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Nonnegative Multiplicative Controllability for Semilinear Multidimensional Reaction-Diffusion Equations

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We consider a multidimensional semilinear reaction-diffusion equation and we obtain at any arbitrary time an approximate controllability result between nonnegative states using as control term the reaction coefficient, that is, via multiplicative controls.

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1. Introduction

Let $n \in \mathbb{N}$, Ω be a bounded open subset of $\mathbb{R}^n$, $T > 0$, $Q_T := \Omega \times (0,T)$, and let us denote by (x,t) the generic element of the Cartesian product Q_T. Let us consider the following semilinear parabolic Cauchy-Dirichlet boundary value problem

$$\begin{cases} u_t \;=\; \Delta u + \; v(x,t)u \;+\; f(u) & \text{in} \quad Q_T = \Omega \times (0,T)\,, \\ u \mid_{\partial\Omega} = 0, & \qquad t \in (0,T), \\ u \mid_{t=0} = u_0 \in L^2(\Omega), & \end{cases} \tag{1}$$

where $v \in L^\infty(Q_T)$, and the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is supposed to be a Lipschitz function, that is, there exists a positive constant L such that

$$\mid f(u_1) - f(u_2) \mid \leq \; L \mid u_1 - u_2 \mid, \quad \forall u_1, u_2 \in \mathbb{R}. \tag{2}$$

Moreover, we assume that
$$f(0) = 0. \tag{3}$$

In this paper we study the global approximate controllability properties of the semilinear problem (1). The control function, that is the variable coefficient through which we can act on the process, is the reaction coefficient $v(x,t)$, that in literature is called *multiplicative control* (see, e.g., [8], [13], [16], and [22]).

Let us recall briefly the classical *well-posedness* of the system (1). So, we need to consider the standard Sobolev spaces:

$$H^1(\Omega) = \{\phi \in L^2(\Omega) \mid \; \phi_x \in L^2(\Omega)\}$$
$$H^1_0(\Omega) = \{\phi \in H^1(\Omega) \mid \phi|_{\partial\Omega} = 0\}$$
$$H^2(\Omega) = \{\phi \in H^1(\Omega) \mid \; \phi_{x_i x_i} \in L^2(\Omega),\, i = 1, \ldots, n\}.$$

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By classical well-posedness results (see, for instance, [24, Theorem 6.1, pages 466–467]) problem (1) with initial data $u_0 \in L^2(\Omega)$ admits a unique solution

$$u \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega)).$$

Furthermore, if $u_0 \in H_0^1(\Omega)$, then the solution u of problem (1) satisfies

$$u \in H^1(0,T; L^2(\Omega)) \cap C([0,T]; H_0^1(\Omega)) \cap L^2(0,T; H^2(\Omega)).$$

The above functional spaces are equipped with the standard norms. Moreover, for simplicity in this paper we will use the following notation $\| \cdot \|_\infty$ and $\langle \cdot, \cdot \rangle$, for the norm $\| \cdot \|_{L^\infty(Q_T)}$ and the inner product $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$, respectively.

Now, we can present the main result in Theorem 1.2, where we prove that system (1) is *nonnegatively globally approximately controllable in* $L^2(\Omega)$ *at any time* $T > 0$, *by means of multiplicative controls* v. We will see that the multiplicative controls v have a simple structure, that is, v are piecewise static functions, in the sense of the following definition.

Definition 1.1. We say that a function $v \in L^\infty(Q_T)$ is *piecewise static* (or a *simple function* with respect to the variable t), if there exist $m \in \mathbb{N}$, $v_k = v_k(x)$, $v_k \in L^\infty(\Omega)$, and $t_k \in [0,T]$, $t_{k-1} < t_k$, $k = 1, \dots, m$ with $t_0 = 0$ and $t_m = T$, such that

$$v(x,t) = v_1(x)\mathcal{X}_{[t_0,t_1]}(t) + \sum_{k=2}^{m} v_k(x)\mathcal{X}_{(t_{k-1},t_k]}(t), \tag{4}$$

where $\mathcal{X}_{[t_0,t_1]}$ and $\mathcal{X}_{(t_{k-1},t_k]}$ are the indicator function of $[t_0,t_1]$ and $(t_{k-1},t_k]$, respectively. Sometime, for clarity purposes, we will call the function v in (4) a m-steps piecewise static function.

Finally, we can give the main result.

Theorem 1.2. *For any nonnegative* $u_0, u^* \in L^2(\Omega)$ *with* $u_0 \neq 0_{L^2(\Omega)}$, *for every* $\varepsilon > 0$, *and any* $T > 0$ *there exists a piecewise static multiplicative control* $v = v(\varepsilon, T, u_0, u^*)$, $v \in L^\infty(Q_T)$, *such that the corresponding solution* $u(x,t)$ *of* (1) *satisfies* $\|u(\cdot,T) - u^*\|_{L^2(\Omega)} < \varepsilon$.

Theorem 1.2 is proved in Section 3. It is useful to have the following remark.

Remark 1.3. We note that, as a consequence of assumptions (2) and (3) on the nonlinear function f, the following inequality holds

$$|f(u)| \leq L|u|, \quad \forall u \in \mathbb{R}, \tag{5}$$

where L is the Lipschitz constant in (2).

We observe that the nonnegative control result, given in Theorem 1.2, is consistent with the constraints given by the PDE in (1). Indeed, from (5) it follows that

$$\frac{f(u)}{u} \in L^\infty(Q_T).$$

Thus,we can extend the strong maximum principle from linear parabolic PDEs (see, e.g, [21, Chapter 2, page 34]) to the semilinear parabolic problem (1), since the terms $v(x,t)u + f(u(x,t))$ can be written as $\widetilde{v}(x,t)u(x,t)$, where

$$\widetilde{v} := v + \frac{f(u)}{u} \in L^\infty(Q_T).$$

So, the strong maximum principle implies that system (1) cannot be steered anywhere from $u_0 \equiv 0$, and if $u_0(x) \geq 0$ in Ω, then the corresponding solution to (1) remains nonnegative at any time, regardless of the possible choice of the multiplicative control v. Hence, system (1) cannot be steered from any nonnegative $u_0 \in L^2(\Omega)$ to a target state $u^* \in L^2(\Omega)$ which is negative on a nonzero measure set in the space domain.

To prove Theorem 1.2 we need an intermediate and crucial controllability result given in Theorem 1.4, obtained under further regularity assumptions and constraints on the initial and target states.

Theorem 1.4. *Let* $u_0,\, u^* \in C^2(\Omega)$ *be such that* $u_0(x) \neq 0$ *for every* $x \in \Omega$, *and*

$$\exists \nu > 0 : \; \nu \leq \frac{u^*(x)}{u_0(x)} \leq 1 \quad \forall\, x \in \Omega\,. \tag{6}$$

Then, for every $\varepsilon > 0$ *and any* $T > 0$ *there exists a piecewise static multiplicative control* $v = v(\varepsilon, T, u_0,\, u^*) \in L^\infty(Q_T)$ *such that*

$$\|u(\cdot, T) - u^*(\cdot)\|_{L^2(\Omega)} \leq \varepsilon\,, \tag{7}$$

where u *is the corresponding solution to* (1) *on* Q_T. *Moreover, for* $T > 0$ *small enough the multiplicative control* v *is the following static function*

$$v(x,t) = \frac{v_0^*(x)}{T} \quad \forall (x,t) \in Q_T,$$

with $v_0^*(x) := \ln\left(\dfrac{u^*(x)}{u_0(x)}\right)$, *for every* $x \in \Omega$.

Remark 1.5. Theorem 1.4 includes also the case of both u_0 and u^* strictly negative on Ω because of condition (6). So, for this result it is only necessary that both initial and target state have the same sign.

Structure of the paper

We prove the main result, that is Theorem 1.2, in Section 3. The proof of Theorem 1.2 needs the intermediate and crucial result given in Theorem 1.4, that is proved in Section 2 together with some useful PDE estimates.

State of the art

The nonnegative approximate controllability results for reaction-diffusion equations are consistent with the strong maximum principle constraints. In the literature, for this kind of results we refer to the pioneering papers by A.Y. Khapalov, contained in the book [22], where nonnegative approximate controllability is obtained for reaction diffusion equations via multiplicative controls, first in large time and then in small time under very strong assumptions on the initial and target states. In this paper, we are able to remove those strong constraints on the data in our general Theorem 1.2. Moreover, the main result of this paper permits us to obtain nonnegative controllability first in *arbitrary small time*, then at any time by an iterative argument.

This proof is inspired by the recent paper [14] of the author, where a similar result is proved in the unidimensional setting for degenerate reaction-diffusion equations. About nonnegative controllability results for degenerate parabolic equations we also mention [13]. Moreover, in [25] Vancostenoble proved a nonnegative controllability result in large time for a linear parabolic equation with singular potential, following the approach of [5] and [6].

For completeness, we recall some recent results about approximate multiplicative controllability of unidimensional reaction-diffusion equations between sign-changing states, see [8], by the author with Cannarsa and Khapalov regarding a semilinear uniformly parabolic system, and [16], by the author with Nitsch and Trombetti, about degenerate parabolic equations.

Recently, results of exact controllability for evolution equations via bilinear controls have been obtained. See, e.g., [1] and [2] by Alabau-Boussouira, Cannarsa and Urbani, [10] by Cannarsa and Urbani, and [12] by Duprez and Lissy.

Finally, an interesting work in progress, related to this paper is the problem of the approximate controllability via multiplicative control for nonlocal operators, applied to the fractional heat equation studied in [4] by Biccari, Warma and Zuazua. For other problems related to PDEs and applications see also [3], [5]–[9], [15]–[20] and [23].

2. Some PDE estimates and proof of Theorem 1.4

We prove the crucial intermediate result given by Theorem 1.4 in Section 2.2. For the proof of Theorem 1.4 and Theorem 1.2, we need some general PDE estimates for the solution of problem (1), that we present in Section 2.1.

2.1. Some PDE estimates

Let us start this section by the statement of Proposition 2.1 that we prove immediately below.

Proposition 2.1. *Let* $T \in (0, \frac{1}{4L}]$, *where* L *is the Lipschitz constant in (2). Let* $u_0\, H^1_0(\Omega)$, $v \in C^2(Q_T)$ *with* $v(x,t) \leq 0$ *on* Q_T, *and let* u *be the corresponding unique solution to* (1). *Then, we have*

$$f(u) \in C([0,T]; L^2(\Omega))$$

and the following estimates hold:

(1) $\|u\|_{C([0,T];L^2(\Omega))} \leq \sqrt{2}\|u_0\|_{L^2(\Omega)}$,

(2) $\|f(u)\|_{C([0,T];L^2(\Omega))} \leq \sqrt{2}L\|u_0\|_{L^2(\Omega)}$,

(3) $\|\Delta u\|_{L^2(Q_T)} \leq C(L,T,v)\|u_0\|_{H^1_0(\Omega)}$,

where $$C(L,T,v) := \sqrt{1 + 2T\max_{x\in\overline{\Omega}}|\Delta v| + 2L^2T}\,.$$

Proof. We start by estimating $\|u\|_{C([0,T];L^2(\Omega))}$. Since $v(x,t) \leq 0$ for a.e. $(x,t) \in Q_T$, multiplying by u the equation in (1), integrating by parts and using (5) yields

$$\begin{aligned}\frac{1}{2}\int_0^t\int_\Omega (u^2)_t dxds &= \int_0^t\int_\Omega u\, u_t dxds \\ &= \int_0^t\int_\Omega u\,\Delta u\, dxds + \int_0^t\int_\Omega v\, u^2 dxds + \int_0^t\int_\Omega f(u)u\, dxds \\ &\le -\int_0^t\int_\Omega |\nabla u|^2\, dxds + L\int_0^T\int_\Omega u^2 dxdt \le L\int_0^T\int_\Omega u^2 dxdt,\end{aligned}$$

where L is the Lipschitz constant in (2). Then, since $T \in \left(0, \frac{1}{4L}\right)$ we deduce

$$\begin{aligned}\int_\Omega u^2(x,t)dx &\le \int_\Omega u_0^2(x)dx + 2L\int_0^T\int_\Omega u^2 dxdt \\ &\le \int_\Omega u_0^2(x)dx + 2LT\|u\|^2_{C([0,T];L^2(\Omega))} \\ &\le \|u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{C([0,T];L^2(\Omega))}, \quad t\in(0,T)\,.\end{aligned}$$

Thus
$$\|u\|_{C([0,T];L^2(\Omega))} \ \le\ \sqrt{2}\|u_0\|_{L^2(\Omega)}, \tag{8}$$

proving the first estimate in the statement of Proposition 2.1.

Now, we estimate $\|f(u)\|_{C([0,T];L^2(\Omega))}$. From (5) and (8) it easily follows that

$$f(u) \in C([0,T];L^2(\Omega)),$$

and, thanks to (8), the following estimate holds

$$\|f(u)\|_{C([0,T];L^2(\Omega))} \ \le\ L\|u\|_{C([0,T];L^2(\Omega))} \le \sqrt{2}L\|u_0\|_{L^2(\Omega)}, \tag{9}$$

that proves the second estimate in Proposition 2.1.

Finally, we estimate $\|\Delta u\|_{L^2(Q_T)}$. Multiplying by Δu the equation in (1), integrating over Q_T, and applying Young's inequality we obtain

$$\begin{aligned}\|\ \Delta u\ \|^2_{L^2(Q_T)} =& \int_0^T\int_\Omega u_t\Delta u dxdt - \int_0^T\int_\Omega vu\Delta u dxdt - \int_0^T\int_\Omega f(u)\Delta u\, dxdt \\ \le& \int_0^T\int_\Omega u_t\Delta u dxdt\ - \int_0^T\int_\Omega vu\Delta u dxdt \\ &+\ \frac{1}{2}\int_0^T\int_\Omega f^2(u)dxdt + \frac{1}{2}\int_0^T\int_\Omega |\Delta u|^2 dxdt\,.\end{aligned}$$

Thus, integrating by parts, keeping in mind the boundary condition in (1), taking into account that the reaction term v is such that $v(x,t)\le 0$ for every $(x,t)\in Q_T$, and using the estimates (9) and (8) it follows that

$$\begin{aligned}\|\Delta u\|^2_{L^2(Q_T)} \le& 2\int_0^T\int_\Omega u_t\Delta u dxdt\ -\ 2\int_0^T\int_\Omega v\, u\, \Delta u dxdt\ +\ \int_0^T\int_\Omega f^2(u)dxdt \\ \le&\ -\int_0^T\int_\Omega (|\nabla u|^2)_t dxdt\ +\ 2\int_0^T\int_\Omega v|\nabla u|^2 dxdt \\ &+\int_0^T\int_\Omega \nabla v\cdot\nabla(u^2)dxdt\ +\ T\,\|f(u)\|^2_{C([0,T];L^2(\Omega))}\end{aligned}$$

$$
\begin{aligned}
&\le \int_\Omega |\nabla u_0|^2 dx \;-\; \int_0^T \int_\Omega \Delta v\, u^2 dxdt \;+2L^2T\|u_0\|^2_{L^2(\Omega)}\\
&\le \int_\Omega |\nabla u_0|^2 dx + \max_{x\in\overline{\Omega}} |\Delta v| \int_0^T \int_\Omega u^2 dxdt + 2L^2T\|u_0\|^2_{L^2(\Omega)}\\
&\le \|\nabla u_0\|^2_{L^2(\Omega)} + \max_{x\in\overline{\Omega}} |\Delta v| \int_0^T \|u\|^2_{C([0,T];L^2(\Omega))} dt + 2L^2T\|u_0\|^2_{L^2(\Omega)}\\
&\le \left(1 + 2T \max_{x\in\overline{\Omega}} |\Delta v| + 2L^2T\right) \|u_0\|^2_{H^1_0(\Omega)}. \qquad \square
\end{aligned}
$$

In the following Proposition 2.2, we generalize the first estimate in Proposition 2.1 to the case of a general reaction coefficient $v \in L^\infty(Q_T)$.

Proposition 2.2. *Let $T > 0$. Let $u_0 \in H^1_0(\Omega)$, $v \in L^\infty(Q_T)$, and let u be the corresponding unique solution to* (1)*. Then, we have*

$$\|u\|_{C([0,T];L^2(\Omega))} \;\le\; e^{(L+\|v^+\|_\infty)T}\|u_0\|_{L^2(\Omega)},$$

where L is the Lipschitz constant in (2) *and $v^+(x,t) = \max\{v(x,t), 0\}$ is the positive part of v.*

Proof. Proceeding as in the proof of Proposition 2.1, that is, multiplying by u the equation in (1), integrating by parts and using (5) we obtain

$$
\begin{aligned}
\frac{1}{2}\int_0^t \int_\Omega (u^2)_t dxds &= \int_0^t \int_\Omega u\,\Delta u\, dxds + \int_0^t \int_\Omega v\, u^2 dxds + \int_0^t \int_\Omega f(u)u\, dxds\\
&\le -\int_0^t \int_\Omega |\nabla u|^2\, dxds + \int_0^t \int_\Omega v^+\, u^2 dxds + L\int_0^t \int_\Omega u^2 dxdt\\
&\le (L + \|v^+\|_\infty) \int_0^t \int_\Omega u^2 dxds, \quad \forall t \in (0,T).
\end{aligned}
$$

Hence, $\displaystyle\int_\Omega u^2(x,t)dx \le \int_\Omega u_0^2(x)dx + 2\,(L + \|v^+\|_\infty) \int_0^t \int_\Omega u^2 dxds, \quad \forall\, t \in (0,T).$

Thus, applying Grönwall's inequality we deduce

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \le e^{2(L+\|v^+\|_\infty)T}\|u_0\|^2_{L^2(\Omega)},\ t \in (0,T),$$

from which it follows the conclusion of the proof. □

From Proposition 2.2 we can easily obtain the following Corollary 2.3.

Corollary 2.3. *Let $T > 0$. Let u_1^0, $u_2^0 \in H^1_0(\Omega)$, $v \in L^\infty(Q_T)$, and let u_1 and u_2 be the unique solutions of* (1) *corresponding to u_1^0 and u_2^0, respectively. Then, we have*

$$\|u_1 - u_2\|_{C([0,T];L^2(\Omega))} \;\le\; e^{(L+\|v^+\|_\infty)T}\|u_1^0 - u_2^0\|_{L^2(\Omega)},$$

where L is the Lipschitz constant in (2) *and $v^+(x,t) = \max\{v(x,t), 0\}$ is the positive part of v.*

Proof. Let us set $w := u_1 - u_2$. We note that w satisfies the following Cauchy-Dirichlet problem

$$
\begin{cases}
w_t \;=\; \Delta w + \; v(x,t)w \;+\; f(u_1) - f(u_2) & \text{in} \quad Q_T = \Omega \times (0,T)\,,\\
w\mid_{\partial\Omega} = 0, & t \in (0,T),\\
w\mid_{t=0} = u_1^0 - u_2^0 \in H^1_0(\Omega).
\end{cases}
\tag{10}
$$

Following the same idea of the proof of Proposition 2.2 we get

$$\begin{aligned}\frac{1}{2}\int_{\Omega} w^2(x,t)dx =& \frac{1}{2}\int_{\Omega} w^2(x,0)dx + \int_0^t\int_{\Omega} w\,\Delta w\,dxds\\ &+ \int_0^t\int_{\Omega} v\,w^2 dxds + \int_0^t\int_{\Omega}(f(u_1)-f(u_2))\,w\,dxds\\ \leq& \frac{1}{2}\|u_1^0-u_2^0\|^2_{L^2(\Omega)} + (L+\|v^+\|_\infty)\int_0^t\int_{\Omega} w^2 dxds, \quad \forall t\in(0,T),\end{aligned}$$

and therefore, applying Grönwall's inequality, we obtain

$$\|u_1(\cdot,t)-u_2(\cdot,t)\|^2_{L^2(\Omega)} \leq e^{2(L+\|v^+\|_\infty)T}\|u_1^0-u_2^0\|^2_{L^2(\Omega)}, \quad \forall t\in(0,T),$$

from which the conclusion follows. □

2.2. Proof of Theorem 1.4

Let us set $v_0^*(x) := \ln\left(\frac{u^*(x)}{u_0(x)}\right)$, for every $x\in\Omega$.

From (6) we get $v_0^*\in L^\infty(\Omega)$ and, moreover, the static multiplicative control v satisfies

$$v(x,t) := \frac{v_0^*(x)}{T} \leq 0 \quad \text{for every } (x,t)\in Q_T. \tag{11}$$

The solution u of (1) at time T, with v defined as in (11), can be written as

$$u(x,T) \;=\; u^*(x) \;+ \int_0^T e^{v_0^*(x)\frac{(T-t)}{T}}(\Delta u(x,\tau)+f(u(x,\tau)))dt, \quad \forall x\in\Omega. \tag{12}$$

Formula (12) is obtained in the following way: for every fixed $\bar{x}\in\Omega$, consider the non-homogeneous first-order ODE

$$u'(\bar{x},t) = \frac{v_0^*(\bar{x})}{T}u(\bar{x},t) \;+ \big(\Delta u(\bar{x},t)+f(u(\bar{x},t))\big) \quad t\in(0,T),$$

associated to (1). Then, the solution u of the above equation can be represented by the Duhamel's formula

$$u(x,t) \;=\; e^{v_0^*(x)\frac{t}{T}}u_0(x) \;+ \int_0^t e^{v_0^*(x)\frac{(t-\tau)}{T}}(\Delta u(x,\tau)+f(u(x,\tau)))d\tau, \quad \forall (x,t)\in Q_T.$$

Taking $t=T$ we obtain (12). From (12), using (11) and Hölder's inequality, and applying the estimates in Proposition 2.1, we deduce the following inequalities

$$\|u(\cdot,T)-u^*(\cdot)\|^2_{L^2(\Omega)} = \int_{\Omega}\left(\int_0^T e^{v_0^*(x)\frac{(T-\tau)}{T}}(\Delta u(x,\tau)+f(u(x,\tau)))d\tau\right)^2 dx \tag{13}$$

$$\begin{aligned}&\leq T\|\Delta u+f(u)\|^2_{L^2(Q_T)} \leq 2\,T\left(\|\Delta u\|^2_{L^2(Q_T)}+\|f(u)\|^2_{L^2(Q_T)}\right)\\ &\leq 2\,T\left[\left(1+2T\max_{x\in\overline{\Omega}}|\Delta v|+2L^2T\right)+2L^2T\right]\|u_0\|^2_{H_0^1(\Omega)}.\\ &= 2\,T\left(1+2\max_{x\in\overline{\Omega}}|\Delta v_0^*|+4L^2T\right)\|u_0\|^2_{H_0^1(\Omega)}\,.\end{aligned} \tag{14}$$

We note that in the last equality we replaced $v = v_0^*/T$ in the constant $C(L,T,v)$ of Proposition 2.1. For the sequel it is crucial that the constant $C(L,T,v)$ remains bounded, as $T \to 0^+$, despite our "*singular*" (in T) choice of the reaction coefficient v. Finally, fixed $\varepsilon > 0$, since the right-hand side of (13) goes to zero as $T \to 0^+$, there exists $T_0^* \in (0, \frac{1}{4L})$, $T_0^* = T_0^*(\varepsilon, v_0^*)$ such that for every $T \in (0, T_0^*]$ we obtain

$$\|u(\cdot,T) - u^*\|_{L^2(\Omega)} < \varepsilon,$$

that is, the approximate controllability of system (1) at any time $T \in (0, T_0^*]$. Furthermore, if $T > T_0^*$ we can prove the approximate controllability at time T, using an iterative argument introduced by the author in [14, proof of Theorem 1.4] for unidimensional degenerate reaction-diffusion equations. So, we first obtain the approximate controllability at time T_0^*. Then, we start at time T_0^* from a state close to u^*, and we stabilize the system into a neighborhood of u^*, applying the above strategy overall N times, for some $N \in \mathbb{N}$, for N time steps of lenght $\frac{T-T_0^*}{N}$. From (11) it follows that the multiplicative control is chosen $v \equiv 0$ in any interval after T_0^*. This iterative method find its origin, in a more complex setting, in [8, Proof of Lemma 3.1, see in particular the iterative inequality (18)]. □

3. Proof of the main result

Let us prove the main result of this paper. In the proof of Theorem 1.2 for simplicity we will use $\|\cdot\|$ for the norm $\|\cdot\|_{L^2(\Omega)}$.

Proof. (Proof of Theorem 1.2). Let us fix $\varepsilon > 0$. Since u_0, $u^* \in L^2(\Omega)$, there exist u_0^ε, $u_\varepsilon^* \in C^2(\overline{\Omega})$ such that:

- u_0^ε, $u_\varepsilon^* > 0$ on Ω, and the quotient function $u_\varepsilon^*/u_0^\varepsilon$ is bounded on Ω, that is,
$$\exists\, M_\varepsilon := M(\varepsilon, u_0, u^*) > 0 \;:\; 0 < \frac{u_\varepsilon^*(x)}{u_0^\varepsilon(x)} \le M_\varepsilon, \qquad \forall x \in \Omega, \tag{15}$$
without loss of generality, we can choose $M_\varepsilon > 1$;
- u_ε^* and u_0^ε satisfy the following approximation conditions
$$\|u_\varepsilon^* - u^*\| < \frac{\varepsilon}{4} \quad \text{and} \quad \|u_0^\varepsilon - u_0\| < \frac{\varepsilon}{16 e^L M_\varepsilon}, \tag{16}$$
where L is the Lipschitz constant in (2).

For every $x \in \Omega$ and any $\rho > 0$, let us define $\operatorname{dist}(x, \partial\Omega) := \inf\limits_{y \in \partial\Omega} |x - y|$ (> 0) and

$$\Omega_\rho := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \rho\}.$$

Let $\eta > 0$ be such that Ω_η is a not empty open set, so that $\Omega_\eta \subset \Omega_{\frac{\eta}{2}} \subset \Omega$.
From (15) we deduce that

$$0 < \min_{x \in \overline{\Omega}_{\frac{\eta}{2}}} \left\{ \frac{u_\varepsilon^*(x)}{u_0^\varepsilon(x)} \right\} \le \frac{u_\varepsilon^*(x)}{u_0^\varepsilon(x)} \le M_\varepsilon, \qquad \forall x \in \overline{\Omega}_{\frac{\eta}{2}}.$$

Then, there exists $\nu(\eta) > 0$ $\left(\nu(\eta) := \frac{1}{M_\varepsilon} \min\limits_{x \in \overline{\Omega}_{\frac{\eta}{2}}} \left\{ \frac{u_\varepsilon^*(x)}{u_0^\varepsilon(x)} \right\}\right)$ such that

$$\nu(\eta) \le \frac{u_\varepsilon^*(x)}{M_\varepsilon\, u_0^\varepsilon(x)} \le 1, \qquad \forall x \in \overline{\Omega}_{\frac{\eta}{2}}. \tag{17}$$

Let us note that (17) has the same structure of assumption (6) in Theorem 1.4. Therefore, we proceed with the following two control actions: first, we drive the system from the initial state u_0 to the intermediate target state $M_\varepsilon u_0^\varepsilon$. Then, we steer the system from such intermediate state to u^* applying Theorem 1.4.

We consider a further approximation of u^*: indeed there exist $\eta = \eta(\varepsilon) > 0$ and $u^*_\eta, v_0^\eta \in C^2(\overline{\Omega})$, with $0 < u^*_\eta \leq u^*_\varepsilon$ and $v_0^\eta \leq 0$ on $\Omega_{\frac{\eta}{2}}$, such that:

$$\text{(i)} \quad u^*_\eta(x) \ = \begin{cases} u^*_\varepsilon(x), & x \in \Omega_\eta, \\ 0, & x \in \overline{(\Omega \backslash \Omega_{\frac{\eta}{2}})}, \end{cases} \qquad \text{and}$$

$$\|u^*_\eta - u^*_\varepsilon\|_{L^2(\Omega)} < \frac{\varepsilon}{4}, \tag{18}$$

$$\text{(ii)} \quad v_0^\eta(x) = \begin{cases} \ln\left(\frac{u^*_\varepsilon(x)}{M_\varepsilon\, u_0^\varepsilon(x)}\right), & x \in \Omega_\eta, \\ 0, & x \in \overline{(\Omega \backslash \Omega_{\frac{\eta}{2}})}. \end{cases}$$

Now, keeping in mind that $M_\varepsilon > 1$, we can select the positive constant reaction coefficient

$$v_1 := \frac{\log M_\varepsilon}{T_1} > 0, \qquad \forall\, (x,t) \in \overline{\Omega} \times (0, T_1), \quad \text{for some } T_1 > 0. \tag{19}$$

Then, we define the piecewise static (see Definition 1.1) multiplicative control as

$$v(x,t) \ = \begin{cases} v_1\,, & (x,t) \in \overline{\Omega} \times (0, T_1), \\ \dfrac{v_0^\eta(x)}{T - T_1}\,, & (x,t) \in \overline{\Omega} \times (T_1, T)\,, \end{cases} \tag{20}$$

where T_1 and T will be determined below.

Let u be the solution to (1) corresponding to the above choice of the multiplicative control v and to the initial state u_0.

Step 1: *Steering the system from u_0^ε to $M_\varepsilon u_0^\varepsilon$, at time $T_1 > 0$.*

Let us denote by $u^\varepsilon(x,t)$ the solution of (1) with initial state u_0^ε. Thus, the solution $u^\varepsilon(x,t)$, at some time T_1, is represented in Fourier series in the following way

$$u^\varepsilon(x, T_1) = \ e^{v_1 T_1} \sum_{k=1}^{\infty} e^{-\lambda_k T_1} \langle u_0^\varepsilon, \varphi_k \rangle \varphi_k(x) + F_\varepsilon(x, T_1) \tag{21}$$

with $$F_\varepsilon(x, T_1) := \sum_{k=1}^{\infty} \left[\int_0^{T_1} e^{(v_1 - \lambda_k)(T_1 - t)} \langle f(u^\varepsilon(\cdot, t)), \varphi_k \rangle dt \right] \varphi_k(x)$$

where $\{-\lambda_k\}_{k \in \mathbb{N}}$ are the eigenvalues of the Laplacian operator $A_0 u := \Delta u$ (we note that $\lambda_k \geq 0$ and $\lambda_k \leq \lambda_{k+1}$, for every $k \in \mathbb{N}$, and $\lambda_k \to +\infty$, as $k \to +\infty$), and $\{\varphi_k\}_{k \in \mathbb{N}}$ are the corresponding eigenfunctions that form a complete orthonormal system in $L^2(\Omega)$. We observe that the eigenvalues of the operator $Au := \Delta u + v_1 u$ are the values $\{-\lambda_k + v_1\}_{k \in \mathbb{N}}$ and the corresponding eigenfunctions are the same of the Laplacian operator $\{\varphi_k\}_{k \in \mathbb{N}}$.

By the strong continuity semigroup property of the heat equation, we deduce

$$\sum_{k=1}^{\infty} e^{-\lambda_k T_1} \langle u_0^\varepsilon, \varphi_k \rangle \varphi_k(x) \longrightarrow u_0^\varepsilon \quad \text{in } L^2(\Omega) \quad \text{as} \quad T_1 \to 0.$$

So, there exists a small time $T_1' \in (0,1)$, $T_1' = T_1'(\varepsilon, u_0, u^*)$, such that, keeping also in mind that $e^{v_1 T_1} = M_\varepsilon$ by (19), we deduce

$$\|u^\varepsilon(\cdot, T_1) - M_\varepsilon u_0^\varepsilon(\cdot)\| < \frac{\varepsilon}{32} + \|F_\varepsilon(\cdot, T_1)\|, \qquad \forall\, T_1 \in (0, T_1']. \tag{22}$$

Using Hölder's inequality, Parseval's identity, inequality (5), and Proposition 2.2 we deduce

$$\begin{aligned} \|F_\varepsilon(\cdot, T_1)\|^2 &= \sum_{k=1}^{\infty} \left| \int_0^{T_1} e^{(v_1 - \lambda_k)(T_1 - t)} \langle f(u^\varepsilon(\cdot, t)), \varphi_k \rangle dt \right|^2 \\ &\le \sum_{k=1}^{\infty} \left(\int_0^{T_1} e^{2(v_1 - \lambda_k)(T_1 - t)} dt \right) \cdot \left(\int_0^{T_1} |\langle f(u^\varepsilon(\cdot, t)), \varphi_k \rangle|^2 dt \right) \\ &\le e^{2v_1 T_1} T_1 \int_0^{T_1} \sum_{k=1}^{\infty} |\langle f(u^\varepsilon(\cdot, t)), \varphi_k \rangle|^2 dt = M_\varepsilon^2 T_1 \int_0^{T_1} \|f(u^\varepsilon(\cdot, t))\|^2 dt \\ &\le L^2 M_\varepsilon^2 T_1 \int_0^{T_1} \|u^\varepsilon(\cdot, t)\|^2 dt \le L^2 M_\varepsilon^2 T_1^2 e^{2LT_1} \|u_0^\varepsilon\|^2_{L^2(\Omega)} \,. \end{aligned} \tag{23}$$

From (22) using (23) it follows that there exists $T_1^* \in (0, T_1']$, $T_1^* = T_1^*(\varepsilon, u_0, u^*)$, such that

$$\|u^\varepsilon(\cdot, T_1) - M_\varepsilon u_0^\varepsilon(\cdot)\| < \frac{\varepsilon}{32} + \|F_\varepsilon(\cdot, T_1)\| \le \frac{\varepsilon}{16}, \qquad \forall\, T_1 \in (0, T_1^*]. \tag{24}$$

Using Corollary 2.3 and the inequaity (24), keeping in mind (19) and (16), for every $T_1 \in (0, T_1^*]$, we obtain

$$\begin{aligned} \|u(\cdot, T_1) - M_\varepsilon u_0^\varepsilon(\cdot)\| &\le \|u(\cdot, T_1) - u^\varepsilon(\cdot, T_1)\| + \|u^\varepsilon(\cdot, T_1) - M_\varepsilon u_0^\varepsilon(\cdot)\| \\ &< e^{(L + \|v_1^+\|_\infty) T_1} \|u_0 - u_0^\varepsilon\| + \frac{\varepsilon}{16} \le e^{(L + v_1 T_1)} \frac{\varepsilon}{16\, e^L M_\varepsilon} + \frac{\varepsilon}{16} = \frac{\varepsilon}{8}. \end{aligned} \tag{25}$$

Let us set $0^\varepsilon(x) := u(x, T_1) - M_\varepsilon u_0^\varepsilon(x)$. Then, from (25) we have

$$\|\delta_0^\varepsilon\| < \frac{\varepsilon}{4\sqrt{2}} \,. \tag{26}$$

Step 2: *Steering the system from $M_\varepsilon u_0^\varepsilon + \delta_0^\varepsilon$ (at time $T_1 > 0$) to u^* at time T for $T > T_1$.*

Now, we start at time T_1 from $M_\varepsilon u_0^\varepsilon + 0^\varepsilon$, and our goal is to steer the system arbitrarily close to u^*.

Let us consider the following semilinear Dirichlet problem

$$\begin{cases} u_t - \Delta u = \dfrac{v_0^\eta(x)}{T - T_1} u + f(u) & \text{in } \widetilde{Q}_T := \Omega \times (T_1, T) \\ u \,|_{\partial\Omega} = 0, & t \in (T_1, T)\ , \end{cases} \tag{27}$$

and we denote by $\tilde{u}(x,t)$ the unique solution to (27) with the initial condition $\tilde{u}(x, T_1) = M_\varepsilon u_0^\varepsilon(x)$. Of course, the restriction on $\widetilde{Q}_T$ of the solution u of (1), corresponding to the multiplicative control v, given in (20), and to the initial state u_0, solves (27) with the initial state $u(x, T_1) = M_\varepsilon u_0^\varepsilon(x) + 0^\varepsilon(x)$.

Since the inequality (17) holds we can apply Theorem 1.4 to steer system (27) from $M_\varepsilon u_0^\varepsilon$ to the approximation u_η^*. Thus, for $T > T_1$ we have

$$\|\widetilde{u}(\cdot,T) - u_\eta^*(\cdot)\| < \frac{\varepsilon}{4}. \tag{28}$$

Then, using Corollary 2.3 (see also Proposition 2.1), from (28), (18), (16), and (26), for any T such that $T - T_1 > 0$ is sufficiently small, we have

$$\begin{aligned} \|u(\cdot,T) - u^*(\cdot)\| &\le \|u(\cdot,T) - u_\eta^*(\cdot)\| + \|u_\eta^* - u^*\| \\ &\le \|u(\cdot,T) - \widetilde{u}(\cdot,T)\| + \|\widetilde{u}(\cdot,T) - u_\eta^*(\cdot)\| + \|u_\eta^* - u_\varepsilon^*\| + \|u_\varepsilon^* - u^*\| \\ &< \sqrt{2}\,\|M_\varepsilon u_0^\varepsilon + 0^\varepsilon - M_\varepsilon u_0^\varepsilon\| + \frac{3}{4}\,\varepsilon < \varepsilon\,. \end{aligned}$$

From which it follows the conclusion and the approximate controllability at any time $T > 0$, using the same approach of the end of the proof of Theorem 1.4. ☐

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References

[1] F. Alabau-Boussouira, P. Cannarsa, C. Urbani: *Superexponential stabilizability of evolution equations of parabolic type via bilinear control*, J. Evolution Equations (2020) https://doi.org/10.1007/s00028-020-00611-z.

[2] F. Alabau-Boussouira, P. Cannarsa, C. Urbani: *Exact controllability to the ground state solution for evolution equations of parabolic type via bilinear control*, preprint.

[3] P. Baldi, G. Floridia, E. Haus: *Exact controllability for quasi-linear perturbations of KdV*, Analysis & PDE 10/2 (2017) 281–322.

[4] U. Biccari, M. Warma, E. Zuazua: *Local regularity for fractional heat equations*, in: *Recent Advances in PDEs: Analysis, Numerics and Control*, SEMA SIMAI 17 (2018) 233–249.

[5] P. Cannarsa, G. Floridia: *Approximate controllability for linear degenerate parabolic problems with bilinear control*, Proc. Evolution Equations and Materials with Memory 2010, Casa Editrice La Sapienza, Roma (2011) 19–36.

[6] P., Cannarsa, G. Floridia: *Approximate multiplicative controllability for degenerate parabolic problems with Robin boundary conditions*, Comm. Appl. Ind. Math. (2011) 1–16.

[7] P. Cannarsa, G. Floridia, F. Gölgeleyen, M. Yamamoto: *Inverse coefficient problems for a transport equation by local Carleman estimate*, Inverse Problems (IOS Science) 35/10 (2019) https://doi.org/10.1088/1361-6420/ab1c69.

[8] P. Cannarsa, G. Floridia, A.Y. Khapalov: *Multiplicative controllability for semilinear reaction-diffusion equations with finitely many changes of sign*, J. Math. Pures Appl. 108 (2017) 425–458.

[9] P. Cannarsa, G. Floridia, M. Yamamoto: *Observability inequalities for transport equations through Carleman estimates*, in: *Trends in Control Theory and Partial Differential Equations*, F. Alabau-Boussouira, F. Ancona, A. Porretta, C. Sinestrari (eds.), Springer INdAM Series 32, Springer, Cham (2019) 69–87

[10] P. Cannarsa, C. Urbani: *Superexponential stabilizability of degenerate parabolic equations via bilinear control*, in: *Inverse Problems and Related Topics*, ICIP2 2018, J. Cheng, S. Lu, M. Yamamoto (eds.), Springer Proceedings in Mathematics and Statistics 310, Springer, Singapore (2020).

[11] Y. Chen, J. Cheng, G. Floridia, Y. Wada, M. Yamamoto: *Conditional stability for an inverse source problem and an application to the estimation of air dose rate of radioactive substances by drone data*, in: *Special Issue: Inverse problems in imaging and engineering science*, L. Oksanen, M. Salo (eds.), Mathematics in Engineering (AIMS) 2/1 (2020) 26–33.

[12] M. Duprez, P. Lissy: *Bilinear local controllability to the trajectories of the Fokker-Planck equation with a localized control*, arXiv: 1909.02831 (2019).

[13] G. Floridia: *Approximate controllability for nonlinear degenerate parabolic problems with bilinear control*, J. Diff. Equations 9 (2014) 3382–3422.

[14] G. Floridia: *Nonnegative controllability for a class of nonlinear degenerate parabolic equations with application to climate science*, Electr. J. Diff. Equations 59 (2020) 1–27.

[15] G. Floridia, Z. Li, M. Yamamoto: *Well-posedness for the backward problems in time for general time-fractional diffusion equation*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. 31/3 (2020) 593–610.

[16] G. Floridia, C. Nitsch, C. Trombetti: *Multiplicative controllability for nonlinear degenerate parabolic equations between sign-changing states*, ESAIM: COCV 26/18 (2020) 1–34.

[17] G. Floridia, M. A. Ragusa: *Differentiability of solutions of nonlinear elliptic systems of order $2m$*, AIP Conference Proceedings 2 (2010) 278–281.

[18] G. Floridia, M. A. Ragusa: *Interpolation inequalities for weak solutions of nonlinear parabolic systems*, J. Inequalities Appl. 20112 (2011), art. no. 42.

[19] G. Floridia, M. A. Ragusa: *Differentiability and partial Hölder continuity of solutions of nonlinear elliptic systems*, J. Convex Analysis 19/1 (2012) 63–90.

[20] G. Floridia, M. Yamamoto: *Backward problems in time for fractional diffusion-wave equation*, Inverse Problems (IOS Science) 36 (2020) https://doi.org/10.1088/1361-6420/abbc5e.

[21] A. Friedman: *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs (1964).

[22] A. Y. Khapalov: *Controllability of Partial Differential Equations Governed by Multiplicative Controls*, Lecture Series in Mathematics 1995, Springer, Berlin (2010).

[23] A. Y. Khapalov, P. Cannarsa, F. S. Priuli, G. Floridia: *Wellposedness of a 2-D and 3-D swimming models in the incompressible fluid governed by Navier-Stokes equation*, J. Math. Analysis Appl. 429/2 (2015) 1059–1085.

[24] O. H. Ladyzhenskaya, V. A. Solonikov, N. N. Uralceva: *Linear and Quasi-linear Equations of Parabolic Type*, American Mathematical Society, Providence (1968).

[25] J. Vancostenoble: *Global Non-negative approximate controllability of parabolic equations with singular potentials*, in: *Trends in Control Theory and Partial Differential Equations*, F. Alabau-Boussouira, F. Ancona, A. Porretta, C. Sinestrari (eds.), Springer INdAM Series 32, Springer, Cham (2019) 255–276.