^\varepsilon))\, dx = \int_{\{m^\varepsilon = 0\}} F^*(x, \varepsilon\bar{u}^\varepsilon + H(x, D\bar{u}^\varepsilon))\, dx. \tag{38}$$

Here, we suppose that $\mathcal{L}^d(\{u_1^\varepsilon \neq u_2^\varepsilon\}) > 0$. By (7) and (38), we have

$$\begin{aligned}\mathcal{A}^\varepsilon(\bar{u}^\varepsilon) &< \int_{\mathbb{T}^d} F^*(x, \varepsilon\bar{u}^\varepsilon + H(x, D\bar{u}^\varepsilon)) - \varepsilon u_1^\varepsilon\, dx \\ &= \int_{\mathbb{T}^d} F^*(x, \varepsilon u_1^\varepsilon + H(x, Du_1^\varepsilon)) - \varepsilon u_1^\varepsilon\, dx = \mathcal{A}^\varepsilon(u_1^\varepsilon),\end{aligned}$$

which contradicts that $u_1^\varepsilon$ is a minimizer of (22). Thus, we have $u_1^\varepsilon = u_2^\varepsilon$ on $\mathbb{T}^d$.

Finally, we assume $pr > d$. Because of the uniqueness of weak solutions, $u^\varepsilon$ coincides the minimizer $\bar{\phi}$ obtained in Proposition 2.2, which satisfies (10) in the viscosity sense. □

## 3. Weak compactness and stability

In this section we prove Theorem 1.3. We first recall the definition of weak solutions to (2) introduced by [5].

**Definition 3.1.** We call a triple of $(u, m, \lambda) \in W^{1,pr}(\mathbb{T}^d) \times L^q(\mathbb{T}^d) \times \mathbb{R}$ a *weak solution* to (2) if

(i) $m \ge 0$ a.e. in $\mathbb{T}^d$, $\int_{\mathbb{T}^d} m\, dx = 1$ and $mD_pH(\cdot, Du) \in L^1(\mathbb{T}^d)$,

(ii) the first equation of (2) holds in the following sense:

$$H(x, Du) = f(x, m) + \lambda \quad \text{a.e. in } \{m > 0\}, \tag{39}$$

and

$$H(x, Du) \leq f(x, m) + \lambda \quad \text{a.e. in } \mathbb{T}^d, \tag{40}$$

(iii) the second equation of (2) holds

$$-\mathrm{div}(mD_pH(x, Du)) = 0 \quad \text{in } \mathbb{T}^d, \tag{41}$$

in the sense of distribution.

### 3.1. Discounted aproximations

The existence of weak solutions to (2) is firstly proved by [5]. In [5], they directly prove the existence by considering the optimization problem as in Section 2. In this paper, we prove it by considering weak compactness and stability for the vanishing discount problem.

**Proposition 3.2.** *Let $(u^\varepsilon, m^\varepsilon) \in W^{1,pr}(\mathbb{T}^d) \times L^q(\mathbb{T}^d)$ be the weak solution to (1), and $w^\varepsilon = -m^\varepsilon D_pH(x, Du^\varepsilon)$. There exists a constant $C > 0$ independent of $\varepsilon > 0$ such that*

$$\|\langle u^\varepsilon\rangle\|_{W^{1,pr}(\mathbb{T}^d)} + \|m^\varepsilon\|_{L^q(\mathbb{T}^d)} + \|w^\varepsilon\|_{L^{\frac{r'q}{r'+q-1}}(\mathbb{T}^d;\mathbb{R}^d)} + \left|\varepsilon \int_{\mathbb{T}^d} u^\varepsilon \, dx\right| \leq C. \tag{42}$$

**Proof.** Firstly, as in the proof of Proposition 2.2, we can prove that both $|\varepsilon \int u^\varepsilon dx|$ and $\|Du^\varepsilon\|_{L^{pr}}$ are bounded uniformly in $\varepsilon$. It follows from the Poincaré-Wirtinger inequality that $\|\langle u^\varepsilon\rangle\|_{W^{1,pr}}$ is bounded uniformly in $\varepsilon$.

Finally, using (30) and (31), we have

$$\|m^\varepsilon\|_{L^q(\mathbb{T}^d)} + \|w^\varepsilon\|_{L^{\frac{r'q}{r'+q-1}}(\mathbb{T}^d;\mathbb{R}^d)} \leq C,$$

which gives (42). □

Finally, we study the stability of weak solutions using Γ-convergence type arguments. To prove Theorem 1.3, we introduce several notations. We define

$$\mathcal{A} : W^{1,pr}(\mathbb{T}^d) \times \mathbb{R} \to \mathbb{R} \ \text{ by } \ \mathcal{A}(\phi, \lambda) := \int_{\mathbb{T}^d} F^*(x, -\lambda + H(x, D\phi)) + \lambda \, dx.$$

Let $K$ be the set of pairs $(m, w) \in L^q(\mathbb{T}^d) \times L^{\frac{r'q}{r'+q-1}}(\mathbb{T}^d; \mathbb{R}^d)$ such that $m \geq 0$ a.e. in $\mathbb{T}^d$, $\int_{\mathbb{T}^d} m = 1$, and satisfies

$$\mathrm{div}(w) = 0,$$

in the sense of distribution. Next, we prove the following two lemmas.

**Lemma 3.3.** *It holds that*

$$\limsup_{\varepsilon\to 0} \min_{\phi\in W^{1,pr}(\mathbb{T}^d)} \mathcal{A}^\varepsilon(\phi) \leq \inf_{(\phi,c)\in W^{1,pr}(\mathbb{T}^d)\times\mathbb{R}} \mathcal{A}(\phi, c).$$

**Proof.** Take any $(\phi, c) \in C^1(\mathbb{T}^d) \times \mathbb{R}$. Set $\phi^\varepsilon := \phi - c/\varepsilon$. Then, we get

$$\begin{aligned}\limsup_{\varepsilon\to 0} \inf_{\phi\in W^{1,pr}(\mathbb{T}^d)} \mathcal{A}^\varepsilon(\phi) &\le \limsup_{\varepsilon\to 0} \int_{\mathbb{T}^d} F^*(x, \varepsilon\phi^\varepsilon + H(x, D\phi^\varepsilon)) - \varepsilon\phi^\varepsilon \, dx \\ &= \limsup_{\varepsilon\to 0} \int_{\mathbb{T}^d} F^*(x, \varepsilon\phi - c + H(x, D\phi)) - \varepsilon\phi + c \, dx \\ &\le \int_{\mathbb{T}^d} F^*(x, -c + H(x, D\phi)) + c \, dx = \mathcal{A}(\phi, c).\end{aligned}$$

Taking the infimum on $(\phi, c)$ yields the conclusion. □

**Lemma 3.4.** *Assume either $q \ge d$ or $r' \le \frac{qd}{d-q}$ if $q < d$. Then, it holds that*

$$\limsup_{\varepsilon\to 0} \min_{(m,w)\in K_\varepsilon} \mathcal{B}(m, w) \le \inf_{(m,w)\in K} \mathcal{B}(m, w). \tag{43}$$

**Proof.** We only need to consider the case of $\inf_K \mathcal{B} < +\infty$. Take any $(m_n, w_n) \in K$ so that $\mathcal{B}(m_n, w_n) \to \inf_K \mathcal{B}$ as $n \to \infty$. Consider

$$-\Delta v_n = m_n - 1 \quad \text{in } \mathbb{T}^d.$$

Since $m_n \in L^q(\mathbb{T}^d)$, we have $v_n \in W^{2,q}(\mathbb{T}^d)$. In particular, $v_n$ satisfies

$$\int_{\mathbb{T}^d} D\psi \cdot Dv_n \, dx = \int_{\mathbb{T}^d} (m_n - 1)\psi \, dx, \quad \text{for all } \psi \in C^1(\mathbb{T}^d).$$

Set $$m_n^\varepsilon := \varepsilon + (1-\varepsilon)m_n, \quad \text{and} \quad w_n^\varepsilon := w_n - \varepsilon(\varepsilon - 1)Dv_n.$$

Note that if $q \ge d$ then we can easily check that $(m_n^\varepsilon, w_n^\varepsilon) \in L^q(\mathbb{T}^d) \times L^{\frac{r'q}{r'+q-1}}(\mathbb{T}^d)$. If $q < d$ then since we assume $r' \le \frac{qd}{d-q}$, we have

$$\frac{r'q}{r'+q-1} \le \frac{q^2 d}{(d-q)(r'+q-1)} < \frac{qd}{d-q}.$$

By the Sobolev inequality we get $w_n^\varepsilon \in L^{\frac{r'q}{r'+q-1}}(\mathbb{T}^d)$. Moreover, it is easily seen that

$$\begin{aligned}\int_{\mathbb{T}^d} w_n^\varepsilon \cdot D\psi \, dx &= \int_{\mathbb{T}^d} w_n \cdot D\psi - \varepsilon(\varepsilon-1) D\psi \cdot Dv_n \, dx \\ &= \int_{\mathbb{T}^d} -\varepsilon(\varepsilon-1)(m_n - 1)\psi \, dx = \int_{\mathbb{T}^d} \varepsilon(m_n^\varepsilon - 1)\psi \, dx\end{aligned}$$

for all $\psi \in C^1(\mathbb{T}^d)$, which implies $(m_n^\varepsilon, w_n^\varepsilon) \in K_\varepsilon$.

Note that $F(x, m_n^\varepsilon) \le C(|m_n|^q + 1) \in L^1(\mathbb{T}^d)$, and

$$\begin{aligned}m_n^\varepsilon H^*\left(x, -\frac{w_n^\varepsilon}{m_n^\varepsilon}\right) &\le C\left(\frac{|w_n - \varepsilon(\varepsilon-1)Dv_n|^{r'}}{|\varepsilon + (1-\varepsilon)m_n|^{r'-1}} + m_n^\varepsilon\right) \\ &\le C\left(m_n \left|\frac{w_n}{m_n}\right|^{r'} + |Dv_n|^{r'} + m_n\right).\end{aligned} \tag{44}$$

By a similar argument to Proposition 2.3 and using assumption $r' \le \frac{qd}{d-q}$, we can see that the right hand side of (44) is in $L^1(\mathbb{T}^d)$. By the Fatou lemma, we obtain

$$\limsup_{\varepsilon\to 0} \min_{(m,w)\in K_\varepsilon} \mathcal{B}(m,w) \le \limsup_{\varepsilon\to 0} \int_{\mathbb{T}^d} m_n^\varepsilon H^*(x, -\frac{w_n^\varepsilon}{m_n^\varepsilon}) + F(x, m_n^\varepsilon)\, dx$$
$$\le \int_{\mathbb{T}^d} m_n H^*(x, -\frac{w_n}{m_n}) + F(x, m_n)\, dx = \mathcal{B}(m_n, w_n).$$

Sending $n \to \infty$ yields the conclusion. □

We are ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** By Lemmas 3.3, 3.4 and the duality on $\mathcal{A}$ and $\mathcal{B}$ by [5, Lemma 4.3], we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \inf_{\phi\in W^{1,pr}(\mathbb{T}^d)} \mathcal{A}^\varepsilon(\phi) \le \inf_{(\phi,c)\in W^{1,pr}(\mathbb{T}^d)\times\mathbb{R}} \mathcal{A}(\phi, c) = -\inf_{(m,w)\in K} \mathcal{B}(m,w)\\
&\le -\limsup_{\varepsilon\to 0} \inf_{(m,w)\in K_\varepsilon} \mathcal{B}(m,w) = \liminf_{\varepsilon\to 0} \left(-\inf_{(m,w)\in K_\varepsilon} \mathcal{B}(m,w)\right)\\
&= \liminf_{\varepsilon\to 0} \inf_{\phi\in W^{1,pr}(\mathbb{T}^d)} \mathcal{A}^\varepsilon(\phi).
\end{aligned}$$

Thus, equalities hold in the above estimates, which implies

$$\begin{aligned}
\lim_{\varepsilon\to 0} \inf_{\phi\in W^{1,pr}(\mathbb{T}^d)} \mathcal{A}^\varepsilon(\phi) &= \inf_{(\phi,c)\in W^{1,pr}(\mathbb{T}^d)\times\mathbb{R}} \mathcal{A}(\phi, c)\\
&= -\inf_{(m,w)\in K} \mathcal{B}(m,w) = -\lim_{\varepsilon\to 0} \inf_{(m,w)\in K_\varepsilon} \mathcal{B}(m,w).
\end{aligned}$$

Let $(m^\varepsilon, w^\varepsilon)$ be the minimizer of (29). By Proposition 3.2, up to extracting a subsequence, $(m^\varepsilon, w^\varepsilon)$ converges weakly in $L^q(\mathbb{T}^d) \times L^{\frac{r'q}{r'+q-1}}(\mathbb{T}^d; \mathbb{R}^d)$ for some $(m,w) \in K$. Then, we observe

$$\inf_{(m,w)\in K} \mathcal{B}(m,w) = \lim_{\varepsilon\to 0} \inf_{(m,w)\in K_\varepsilon} \mathcal{B}(m,w) = \liminf_{\varepsilon\to 0} \mathcal{B}(m^\varepsilon, w^\varepsilon) \ge \mathcal{B}(m,w),$$

which implies that $(m,w)$ minimizes $\mathcal{B}$ over $K$.

On the other hand, let $u^\varepsilon$ be the minimizer of (16). In view of Proposition 3.2, $\langle u^\varepsilon\rangle \rightharpoonup u$ weakly in $W^{1,pr}(\mathbb{T}^d)$ and $\int \varepsilon u^\varepsilon \to -\lambda$ as $\varepsilon \to 0$ along subsequences. Similarly, we can see that $(u, \lambda)$ minimize $\mathcal{A}$ over $W^{1,pr}(\mathbb{T}^d) \times \mathbb{R}$. By [5, Proposition 4.5], $(u, m, \lambda)$ is a weak solution to (2).

Finally, if we assume $pr > d$, by the Rellich-Kondrachov compact embedding theorem, $\langle u^\varepsilon\rangle \to u$ uniformly on $\mathbb{T}^d$. In view of the stability of viscosity solutions and (10), $u$ satisfies (11) in the viscosity sense. □

### 3.2. Continuity of density $m$

In this section, we give some conditions to obtain the continuity of $m$.

**Proposition 3.5.** *Let $d = 1$. Assume that there exists $\tilde{J} : [0,\infty) \to \mathbb{R}$ and $c_0 > 0$ such that*

$$F(x,m) - F^*(x,a) \geq ma + c_0|m - \tilde{J}(a)|^2 \quad \text{for all } m, a \in [0,\infty) \text{ and } x \in \mathbb{T}. \tag{45}$$

*Then, we have*

$$m \in H^1(\mathbb{T}) \subset C(\mathbb{T}). \tag{46}$$

*Furthermore, assume that $H^*$ and $F$ are twice differentiable in the first variable and*

$$H^*_{xx}(x,p) \leq C(|p|^{r'} + 1) \quad \text{and} \quad F_{xx}(x,m) \leq C(|m|^q + 1),$$

*for all $(x,p) \in \mathbb{T} \times \mathbb{R}$ and $m \geq 0$. Then, we have*

$$\|m^\varepsilon\|_{H^1(\mathbb{T})} \leq C \tag{47}$$

*for some $C \geq 0$, which is independent of $\varepsilon$.*

**Proof.** The continuity of $m$, that is (46), is due to [14, Theorem 3.4]. Estimate (47) is a straightforward result of a Sobolev estimate in Appendix, Proposition A.1. □

For instance, if we have $F(x,m) = F(m)$ and $F''(m) \geq c_0 > 0$ for all values $(x,m) \in \mathbb{T} \times [0,\infty)$, then taking $\tilde{J}(a) := (F^*)'(a) = f^{-1}(a)$, we can check that (45) holds. See [26, Section 3] for more details.

**Remark 3.6.** We point out here that due to (47) and the uniqueness of $m$ we have

$$m^\varepsilon \to m \quad \text{in } C(\mathbb{T}) \quad \text{as } \varepsilon \to 0,$$

which is of independent interest.

In a higher dimension, we give another type of results.

**Proposition 3.7.** *Let $c \in \mathbb{R}$ and assume that $(c, m, \lambda)$ is a weak solution to ergodic problem* (2). *Then, $m \in C(\mathbb{T}^d)$.*

**Proof.** From (39), it holds that

$$H(x,0) = f(x,m) + \lambda \quad \text{a.e. in } \{m > 0\}.$$

Since $\lambda \mapsto \int_{\mathbb{T}^d} \max\{f^{-1}(x, H(x,0) - \lambda), 0\}\, dx$ is monotone, we can choose $\lambda$ so that

$$\int_{\mathbb{T}^d} \max\{f^{-1}(x, H(x,0) - \lambda), 0\}\, dx = 1.$$

Hence, we can explicitly get $m(x) = \max\{f^{-1}(x, H(x,0) - \lambda), 0\}$. □

For instance, letting $H(x,p) := \frac{1}{2}|p|^2 + V(x)$ for some $V \in C(\mathbb{T}^d)$, we can easily check that $(0, m, \lambda)$ is a weak solution to ergodic problem (2).

## 4. Uniqueness issue of the ergodic problem

Here, we study uniqueness and non-uniqueness of solutions to the ergodic problem (2). We prove that $(m, \lambda) \in L^q(\mathbb{T}^d) \times \mathbb{R}$ is unique. On the other hand, we give an example which shows the multiplicity of $u \in W^{1,pr}(\mathbb{T}^d)$.

### 4.1. Uniqueness of $m$ and $\lambda$

First, we prove the uniqueness of $(m, \lambda)$. This is known in [5], but we give it for the completeness of the paper.

**Proposition 4.1.** *Let $(u_1, m_1, \lambda_1)$ and $(u_2, m_2, \lambda_2)$ be weak solutions to* (2). *Then, $\lambda_1 = \lambda_2$, $m_1 = m_2$ a.e. in $\mathbb{T}^d$ and $Du_1 = Du_2$ a.e. in $\{x \in \mathbb{T}^d \mid m(x) > 0\}$.*

**Proof.** Just as in Proposition 2.3, $\inf_{(m,w)\in K} \mathcal{B}(m, w)$ has the unique minimizer $(m, w) \in K$. As in the proof of Proposition 2.5, we have

$$w = -mD_pH(x, Du_i),$$

and combining this with (39) we obtain

$$-w \cdot Du_i = m\{H^*(x, -\frac{w}{m}) + H(x, Du_i)\} = m\{H^*(x, -\frac{w}{m}) + (f(x, m) + \lambda_i)\},$$

for $i = 1, 2$. Then, we have

$$m\lambda_i = -mH^*(x, -\frac{w}{m}) - w \cdot Du_i - mf(x, m) \quad \text{a.e. in } \mathbb{T}^d.$$

Therefore, integrating over $\mathbb{T}^d$, we get

$$\begin{aligned} \lambda_i &= \int_{\mathbb{T}^d} -mH^*(x, -\frac{w}{m}) - w \cdot Du_i - mf(x, m)\, dx \\ &= \int_{\mathbb{T}^d} -mH^*(x, -\frac{w}{m}) - mf(x, m)\, dx, \end{aligned}$$

since we have $\operatorname{div}(w) = 0$ in the sense of distribution, which implies $\lambda_1 = \lambda_2$. Set $\lambda := \lambda_1 = \lambda_2$.

Next, suppose that

$$\mathcal{L}^d\left(\{x \in \mathbb{T}^d \mid m(x) > 0 \text{ and } Du_1(x) \neq Du_2(x)\}\right) > 0.$$

In light of the strict convexity of $H(x, \cdot)$, by a similar argument to that of the proof of Proposition 2.5, we have

$$\begin{aligned} \mathcal{A}(u_2, \lambda) &\geq \int_{\mathbb{T}^d} F^*(x, -\lambda + H(x, Du_1)) + m(H(x, Du_2) - H(x, Du_1)) + \lambda\, dx. \\ &> \int_{\mathbb{T}^d} F^*(x, -\lambda + H(x, Du_1)) + mD_pH(x, Du_1) \cdot D(u_2 - u_1) + \lambda\, dx \\ &= \int_{\mathbb{T}^d} F^*(x, -\lambda + H(x, Du_1)) + \lambda\, dx = \mathcal{A}(u_1, \lambda), \end{aligned}$$

where we use the fact that $(m, w)$ satisfies (41). Hence, $(u_2, \lambda)$ does not minimize $\mathcal{A}$ over $W^{1,pr}(\mathbb{T}^d) \times \mathbb{R}$, which contradicts the result in [5, Proposition 4.5]. Thus we conclude that $Du_1 = Du_2$ a.e. in $\{x \in \mathbb{T}^d \mid m(x) > 0\}$. □

### 4.2. Lack of uniqueness

In this section, we point out an example to demonstrate the non-uniqueness issue of $u \in W^{1,pr}(\mathbb{T}^d)$, where $(u, m, \lambda)$ is a weak solution to (2) satisfying (11).

**Example 4.2.** Let $d = 1$, $f(x,m) = m$ and $H(x,p) = \frac{1}{2}|p|^2 + W(x)$, where the map $W : \mathbb{T} \to \mathbb{R}$ is given by

$$W(x) := \begin{cases} -32x + 4 & \text{for } \ 0 \le x \le \frac{1}{4} \\ 32x - 12 & \text{for } \ \frac{1}{4} \le x \le \frac{1}{2} \\ -32x + 20 & \text{for } \ \frac{1}{2} \le x \le \frac{3}{4} \\ 32x - 28 & \text{for } \ \frac{3}{4} \le x \le 1. \end{cases}$$

Then, (2) becomes

$$\begin{cases} \frac{1}{2}|u_x|^2 + W(x) = m + \lambda & \text{in } \mathbb{T}, \\ -(mu_x)_x = 0 & \text{in } \mathbb{T}. \end{cases} \tag{48}$$

In this case, $p = q = r = 2$. Thus $u \in W^{1,4}(\mathbb{T}) \subset C^{0,\frac{3}{4}}(\mathbb{T})$. Set $\lambda = 0$ and $m(x) = \max\{W(x), 0\}$. Then, $(0, m, \lambda)$ is a weak solution to (48). Indeed, clearly (39) and (40) hold and also we can easily check that for all $\psi \in C^1(\mathbb{T})$,

$$\int_{\mathbb{T}} -(mu_x)\psi_x \, dx = 0,$$

because $u_x = 0$ in $\mathbb{T}$, which implies (41). By Proposition 4.1, $(m, \lambda)$ is uniquely determined. Moreover, all weak solutions $u \in C^{0,\frac{3}{4}}(\mathbb{T})$ satisfy

$$u_x = 0 \quad \text{a.e. in } \{m > 0\} = [0, 1/8] \cup [3/8, 5/8] \cup [7/8, 1].$$

From the first equation of (48), all weak solutions $u$ satisfy

$$|u_x| \le \sqrt{2\max\{-W(x), 0\}} \quad \text{a.e. in } \mathbb{T}.$$

We consider another type weak solution satisfying

$$\frac{1}{2}|u_x|^2 + W(x) \ge 0 \quad \text{ in } \mathbb{T} \ \text{ in the sense of viscosity solutions,} \tag{49}$$

which corresponds to (11). In particular, due to (49), $u_x$ cannot jump from a negative value to a positive value in $\{m = 0\}$. For any $\theta \in [\frac{1}{8}, \frac{3}{8}]$, we set

$$\begin{aligned} u_x^\theta(x) = &\sqrt{2\max\{-W(x), 0\}} \cdot \chi_{\{\frac{1}{8} < x < \theta\} \cup \{\frac{5}{8} < x < 1-\theta\}} \\ &- \sqrt{2\max\{-W(x), 0\}} \cdot \chi_{\{\theta < x < \frac{3}{8}\} \cup \{1-\theta < x < \frac{7}{8}\}}, \end{aligned} \tag{50}$$

where $\chi_A$ is the characteristic function of a set $A \subset \mathbb{T}$, which was used as in the proof of Theorem 1.2. Set

$$u^\theta(x) := \int_0^x u_x^\theta(y) \, dy + C, \tag{51}$$

with any $C \in \mathbb{R}$. We can easily check $u^\theta \in W^{1,4}(\mathbb{T}) \subset C^{0,\frac{3}{4}}(\mathbb{T})$ and $(u^\theta, m, 0)$ is a weak solution to (48) and satisfies (49). Also, we emphasize that all weak solutions satisfying (49) are characterized by $u^\theta$ for $\theta \in [\frac{1}{8}, \frac{3}{8}]$.

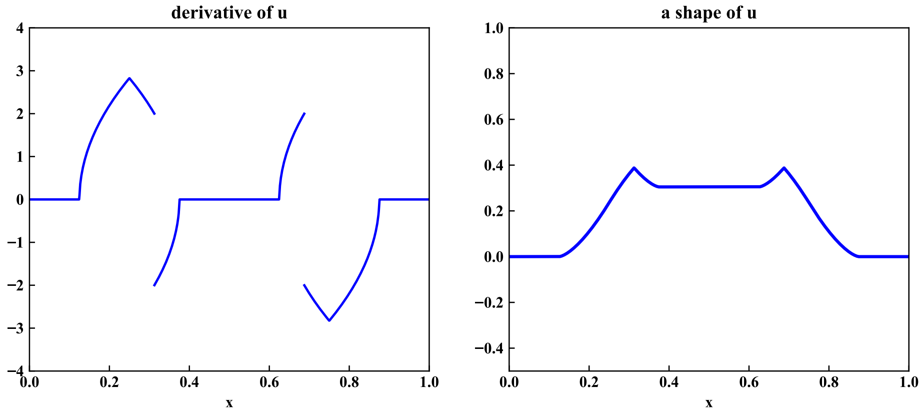


Figure 4.1: Plot of $u^\theta$ in (51) with $\theta = \frac{5}{16}$ and $C = 0$.

**Remark 4.3.** In Example 4.2, we point out that there exist weak solutions which do not satisfy (49). Indeed, $(0, m, 0)$ is a weak solution, but it does not satisfy (49). Moreover, setting

$$\begin{aligned} v_x^\theta(x) = &- \sqrt{2\max\{-W(x), 0\}} \cdot \chi_{\{\frac{1}{8}<x<\theta\}\cup\{\frac{5}{8}<x<1-\theta\}} \\ &+ \sqrt{2\max\{-W(x), 0\}} \cdot \chi_{\{\theta<x<\frac{3}{8}\}\cup\{1-\theta<x<\frac{7}{8}\}}, \end{aligned}$$

we can easily check that $v^\theta$ is a weak solution for all $\theta \in [\frac{1}{8}, \frac{3}{8}]$, but it does not satisfy (49).

## 5. Selection criterion

### 5.1. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. This criterion is inspired by the works in [10, 22, 11].

**Proof.** We normalize $\lambda = 0$ in the proof. By (7) and (40) we have for almost every $x \in \{m^\varepsilon > 0\}$,

$$\begin{aligned} f(x, m) - f(x, m^\varepsilon) &\geq -\varepsilon u^\varepsilon + H(x, Du) - H(x, Du^\varepsilon) \\ &\geq -\varepsilon u^\varepsilon + D_p H(x, Du^\varepsilon) \cdot D(u - u^\varepsilon). \end{aligned}$$

Multiplying this by $m^\varepsilon$ and integrating on $\mathbb{T}^d$, we get

$$-\varepsilon \int_{\mathbb{T}^d} u^\varepsilon m^\varepsilon \, dx - \int_{\mathbb{T}^d} \operatorname{div}(D_p H(x, Du^\varepsilon) m^\varepsilon)(u - u^\varepsilon) \, dx \leq \int_{\mathbb{T}^d} (f(x, m) - f(x, m^\varepsilon)) m^\varepsilon \, dx,$$

which implies

$$\varepsilon \int_{\mathbb{T}^d} (u - u^\varepsilon) \, dx - \varepsilon \int_{\mathbb{T}^d} u m^\varepsilon \, dx \leq \int_{\mathbb{T}^d} (f(x, m) - f(x, m^\varepsilon)) m^\varepsilon \, dx. \tag{52}$$

Similarly, by (8), (39), for almost every $x \in \{m > 0\}$, we have

$$\varepsilon u^\varepsilon + D_p H(x, Du^\varepsilon) \cdot D(u^\varepsilon - u) \leq f(x, m^\varepsilon) - f(x, m).$$

Multiplying this by $m$ and integrating over $\mathbb{T}^d$ yields

$$\varepsilon \int_{\mathbb{T}^d} u^\varepsilon m \, dx \le \int_{\mathbb{T}^d} (f(x,m^\varepsilon) - f(x,m)) m \, dx. \tag{53}$$

Combining (52) and (53) together, we get

$$\varepsilon \int_{\mathbb{T}^d} (u - u^\varepsilon) \, dx - \varepsilon \int_{\mathbb{T}^d} u m^\varepsilon \, dx + \varepsilon \int_{\mathbb{T}^d} u^\varepsilon m \, dx \le \int_{\mathbb{T}^d} (f(x,m^\varepsilon) - f(x,m))(m - m^\varepsilon) \, dx \le 0.$$

Therefore, dividing this by $\varepsilon > 0$, and sending $\varepsilon = \varepsilon_n \to 0$, we obtain

$$\int_{\mathbb{T}^d} \overline{u} m \, dx - \int_{\mathbb{T}^d} \overline{u} \, dx \le \int_{\mathbb{T}^d} u m \, dx - \int_{\mathbb{T}^d} u \, dx,$$

which finishes the proof. □

### 5.2. Convergence result

In this section, as an application of our criterion, Theorem 1.4, we show a nontrivial example where convergence of the whole sequense of weak solutions to (1) holds. In Example 4.2, we show the multiplicity of weak solutions satisfying (49). Here, we prove that the minimizer of (12) is unique. In this example, it holds $pr > d$. Hence, any limit of $\langle u^{\varepsilon_n} \rangle$ satisfies (49) as in Theorem 1.3. First we notice that

$$\inf_{u \in \mathcal{E}} \int_{\mathbb{T}} \langle u \rangle m \; dx = \inf_{\theta \in [\frac{1}{8}, \frac{3}{8}]} \int_{\mathbb{T}} \langle u^\theta \rangle m \; dx, \tag{54}$$

where $u^\theta$ is the function defined by (51). Because $\langle u^\theta \rangle$ is invariant with respect to adding constants, we can assume that $u^\theta(0) = 0$ without loss of generality. Moreover, because $u^\theta$ and $m$ are symmetric with respect to $x = \frac{1}{2}$, it suffices to consider the interval $[0, \frac{1}{2}]$. Recall that $m(x) = \max\{W(x), 0\}$ and

$$u^\theta_x = 0 \quad \text{a.e. in } \{m > 0\} = [0, 1/8] \cup [3/8, 5/8] \cup [7/8, 1].$$

Thus,

$$\begin{aligned} \int_0^{\frac{1}{2}} \langle u^\theta \rangle m \; dx &= \int_0^{\frac{1}{2}} u^\theta m \; dx - \int_0^{\frac{1}{2}} u^\theta \; dx = u^\theta \left( \frac{3}{8} \right) \int_{\frac{3}{8}}^{\frac{1}{2}} m \; dx - \int_0^{\frac{1}{2}} u^\theta \; dx \\ &= \frac{1}{4} u^\theta \left( \frac{3}{8} \right) - \int_0^{\frac{1}{2}} u^\theta \; dx. \end{aligned}$$

Then, we consider the following two cases.

$$\textbf{Case 1:} \ \ \frac{1}{8} \le \theta \le \frac{1}{4}, \quad \textbf{Case 2:} \ \ \frac{1}{4} \le \theta \le \frac{3}{8}.$$

**Case 1.** Here, (50) can be written by

$$u^\theta_x(x) = 2\sqrt{2} \left\{ \sqrt{8x - 1} \left( \chi_{\{\frac{1}{8} < x < \theta\}} - \chi_{\{\theta < x < \frac{1}{4}\}} \right) - \sqrt{3 - 8x} \cdot \chi_{\{\frac{1}{4} < x < \frac{3}{8}\}} \right\},$$

and $u^\theta$ is given by

$$u^\theta(x) = \begin{cases} 0 & \text{for} \quad 0 \le x \le \frac{1}{8} \\ \frac{16}{3}(x-\frac{1}{8})^{\frac{3}{2}} & \text{for} \quad \frac{1}{8} \le x \le \theta \\ \frac{32}{3}(\theta-\frac{1}{8})^{\frac{3}{2}} - \frac{16}{3}(x-\frac{1}{8})^{\frac{3}{2}} & \text{for} \quad \theta \le x \le \frac{1}{4} \\ \frac{32}{3}(\theta-\frac{1}{8})^{\frac{3}{2}} - \frac{32}{3}(\frac{1}{8})^{\frac{3}{2}} + \frac{16}{3}(\frac{3}{8}-x)^{\frac{3}{2}} & \text{for} \quad \frac{1}{4} \le x \le \frac{3}{8} \\ \frac{32}{3}(\theta-\frac{1}{8})^{\frac{3}{2}} - \frac{32}{3}(\frac{1}{8})^{\frac{3}{2}} & \text{for} \quad \frac{3}{8} \le x \le \frac{1}{2}. \end{cases}$$

In particular, we have

$$\frac{\partial}{\partial\theta}u^\theta(x) = \begin{cases} 0 & \text{for} \quad 0 \le x \le \theta \\ 16(\theta-\frac{1}{8})^{\frac{1}{2}} & \text{for} \quad \theta \le x \le \frac{1}{2}. \end{cases}$$

Thus

$$\begin{aligned} \frac{\partial}{\partial\theta}\left(\int_0^{\frac{1}{2}} \langle u^\theta \rangle m \, dx\right) &= \frac{\partial}{\partial\theta}\left\{\frac{1}{4}u^\theta\left(\frac{3}{8}\right) - \int_0^{\frac{1}{2}} u^\theta \, dx\right\} \\ &= \frac{1}{4}\frac{\partial}{\partial\theta}u^\theta\left(\frac{3}{8}\right) - \int_0^{\frac{1}{2}} \frac{\partial}{\partial\theta}u^\theta \, dx = 16(\theta-\frac{1}{8})^{\frac{1}{2}}(\theta-\frac{1}{4}) < 0, \end{aligned}$$

if $\theta \in (\frac{1}{8}, \frac{1}{4})$, which implies $\int_{\mathbb{T}} \langle u^\theta \rangle m \, dx$ is strictly decreasing in $\frac{1}{8} < \theta < \frac{1}{4}$.

**Case 2.** Here, (50) can be written by

$$u_x^\theta(x) = 2\sqrt{2}\left\{\sqrt{8x-1}\chi_{\{\frac{1}{8}<x<\frac{1}{4}\}} + \sqrt{3-8x}\left(\chi_{\{\frac{1}{4}<x<\theta\}} - \chi_{\{\theta<x<\frac{3}{8}\}}\right)\right\},$$

and $u^\theta$ is given by

$$u^\theta(x) = \begin{cases} 0 & \text{for} \quad 0 \le x \le \frac{1}{8} \\ \frac{16}{3}(x-\frac{1}{8})^{\frac{3}{2}} & \text{for} \quad \frac{1}{8} \le x \le \frac{1}{4} \\ -\frac{16}{3}(\frac{3}{8}-x)^{\frac{3}{2}} + \frac{32}{3}(\frac{1}{8})^{\frac{3}{2}} & \text{for} \quad \frac{1}{4} \le x \le \theta \\ -\frac{32}{3}(\frac{3}{8}-\theta)^{\frac{3}{2}} + \frac{32}{3}(\frac{1}{8})^{\frac{3}{2}} + \frac{16}{3}(\frac{3}{8}-x)^{\frac{3}{2}} & \text{for} \quad \theta \le x \le \frac{3}{8} \\ -\frac{32}{3}(\frac{3}{8}-\theta)^{\frac{3}{2}} + \frac{32}{3}(\frac{1}{8})^{\frac{3}{2}} & \text{for} \quad \frac{3}{8} \le x \le \frac{1}{2}. \end{cases}$$

In particular, we get

$$\frac{\partial}{\partial\theta}u^\theta(x) = \begin{cases} 0 & \text{for} \quad 0 \le x \le \theta \\ 16(\frac{3}{8}-\theta)^{\frac{1}{2}} & \text{for} \quad \theta \le x \le \frac{1}{2}. \end{cases}$$

Thus

$$\begin{aligned} \frac{\partial}{\partial\theta}\left(\int_0^{\frac{1}{2}} \langle u^\theta \rangle m \, dx\right) &= \frac{\partial}{\partial\theta}\left\{\frac{1}{4}u^\theta\left(\frac{3}{8}\right) - \int_0^{\frac{1}{2}} u^\theta \, dx\right\} \\ &= \frac{1}{4}\frac{\partial}{\partial\theta}u^\theta\left(\frac{3}{8}\right) - \int_0^{\frac{1}{2}} \frac{\partial}{\partial\theta}u^\theta \, dx = 16(\frac{3}{8}-\theta)^{\frac{1}{2}}(\theta-\frac{1}{4}) > 0, \end{aligned}$$

which implies $\int_{\mathbb{T}} \langle u^\theta \rangle m \, dx$ is strictly increasing in $\frac{1}{4} < \theta < \frac{3}{8}$.

As observed above, the minimization (54) attains at $\theta = \frac{1}{4}$ uniquely. Note that the limit $\tilde{u}$ of $\langle u^\varepsilon \rangle$ satisfies $\int_{\mathbb{T}^d} \tilde{u}\, dx = 0$ because $\int_{\mathbb{T}^d} \langle u^\varepsilon \rangle\, dx = 0$. In conclusion, we get the following result.

**Proposition 5.1.** *Let $W$ be the function defined in Example 4.2, and let $(u^\varepsilon, m^\varepsilon)$ be the weak solution to*

$$\begin{cases} \varepsilon u^\varepsilon + \frac{1}{2}|u^\varepsilon_x|^2 + W(x) = m^\varepsilon & \text{in } \mathbb{T}, \\ \varepsilon m^\varepsilon - (m^\varepsilon u^\varepsilon_x)_x = \varepsilon & \text{in } \mathbb{T}. \end{cases}$$

*Let $\tilde{u}$ be defined by (51) with $\theta = \frac{1}{4}$ and choosing $C \in \mathbb{R}$ such that $\int_{\mathbb{T}^d} \tilde{u}\, dx = 0$. Then, $\langle u^\varepsilon \rangle \to \tilde{u}$ as $\varepsilon \to 0$ uniformly on $\mathbb{T}$.*

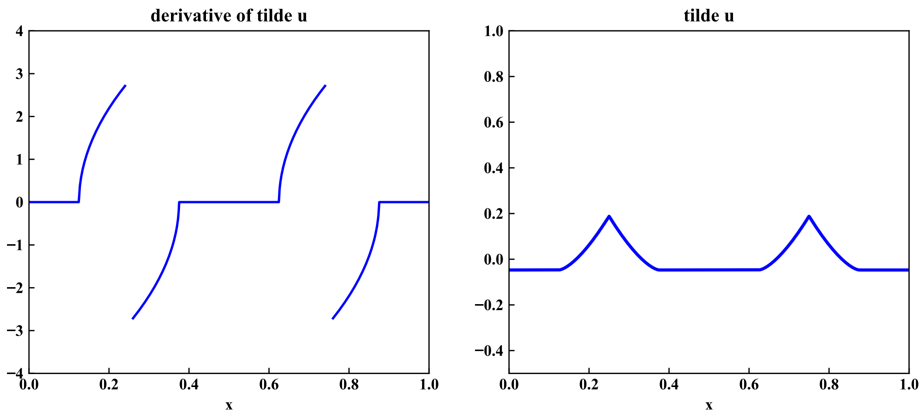


Figure 5.1: The limit of $\langle u^\varepsilon \rangle$ in Proposition 5.1.

## 6. Uniqueness set

In this section we consider a uniqueness set for weak solutions to the ergodic problem (2). We call $\mathcal{Z} \subset \mathbb{T}^d$ a *uniqueness set* if two weak solutions $(u_1, m, \lambda), (u_2, m, \lambda) \in W^{1,pr}(\mathbb{T}^d) \times L^q(\mathbb{T}^d) \times \mathbb{R}$ with (11) satisfy $u_1 = u_2$ on $\mathcal{Z}$, then $u_1 = u_2$ on $\mathbb{T}^d$.

### 6.1. Classical comparison principle

We prove that $\mathcal{Z}$ given by (13) is a uniqueness set of the ergodic problem (2). We first notice that $m$ may not be continuous in general, so the notion of viscosity solutions may not work in $\mathbb{T}^d$. However, if $\mathcal{Z}^c := \mathbb{T}^d \setminus \mathcal{Z} \neq \emptyset$, then $u$ satisfies

$$H(x, Du) \leq f(x, 0) + \lambda \quad \text{a.e. in } \mathcal{Z}^c.$$

Then, $u \in W^{1,\infty}(\mathcal{Z}^c)$. Due to the convexity of Hamiltonian, $u$ satisfies

$$H(x, Du) \leq f(x, 0) + \lambda \quad \text{in } \mathcal{Z}^c, \quad \text{in the viscosity sense.}$$

Therefore, we can use a standard technique in the theory of viscosity solutions. Indeed, by a simple adaptation of the argument in [16], we can easily prove the following comparison result. We give it for the completeness of the paper.

**Proof of Theorem 1.5.** Let $0 < \mu < 1$. Suppose that

$$\max_{x\in\mathbb{T}^d}\{\mu u(x) - v(x)\} > 0,$$

and this maximum is attained at some $x_0 \in \mathcal{Z}^c$. For $\delta > 0$ define $\Psi : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}$ by

$$\Psi(x, y) := \mu u(x) - v(y) - \frac{|x-y|^2}{2\delta} - |x - x_0|^2.$$

Take $(x_\delta, y_\delta)$ so that $\Psi(x_\delta, y_\delta) = \max_{\mathbb{T}^d\times\mathbb{T}^d} \Psi$. By a standard argument, we can prove $x_\delta \to x_0$ and $y_\delta \to x_0$ as $\delta \to 0$.

Let $U_{x_0}$ be an open neighborhood of $x_0$ with $U_{x_0} \subset \mathcal{Z}^c$. Then, by (40), $u$ is a viscosity subsolution to

$$H(x, Du) \le f(x, 0) + \lambda \quad \text{in } U_{x_0}.$$

Taking $\delta > 0$ so small that $x_\delta, y_\delta \in U_{x_0}$, by the definition of viscosity solution, we get

$$\mu H\left(x_\delta, \frac{p_\delta + 2(x_\delta - x_0)}{\mu}\right) \le \mu f(x_\delta, 0) + \mu\lambda \tag{55}$$

and

$$H(y_\delta, p_\delta) \ge f(y_\delta, 0) + \lambda, \tag{56}$$

where $p_\delta := \frac{x_\delta - y_\delta}{\delta}$. By the coercivity of $H(x, \cdot)$, inequality (55) implies that $|p_\delta| \le R$ for some $R > 0$ independent of $\delta$. Subtracting (55) from (56) yields

$$\mu H\left(x_\delta, \frac{p_\delta + 2(x_\delta - x_0)}{\mu}\right) - H(y_\delta, p_\delta) \le \mu f(x_\delta, 0) - f(y_\delta, 0) + (\mu - 1)\lambda.$$

By passing to a subsequence if necessary, we have $p_{\delta_j} \to p \in B(0, R)$.

Sending $\delta = \delta_j \to 0$ yields

$$(\mu - 1)\{f(x_0, 0) + \lambda\} \ge \mu H\left(x_0, \frac{p}{\mu}\right) - H(x_0, p) \ge -(1 - \mu)H(x_0, 0),$$

by the convexity of Hamiltonian, which implies

$$H(x_0, 0) - \lambda - f(x_0, 0) \ge 0.$$

This contradicts a fact $x_0 \in \mathcal{Z}^c$. □

**Example 6.1.** Let $H(x, p) = \frac{1}{2}|p|^2 + V(x)$ for $V \in C(\mathbb{T}^d)$ and $f \in C([0, \infty))$. Then,

$$\begin{cases} \frac{1}{2}|Du|^2 + V(x) = f(m) + \lambda & \text{in } \mathbb{T}^d, \\ -\mathrm{div}(mDu) = 0 & \text{in } \mathbb{T}^d. \end{cases}$$

Then, for all $c \in \mathbb{R}$, $(c, m, \lambda)$ is a weak solution, where $m$ is uniquely determined by

$$m(x) := \max\{f^{-1}(-\lambda + V(x)), 0\}$$

and we choose $\lambda \in \mathbb{R}$ such that $\int_{\mathbb{T}^d} m(x) = 1$. Thus, (13) becomes

$$\mathcal{Z} = \{x \in \mathbb{T}^d \mid -\lambda + V(x) - f(0) \ge 0\}.$$

Combining Proposition 4.1 and Theorem 1.5, we get a sufficient condition to obtain the convergence of the whole sequence $\langle u^\varepsilon \rangle$.

**Corollary 6.2.** *Let $c \in \mathbb{R}$ and assume that $(c, m, \lambda)$ is a weak solution to the ergodic problem* (2)*. Furthermore assume that the set defined by*

$$\{x \in \mathbb{T}^d \mid -\lambda + H(x, 0) - f(x, 0) > 0\}$$

*is connected and satisfies*

$$\{x \in \mathbb{T}^d \mid -\lambda + H(x, 0) - f(x, 0) \geq 0\} = \overline{\{x \in \mathbb{T}^d \mid -\lambda + H(x, 0) - f(x, 0) > 0\}}.$$

*Then, a weak solution $u \in W^{1,\infty}(\mathbb{T}^d)$ to* (2) *with* (11) *is unique up to constants.*

*Moreover, if $pr > d$, then $\langle u^\varepsilon \rangle$ uniformly converges to a unique limit on $\mathbb{T}^d$ as $\varepsilon \to 0$.*

**Proof.** By Proposition 3.7, we have $m \in C(\mathbb{T}^d)$, which implies $u \in W^{1,\infty}(\mathbb{T}^d)$. Because $(c, m, \lambda)$ is a weak solution we have

$$\begin{aligned}\{x \in \mathbb{T}^d \mid m(x) > 0\} &= \{x \in \mathbb{T}^d \mid f^{-1}(x, -\lambda + H(x, 0)) > 0\} \\ &= \{x \in \mathbb{T}^d \mid -\lambda + H(x, 0) - f(x, 0) > 0\}.\end{aligned}$$

Let $(u_1, m, \lambda)$ and $(u_2, m, \lambda)$ be weak solutions to (2) and satisfy (11). By Proposition 4.1, $Du_1 = Du_2 = 0$ in $\{m > 0\}$. Because $\{m > 0\}$ is connected, there exists a unique constant $M \in \mathbb{R}$ such that $u_1 = u_2 + M$ in $\{m > 0\}$. Since

$$\begin{aligned}\overline{\{x \in \mathbb{T}^d \mid m(x) > 0\}} &= \overline{\{x \in \mathbb{T}^d \mid -\lambda + H(x, 0) - f(x, 0) > 0\}} \\ &= \{x \in \mathbb{T}^d \mid -\lambda + H(x, 0) - f(x, 0) \geq 0\},\end{aligned}$$

it holds that $\mathcal{Z} = \overline{\{m(x) > 0\}}$. Hence, by Theorem 1.5, we have $u_1 = u_2 + M$ in $\mathbb{T}^d$.

Under assumption $pr > d$, $\langle u^\varepsilon \rangle$ uniformly converges to $u$, which is a unique (up to constants) weak solution to (2) and satisfies (11), along subsequences. Because $\int_{\mathbb{T}^d} \langle u^\varepsilon \rangle \, dx = 0$, one has $\langle u^\varepsilon \rangle \to \bar{u}$ uniformly, where $\bar{u}$ satisfies $\int_{\mathbb{T}^d} \bar{u} \, dx = 0$. □

### 6.2. Comparison in terms of Mather measures

In this section, we first prove Theorem 1.6, and as a corollary, we give a comparison principle in terms of Mather measures.

**Proof of Theorem 1.6.** We first prove (i). We notice that under the continuity assumption on $m$ we can easily see that $\lambda$ coincides with the ergodic constant for the Hamilton-Jacobi equation. Thus, (15) admits at least one viscosity solution $v \in \mathrm{Lip}(\mathbb{T}^d)$. Note that $v$ satisfies

$$H(x, Dv) = f(x, m) + \lambda \quad \text{a.e. in } \mathbb{T}^d.$$

Thus, we obtain

$$\begin{aligned}\mathcal{A}(v, \lambda) &= \int_{\mathbb{T}^d} F^*(x, -\lambda + H(x, Dv)) + \lambda \, dx = \int_{\mathbb{T}^d} F^*(x, f(x, m)) + \lambda \, dx \\ &= \int_{\{m>0\}} F^*(x, -\lambda + H(x, Du)) + \lambda \, dx + \int_{\{m=0\}} F^*(x, f(x, 0)) + \lambda \, dx.\end{aligned}$$

As in (28), because

$$F^*(x, -\lambda + H(x, Du)) = F^*(x, f(x, 0)) = 0 \quad \text{on } \{m = 0\},$$

we have

$$\mathcal{A}(v, \lambda) = \int_{\mathbb{T}^d} F^*(x, -\lambda + H(x, Du)) + \lambda \, dx = \mathcal{A}(u, \lambda).$$

Therefore, $(v, \lambda)$ is a minimizer of $\inf_{(\phi,c)\in W^{1pr}\times\mathbb{R}} \mathcal{A}(\phi, c)$. As in Proposition 2.5, $(v, m, \lambda)$ is a weak solution to (2). Because $f(x, \cdot)$ is increasing, $v$ satisfies (11) in the sense of viscosity solutions.

Next, we prove (ii). In light of (40) and $m \in C(\mathbb{T}^d)$, we have $u \in W^{1,\infty}(\mathbb{T}^d)$. Due to the convexity of $H(x, \cdot)$, we can easily check that $u$ is a viscosity subsolution to (2). We prove that $u$ is a viscosity supersolution by contradiction. Suppose that there exists $x_0 \in \mathbb{T}^d$ and $\phi \in C^1(\mathbb{T}^d)$ such that

$$(u - \phi)(x_0) < (u - \phi)(x) \quad \text{for all } x \in U_{x_0},$$

and

$$H(x_0, D\phi(x_0)) < f(x_0, m(x_0)) + \lambda, \tag{57}$$

where $U_{x_0}$ is a sufficiently small open neighborhood of $x_0$. Because $u$ satisfies (11), it is enough to consider the case $x_0 \in \{m > 0\}$ and $U_{x_0} \subset \{m > 0\}$.

Here, let $v \in \mathrm{Lip}(\mathbb{T}^d)$ be a viscosity solution to (15). Then, as we proved above, $(v, m, \lambda)$ is a weak solution to (2). Note that $Du = Dv$ a.e. in $\{m > 0\}$ due to Proposition 4.1. Thus, there exists a constant $\hat{c} = u(x) - v(x)$ for all $x \in U_{x_0}$. In particular, we have

$$(v + \hat{c} - \phi)(x_0) < (v + \hat{c} - \phi)(x) \quad \text{for all } x \in U_{x_0}.$$

Because $v + \hat{c}$ is a viscosity solution to (15), it holds

$$H(x_0, D\phi(x_0)) \geq f(x_0, m(x_0)) + \lambda,$$

which contradicts (57). □

**Corollary 6.3.** *Let $(u_1, m, \lambda), (u_2, m, \lambda) \in W^{1,pr}(\mathbb{T}^d) \times L^q(\mathbb{T}^d) \times \mathbb{R}$ be weak solutions to the ergodic problem (2) and satisfy (11). Assume that $m \in C(\mathbb{T}^d)$. If*

$$\iint_{\mathbb{T}^d\times\mathbb{R}^d} u_1 \, d\mu(x, v) \leq \iint_{\mathbb{T}^d\times\mathbb{R}^d} u_2 \, d\mu(x, v) \quad \textit{for all } \mu \in \mathcal{M},$$

*then, $u_1 \leq u_2$ in $\mathbb{T}^d$. Here, we denote by $\mathcal{M} \subset \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ all minimizing measures of*

$$\inf_{\mu\in\mathcal{H}} \iint_{\mathbb{T}^d\times\mathbb{R}^d} L(x, v) \, d\mu(x, v),$$

*where we set $L(x, v) := \sup_{p\in\mathbb{R}^d}\{v \cdot p - H(x, p) + f(x, m)\}$, and*

$$\mathcal{H} := \left\{ \mu \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d) \mid \iint_{\mathbb{T}^d\times\mathbb{R}^d} v \cdot D\phi \, d\mu(x, v) = 0 \quad \textit{for all } \phi \in C^1(\mathbb{T}^d) \right\}.$$

**Proof.** By Theorem 1.6, $u_1$ and $u_2$ are viscosity solutions to (15). Thus, this is a straightforward result of [23, Theorem 1.1]. □

## A. Sobolev estimate

In this appendix, we obtain a $H^1$-estimate for $m^\varepsilon$ inspired by the works in [14, 25, 26]. In this section, we additionally assume that $H^*$ and $F$ are twice-differentiable with respect to the first variable. Here is the goal of this appendix.

**Proposition A.1.** *Assume that there exist $J, \tilde{J} : [0,\infty) \to \mathbb{R}$ and $c_0 > 0$, such that*

$$F(x,m) + F^*(x,a) - ma \geq c_0 |J(m) - \tilde{J}(a)|^2. \tag{58}$$

*for all $m, a \in [0,\infty)$. Further, assume that there exists $C > 0$ such that*

$$\sum_{i,j=1}^{d} \frac{\partial^2 H^*}{\partial x_i \partial x_j}(x,p) h_i h_j \leq C\left(|p|^{r'} + 1\right)|h|^2, \tag{59}$$

$$\sum_{i,j=1}^{d} \frac{\partial^2 F}{\partial x_i \partial x_j}(x,m) h_i h_j \leq C\left(|m|^q + 1\right)|h|^2, \tag{60}$$

*for all $h = (h_1, \ldots, h_d) \in \mathbb{R}^d$, $(x,p) \in \mathbb{T}^d \times \mathbb{R}^d$ and $m \geq 0$. Let $(m^\varepsilon, w^\varepsilon) \in K_\varepsilon$ be the minimizer of* (29)*. Then there exists a constant $C > 0$, which is independent of $\varepsilon$, such that*

$$\|J(m^\varepsilon)\|_{H^1(\mathbb{T}^d)} \leq C.$$

**Lemma A.2.** *Let us define $B^\varepsilon : \mathbb{T}^d \to \mathbb{R}$ by*

$$B^\varepsilon(h) := \int_{\mathbb{T}^d} m^\varepsilon(x) H^*\left(x+h, \frac{-w^\varepsilon(x)}{m^\varepsilon(x)}\right) + F(x+h, m^\varepsilon(x))\, dx,$$

*where $(m^\varepsilon, w^\varepsilon)$ is obtained in Proposition* 2.3*. There exists a constant $C > 0$ such that for all $h \in \mathbb{T}^d$ and $\varepsilon > 0$,*

$$B^\varepsilon(h) \leq B^\varepsilon(0) + C|h|^2.$$

**Proof.** In the proof, we write $f_h(x) := f(x-h)$ for all $h \in \mathbb{T}^d$ and $f : \mathbb{T}^d \to \mathbb{R}$. We first notice that $B^\varepsilon : \mathbb{T}^d \to \mathbb{R}$ takes a minimum at $h = 0$. Indeed, since $(m_h^\varepsilon, w_h^\varepsilon) \in K_\varepsilon$, we have

$$B^\varepsilon(h) = \mathcal{B}(m_h^\varepsilon, w_h^\varepsilon) \geq \mathcal{B}(m^\varepsilon, w^\varepsilon) = B^\varepsilon(0). \tag{61}$$

Let $y \in \mathbb{T}^d$ and $h = (h_1, \ldots, h_d) \in \mathbb{T}^d$. By Taylor's expansion, there exists a constant $\theta \in (0,1)$ such that

$$\begin{aligned}
&B^\varepsilon(y+h) - B^\varepsilon(y) \\
&= \int_{\mathbb{T}^d} m^\varepsilon D_x H^*(x+y, -\frac{w^\varepsilon}{m^\varepsilon}) \cdot h + D_x F(x+y, m^\varepsilon) \cdot h\ dx \\
&\quad + \frac{1}{2}\int_{\mathbb{T}^d} \sum_{i,j=1}^{d} \left\{ m^\varepsilon \frac{\partial^2 H^*}{\partial x_i \partial x_j}(x+y+\theta h, -\frac{w^\varepsilon}{m^\varepsilon}) + \frac{\partial^2 F}{\partial x_i \partial x_j}(x+y+\theta h, m^\varepsilon) \right\} h_i h_j\ dx.
\end{aligned}$$

It follows that $B^\varepsilon \in C^1(\mathbb{T}^d)$ with

$$DB^\varepsilon(y) = \int_{\mathbb{T}^d} m^\varepsilon D_x H^*(x+y, -\frac{w^\varepsilon}{m^\varepsilon}) + D_x F(x+y, m^\varepsilon)\ dx.$$

In view of (61), $DB^\varepsilon(0) = 0$. Hence, it follows from (59) and (60) that

$$\begin{aligned} &B^\varepsilon(h) - B^\varepsilon(0) \\ &= \frac{1}{2}\int_{\mathbb{T}^d} \sum_{i,j=1}^{d} \left\{ m^\varepsilon \frac{\partial^2 H^*}{\partial x_i \partial x_j}(x+y+\theta h, -\frac{w^\varepsilon}{m^\varepsilon}) + \frac{\partial^2 F}{\partial x_i \partial x_j}(x+y+\theta h, m^\varepsilon) \right\} h_i h_j\ dx \\ &\leq \frac{C|h|^2}{2} \int_{\mathbb{T}^d} m^\varepsilon \left| \frac{w^\varepsilon}{m^\varepsilon} \right|^{r'}\ dx + |m^\varepsilon|^q + 1\ dx. \end{aligned} \tag{62}$$

Noting that $\|m^\varepsilon\|_{L^q(\mathbb{T}^d)}$ is bounded uniformly in $\varepsilon$ by Proposition 3.2, and as in estimate (30), we have

$$\int_{\mathbb{T}^d} m^\varepsilon \left| \frac{w^\varepsilon}{m^\varepsilon} \right|^{r'}\ dx \leq C\left(\mathcal{B}(1,0) + 1\right).$$

Hence, combining the above with (62), we have (A.2). □

**Lemma A.3.** *We have*

$$\mathcal{A}^\varepsilon(u) + \mathcal{B}(m,w) \geq c_0 \int_{\mathbb{T}^d} |J(m) - \tilde{J}(\varepsilon u + H(x,Du))|^2 dx$$

*for all $u \in W^{1,pr}(\mathbb{T}^d)$, and $(m,w) \in K_\varepsilon$.*

**Proof.** Using (58), and the convexity of $H^*(x,\cdot)$, we have

$$\begin{aligned} &\mathcal{A}^\varepsilon(u) + \mathcal{B}(m,w) \\ &\geq c_0 \int_{\mathbb{T}^d} |J(m) - \tilde{J}(\varepsilon u + H(x,Du))|^2 dx \\ &\quad + \int_{\mathbb{T}^d} m\left(H(x,Du) + H^*\left(x, -\frac{w}{m}\right)\right) - \varepsilon u + \varepsilon u m\, dx \\ &\geq c_0 \int_{\mathbb{T}^d} |J(m) - \tilde{J}(\varepsilon u + H(x,Du))|^2\, dx + \int_{\mathbb{T}^d} -Du\cdot w - \varepsilon u + \varepsilon u m\, dx \\ &= c_0 \int_{\mathbb{T}^d} |J(m) - \tilde{J}(\varepsilon u + H(x,Du))|^2\, dx. \end{aligned}$$

□

**Proof of Proposition A.1.** Let $u^\varepsilon$ be a minimizer of (22). Using Lemmas A.3, A.2 and Proposition 2.4, we get

$$\begin{aligned} &\frac{1}{2}\|J(m^\varepsilon) - J(m^\varepsilon_h)\|^2_{L^2(\mathbb{T}^d)} \\ &\leq \|J(m^\varepsilon) - \tilde{J}(\varepsilon u^\varepsilon + H(x,Du^\varepsilon))\|^2_{L^2} + \|\tilde{J}(\varepsilon u^\varepsilon + H(x,Du^\varepsilon)) - J(m^\varepsilon_h)\|^2_{L^2} \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{c_0}\{\mathcal{A}^\varepsilon(u^\varepsilon)+\mathcal{B}(m^\varepsilon,w^\varepsilon)\}+\frac{1}{c_0}\{\mathcal{A}^\varepsilon(u^\varepsilon)+\mathcal{B}(m_h^\varepsilon,w_h^\varepsilon)\}\\
&=\frac{1}{c_0}\{-\mathcal{B}(m^\varepsilon,w^\varepsilon)+\mathcal{B}(m_h^\varepsilon,w_h^\varepsilon)\}=\frac{1}{c_0}\{B(h)-B(0)\}\le C|h|^2,
\end{aligned}
$$

which gives the uniform boundness of $J(m^\varepsilon)$ in $H^1(\mathbb{T}^d)$. □

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