Minimax Theory and its Applications
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Duality Minimax and Applications

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The paper is devoted to the strong duality minimax theory, that works in infinite dimensional settings, and to its applications. In particular, we deal with the nonconstant gradient constrained problem and with the random traffic equilibrium problem. By means of this theory, we are able to show that, for both problems, the associated infinite dimensional variational inequalitiy on a convex feasible set is equivalent to a system of equations.

Keywords: Duality theory, Lagrange multipliers, Nonconstant gradient constraints, Random traffic equilibrium problem.

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1. Introduction

The paper deals with the problem of strong duality minimax between an infinite-dimensional convex optimization problem with nonlinear constraints and its Lagrange dual formulation.

The classical results in literature on the strong duality are based on the nonemptiness of the interior, on the core, on the intrinsic core or on the strong quasi-relative interior of the ordering cone (see [1, 16, 21]); therefore, they cannot be applied if the ordering cone of the problem has an empty interior (or if any of the above-mentioned generalized interior concepts is empty). However, this situation occurs in many equilibrium problems, stated in infinite-dimensional spaces, as the traffic equilibrium problem, the financial equilibrium problem, the Warlas problem, the obstacle problem, the elastic-plastic torsion problem and many others (see [20] and the references therein).

In order to overcome this drawback, in [8] the authors introduced a necessary and sufficient condition, called Assumption S (for Separation), which ensures the strong duality and it is successfull in the applications.

In this paper we present the new strong duality minimax theory (see [5, 6, 9, 10, 17, 19]). Moreover, we show that the strong duality theory, in particular the properties of the Lagrange functional, may be applied in order to transform a constrained minimization problem or a variational inequality on a convex feasible set in a system of equations. This result is obtained for a nonconstant gradient constrained problem and for the random traffic equilibrium problem. We stress that the classical strong duality theory cannot be applied in both problems, since the interior of the ordering cone is empty.

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In both the problems under consideration, the equivalent system of equations (see (9), (11) and (32)–(34)) is expressed in terms of the Lagrange multipliers associated to the problems.

The paper is organized as follows: in Section 2 the Assumption S and the strong duality minimax are presented. Section 3 is devoted to the applications, in particular, in Subsection 3.1 we study a nonconstant gradient constrained problem, whereas in Subsection 3.2 we investigate the random traffic equilibrium problem. Finally, Section 4 summarizes our results and future work.

2. The strong duality minimax

Let $f\colon S \to \mathbb{R},\ g\colon S \to Y,\ h\colon S \to Z$ be three mappings, where S is a convex subset of a real normed space X, Y is a real normed space ordered by a convex cone C, Z is a real normed space and consider the optimization problem:

$$\begin{cases} f(x_0) = \min\limits_{x \in \mathbb{K}} f(x) \\ x_0 \in \mathbb{K} = \{x \in S : g(x) \in -C,\ h(x) = \theta_Z\}, \end{cases} \tag{1}$$

where θ_Z is the zero element in the space Z.

Under the additional assumption that the ordering cone is closed, it is possible to prove that the primal problem (1) is equivalent to the optimization problem

$$\min_{x \in S} \sup_{u \in C^*,\, v \in Z^*} \left[f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle \right], \tag{2}$$

where $$C^* := \{u \in Y^* : \langle u, y \rangle \geq 0,\ \forall y \in C\}$$

is the dual cone of C and Z^* is the dual space of Z (see Lemma 6.5 in [18]).

Moreover, we associate to the primal problem (1) the following problem

$$\max_{u \in C^*,\, v \in Z^*} \inf_{x \in S} \left[f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle \right]. \tag{3}$$

As it is well known, optimization problem (3) is called the dual problem, associated with primal problem (1). The duality theory investigates the relationships between primal problem (1) and dual problem (3). In general, the above problems are not equivalent, but the following weak duality property always holds:

$$\begin{aligned} &\max_{u \in C^*,\, v \in Z^*} \inf_{x \in S} \left[f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle \right] \leq \\ &\leq \min_{x \in S} \sup_{u \in C^*,\, v \in Z^*} \left[f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle \right]. \end{aligned} \tag{4}$$

The weak-duality property means that the maximal value of the dual problem is bounded from above by the minimal value of the primal problem.

If the primal and the dual problems are solvable, the extremal values of the two problems cannot be, in general, equal. If the two problems are solvable and their extremal values are not equal, one speaks of a duality gap.

Then, we say that the strong duality holds for problems (1) and (3) if and only if problems (1) and (3) admit a solution and their optimal values coincide.

In literature a classical theorem establishes, under the assumption of nonemptiness of the ordering cone C, that the strong duality holds (see [18]).

However, the ordering cone of almost all the known problems, stated in infinite dimensional spaces, has the interior empty. Hence, the above theorem cannot guarantee the strong duality in these applications.

In [8] the authors introduced a new conditions called Assumption S, which turns out to be a necessary and sufficient condition for the strong duality (see also [2]) and is really useful in the applications. This condition does not require the nonemptiness of the interior of the ordering cone.

Now, we recall some definitions in order to present in detail the Assumption S.

Let us first recall that for a subset $C \subseteq X$ and $x \in X$ the tangent cone to C at x is defined as

$$T_C(x) = \{y \in X : \; y = \lim_{n\to\infty} \lambda_n(x_n - x), \; \lambda_n > 0, \; x_n \in C, \; \lim_{n\to\infty} x_n = x\}.$$

If $x \in clC$ (the closure of C) and C is convex, it results

$$T_C(x) = clcone(C - \{x\}),$$

where the $coneA = \{\lambda x : \; x \in A, \; \lambda \in \mathbb{R}^+\}$ denotes the cone hull of a general subset A of the space.

Definition 2.1. (see Definition 2.2 in [10]) Given the mappings f, g, h and the set $\mathbb{K}$ as above, we say that Assumption S is fulfilled at a point $x_0 \in \mathbb{K}$ if and only if

$$T_{\widetilde{M}}(0, \theta_Y, \theta_Z) \cap (\mathbb{R}^{--} \times \theta_Y \times \theta_Z) = \emptyset$$

where $\mathbb{R}^{--} = \{\lambda \in \mathbb{R} : \lambda < 0\}$ and

$$\widetilde{M} = \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)) : x \in S \backslash \mathbb{K}, \; \alpha \geq 0, \; y \in C\}.$$

The following statement is the main theorem on strong duality based on Assumption S (see [8, 10]).

Theorem 2.2. (see Theorem 2.2 in [10]) *Assume that the functions $f : S \longrightarrow \mathbb{R}$, $g : S \longrightarrow Y$ are convex and that $h : S \longrightarrow Z$ is an affine-linear mapping. Assume that the Assumption* S *is fulfilled at the optimal solution $x_0 \in \mathbb{K}$ of problem* (1). *Then, also problem* (3) *is solvable and if $\overline{u} \in C^*$, $\overline{v} \in Z^*$ are optimal solutions to* (3), *we have*

$$\langle \overline{u}, g(x_0) \rangle = 0 \tag{5}$$

and the optimal values of the two problems coincide; namely,

$$\begin{aligned} f(x_0) = \min_{x \in \mathbb{K}} f(x) &= f(x_0) + \langle \bar{u}, g(x_0) \rangle + \langle \bar{v}, h(x_0) \rangle \\ &= \max_{\substack{u \in C^* \\ v \in Z^*}} \inf_{x \in S} \left[f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle \right]. \end{aligned}$$

A very important consequence of the strong duality is the usual relationship between a saddle point of the Lagrange functional

$$L(x,u,v) = f(x) + \langle u, g(x)\rangle + \langle v, h(x)\rangle, \quad \forall x \in S,\ \forall u \in C^*,\ \forall v \in Z^*,$$

and the solutions to (1) and (3). Indeed, we have the following theorem.

Theorem 2.3. (see Theorem 2.3 in [10]) *Let the assumptions of Theorem 2.2 be fulfilled. Then, $x_0 \in \mathbb{K}$ is an optimal solution to (1) if and only if there exist $\bar{u} \in C^*$, $\bar{v} \in Z^*$ such that $(x_0, \bar{u}, \bar{v})$ is a saddle point of the Lagrange functional, namely:*

$$L(x_0,u,v) \le L(x_0,\bar{u},\bar{v}) \le L(x,\bar{u},\bar{v}), \quad \forall x \in S,\ \forall u \in C^*,\ \forall v \in Z^* \tag{6}$$

and $\langle \bar{u}, g(x_0)\rangle = 0.$

Theorem 2.3 is very useful in the applications. In the following section we will show how inequalities (6) allow us to transform a constrained minimization problem or a variational inequality on a convex feasible set in a system of equations.

3. Some applications

3.1. A nonconstant gradient constrained problem

In this subsection we deal with the following nonconstant gradient constrained problem, formulated by means of the variational inequality:

Find $u \in K = \left\{ v \in H_0^{1,2}(\Omega) : |Dv|^2 = \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2 \le g(x), \text{ a.e. in } \Omega \right\}$ *such that*:

$$\int_\Omega \mathcal{L}u(v-u)\,dx \ge \int_\Omega f(v-u)\,dx, \quad \forall v \in K, \tag{7}$$

with $g(x) \in C^2(\overline{\Omega})$, $g(x) > 0$. $\Omega \subset \mathbb{R}^n$ is an open bounded convex set with C^2-boundary $\partial\Omega$, and $\mathcal{L}$ is the linear operator

$$\mathcal{L}u = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} + cu$$

with associated bilinear form on $H_0^{1,2}(\Omega) \times H_0^{1,2}(\Omega)$ given by

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} v + cuv \right) dx.$$

We assume the following conditions on the coefficients

$$\begin{cases} \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2 \text{ a.e. in } \Omega, \forall \xi \in \mathbb{R}^n \\ \nu > 0, a_{ij} \in C^2(\overline{\Omega}), b_i \in C^1(\overline{\Omega}) \\ c > 0 \text{ such large that } a(u,u) \ge \alpha \|u\|^2_{H_0^{1,2}(\Omega)},\ \alpha > 0,\ \forall u \in H_0^{1,2}(\Omega). \end{cases} \tag{8}$$

Let us remark that in the case $g \equiv 1$, problem (7) is the well known elastic-plastic torsion problem (see [3], [4], [13], [14], [15] and the references therein).

Now, we show how variational inequality (7) may be rewritten in terms of a system of equations.

Indeed, the following result holds:

Theorem 3.1. (see Theorem 4 in [12]) *Under the above assumptions on* Ω *and under conditions* (8)*, let* $f \in L^p(\Omega)$*,* $p > 1$*, and* $u \in K$ *be the solution to problem* (7)*. Then, there exists* $\overline{\mu} \in (L^\infty(\Omega))^*$ *such that*

$$\begin{cases} \langle \overline{\mu}, y \rangle \geq 0 \quad \forall y \in L^\infty(\Omega),\ y \geq 0 \quad \text{a.e. in } \Omega; \\ \langle \overline{\mu}, \left(\displaystyle\sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 - g(x) \right) \rangle = 0; \\ \displaystyle\int_\Omega (\mathcal{L}u - f)\varphi \, dx = \langle \overline{\mu}, -2 \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \rangle, \quad \forall \varphi \in H_0^{1,\infty}(\Omega). \end{cases} \tag{9}$$

The previous result may be generalized, ensuring the existence of the Lagrange multiplier in L^2, but assuming $f = const > 0$ and an extra condition on the gradient constraint g.

Theorem 3.2. (see Theorem 1 in [12]) *Under the above assumptions on* Ω *and under conditions (8), let* $f \equiv const. > 0$*, and the following conditions be satisfied*

$$-\sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial g}{\partial x_j} \right) \geq 0 \quad \text{in } \Omega$$

and

$$c \geq \|Db\|_{L^\infty} + \|D^2 a_{ij}\|_{L^\infty} + \frac{(3\|Da_{ij}\|_{L^\infty} + \|b\|_{L^\infty})^2}{4\nu}. \tag{10}$$

If $u \in K \cap W^{2,p}(\Omega)$ *is the solution to problem* (7)*, then, there exists* $\bar{\mu} \in L^2(\Omega)$ *with*

$$\begin{cases} \bar{\mu} \geq 0 \ \ \textit{a.e. in } \Omega \\ \bar{\mu} \left(g(x) - \displaystyle\sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 \right) = 0 \ \ \textit{a.e. in } \Omega \\ \mathcal{L}u - f + \bar{\mu} = 0 \ \ \textit{a.e. in } \Omega. \end{cases} \tag{11}$$

Moreover, let us note the Lagrange multiplier $\bar{\mu} \in L^2(\Omega)$ is unique. It easily follows from the third equation in (11).

Theorem 3.1 has been proved in [12], using the classical duality theory, since, setting

$$S = X = H_0^{1,\infty}(\Omega), \ \ Y = L^\infty(\Omega), \ \ G(v) = \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i} \right)^2 - g(x) : H_0^{1,\infty}(\Omega) \to L^\infty(\Omega),$$

the ordering cone $C = \{w \in L^\infty(\Omega) \ : \ w(t) \geq 0 \text{ a.e. in } \Omega\}$ has a nonempty interior. Moreover, since

$$K = \left\{ v \in H_0^{1,\infty}(\Omega) : \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i} \right)^2 \leq g(x), \text{ a.e. on } \Omega \right\},$$

the variational inequality (7) may be rewritten as the minimum problem

$$\min_{v\in K}\psi(v)=\psi(u)=0 \tag{12}$$

where
$$\psi(v)=\int_\Omega(\mathcal{L}u-f)(v-u)dx. \tag{13}$$

So, we can apply classical strong duality property and, consequently, if u is a solution to (7), then to problem (12), there exists $\overline{\mu}\in C^*$ solution to the dual problem

$$\max_{\mu\in C^*}\inf_{v\in S}[\psi(v)+\langle\mu,G(v)\rangle] \tag{14}$$

and $(u,\overline{\mu})$ is a saddle point of the so called Lagrange functional

$$L(v,\mu)=\psi(v)+\langle\mu,G(v)\rangle,\ \forall v\in H_0^{1,\infty}(\Omega),\forall\mu\in C^*,$$

namely,
$$L(u,\mu)\le L(u,\overline{\mu})\le L(v,\overline{\mu}),\ \forall v\in H_0^{1,\infty}(\Omega),\forall\mu\in C^*. \tag{15}$$

From the right hand side of (15) it follows, $\forall v\in H_0^{1,\infty}(\Omega)$,

$$L(v,\overline{\mu})=\int_\Omega(\mathcal{L}u-f)(v-u)dx+\langle\overline{\mu},G(v)\rangle\ge L(u,\overline{\mu})=\langle\overline{\mu},G(u)\rangle=0.$$

Choosing $v=u\pm\varphi$, $\forall\varphi\in H_0^{1,\infty}(\Omega)$ we obtain

$$\int_\Omega(\mathcal{L}u-f)\varphi\,dx+2\langle\overline{\mu},\sum_{i=1}^n\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_i}\rangle\ +\ \langle\overline{\mu},\sum_{i=1}^n\left(\frac{\partial\varphi}{\partial x_i}\right)^2\rangle\ge0$$

and
$$\int_\Omega(\mathcal{L}u-f)\varphi\,dx+2\langle\overline{\mu},\sum_{i=1}^n\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_i}\rangle\ -\ \langle\overline{\mu},\sum_{i=1}^n\left(\frac{\partial\varphi}{\partial x_i}\right)^2\rangle\le0.$$

Then, we consider the test function $\varepsilon\,\varphi$, $\varepsilon>0$, in both the inequalities. If ε tends to zero, we get:

$$\int_\Omega(\mathcal{L}u-f)\varphi\,dx=\langle\overline{\mu},-2\sum_{i=1}^n\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_i}\rangle. \tag{16}$$

In conclusion, from (5), (16), taking into account that

$$C^*=\{\mu\in(L^\infty(\Omega))^*:\ \mu(y)\ge0\ \forall y\in L^\infty(\Omega),\ y(x)\ge0\ a.a.\ x\ \in\Omega\}$$

we obtain conditions (9) and Theorem 3.1 is achieved.

In order to obtain the existence of the multiplier in $L^2(\Omega)$, it is no longer possible to apply the classical strong duality theory, since in this case we have $X=Y=L^2(\Omega)$, $C=C^*=\{v\in L^2(\Omega):v(x)\ge0\text{ a.e. in }\Omega\}$, and the interior of C is the empty set.

In [12] it is proved that, under the assumptions of Theorem 3.2, the solution u to problem (7) coincides with the solution to the obstacle problem

$$\text{Find }u\in K_1:\int_\Omega\mathcal{L}u(v-u)dx\ge\int_\Omega f(v-u)dx,\quad\forall v\in K_1, \tag{17}$$

where
$$K_1=\left\{v\in H_0^1(\Omega):0\le v(x)\le w(x)\text{ a.e. on }\Omega\right\},$$

and the obstacle $w \in H^{1,\infty}(\Omega)$ is the viscosity solution to the Hamilton-Jacobi equation

$$\begin{cases} |Dw| = \sqrt{g(x)} & a.e. \text{ in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{18}$$

defined by
$$w(x) = inf_{x_0 \in \partial\Omega} L(x, x_0) \tag{19}$$

with

$$L(x, x_0) = inf \left\{ \begin{matrix} \displaystyle\int_0^{T_0} \sqrt{g(\xi(s))} ds : \xi\colon [0, T_0] \to \overline{\Omega}, \\ \xi(0) = x,\ \xi(T_0) = x_0,\ |\xi'(s)| \le 1 \ a.e. \text{ in } [0, T_0] \end{matrix} \right\}. \tag{20}$$

Problem (17) may be rewritten as the optimization problem

$$\min_{v \in K_1} \psi(v) = \psi(u) = 0. \tag{21}$$

with $\psi(v) = \int_\Omega (\mathcal{L}u - f)(v - u)\, dx$. The optimization problem (21) fulfills *Assumption S*. We report the proof for the reader's convenience. We have

$$X = Y = L^2(\Omega),\ C = C^* = \left\{ v \in L^2(\Omega) : v(x) \ge 0 \text{ a.e. in } \Omega \right\},\ G(v) = v - w,$$

and we must show that, if we have

$$\left(l, \theta_{L^2(\Omega)}\right) = \left(\lim_n \nu_n(\psi(v_n) + \alpha_n), \lim_n \nu_n(v_n - w(x) + y_n) \right)$$

with $\{\nu_n\}$ a positive real sequence, $\alpha_n \in \mathbb{R}_+$, $v_n \in L^2(\Omega) \setminus K_1$, $\lim_n \psi(v_n) + \alpha_n = 0$, $\lim_n (v_n - w(x) + y_n) = 0$ *in* $L^2(\Omega)$, $y_n \in C$, then l must be nonnegative. We obtain

$$\lim_n \nu_n \psi(v_n) = \lim_n \nu_n \int_\Omega (\mathcal{L}u - f)(v_n - u)\, dx$$
$$= \lim_n \left[\int_\Omega (\mathcal{L}u - f)\nu_n(v_n - w(x) + y_n)\, dx + \nu_n \int_\Omega (\mathcal{L}u - f)(w(x) - y_n - u)\, dx \right].$$

Since $w(x) - y_n \le w(x)$, taking into account that

$$\lim_n \nu_n(v_n - w(x) + y_n) = 0 \text{ in } L^2(\Omega) \quad \text{and} \quad \int_\Omega (\mathcal{L}u - f)(w(x) - y_n - u)\, dx \ge 0,$$

we get $l \ge 0$, that is the strong duality holds. Then, Theorem 2.3 holds too, namely, setting

$$L(v, \mu) = \int_\Omega (\mathcal{L}u - f)(v - u)\, dx + \int_\Omega \mu(v(x) - w(x))\, dx,\ \ v \in L^2(\Omega),\ \mu \in C, \tag{22}$$

there exists $\overline{\mu} \in C$ such that
$$\int_\Omega \overline{\mu}(u(x) - w(x))\, dx = 0 \tag{23}$$

and
$$L(u, \mu) \le L(u, \overline{\mu}) \le L(v, \overline{\mu}) \quad \forall v \in L^2(\Omega),\ \forall \mu \in C. \tag{24}$$

Considering the right-hand side of (24), we get

$$\int_\Omega (\mathcal{L}u - f + \overline{\mu})(v-u)\,dx \geq 0 \quad \forall v \in L^2(\Omega), \tag{25}$$

from which we get
$$\mathcal{L}u - f + \overline{\mu} = 0 \text{ a.e. in } \Omega. \tag{26}$$

Moreover, from (23) we also have

$$\overline{\mu}(x)(u(x) - w(x)) = 0 \text{ a.e. in } \Omega. \tag{27}$$

The second one of conditions (11) follows from Theorem 2 in [12].

3.2. The Random Traffic Equilibrium Problem

In this subsection we deal with a general random traffic equilibrium problem, namely, a traffic problem where the data are affected by a certain degree of uncertainty. The path flows as well as the travel demand often vary over time in a non-regular and predictable manner. Then, a random model may be very useful in order to describe many realistic features of the traffic equilibrium problem. Moreover, since the demand itself is dynamic and can change randomly, this framework is able to handle random constraints.

In [7] the authors give for this equilibrium problem a random generalized Wardrop equilibrium condition and show that the equilibrium conditions are equivalent to a random variational inequality.

The framework of the model is a Hilbert space setting, in which it is possible to obtain, under general assumptions, existence and uniqueness results and to perform a complete duality theory.

For the reader's convenience, we introduce in detail the random traffic equilibrium model. A traffic network consists of a triple (N, A, W), where $N = (N_1, N_2, \ldots, N_p)$ is the set of nodes, $A = (A_1, \ldots, A_n)$ represents the set of the directed arcs connecting couples of nodes and $W = \{w_1, \ldots, w_l\} \subset N \times N$ is the set of the origin-destination (O/D) pairs. The flow on the arc A_i is denoted by f_i and the uncertainty, which affects the knowledge of f_i, is given by the dependence of f_i on ω, namely, $f_i = f_i(\omega)$, where $\omega \in \Omega$ and $(\Omega, \mathcal{A}, P)$ is a probability space. We will set $f(\omega) := (f_1(\omega), \ldots, f_n(\omega))$. A path is a set of consecutive arcs and we assume that each O-D pair w_j is connected by $r_j \geq 1$ paths, whose set is denoted by $\mathcal{R}_j$, $j = 1, \ldots, l$. All the paths in the network are grouped into a vector $(R_1, \ldots, R_m)$.

We can describe the arc structure of the path by using the arc-path incidence matrix $\Delta := \{\delta_{ir}\}$, $i = 1, \ldots, n$, $r = 1, \ldots, m$, whose entries take the value 1 if $A_i \in R_r$ and 0 if $A_i \notin R_r$. To each path R_r there corresponds a flow $F_r(\omega)$, $\omega \in \Omega$, and the path flows are grouped into a vector $(F_1(\omega), \ldots, F_m(\omega))$, which is called the path flow vector. The flow f_i on the arc A_i is equal to the sum of the flows on the paths containing A_i, so that $f(\omega) = \Delta F(\omega)$, $\omega \in \Omega$. Let us now introduce the unit cost of going through A_i as a function $c_i(f(\omega)) \geq 0$ of the flows on the network, so that $c(f(\omega)) := (c_1(f(\omega)), \ldots, c_n(f(\omega)))$ denotes the arc cost on the network. Analogously $C(F(\omega)) := (C_1(F(\omega)), \ldots, C_m(F(\omega)))$ will denote the cost on the paths.

Usually $C_r(F(\omega))$ is given by the sum of the costs on the arcs building the path:

$$C_r(F(\omega)) = \sum_{i=1}^{n} \delta_{ir} c_i(f(\omega)), \quad \omega \in \Omega \quad \text{or} \quad C(F(\omega)) = \Delta^T c(\Delta F(\omega)).$$

Instead of assuming paths with infinite capacity, we suppose that there exist two random capacity vectors $\lambda(\omega)$, $\mu(\omega)$, $\lambda(\omega) < \mu(\omega)$, such that

$$0 \le \lambda(\omega) \le F(\omega) \le \mu(\omega), \ P - a.s.$$

For each pair w_j there is a given random traffic demand $D_j(\omega) \ge 0$, so that $(D_1(\omega), \dots, D_l(\omega))$ is the demand vector. We require that the so-called traffic conservation law is fulfilled, namely, that the demand $D_j(\omega)$ verifies the conservation law

$$\sum_{r=1}^{m} \varphi_{jr} F_r(\omega) = D_j(\omega) \quad j = 1, \dots, l, \ P - a.s.,$$

where $\Phi := \{\varphi_{jr}\}$, $j = 1, \dots, l$, $r = 1, \dots, m$, is the pair-incidence matrix whose elements φ_{jr} are equal to 1, if the path R_r connects the pair w_j, and equal to 0 otherwise. We assume that $F(\omega) \in L^2(\Omega, P, \mathbb{R}^m)$, $D(\omega) \in L^2(\Omega, P, \mathbb{R}^l)$ and the random cost $C(F(\omega)) : L^2(\Omega, P, \mathbb{R}^m) \to L^2(\Omega, P, \mathbb{R}^m)$. By $L^2(\Omega, P, \mathbb{R}^m)$ we denote the class of $\mathbb{R}^m$-valued functions defined in Ω, which are square integrable with respect to the probability measure P, while the symbol $\langle \cdot, \cdot \rangle$ will denote the standard scalar product in $\mathbb{R}^m$. Moreover we set

$$\langle\langle G, F \rangle\rangle := \int_\Omega \langle G(\omega), F(\omega) \rangle dP_\omega \quad \forall F, G \in L^2(\Omega, P, \mathbb{R}^m).$$

Then, the set of random feasible flows is given by

$$\mathbb{K}_P = \{F(\omega) \in L^2(\Omega, P, \mathbb{R}^m) : \ \lambda(\omega) \le F(\omega) \le \mu(\omega), \Phi F(\omega) = D(\omega), P - a.s.\},$$

which is a closed, bounded and convex subset of $L^2(\Omega, P, \mathbb{R}^m)$. Setting $\forall \omega \in \Omega$,

$$\mathbb{K}(\omega) := \{F(\omega) \in \mathbb{R}^m : \ \lambda(\omega) \le F(\omega) \le \mu(\omega), \Phi F(\omega) = D(\omega)\},$$

and, assuming, in order to ensure the nonemptiness of $\mathbb{K}(\omega)$,

$$\Phi\lambda(\omega) \le \Phi F(\omega) \le \Phi\mu(\omega),$$

$\mathbb{K}_P$ may be rewritten as

$$\mathbb{K}_P = \{F(\omega) \in L^2(\Omega, P, \mathbb{R}^m) : F(\omega) \in \mathbb{K}(\omega), P - a.s.\}.$$

We can give the following equilibrium definition, that generalizes Wardrop equilibrium condition.

Definition 3.3. (see Definiton 2.1 in [7]) A distribution $H \in \mathbb{K}_P$ is an *equilibrium distribution* from the user's point of view iff

$$\forall w_j \in W, \ \forall R_q, \ R_s \in \mathcal{R}_j \text{ and P-a.s. there holds} \tag{28}$$

$$C_q(H(\omega)) < C_s(H(\omega)) \Longrightarrow H_q(\omega) = \mu_q(\omega) \text{ or } H_s(\omega) = \lambda_s(\omega).$$

An equilibrium distribution can be characterized by means of a variational inequality.

Theorem 3.4. (see Theorem 1 in [11]) *$H \in \mathbb{K}_P$ is an equilibrium flow according to Definition* 3.3 *iff it is a solution to the variational inequality:*

$$\langle\langle C(H), F - H\rangle\rangle = \int_\Omega \langle C(H(\omega)), F(\omega) - H(\omega)\rangle \, dP_\omega \geq 0, \forall F \in \mathbb{K}_P. \tag{29}$$

Let us now introduce the Lagrange functional

$$L(F, \rho_1, \rho_2, \delta) := \Gamma(F) + \ll \rho_1, \lambda – F \gg + \ll \rho_2, F – \mu \gg + \ll \delta, \Phi F(\omega) – D(\omega) \gg, \tag{30}$$

where $$\Gamma(F) = \ll C(H), F - H \gg, \; F \in L^2(\Omega, P, \mathbb{R}^m)$$

and $H \in \mathbb{K}_p$ is a solution to variational inequality (29).

The following Theorem guarantees that the random variational inequality (29) may be expressed in terms of the system of equations (32), (33), (34):

Theorem 3.5. (see Theorem 2.3 in [7]) *Let $H \in \mathbb{K}_p$ be a solution to* (29)*. We consider the associated Lagrange functional* (30)*, then there exist ρ^{1*}, $\rho^{2*} \in L^2(\Omega, P, \mathbb{R}^m_+)$ and $\delta^* \in L^2(\Omega, P, \mathbb{R}^l)$ such that $(H, \rho^{1*}, \rho^{2*}, \delta^*)$ is a saddle point of* (30)*, namely:*

$$L(H, \rho_1, \rho_2, \delta) \leq L(H, \rho^{1*}, \rho^{2*}, \delta^*) \leq L(F, \rho^{1*}, \rho^{2*}, \delta^*) \tag{31}$$

$$\forall F \in L^2(\Omega, P, \mathbb{R}^m), \;\; \forall \rho_1, \; \rho_2 \in L^2(\Omega, P, \mathbb{R}^m_+), \;\; \forall \delta \in L^2(\Omega, P, \mathbb{R}^l)$$

and P-a.s.

$$C_r(H(\omega)) - \rho_r^{1*}(\omega) + \rho_r^{2*}(\omega) + \sum_{j=1}^{l} \delta_j^*(\omega)\varphi_{jr} = 0 \quad \forall r = 1, \ldots, m \tag{32}$$

$$\rho_r^{1*}(\omega)\,(\lambda_r(\omega) - H_r(\omega)) = 0 \quad \forall r = 1, \ldots, m \tag{33}$$

$$\rho_r^{2*}(\omega)\,(H_r(\omega) - \mu_r(\omega)) = 0 \quad \forall r = 1, \ldots, m. \tag{34}$$

It is easily proved also in this case that the Lagrange multipliers ρ^{1*}, ρ^{2*} are unique.

From formula (32), we derive the meaning of the Lagrange multiplier $\delta_j^*(\omega)$. Indeed, $-\delta_j^*(\omega)$ represents the generalized equilibrium cost for each O/D pair. If $\lambda_r(\omega) < H_r(\omega) < \mu_r(\omega)$, $\Phi_{jr} = 1$, then $-\delta_j^*(\omega)$ is exactly the equilibrium cost. On the contrary, the equilibrium cost $-\delta_j^*(\omega)$ increases or decreases when $H_r(\omega)$ coincides with the lower or the upper bound.

Let us stress that inequalities (6), namely, inequalities (31), allow us to solve the problem we are dealing with, that is transforming variational inequality (29) in the system of equations (32), (33), (34).

It is not possible to use the classical strong duality theory, since the ordering cone of $L^2(\Omega, P, \mathbb{R}^m)$ is $C = \{F \in L^2(\Omega, P, \mathbb{R}^m) : F \geq 0 \text{ P-a.s.}\}$, that has an empty interior.

On the contrary, Assumption S is verified (see [7]).

Then, the strong duality holds and, in virtue of Theorem 2.3, there exist $\rho^{1*}(\omega)$, $\rho^{2*}(\omega) \in L^2(\Omega, P, \mathbb{R}^m_+)$ and $\delta^*(\omega) \in L^2(\Omega, P, \mathbb{R}^l)$ such that $(H, \rho^{1*}, \rho^{2*}, \delta^*)$ is a saddle point of the Lagrange functional, namely:

$$L(H, \rho_1, \rho_2, \delta) \leq L(H, \rho^{1*}, \rho^{2*}, \delta^*) \leq L(F, \rho^{1*}, \rho^{2*}, \delta^*)$$

$$\forall F \in L^2(\Omega, P, \mathbb{R}^m), \;\; \forall \rho_1, \; \rho_2 \in L^2(\Omega, P, \mathbb{R}^m_+), \;\; \forall \delta \in L^2(\Omega, P, \mathbb{R}^l)$$

and $$\ll \rho^{1*}, \lambda - H \gg = 0, \quad \ll \rho^{2*}, H - \mu \gg = 0. \tag{35}$$

From (35) it is easy to derive (33) and (34). Considering the right-hand side of (31), we may prove (32). Indeed,

$$L(F, \rho^{1*}, \rho^{2*}, \delta^*) \geq L(H, \rho^{1*}, \rho^{2*}, \delta^*) = 0, \quad \forall F \in L^2(\Omega, P, \mathbb{R}^m),$$

namely, taking into account (35):

$$\ll C(H), F - H \gg - \ll \rho^{1*}, F - H \gg + \ll \rho^{2*}, F - H \gg + \ll \delta^*, \Phi F - D \gg \geq 0$$

for all $F \in L^2(\Omega, P, \mathbb{R}^m)$. Since $\Phi H(\omega) = D(\omega)$, we can rewrite the term

$$\ll \delta^*, \Phi(F - H) \gg = \ll \Phi^T \delta^*, F - H \gg$$

and, hence, we obtain:

$$\ll C(H) - \rho^{1*} + \rho^{2*} + \Phi^T \delta^*, F - H \gg \geq 0, \quad \forall F \in L^2(\Omega, P, \mathbb{R}^m). \tag{36}$$

From (36), choosing $F = H \pm \varphi \ \forall \varphi \in L^2(\Omega, P, \mathbb{R}^m)$, we get

$$\ll C(H) - \rho^{1*} + \rho^{2*} + \Phi^T \delta^*, \varphi \gg = 0, \quad \forall \varphi \in L^2(\Omega, P, \mathbb{R}^m). \tag{37}$$

Finally, choosing $\varphi = C(H) - \rho^{1*} + \rho^{2*} + \Phi^T \delta^*$, we obtain the desired equation

$$C(H(\omega)) - \rho^{1*}(\omega) + \rho^{2*}(\omega) + \Phi^T \delta^*(\omega) = 0 \quad \text{P-a.s.}$$

4. Conclusions

In this paper we point out the importance of a new strong duality, which works in infinite-dimensional settings, where the classical theory cannot be applied. The theory allows to transform a variational inequality on a convex feasible set in terms of a system of equations. As applications, we present a nonconstant gradient constrained problem and the random traffic equilibrium problem. In the future we would like to continue the study of this topic and, in particular, we will study the existence of Lagrange multipliers for the problem with non-constant gradient constraints associated with a nonlinear monotone operator. Moreover, we will take into account uncertainty on the data in other equilibrium problems, which leads to a random formulation of the model, for example in a cybersecurity investment supply chain game theory model.

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