

Let us assume that coefficients $a_{i,j}^{\alpha,\beta}(x,y)$ are measurable with respect to x , continuous with respect to y , bounded and elliptic. When $N = 1$, we have only one equation and the celebrated De Giorgi-Nash-Moser theorem forces weak solutions $u \in W^{1,2}(\Omega)$ to be locally bounded and even Hölder continuous, see section 2.1 in [17]. The result is no longer true, in general, for systems: De Giorgi's counterexample shows that $u(x) = x/|x|^\gamma$ is a weak solution to a particular system (2) where Ω is the ball centered at the origin with radius 1 and $\gamma > 1$ is a suitable exponent; it turns out that u cannot be bounded inside Ω near the origin; see [3], section 3 in [17] and the recent paper [18]; see also [21] and [11]. Now the effort is finding additional restrictions on the coefficients $a_{i,j}^{\alpha,\beta}$ that keep away De Giorgi's counterexample and allow for local boundedness of weak solutions u . The easiest case happens when off-diagonal coefficients vanish, that is

$$a_{i,j}^{\alpha,\beta} = 0 \quad \text{for} \quad \beta \neq \alpha. \tag{3}$$

In such a case, the α row of the system is

$$-\sum_{i=1}^n D_i \left(\sum_{j=1}^n a_{i,j}^{\alpha,\alpha}(x, u(x)) D_j u^\alpha(x) \right) = 0, \tag{4}$$

so, if $b_{i,j}(x) = a_{i,j}^{\alpha,\alpha}(x, u(x))$ and $v(x) = u^\alpha(x)$, then we are in the linear scalar case

$$-\sum_{i=1}^n D_i \left(\sum_{j=1}^n b_{i,j}(x) D_j v(x) \right) = 0 \tag{5}$$

and v turns out to be locally bounded and Hölder continuous. A further step has been made in [22]: the system (2) is assumed to be tridiagonal, that is

$$a_{i,j}^{\alpha,\beta} = 0 \quad \text{for} \quad \beta > \alpha. \tag{6}$$

In such a case the system (2) becomes

$$\begin{aligned} &-\sum_{i=1}^n D_i \left(\sum_{j=1}^n a_{i,j}^{1,1}(x, u) D_j u^1 \right) = 0 \\ &-\sum_{i=1}^n D_i \left(\sum_{j=1}^n a_{i,j}^{2,1}(x, u) D_j u^1 + \sum_{j=1}^n a_{i,j}^{2,2}(x, u) D_j u^2 \right) = 0 \\ &\dots\dots\dots \\ &-\sum_{i=1}^n D_i \left(\sum_{j=1}^n a_{i,j}^{N,1}(x, u) D_j u^1 + \sum_{j=1}^n a_{i,j}^{N,2}(x, u) D_j u^2 + \dots + \sum_{j=1}^n a_{i,j}^{N,N}(x, u) D_j u^N \right) = 0 \end{aligned} \tag{7}$$

Then we can apply to the first equation the regularity for scalar case: Hölder continuity for u^1 and suitable decay on balls for Du^1 . Now the second row can be written as follow

$$-\sum_{i=1}^n D_i \left(\sum_{j=1}^n a_{i,j}^{2,2}(x, u) D_j u^2 \right) = \sum_{i=1}^n D_i \left(\sum_{j=1}^n a_{i,j}^{2,1}(x, u) D_j u^1 \right); \tag{8}$$

the good behaviour of Du^1 on the right hand side can be transferred to the left hand side so that u^2 inherits Hölder continuity and Du^2 gets a suitable decay on balls.

The procedure can be iterated until we arrive at u^N . Another step has been made in [16] where the local boundedness is obtained under the following structure assumption: there exist numbers $\lambda > 0$, $L \geq 0$ and two nonnegative functions $d(x), g(x)$, such that

$$\sum_{\alpha=1}^N \sum_{\gamma=1}^N \frac{y^\alpha y^\gamma}{|y|^2} \left(\sum_{i=1}^n p_i^\gamma \sum_{\beta=1}^N \sum_{j=1}^n a_{i,j}^{\alpha,\beta}(x, y) p_j^\beta \right) \geq - \left\{ \delta |p|^2 + \left(\frac{1}{\delta} \right)^\lambda [d(x)|y|^2 + g(x)] \right\} \quad (9)$$

is fulfilled for all $\delta \in (0, 1)$ and all (x, y, p) , with $|y| > L$. In the present work we assume a condition on the support of off-diagonal coefficients: there exists $L_0 \in (0, +\infty)$ such that $\forall L \geq L_0$, when $\alpha \neq \beta$,

$$\begin{aligned} (a_{i,j}^{\alpha,\beta}(x, y) \neq 0 \text{ and } y^\alpha > L) &\Rightarrow y^\beta > L, \\ (a_{i,j}^{\alpha,\beta}(x, y) \neq 0 \text{ and } y^\alpha < -L) &\Rightarrow y^\beta < -L \end{aligned} \quad (10)$$

(see Figure 2.1). Under such a restriction we are able to prove local boundedness of weak solutions. All the necessary assumptions and the result will be listed in section 2 while proofs will be performed in section 3. Let us mention that off-diagonal coefficients with a particular support have been successfully used when proving maximum principles in [12] and when obtaining existence for measure data problems in [14], [15]. It is worth mentioning that, when the ratio between the largest and the smallest eigenvalues of $a_{i,j}^{\alpha,\beta}$ is close to 1, then regularity of u is studied at page 183 of [7]; see also [20], [10], [9], [13]. Let us also say that proving boundedness for weak solutions could be an important tool for getting fractional differentiability, see the estimate after (4.15) in [4]; sometimes, a gain in fractional differentiability can be iterated as in Theorem 3.III of [1] and in Theorem 3.3 of [5].

2. Assumptions and result

Assume Ω is an open bounded subset of \mathbb{R}^n , with $n \geq 2$. Consider the system of $N \geq 2$ equations

$$- \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sum_{\beta=1}^N \sum_{j=1}^n a_{i,j}^{\alpha,\beta}(x, u) \frac{\partial}{\partial x_j} u^\beta \right) = 0 \text{ in } \Omega, \text{ for } \alpha = 1, \dots, N. \quad (11)$$

Note that u^β is the β component of $u = (u^1, u^2, \dots, u^N)$. We list our structural conditions.

- (A) For all $i, j \in \{1, \dots, n\}$ and all $\alpha, \beta \in \{1, \dots, N\}$, we require that $a_{i,j}^{\alpha,\beta} : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies the following conditions:
- (A₀) $x \mapsto a_{i,j}^{\alpha,\beta}(x, y)$ is measurable and $y \mapsto a_{i,j}^{\alpha,\beta}(x, y)$ is continuous;
- (A₁) (boundedness of all the coefficients) for some positive constant $c > 0$, we have

$$|a_{i,j}^{\alpha,\beta}(x, y)| \leq c$$

for almost all $x \in \Omega$ and for all $y \in \mathbb{R}^N$;

(\mathcal{A}_2) (ellipticity of all the coefficients) for some positive constant $\nu > 0$, we have

$$\sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x,y) \xi_i^\alpha \xi_j^\beta \geq \nu |\xi|^2$$

for almost all $x \in \Omega$, for all $y \in \mathbb{R}^N$ and for all $\xi \in \mathbb{R}^{N \times n}$;

(\mathcal{A}_3) (support of off-diagonal coefficients) there exists $L_0 \in (0, +\infty)$ such that for all $L \geq L_0$, when $\alpha \neq \beta$,

$$(a_{i,j}^{\alpha,\beta}(x,y) \neq 0 \text{ and } y^\alpha > L) \Rightarrow y^\beta > L, \quad (\mathcal{A}'_3)$$

$$(a_{i,j}^{\alpha,\beta}(x,y) \neq 0 \text{ and } y^\alpha < -L) \Rightarrow y^\beta < -L. \quad (\mathcal{A}''_3)$$

(see Figure 2.1).

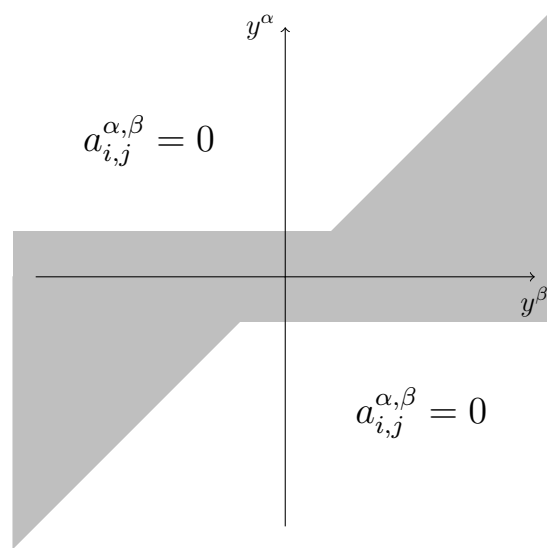


Figure 2.1: Assumption (\mathcal{A}_3): off-diagonal entries $a_{i,j}^{\alpha,\beta}$ vanish on the white part of the picture; they might be non zero only on the grey part.

We say that a function $u : \Omega \rightarrow \mathbb{R}^N$ is a *weak solution* of the system (11), if $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ and

$$\int_{\Omega} \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, u(x)) D_j u^\beta(x) D_i \varphi^\alpha(x) dx = 0, \quad (12)$$

for all $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$.

Theorem 2.1. *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution of system (11) under the set (\mathcal{A}) of assumptions. Then $u \in L_{loc}^\infty(\Omega, \mathbb{R}^N)$.*

3. Proof of the result

The proof of the Theorem 2.1 will be performed in several steps.

Step 1. Caccioppoli inequality

Theorem 3.1. (Caccioppoli inequality on superlevel sets) Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution of system (11) under the assumptions (\mathcal{A}_0) , (\mathcal{A}_1) , (\mathcal{A}_2) , (\mathcal{A}'_3) . For $0 < s < t$, let $B(x_0, s)$ and $B(x_0, t)$ be concentric open balls centered at x_0 with radii s and t respectively. Assume that $B(x_0, t) \subset \Omega$ and $L \geq L_0$. Then

$$\sum_{\alpha=1}^N \int_{\{u^\alpha > L\} \cap B(x_0, s)} |Du^\alpha|^2 dx \leq \frac{16c^2 n^4 N^4}{\nu^2} \sum_{\alpha=1}^N \int_{\{u^\alpha > L\} \cap B(x_0, t)} \left(\frac{u^\alpha - L}{t - s}\right)^2 dx, \quad (13)$$

where c is the constant involved in assumption (\mathcal{A}_1) , ν is given in (\mathcal{A}_2) and L_0 appears in (\mathcal{A}_3) .

Proof of Theorem 3.1. Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution of system (11). Let $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$ be the standard cut-off function such that $0 \leq \eta \leq 1$, $\eta \in C_0^1(B(x_0, t))$, with $B(x_0, t) \subset \Omega$ and $\eta = 1$ in $B(x_0, s)$. Moreover, $|D\eta| \leq 2/(t - s)$ in \mathbb{R}^n . For every level $L \geq L_0$, consider the test function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^N$ with $\varphi = (\varphi^1, \dots, \varphi^N)$,

where
$$\varphi^\alpha(x) := \eta^2(x) \max\{0, u^\alpha(x) - L\}, \quad \text{for all } \alpha \in \{1, \dots, N\}.$$

Then $D_i \varphi^\alpha = \eta^2 \mathbf{1}_{\{u^\alpha > L\}} D_i u^\alpha + 2\eta(D_i \eta) \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L)$ for all $i \in \{1, \dots, n\}$ and $\alpha \in \{1, \dots, N\}$.

Using this test function in the weak formulation (12) of system (11), we obtain

$$\begin{aligned} 0 &= \int_{\Omega} \sum_{\alpha, \beta=1}^N \sum_{i, j=1}^n a_{i, j}^{\alpha, \beta} D_j u^\beta D_i \varphi^\alpha dx = \int_{\Omega} \sum_{\alpha, \beta=1}^N \sum_{i, j=1}^n a_{i, j}^{\alpha, \beta} D_j u^\beta \eta^2 \mathbf{1}_{\{u^\alpha > L\}} D_i u^\alpha dx \\ &\quad + \int_{\Omega} \sum_{\alpha, \beta=1}^N \sum_{i, j=1}^n a_{i, j}^{\alpha, \beta} D_j u^\beta 2\eta(D_i \eta) \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L) dx. \end{aligned}$$

Now, assumption (\mathcal{A}'_3) guarantees that

$$a_{i, j}^{\alpha, \beta}(x, u(x)) \mathbf{1}_{\{u^\alpha > L\}}(x) = a_{i, j}^{\alpha, \beta}(x, u(x)) \mathbf{1}_{\{u^\beta > L\}}(x) \mathbf{1}_{\{u^\alpha > L\}}(x) \quad (14)$$

when $\beta \neq \alpha$ and $L \geq L_0$. It is worthwhile to note that (14) holds true when $\alpha = \beta$ as well; then

$$\begin{aligned} &\int_{\Omega} \sum_{\alpha, \beta=1}^N \sum_{i, j=1}^n a_{i, j}^{\alpha, \beta} \mathbf{1}_{\{u^\beta > L\}} D_j u^\beta \eta^2 \mathbf{1}_{\{u^\alpha > L\}} D_i u^\alpha dx \\ &= - \int_{\Omega} \sum_{\alpha, \beta=1}^N \sum_{i, j=1}^n a_{i, j}^{\alpha, \beta} \mathbf{1}_{\{u^\beta > L\}} D_j u^\beta 2\eta(D_i \eta) \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L) dx. \end{aligned} \quad (15)$$

Now we can use the ellipticity assumption (\mathcal{A}_2) with $\xi_i^\alpha = \mathbf{1}_{\{u^\alpha > L\}} D_i u^\alpha$ and we get

$$\nu \int_{\Omega} \eta^2 \sum_{\alpha=1}^N \mathbf{1}_{\{u^\alpha > L\}} |Du^\alpha|^2 dx \leq \int_{\Omega} \sum_{\alpha, \beta=1}^N \sum_{i, j=1}^n a_{i, j}^{\alpha, \beta} \mathbf{1}_{\{u^\beta > L\}} D_j u^\beta \eta^2 \mathbf{1}_{\{u^\alpha > L\}} D_i u^\alpha dx. \quad (16)$$

Moreover

$$\begin{aligned}
 & - \int_{\Omega} \sum_{\alpha, \beta=1}^N \sum_{i, j=1}^n a_{i, j}^{\alpha, \beta} \mathbf{1}_{\{u^\beta > L\}} D_j u^\beta 2\eta(D_i \eta) \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L) dx \\
 & \leq \int_{\Omega} c \sum_{\beta=1}^N \sum_{j=1}^n \mathbf{1}_{\{u^\beta > L\}} |D_j u^\beta| \sum_{\alpha=1}^N \sum_{i=1}^n 2\eta |D_i \eta| \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L) dx \\
 & \leq \int_{\Omega} c \sum_{\beta=1}^N n \mathbf{1}_{\{u^\beta > L\}} |Du^\beta| \sum_{\alpha=1}^N n 2\eta |D\eta| \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L) dx \\
 & \leq \int_{\Omega} cn^2 \epsilon \eta^2 \left(\sum_{\beta=1}^N \mathbf{1}_{\{u^\beta > L\}} |Du^\beta| \right)^2 + \int_{\Omega} \frac{cn^2}{\epsilon} |D\eta|^2 \left(\sum_{\alpha=1}^N \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L) \right)^2 dx \\
 & \leq \int_{\Omega} cn^2 N^2 \epsilon \eta^2 \sum_{\beta=1}^N \mathbf{1}_{\{u^\beta > L\}} |Du^\beta|^2 + \int_{\Omega} \frac{cn^2 N^2}{\epsilon} |D\eta|^2 \sum_{\alpha=1}^N \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L)^2 dx, \tag{17}
 \end{aligned}$$

where we used the inequality $2ab \leq \epsilon a^2 + b^2/\epsilon$, provided $\epsilon > 0$. Merging (16) and (17) into (15) we get

$$\begin{aligned}
 & \nu \int_{\Omega} \eta^2 \sum_{\alpha=1}^N \mathbf{1}_{\{u^\alpha > L\}} |Du^\alpha|^2 dx \\
 & \leq \int_{\Omega} cn^2 N^2 \epsilon \eta^2 \sum_{\beta=1}^N \mathbf{1}_{\{u^\beta > L\}} |Du^\beta|^2 + \int_{\Omega} \frac{cn^2 N^2}{\epsilon} |D\eta|^2 \sum_{\alpha=1}^N \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L)^2 dx.
 \end{aligned}$$

We take $\epsilon = \nu/(2cn^2N^2)$ and we obtain

$$\frac{\nu}{2} \int_{\Omega} \eta^2 \sum_{\alpha=1}^N \mathbf{1}_{\{u^\alpha > L\}} |Du^\alpha|^2 dx \leq \int_{\Omega} \frac{2c^2 n^4 N^4}{\nu} |D\eta|^2 \sum_{\alpha=1}^N \mathbf{1}_{\{u^\alpha > L\}} (u^\alpha - L)^2 dx.$$

Using the properties of the cut off function η we get

$$\sum_{\alpha=1}^N \int_{\{u^\alpha > L\} \cap B(x_0, s)} |Du^\alpha|^2 dx \leq \frac{16c^2 n^4 N^4}{\nu^2} \sum_{\alpha=1}^N \int_{\{u^\alpha > L\} \cap B(x_0, t)} \left(\frac{u^\alpha - L}{t - s} \right)^2 dx. \tag{18}$$

This ends the proof of Theorem 3.1. □

The “excess” on superlevel sets

In the previous Caccioppoli inequality the following sum appears on the right hand side:

$$\sum_{\alpha=1}^N \int_{\{u^\alpha > L\} \cap B(x_0, t)} (u^\alpha - L)^2 dx. \tag{19}$$

Note that the sum (19) is zero if and only if all the superlevel sets have zero measure, that is $|\{u^1 > L\}| = 0$, $|\{u^2 > L\}| = 0$, ... , $|\{u^N > L\}| = 0$, where $|A|$ is the n -dimensional Lebesgue measure of $A \subset \mathbb{R}^n$.

This happens when $L \geq \max\{\text{esssup } u^1, \text{esssup } u^2, \dots, \text{esssup } u^N\}$. On the contrary, if $L < \max\{\text{esssup } u^1, \text{esssup } u^2, \dots, \text{esssup } u^N\}$, then the sum (19) is positive. Moreover,

$$L \mapsto \sum_{\alpha=1}^N \int_{\{u^\alpha > L\} \cap B(x_0, t)} (u^\alpha - L)^2 dx \quad \text{decreases.} \tag{20}$$

A sum of the form of (19) measures how much L is far from

$$\max\{\text{esssup } u^1, \text{esssup } u^2, \dots, \text{esssup } u^N\}.$$

Let us call (19) the *excess* of u with respect to the level L , the “excess” for short. We aim to show that the excess is zero for a suitable level L . We first show that the excess at level L_2 can be estimated by means of a power σ of the excess at level L_1 , for a suitable pair of levels $L_2 > L_1$. Then we iterate the procedure.

Step 2. Decay of the “excess” on superlevel sets

In general, we consider a vector valued function $v : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$, $n \geq 2$, with $v = (v^1, \dots, v^N)$ and $v \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^N)$, $p \geq 1$, where Ω is an open set in \mathbb{R}^n . We fix $B_{R_0} = B(x_0, R_0) \Subset \Omega$, with $R_0 < 1$ small enough so that

$$|B_{R_0}| < 1 \quad \text{and} \quad \sum_{\alpha=1}^N \int_{B_{R_0}} |v^\alpha|^{p^*} dx < 1, \tag{21}$$

where $A \Subset \Omega$ means that the closure \bar{A} is a compact set contained in Ω ; moreover, $p^* = \frac{np}{n-p}$, if $p < n$, and p^* is any $q > p$, else. For every $R \in (0, R_0]$ we define the decreasing sequences

$$\rho_h := \frac{R}{2} + \frac{R}{2^{h+1}} = \frac{R}{2} \left(1 + \frac{1}{2^h}\right), \quad \bar{\rho}_h := \frac{\rho_h + \rho_{h+1}}{2} = \frac{R}{2} \left(1 + \frac{3}{4 \cdot 2^h}\right).$$

Fix a positive constant $d \geq 1$ and define the increasing sequence of positive real numbers

$$k_h := d \left(1 - \frac{1}{2^{h+1}}\right), \quad h \in \{0, 1, 2, \dots\}. \tag{22}$$

Moreover, define the sequence (J_h) ,

$$J_h := \sum_{\alpha=1}^N \int_{A_{k_h, \rho_h}^\alpha} (v^\alpha - k_h)^{p^*} dx,$$

where $A_{k, \rho}^\alpha = \{v^\alpha > k\} \cap B_\rho$ and $B_\rho = B(x_0, \rho)$. The following result holds.

Proposition 3.2. (Decay of the excess from step h to step $h+1$) *Let $v \in W_{\text{loc}}^{1,p}(\Omega; \mathbb{R}^N)$, $p \geq 1$. Fix $B(x_0, R_0) \Subset \Omega$, with $R_0 < 1$ small enough such that (21) holds. If there exist $0 \leq \vartheta \leq 1$ and $c_0 > 0$ such that, for every $0 < s < t \leq R_0$ and for every $h \in \{0, 1, 2, \dots\}$,*

$$\sum_{\alpha=1}^N \int_{A_{k_h, s}^\alpha} |Dv^\alpha|^p dx \leq c_0 \sum_{\alpha=1}^N \left\{ \int_{A_{k_h, t}^\alpha} \left(\frac{v^\alpha - k_h}{t - s}\right)^{p^*} dx + |A_{k_h, t}^\alpha|^\vartheta \right\}, \tag{23}$$

then, for every $R \in (0, R_0]$ and for every $h \in \{0, 1, 2, \dots\}$,

$$J_{h+1} \leq c(\vartheta, R) \left(2^{\frac{p^* p^*}{p}}\right)^h (J_h)^{\vartheta \frac{p^*}{p}},$$

with the positive constant $c(\vartheta, R)$ independent of h .

Remark 3.3. We want to stress that the exponent on the right hand side is p^* larger than the exponent p on the left hand side: this situation has been studied in the scalar case $N = 1$ in [2], [6], [19].

Proof. Notice that (J_h) is a decreasing sequence, since the following chain of inequalities holds:

$$\begin{aligned} J_{h+1} &= \sum_{\alpha=1}^N \int_{A_{k_{h+1}, \rho_{h+1}}^\alpha} (v^\alpha - k_{h+1})^{p^*} dx \leq \sum_{\alpha=1}^N \int_{A_{k_{h+1}, \rho_h}^\alpha} (v^\alpha - k_{h+1})^{p^*} dx \\ &\leq \sum_{\alpha=1}^N \int_{A_{k_h, \rho_h}^\alpha} (v^\alpha - k_h)^{p^*} dx \leq \sum_{\alpha=1}^N \int_{A_{k_h, \rho_h}^\alpha} (v^\alpha - k_h)^{p^*} dx = J_h. \end{aligned} \tag{24}$$

Let us now define a sequence (ζ_h) of cut-off functions in $C_0^1(B(x_0, \bar{\rho}_h))$, such that $0 \leq \zeta_h \leq 1$, $\zeta_h \equiv 1$ in $B_{\rho_{h+1}}$, $|D\zeta_h| \leq \frac{2^{h+4}}{R}$. If we denote $(v - k_{h+1})_+ = \max\{v - k_{h+1}, 0\}$ we get

$$\begin{aligned} J_{h+1} &= \sum_{\alpha=1}^N \int_{A_{k_{h+1}, \rho_{h+1}}^\alpha} (v^\alpha - k_{h+1})^{p^*} \zeta_h^{p^*} dx \leq \sum_{\alpha=1}^N \int_{A_{k_{h+1}, \bar{\rho}_h}^\alpha} (v^\alpha - k_{h+1})^{p^*} \zeta_h^{p^*} dx \\ &= \sum_{\alpha=1}^N \int_{B_R} (\zeta_h (v^\alpha - k_{h+1})_+)^{p^*} dx. \end{aligned} \tag{25}$$

The Sobolev Embedding Theorem and the properties of ζ_h yield

$$\begin{aligned} \int_{B_R} (\zeta_h (v^\alpha - k_{h+1})_+)^{p^*} dx &\leq c \left(\int_{B_R} |D(\zeta_h (v^\alpha - k_{h+1})_+)|^p dx \right)^{\frac{p^*}{p}} \\ &\leq c \left\{ \left(\int_{B_R} |Dv^\alpha \zeta_h|^p 1_{\{v^\alpha > k_{h+1}\}} dx \right)^{\frac{1}{p}} + \left(\int_{B_R} |(v^\alpha - k_{h+1})_+ D\zeta_h|^p dx \right)^{\frac{1}{p}} \right\}^{p^*} \\ &\leq c \left\{ \left(\int_{A_{k_{h+1}, \bar{\rho}_h}^\alpha} |Dv^\alpha|^p dx \right)^{\frac{1}{p}} + \left(\left(\frac{2^h}{R}\right)^p \int_{A_{k_{h+1}, \rho_h}^\alpha} (v^\alpha - k_{h+1})^p dx \right)^{\frac{1}{p}} \right\}^{p^*}. \end{aligned} \tag{26}$$

Note that, when $p = n$, we used $|B_R| \leq 1$ (see (21)) in the Sobolev inequality.

Substituting $t = \rho_h$ and $s = \bar{\rho}_h$ in (23) we deduce

$$\sum_{\alpha=1}^N \int_{A_{k_{h+1}, \bar{\rho}_h}^\alpha} |Dv^\alpha|^p dx \leq c \sum_{\alpha=1}^N \left\{ \left(\frac{2^h}{R}\right)^{p^*} \int_{A_{k_{h+1}, \rho_h}^\alpha} |v^\alpha - k_{h+1}|^{p^*} dx + |A_{k_{h+1}, \rho_h}^\alpha|^\vartheta \right\}. \tag{27}$$

Collecting (25), (26), (27), we obtain

$$J_{h+1} \leq c \left\{ \sum_{\alpha=1}^N \left(\frac{2^h}{R}\right)^{p^*} \int_{A_{k_{h+1}, \rho_h}^\alpha} (v^\alpha - k_{h+1})^{p^*} dx + \sum_{\alpha=1}^N |A_{k_{h+1}, \rho_h}^\alpha|^\vartheta + \sum_{\alpha=1}^N \left(\frac{2^h}{R}\right)^p \int_{A_{k_{h+1}, \rho_h}^\alpha} (v^\alpha - k_{h+1})^p dx \right\}^{\frac{p^*}{p}}. \tag{28}$$

Since $z^p \leq z^{p^*} + 1$ for every $z \geq 0$, then

$$\left(\frac{2^h}{R}\right)^p \int_{A_{k_{h+1}, \rho_h}^\alpha} (v^\alpha - k_{h+1})^p dx \leq \left(\frac{2^h}{R}\right)^{p^*} \int_{A_{k_{h+1}, \rho_h}^\alpha} (v^\alpha - k_{h+1})^{p^*} dx + |A_{k_{h+1}, \rho_h}^\alpha|,$$

and we obtain

$$J_{h+1} \leq c \left\{ \sum_{\alpha=1}^N \left(\frac{2^h}{R}\right)^{p^*} \int_{A_{k_{h+1}, \rho_h}^\alpha} (v^\alpha - k_{h+1})^{p^*} dx + \sum_{\alpha=1}^N |A_{k_{h+1}, \rho_h}^\alpha|^\vartheta + \sum_{\alpha=1}^N |A_{k_{h+1}, \rho_h}^\alpha| \right\}^{\frac{p^*}{p}}. \tag{29}$$

Since $\sum_{\alpha=1}^N |A_{k_{h+1}, \rho_h}^\alpha| (k_{h+1} - k_h)^{p^*} \leq \sum_{\alpha=1}^N \int_{A_{k_{h+1}, \rho_h}^\alpha} (v^\alpha - k_h)^{p^*} dx \leq J_h,$

then $|A_{k_{h+1}, \rho_h}^\beta| \leq \sum_{\alpha=1}^N |A_{k_{h+1}, \rho_h}^\alpha| \leq \frac{J_h}{(k_{h+1} - k_h)^{p^*}} = \left(\frac{2^{h+2}}{d}\right)^{p^*} J_h.$

Taking also into account that (see (24))

$$\sum_{\alpha=1}^N \int_{A_{k_{h+1}, \rho_h}^\alpha} (v^\alpha - k_{h+1})^{p^*} dx \leq \sum_{\alpha=1}^N \int_{A_{k_{h+1}, \rho_h}^\alpha} (v^\alpha - k_h)^{p^*} dx \leq J_h,$$

inequality (29) gives

$$J_{h+1} \leq c \left\{ \left(\frac{2^h}{R}\right)^{p^*} J_h + \left(\frac{2^h}{d}\right)^{\vartheta p^*} (J_h)^\vartheta + \left(\frac{2^h}{d}\right)^{p^*} J_h \right\}^{\frac{p^*}{p}}. \tag{30}$$

We keep in mind that J_h is decreasing and $k_0 = d/2 > 0$, so

$$J_h \leq J_0 = \sum_{\alpha=1}^N \int_{A_{k_0, \rho_0}^\alpha} (v^\alpha - k_0)^{p^*} dx \leq \sum_{\alpha=1}^N \int_{A_{k_0, \rho_0}^\alpha} (v^\alpha)^{p^*} dx \leq \sum_{\alpha=1}^N \int_{B_R} |v^\alpha|^{p^*} dx \leq 1,$$

where we used (21). Since $J_h \leq 1$ for every h and recalling that $d \geq 1 > R_0 \geq R$, we get

$$\begin{aligned} & \left(\frac{2^h}{R}\right)^{p^*} J_h + \left(\frac{2^h}{d}\right)^{\vartheta p^*} (J_h)^\vartheta + \left(\frac{2^h}{d}\right)^{p^*} J_h \\ & \leq \left\{ 2 \frac{2^{hp^*}}{R^{p^*}} + \frac{2^{h\vartheta p^*}}{R^{\vartheta p^*}} \right\} (J_h)^\vartheta \leq \left(\frac{2}{R^{p^*}} + \frac{1}{R^{\vartheta p^*}} \right) 2^{hp^*} (J_h)^\vartheta. \end{aligned}$$

By (30) it follows

$$J_{h+1} \leq c \left\{ \left(\frac{2}{R^{p^*}} + \frac{1}{R^{\vartheta p^*}} \right) 2^{hp^*} (J_h)^\vartheta \right\}^{\frac{p^*}{p}} \leq c(\vartheta, R) \left(2^{\frac{p^* \vartheta}{p}} \right)^h (J_h)^{\vartheta \frac{p^*}{p}}. \quad \square$$

Step 3. Iteration

We need the following classical result, see e.g. [8].

Lemma 3.4. *Let $\gamma > 0$ and let $J_h \in [0, +\infty)$ be such that*

$$J_{h+1} \leq A \lambda^h J_h^{1+\gamma} \quad \forall h \in \mathbb{N} \cup \{0\}, \tag{31}$$

with $A > 0$ and $\lambda > 1$. If $J_0 \leq A^{-\frac{1}{\gamma}} \lambda^{-\frac{1}{\gamma^2}}$, then $\lim_{h \rightarrow \infty} J_h = 0$.

Step 4. Conclusion

We have got all we need to give the proof of Theorem 2.1. Fix $B_{R_0} = B(x_0, R_0) \Subset \Omega$, with $R_0 < 1$ small enough such that $|B_{R_0}| < 1$ and $\int_{B_{R_0}} |u|^{p^*} dx \leq 1$. From (13) we have that, for every $0 < s < t \leq R_0$ and every h , u satisfies

$$\sum_{\alpha=1}^N \int_{A_{k_h, s}^\alpha} |D u^\alpha|^2 dx \leq c_0 \sum_{\alpha=1}^N \left\{ \int_{A_{k_h, t}^\alpha} \left(\frac{u^\alpha - k_h}{t - s} \right)^{2^*} dx + |A_{k_h, t}^\alpha| \right\}, \tag{32}$$

where $c_0 > 0$ is independent of s, t, h , provided $L_0 \leq d/2$. Therefore u satisfies (23) of Proposition 3.2 with $p = 2$ and $\vartheta = 1$. Then Proposition 3.2, applied to u , gives

$$J_{h+1} \leq c(R) \left(2^{\frac{2^* 2^*}{2}} \right)^h (J_h)^{\frac{2^*}{2}}, \tag{33}$$

with the positive constant $c(R)$ independent of h . Let us note that

$$J_0 := \sum_{\alpha=1}^N \int_{A_{k_0, \rho_0}^\alpha} (u^\alpha - k_0)^{2^*} dx = \sum_{\alpha=1}^N \int_{A_{\frac{d}{2}, R}^\alpha} \left(u^\alpha - \frac{d}{2} \right)^{2^*} dx \leq \sum_{\alpha=1}^N \int_{A_{\frac{d}{2}, R}^\alpha} |u^\alpha|^{2^*} dx,$$

and so $J_0 \rightarrow 0$ when $d \rightarrow +\infty$.

Therefore, we can choose $d > 0$ large enough such that

$$J_0 < c(R)^{-\frac{1}{\frac{2^*}{2}-1}} \left(2^{\frac{2^* 2^*}{2}} \right)^{-\frac{1}{(\frac{2^*}{2}-1)^2}}.$$

Thus, by Lemma 3.4 we deduce that $\lim_{h \rightarrow \infty} J_h = 0$; since

$$\begin{aligned} J_h &= \sum_{\alpha=1}^N \int_{A_{k_h, \rho_h}^\alpha} (u^\alpha - k_h)^{2^*} dx \geq \sum_{\alpha=1}^N \int_{\{u^\alpha > k_h\} \cap B_{R/2}} (u^\alpha - k_h)^{2^*} dx \\ &\geq \sum_{\alpha=1}^N \int_{\{u^\alpha > d\} \cap B_{R/2}} (u^\alpha - k_h)^{2^*} dx \geq \sum_{\alpha=1}^N \int_{\{u^\alpha > d\} \cap B_{R/2}} (u^\alpha - d)^{2^*} dx, \end{aligned}$$

we deduce that $|\{u^\alpha > d\} \cap B_{R/2}| = 0$, namely $u^\alpha \leq d$ a.e. in $B_{\frac{R}{2}}$. We have so proved that u^α is locally bounded from above.

Now, let $\tilde{u} = -u$. Then $\tilde{u} \in W^{1,2}(\Omega; \mathbb{R}^N)$ and, since u satisfies (12), \tilde{u} satisfies

$$0 = \int_{\Omega} \sum_{\alpha, \beta=1}^N \sum_{i, j=1}^n \tilde{a}_{i, j}^{\alpha, \beta}(x, \tilde{u}(x)) D_j \tilde{u}^\beta(x) D_i \varphi^\alpha(x) dx \tag{34}$$

for every $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$, where

$$\tilde{a}_{i,j}^{\alpha,\beta}(x, y) := a_{i,j}^{\alpha,\beta}(x, -y). \tag{35}$$

We observe that the new coefficients, defined by (35), readily satisfy conditions (\mathcal{A}_0) , (\mathcal{A}_1) , (\mathcal{A}_2) . Moreover, if $\alpha \neq \beta$ the coefficients $\tilde{a}_{i,j}^{\alpha,\beta}(x, y)$ satisfy (\mathcal{A}'_3) provided the $a_{i,j}^{\alpha,\beta}(x, y)$ satisfy (\mathcal{A}''_3) . Therefore, we can argue as above on \tilde{u} obtaining the estimate from below for u^α . This ends the proof of Theorem 2.1. \square

4. An example

Let us take $N = 2$ and $n = 3$; we define the matrices $a^{\alpha,\beta} \equiv a^{\alpha,\beta}(y)$ ($\alpha, \beta \in \{1, 2\}$) and $y = (y^1, y^2)$) as

$$\begin{aligned} a^{1,1} &:= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & a^{1,2} &:= \begin{pmatrix} b(y) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ a^{2,1} &:= \begin{pmatrix} 0 & w(y) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & a^{2,2} &:= \begin{pmatrix} 27 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

where $b : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function such that $0 \leq b(y^1, y^2) \leq 2$, $b(k, k+1) = 2$ for every integer $k \geq 2$, $b(0, 0) = 2$ and the support of b is contained in the grey part of Figure 2.1 with $\alpha = 1$ and $\beta = 2$; moreover, $w : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function such that $-10 \leq w(y^1, y^2) \leq 0$, $w(k+1, k) = -10$ for every integer $k \geq 2$, $w(0, 0) = -10$ and the support of w is contained in the grey part of Figure 2.1 with $\alpha = 2$ and $\beta = 1$. It easy to check that assumptions (\mathcal{A}_0) – (\mathcal{A}_3) are satisfied with $c = 27$ and $\nu = 1$. For the convenience of the reader, let us do the calculations for the ellipticity (\mathcal{A}_2) .

$$\begin{aligned} & \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, y) \xi_i^\alpha \xi_j^\beta \\ &= \sum_{i,j=1}^n a_{i,j}^{1,1}(x, y) \xi_i^1 \xi_j^1 + \sum_{i,j=1}^n a_{i,j}^{1,2}(x, y) \xi_i^1 \xi_j^2 + \sum_{i,j=1}^n a_{i,j}^{2,1}(x, y) \xi_i^2 \xi_j^1 + \sum_{i,j=1}^n a_{i,j}^{2,2}(x, y) \xi_i^2 \xi_j^2 \\ &= a_{1,1}^{1,1}(x, y) \xi_1^1 \xi_1^1 + a_{2,2}^{1,1}(x, y) \xi_2^1 \xi_2^1 + a_{3,3}^{1,1}(x, y) \xi_3^1 \xi_3^1 + a_{1,1}^{1,2}(x, y) \xi_1^1 \xi_1^2 \\ & \quad + a_{1,2}^{2,1}(x, y) \xi_1^2 \xi_2^1 + a_{1,1}^{2,2}(x, y) \xi_1^2 \xi_1^2 + a_{2,2}^{2,2}(x, y) \xi_2^2 \xi_2^2 + a_{3,3}^{2,2}(x, y) \xi_3^2 \xi_3^2 \\ &= 2\xi_1^1 \xi_1^1 + 2\xi_2^1 \xi_2^1 + 1\xi_3^1 \xi_3^1 + b(y) \xi_1^1 \xi_1^2 + w(y) \xi_1^2 \xi_2^1 + 27\xi_1^2 \xi_1^2 + 1\xi_2^2 \xi_2^2 + 1\xi_3^2 \xi_3^2 \\ &\geq 2\xi_1^1 \xi_1^1 + 2\xi_2^1 \xi_2^1 + 1\xi_3^1 \xi_3^1 - 2|\xi_1^1| |\xi_1^2| - 10|\xi_1^2| |\xi_2^1| + 27\xi_1^2 \xi_1^2 + 1\xi_2^2 \xi_2^2 + 1\xi_3^2 \xi_3^2 \\ &\geq 2\xi_1^1 \xi_1^1 + 2\xi_2^1 \xi_2^1 + 1\xi_3^1 \xi_3^1 - \xi_1^1 \xi_1^1 - \xi_1^2 \xi_1^2 - \frac{5}{\epsilon} \xi_1^2 \xi_2^1 - 5\epsilon \xi_2^1 \xi_1^1 + 27\xi_1^2 \xi_1^2 + 1\xi_2^2 \xi_2^2 + 1\xi_3^2 \xi_3^2 \\ &= 1\xi_1^1 \xi_1^1 + 2\xi_2^1 \xi_2^1 + 1\xi_3^1 \xi_3^1 - \xi_1^2 \xi_1^2 - 25\xi_1^2 \xi_2^1 - 1\xi_2^1 \xi_1^1 + 27\xi_1^2 \xi_1^2 + 1\xi_2^2 \xi_2^2 + 1\xi_3^2 \xi_3^2 \\ &= 1\xi_1^1 \xi_1^1 + 1\xi_2^1 \xi_2^1 + 1\xi_3^1 \xi_3^1 + 1\xi_1^2 \xi_1^2 + 1\xi_2^2 \xi_2^2 + 1\xi_3^2 \xi_3^2 = |\xi|^2, \tag{36} \end{aligned}$$

where we used the inequality $2AB \leq A^2 + B^2$ and $2AB \leq \frac{A^2}{\epsilon} + \epsilon B^2$ with $\epsilon = \frac{1}{5}$.

On the other hand, this example satisfies neither assumption (3) nor assumption (6) since the two matrices $a^{1,2}$ and $a^{2,1}$ are not zero. Moreover, (9) does not hold true. Indeed, for every integer $k \geq 2$, let us take

$$y^1 = k + 1, \quad y^2 = k, \quad p_1^1 = p_2^1 = t > 0, \quad p_i^2 = 0 = p_3^\alpha \quad (37)$$

and let us compute the left hand side of (9); we have

$$\begin{aligned} I &= \sum_{\alpha=1}^N \sum_{\gamma=1}^N \frac{y^\alpha y^\gamma}{|y|^2} \left(\sum_{i=1}^n p_i^\gamma \sum_{\beta=1}^N \sum_{j=1}^n a_{i,j}^{\alpha,\beta}(x, y) p_j^\beta \right) \\ &= \sum_{\alpha=1}^2 \sum_{\gamma=1}^2 \frac{y^\alpha y^\gamma}{|y|^2} \left(\sum_{i=1}^3 p_i^\gamma \sum_{\beta=1}^2 \left(a_{i,1}^{\alpha,\beta}(y) p_1^\beta + a_{i,2}^{\alpha,\beta}(y) p_2^\beta \right) \right) \\ &= \sum_{\alpha=1}^2 \sum_{\gamma=1}^2 \frac{y^\alpha y^\gamma}{|y|^2} \left(\sum_{i=1}^3 p_i^\gamma \left(a_{i,1}^{\alpha,1}(y) p_1^1 + a_{i,2}^{\alpha,1}(y) p_2^1 \right) \right) \\ &= \sum_{\alpha=1}^2 \sum_{\gamma=1}^2 \frac{y^\alpha y^\gamma}{|y|^2} \left(p_1^\gamma \left(a_{1,1}^{\alpha,1}(y) p_1^1 + a_{1,2}^{\alpha,1}(y) p_2^1 \right) + p_2^\gamma \left(a_{2,1}^{\alpha,1}(y) p_1^1 + a_{2,2}^{\alpha,1}(y) p_2^1 \right) \right) \\ &= \sum_{\alpha=1}^2 \frac{y^\alpha y^1}{|y|^2} \left(p_1^1 \left(a_{1,1}^{\alpha,1}(y) p_1^1 + a_{1,2}^{\alpha,1}(y) p_2^1 \right) + p_2^1 \left(a_{2,1}^{\alpha,1}(y) p_1^1 + a_{2,2}^{\alpha,1}(y) p_2^1 \right) \right) \\ &= \frac{y^1 y^1}{|y|^2} \left(p_1^1 \left(a_{1,1}^{1,1}(y) p_1^1 + a_{1,2}^{1,1}(y) p_2^1 \right) + p_2^1 \left(a_{2,1}^{1,1}(y) p_1^1 + a_{2,2}^{1,1}(y) p_2^1 \right) \right) \\ &\quad + \frac{y^2 y^1}{|y|^2} \left(p_1^1 \left(a_{1,1}^{2,1}(y) p_1^1 + a_{1,2}^{2,1}(y) p_2^1 \right) + p_2^1 \left(a_{2,1}^{2,1}(y) p_1^1 + a_{2,2}^{2,1}(y) p_2^1 \right) \right) \\ &= \frac{y^1 y^1 |t|^2}{|y|^2} \left(\left(a_{1,1}^{1,1}(y) + a_{1,2}^{1,1}(y) \right) + \left(a_{2,1}^{1,1}(y) + a_{2,2}^{1,1}(y) \right) \right) \\ &\quad + \frac{y^2 y^1 |t|^2}{|y|^2} \left(\left(a_{1,1}^{2,1}(y) + a_{1,2}^{2,1}(y) \right) + \left(a_{2,1}^{2,1}(y) + a_{2,2}^{2,1}(y) \right) \right) \\ &= \frac{(k+1)(k+1)|t|^2}{(k+1)^2 + k^2} 4 + \frac{k(k+1)|t|^2}{(k+1)^2 + k^2} (-10) \\ &= \frac{(-6k+4)(k+1)|t|^2}{(k+1)^2 + k^2} = \frac{(-6k^2 - 2k + 4)|t|^2}{2k^2 + 2k + 1}. \end{aligned}$$

Now we compute the right hand side of (9); we have

$$\begin{aligned} Q &= - \left\{ \delta |p|^2 + \left(\frac{1}{\delta} \right)^\lambda [d(x)|y|^2 + g(x)] \right\} \\ &= - \left\{ \delta 2|t|^2 + \left(\frac{1}{\delta} \right)^\lambda [d(x)[(k+1)^2 + k^2] + g(x) \right\} \\ &= -\delta 2|t|^2 \left\{ 1 + \frac{1}{2|t|^2 \delta^{1+\lambda}} [d(x)[(k+1)^2 + k^2] + g(x) \right\}. \end{aligned}$$

Let us take $|t|^2 = \frac{5}{2\delta^{1+\lambda}}[(d(x) + 1)[(k + 1)^2 + k^2] + g(x)]$ so that, since $\delta \in (0, 1)$,

$$\delta 2|t|^2 \left\{ 1 + \frac{1}{2|t|^2\delta^{1+\lambda}}[d(x)[(k + 1)^2 + k^2] + g(x) \right\} \leq 2|t|^2 \left(1 + \frac{1}{5} \right) = \frac{12}{5}|t|^2$$

and $\frac{-12}{5}|t|^2 \leq -\delta 2|t|^2 \left\{ 1 + \frac{1}{2|t|^2\delta^{1+\lambda}}[d(x)[(k + 1)^2 + k^2] + g(x) \right\} = Q$.

For every $L > 0$, we take k so large that $|y|^2 = (k + 1)^2 + k^2 > L^2$ and

$$\frac{-6k^2 - 2k + 4}{2k^2 + 2k + 1} < \frac{-12}{5}.$$

Then
$$I = \frac{(-6k^2 - 2k + 4)|t|^2}{2k^2 + 2k + 1} < \frac{-12}{5}|t|^2 \leq Q$$

and this shows that the example does not satisfy (9). \square

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