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Characterizations of Vector Equilibria Subject to Explicit Constraints *

Adela Capătă
Department of Mathematics, Technical University of Cluj-Napoca, Str. Memorandumului 28, 400114 Cluj-Napoca, Romania
adela.capata@math.utcluj.ro

Gábor Kassay
Faculty of Mathematics and Computer Science, Babeş-Bolyai University, Str. M. Kogălniceanu 1, 400084 Cluj-Napoca, Romania
kassay@math.ubbcluj.ro

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We investigate weak vector equilibrium problems subject to explicit constraints and provide necessary and sufficient conditions for the existence of their solutions. Two different approaches are used by means of two separation results. As a byproduct, we deduce sufficient conditions for the existence of weak saddle points for generalized Nash two-person noncooperative games, where the payoff function is vector-valued. Our results recover earlier statements from the literature.

Keywords: Vector equilibrium problems, optimality conditions, separation theorem, cone constraints.

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1. Introduction

Let X be a real topological vector space, K a nonempty subset of X and $f : K \times K \to \mathbb{R}$ a real-valued bifunction. In 1972, Fan [25] established the existence of solutions for the following problem (called by himself *minimax inequality*):

$$\text{Find } x^* \in K \text{ such that } f(x^*, y) \geq 0, \quad \forall y \in K. \tag{1}$$

The term "equilibrium problem" (abbreviated (EP)) was attributed to (1) by Muu and Oettli [40] in 1992, where three standard examples were considered: the optimization problems, the variational inequalities and the fixed point problems. Further particular cases like saddle point (minimax) problems, Nash equilibria problems, convex differentiable optimization and complementarity problems have been considered in 1994 by Blum and Oettli [12].

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Equilibrium problems in the above format started to gain real interest only after the publication of the seminal paper of Blum and Oettli. Thanks to its large variety of applications, (EP) has been extensively studied in the last decades (see, for instance [4, 6, 7, 8, 32] and the references therein). A survey covering the main results concerning the existence of equilibria and the solution methods for finding them can be found in the recent paper [10]. Later, the study of equilibrium problems has been extended to vector-valued functions starting from [1] and [5]. By offering a general framework for several problems as vector optimization problems, vector complementarity problems, vector variational inequalities, cone-saddle point problems, etc., the theory of vector equilibrium problems received a great interest from the community of researchers (see, for instance, [2, 37, 38]).

One of the most important particular cases of (EP) is the saddle point (minimax) problem, which has a key role in the theory of noncooperative games. The interest for these results is shown by the numerous papers appeared in the literature (for an extensive survey see [48] and for a more focused one see [26]). Let X_1, X_2 be two nonempty sets and $F : X_1 \times X_2 \to \mathbb{R}$ a given function. (In the framework of two-person noncooperative zero-sum games X_1 and X_2 are the strategy sets of the players, while F is the payoff function of the first player.) The connection between (EP) and the minimax problem can be seen immediately if we consider $K := X_1 \times X_2$ and $f(x, y) := F(x_1, y_2) - F(y_1, x_2)$, where $x = (x_1, x_2)$, $y = (y_1, y_2) \in X_1 \times X_2$. In this setting, the element $\overline{x} = (\overline{x}_1, \overline{x}_2)$ is a solution of (EP) if and only if it is a saddle point of the function F on $X_1 \times X_2$.

For a long time only real-valued payoff functions have been considered in game theory. Motivated by the necessity to describe real-world situations, in the last decades much attention has been attracted to *multicriteria games*, i.e., games with vector-valued payoff functions. For motivations and applications, see, for example, [49] and [53]. Different concepts of solution have been introduced and existence of such solutions has been investigated by various authors (see for instance [11, 18, 39, 47], and the references therein). Let us recall the concepts of *saddle point*, and *weak saddle point* for vector-valued functions. Suppose that Y is a real topological vector space partially ordered by a nontrivial pointed convex cone $C \subset Y$ with nonempty interior, denoted by $\operatorname{int} C$. Let $D \subseteq Y$. By $\operatorname{Min}(D, C)$ (respectively, $\operatorname{Max}(D, C)$) we denote the set of minima (respectively, maxima) of the set D with respect to the cone C, i.e. $d \in \operatorname{Min}(D, C)$ (respectively $d \in \operatorname{Max}(D, C)$) if $d \in D$ and

$$D \cap (d - C) = \{d\} \text{ (respectively, } D \cap (d + C) = \{d\}).$$

Similarly, by $\operatorname{Min}_{\mathrm{w}}(D, C)$ (respectively, $\operatorname{Max}_{\mathrm{w}}(D, C)$) we denote the set of weak minima (respectively, weak maxima) of the set D with respect to the cone C, i.e. $d \in \operatorname{Min}_{\mathrm{w}}(D, C)$ (respectively $d \in \operatorname{Max}_{\mathrm{w}}(D, C)$) if $d \in D$ and

$$D \cap (d - \operatorname{int} C) = \emptyset \text{ (respectively, } D \cap (d + \operatorname{int} C) = \emptyset).$$

Let X_1 and X_2 be nonempty sets and $F : X_1 \times X_2 \to Y$ be a given function. For $x_1 \in X_1$ and $x_2 \in X_2$ we consider the sets

$$F(X_1, x_2) = \{F(x_1, x_2) \mid \ x_1 \in X_1\} \text{ and } F(x_1, X_2) = \{F(x_1, x_2) \mid \ x_2 \in X_2\}.$$

A pair $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ is said to be a *C-saddle point for F* (see [51]) if

$$F(\bar{x}_1, \bar{x}_2) \in \mathrm{Max}(F(X_1, \bar{x}_2), C) \cap \mathrm{Min}(F(\bar{x}_1, X_2), C).$$

Similarly, the pair $(\bar{x}_1, \bar{x}_2) \in X_1 \times X_2$ is said to be a *weak C-saddle point for F* if

$$F(\bar{x}_1, \bar{x}_2) \in \mathrm{Max_w}(F(X_1, \bar{x}_2), C) \cap \mathrm{Min_w}(F(\bar{x}_1, X_2), C).$$

As in the scalar case, the optimal strategies for both players in a multicriteria (vector-valued) game is given by the saddle point (weak saddle point) of the payoff (vector-valued) function F.

The problem of existence of a (weak) saddle point for a vector-valued function can be investigated, like in the scalar case, by considering a more general problem, namely the *vector equilibrium problem.* Let S be a nonempty set and $f : S \times S \to Y$ be a given function. The *strong vector equilibrium problem* is to find an element $x \in S$ such that

$$f(x, y) \notin -C \setminus \{0_Y\} \text{ for all } y \in S, \tag{2}$$

while the *weak vector equilibrium problem* is to find an element $x \in S$ such that

$$f(x, y) \notin -\mathrm{int}\, C \text{ for all } y \in S. \tag{3}$$

The solutions of (2) are usually called *efficient equilibria*, while the solutions of (3) are called *weakly efficient equilibria.* Unlike to the scalar case, the saddle point problem (strong or weak) is not equivalent to the equilibrium problem (strong or weak) when the function f is given by $f(x, y) = F(x_1, y_2) - F(y_1, x_2)$, where $x = (x_1, x_2)$, $y = (y_1, y_2) \in S = X_1 \times X_2$. More precisely, any solution of (2) is a C-saddle point of F but not viceversa (see [9]); similarly, any solution of (3) is a weak C-saddle point of F but not viceversa (see [17]).

In the last few years a growing attention has been given to vector equilibrium problems with explicit constraints. Optimality conditions for such problems have been obtained for instance in [29] and [45]. Let Z be a real vector space, $K \subseteq Z$ a convex cone and $g : S \to Z$ a function. Consider the (feasible) set $A := \{x \in S \mid g(x) \in -K\}$. The corresponding *weak vector equilibrium problem with (explicit) constraints* consists in:

$$(WVEP) \text{ find } x \in A \text{ such that } f(x, y) \notin -\mathrm{int}\, C \text{ for all } y \in A.$$

The aim of this paper is to characterize the existence of solutions of $(WVEP)$, which provide, in particular, sufficient conditions for the existence of weak C-saddle points for generalized multicriteria Nash games. Two different approaches are considered, which lead to different results. The first approach is based on a very general recent separation theorem due to Flores-Bazán and Mastroeni [27], while the second one relies on the classical separation theorem of Eidelheit [23]. We are not supposing that the cone K has a nonempty interior (as in many earlier papers), permitting in this way to cover a larger class of problems. Our results permit us to recover earlier results from the literature.

The paper is organized as follows. In Section 2 we recall some usual concepts and notations from the literature and provide some auxiliary results needed in the sequel. The main results, together with some examples aiming to illustrate and show the independence of them are provided in Section 3. Finally, in Section 4 some applications to generalized Nash games are presented.

2. Preliminaries

Let Y^* be the dual space of a real Hausdorff locally convex topological vector space Y, and let $C \subseteq Y$ be a convex cone. The dual cone of C is defined as

$$C^* := \{y^* \in Y^* \mid y^*(c) \geq 0 \text{ for all } c \in C\}.$$

The closure and the convex hull of a subset M of Y will be denoted by $\overline{M}$ and $\operatorname{co} M$, respectively. The cone generated by the set M is defined by $\operatorname{cone} M := \{ty \mid t \geq 0 \text{ and } y \in M\}$. Obviously, $\overline{\operatorname{cone}}\, M$ is the smallest closed cone containing the set M, and $\overline{\operatorname{cone}}\, M = \overline{\operatorname{cone} \overline{M}}$. Next, we recall the definitions of quasi interior and quasi-relative interior of a convex set.

Definition 2.1. Let M be a nonempty convex subset of Y. The sets

$$\operatorname{qi} M := \{y \in M \mid \overline{\operatorname{cone}}\,(M - y) = Y\}$$

and

$$\operatorname{qri} M := \{y \in M \mid \overline{\operatorname{cone}}\,(M - y) \text{ is a linear subspace of } Y\}$$

are called the *quasi interior* and the *quasi-relative interior* of M, respectively.

The relations below will be useful in the sequel.

Proposition 2.2. *Let M and N be nonempty convex subsets of Y. The following properties hold:*
(i) $\operatorname{qri}(U \times V) = \operatorname{qri} U \times \operatorname{qri} V$.
(ii) *If $0 \in M$, then $0 \in \operatorname{qri} M$ if and only if $0 \in \operatorname{qri}(\operatorname{cone} M)$.*

The proof of these properties can be found in [14] and [27], while for other useful properties of these notions, and other generalizations of the classical interior, respectively, we refer the reader to [15, 31, 36, 46, 54]. Let M be a nonempty and convex set of Y and $y_0 \in M$. The normal cone to M at y_0 is the set

$$N_M(y_0) := \{y^* \in Y^* \mid y^*(y - y_0) \leq 0 \text{ for all } y \in M\}.$$

This notion leads us to the next characterizations of the quasi interior and the quasi-relative interior, respectively.

Theorem 2.3. [16, 21] *Let M be a nonempty convex subset of Y and $y \in M$. Then $y \in \operatorname{qi} M$ if and only if $N_M(y) = \{0^*_{Y^*}\}$.*

Theorem 2.4. [14] *Let M be a nonempty convex subset of Y and $y \in M$. Then $y \in \operatorname{qri} M$ if and only if $N_M(y)$ is a linear subspace of Y^*.*

Remark 2.5. For any nonempty convex subset M of Y, we have $\operatorname{qi} M \subseteq \operatorname{qri} M$, and, if $\operatorname{qi} M \neq \emptyset$ then $\operatorname{qi} M = \operatorname{qri} M$. Moreover, if $\operatorname{int} M \neq \emptyset$, then $\operatorname{int} M = \operatorname{qi} M$ (see, for instance [14]).

We need the following well-known result (see, for instance [22, 35]).

Lemma 2.6. *If $c^* \in C^*$ is a nonzero functional, then $c^*(c) > 0$ for all $c \in \operatorname{int} C$.*

In what follows we recall some separation theorems involving the notions of quasi-relative interior and interior, respectively. The first result is recent and very general, while the second one is classical, widely used along the decades within convex analysis and optimization.

Theorem 2.7. [27] *Let M be a nonempty subset of Y. Then $0 \notin \operatorname{qri}(\operatorname{cone}(\operatorname{co} M))$ (or, equivalently $0 \notin \operatorname{qri}(\operatorname{co}(M \cup \{0\}))$ if and only if there exists $y^* \in Y^* \setminus \{0^*_{Y^*}\}$ such that*

$$y^*(y) \leq 0 \text{ for all } y \in M$$

and with strict inequality for some $\bar{y} \in M$.

The following separation theorem has been given by Eidelheit [23] in normed spaces.

Theorem 2.8. [33] *Let Y be a real topological vector space, and let S_1, S_2 be nonempty convex subsets of Y satisfying:*

(i) S_2 *has nonempty interior;*

(ii) $S_1 \cap (\operatorname{int} S_2) = \emptyset$.

Then, there exist $y^* \in Y^* \setminus \{0^*_{Y^*}\}$ and $r \in \mathbb{R}$ such that

$$y^*(y_1) \leq r \leq y^*(y_2) \text{ for all } y_1 \in S_1 \text{ and } y_2 \in S_2.$$

In optimization theory different generalized convexity concepts have been defined and used along the years. Here we need the following three.

Definition 2.9. Let S be a nonempty subset of a real vector space X. A function $F : S \to Y$ is said to be:

(i) *C-convex on S* if S is a convex set and, for all $x_1, x_2 \in S$ and $t \in [0, 1]$, it holds:
$$tF(x_1) + (1-t)F(x_2) - F(tx_1 + (1-t)x_2) \in C.$$

(ii) *C-convexlike* if, for all $x_1, x_2 \in S$ and all $\lambda \in [0, 1]$, there is $x \in S$ such that
$$\lambda F(x_1) + (1-\lambda)F(x_2) - F(x) \in C;$$

(iii) *C-subconvexlike* if there exists $c \in \operatorname{int} C$ such that, for all $x_1, x_2 \in S$, all $\lambda \in [0, 1]$ and all $\epsilon > 0$, there is $x \in S$ such that
$$\lambda F(x_1) + (1-\lambda)F(x_2) - F(x) + \epsilon c \in C.$$

Definition 2.9 (i) was introduced by Craven [19], (ii) is attributed to Fan [25], while (iii) originates from Jeyakumar [34].

The next lemma can be found in Borwein and Jeyakumar [13], and provides a characterization of C-convexity.

Lemma 2.10. *A function $F : S \to Y$ is C-convexlike (respectively C-subconvexlike) if and only if the set $F(S) + C$ (respectively $F(S) + \operatorname{int} C$) is convex.*

We conclude this section by providing two obvious properties needed in the sequel.

Proposition 2.11. *The following assertions hold.*

(i) *If $U \subseteq Y$ and $V \subseteq Z$ are convex sets such that $0_Y \in U$ and $0_Z \in V$, then*

$$\operatorname{cone}(U \times V) = \operatorname{cone} U \times \operatorname{cone} V.$$

(ii) *For any nonempty subset $U \subseteq Y$, it holds:*

$$U \cap (-\operatorname{int} C) = \emptyset \text{ if and only if } \overline{U} \cap (-\operatorname{int} C) = \emptyset.$$

3. Existence results for solutions of $(WVEP)$

We recall the following constraint qualification that we will use throughout this paper

$$(CQ) \;\; 0_Z \in \operatorname{qri}\Big(\operatorname{co}(g(S)) + K\Big).$$

(CQ) represents a natural generalization of the well-known Slater constraint qualification, namely $0 \in g(S) + \operatorname{int} K$, which plays an important role in obtaining the existence of Lagrange multipliers. Let $a \in A$ and define the function $E_a = (f(a, \cdot), g) : S \to Y \times Z$ by $E_a(x) = (f(a, x), g(x))$ for all $x \in S$. Furthermore, let $f(a, a) = 0$ for all $a \in A$. The latter is generally an important assumption on equilibrium problems and holds in almost all particular cases, although there are some exceptions.

The next statement allows us to obtain necessary and sufficient conditions in a general setting, by using a weak constraint qualification.

Theorem 3.1. *Let Y and Z be real Hausdorff locally convex topological vector spaces. If x_0 is a solution of $(WVEP)$ such that the following conditions are satisfied:*

(i) *the function E_{x_0} is $C \times K$-convexlike;*

(ii) $\operatorname{cone}[E_{x_0}(S) + C \times K] = \operatorname{cone}[(f(x_0, S) + C) \times (g(S) + K)]$;

(iii) *the constraint qualification (CQ) holds.*

then, there exist $y^* \in C^* \setminus \{0^*_{Y^*}\}$ and $z^* \in K^*$ such that

$$\min_{x \in S}\{y^*(f(x_0, x)) + z^*(g(x))\} = 0 \text{ and } z^*(g(x_0)) = 0.$$

If $x_0 \in A$ and there exist $y^* \in C^* \setminus \{0^*_{Y^*}\}$ and $z^* \in K^*$ such that

$$\min_{x \in S}\{y^*(f(x_0, x)) + z^*(g(x))\} = 0, \tag{4}$$

then x_0 is a solution of $(WVEP)$.

Proof. Necessity. Let $x_0 \in A$ be a solution of $(WVEP)$, i.e. $f(x_0, x) \notin -\text{int}\, C$ for all $x \in A$. Assert that

$$\text{cone}\,[E_{x_0}(S) + C \times K] \cap [(-\text{int}\, C) \times (-K)] = \emptyset. \tag{5}$$

If this fails, then there are $t \geq 0$, $x \in S$, $c \in C$ and $k \in K$ such that

$$t[(f(x_0, x), g(x)) + (c, k)] \in (-\text{int}\, C) \times (-K).$$

For $t = 0$, it is obvious that $(0_Y, 0_Z) \notin (-\text{int}\, C) \times (-K)$. Whenever $t > 0$ we get

$$f(x_0, x) + c \in -\text{int}\, C \text{ and } g(x) \in -K.$$

Hence, $x \in A$ and $f(x_0, x) \in -\text{int}\, C$, which is a contradiction to the fact that x_0 is a solution. By, (5), hypothesis (ii) and Proposition 2.11 (ii), we obtain

$$[\overline{\text{cone}}\,(f(x_0, S) + C) \times \text{cone}\,(g(S) + K)] \cap [(-\text{int}\, C) \times (-K)] = \emptyset. \tag{6}$$

Now, we show that $0_Y \notin \text{qi}\,(f(x_0, S) + C)$. If this fails, then

$$\overline{\text{cone}}\,(f(x_0, S) + C) = Y.$$

Therefore,

$$[\overline{\text{cone}}\,(f(x_0, S) + C) \times \text{cone}\,(g(S) + K)] \cap [(-\text{int}\, C) \times (-K)] \neq \emptyset,$$

which is a contradiction to (6).

Thus, by $0_Y \notin \text{qri}\,(f(x_0, S) + C)$, (CQ) and Proposition 2.2 (i), we deduce that

$$(0_Y, 0_Z) \notin \text{qri}\,(f(x_0, S) + C) \times \text{qri}\,(g(S) + K).$$

Due to Proposition 2.2 (ii) we have

$$(0_Y, 0_Z) \notin \text{qri}\,(\text{cone}(f(x_0, S) + C)) \times \text{qri}\,(\text{cone}\,(g(S) + K)).$$

By the properties of the quasi-relative interior and hypothesis (ii), this is equivalent to

$$(0_Y, 0_Z) \notin \text{qri}\,(\text{cone}\,(E_{x_0}(S) + C \times K)) = \text{qri}\,(E_{x_0}(S) + C \times K).$$

Now, we can apply Theorem 2.7, which provides the existence of a nonzero functional $(y^*, z^*) \in Y^* \times Z^*$ such that

$$y^*(y) + z^*(z) \geq 0 \text{ for all } (y, z) \in E_{x_0}(S) + C \times K, \tag{7}$$

and

$$y^*(y) + z^*(z) > 0 \text{ for some } (\bar{y}, \bar{z}) \in E_{x_0}(S) + C \times K. \tag{8}$$

By taking any pair $(y, z) \in E_{x_0}(S) + C \times K$ we have $(y + tc, z)$ and $(y, z + tk) \in E_{x_0}(S) + C \times K$, for all $c \in C$, $k \in K$ and $t > 0$. Applying (7) to these pairs we obtain

$$\frac{y^*(y) + z^*(z)}{t} + y^*(c) \geq 0 \text{ for any } c \in C \text{ and any } t > 0,$$

respectively,

$$\frac{y^*(y) + z^*(z)}{t} + z^*(k) \geq 0 \text{ for any } k \in K \text{ and any } t > 0.$$

Taking the limit as t goes to ∞ we obtain that $y^* \in C^*$ and $z^* \in K^*$, respectively.

Assert that $y^* \neq 0^*_{Y^*}$. If this is not true, then $z^* \neq 0^*_{Z^*}$, while inequalities (7) and (8) become

$$z^*(z) \geq 0 \text{ for all } z \in g(S) + K, \tag{9}$$

and,

$$z^*(\bar{z}) > 0 \text{ for some } \bar{z} \in g(S) + K, \tag{10}$$

respectively. This implies that

$$-z^* \in N_{g(S)+K}(0_Z),$$

and by the constraint qualification and Theorem 2.4, $N_{g(S)+K}(0_Z)$ is a linear subspace of Z^*. Consequently, $z^*(z) \leq 0$ for all $z \in g(S) + K$, contradicting relation (10).

Since $g(x_0) \in g(S) + K$, by (9) we have $z^*(g(x_0)) \geq 0$. On the other hand, by $g(x_0) \in -K$ and $z^* \in K^*$ we get $z^*(g(x_0)) \leq 0$, whence we deduce that $z^*(g(x_0)) = 0$. For every $x \in S$, $c \in C$, $k \in K$ and $t > 0$ we have

$$(f(x_0, x), g(x)) + t(c, k) \in E_{x_0}(S) + C \times K,$$

and by (7) it follows that

$$y^*(f(x_0, x)) + z^*(g(x)) + ty^*(c) + tz^*(z) \geq 0.$$

Taking the limit as t goes to 0 we get

$$0 = y^*(f(x_0, x_0)) + z^*(g(x_0)) = \min_{x \in S}\{y^*(f(x_0, x)) + z^*(g(x))\}.$$

Sufficiency. Suppose that $x_0 \in A$ is not a solution of $(WVEP)$, i.e. there exists $b_0 \in A$ such that

$$f(x_0, b_0) \in -\operatorname{int} C.$$

By the hypothesis we have the existence of the functional $(y^*, z^*) \in (C^* \setminus \{0^*_{Y^*}\}) \times K^*$ such that (4) holds. Then, by Lemma 2.6

$$0 = \min_{x \in S}\{y^*(f(x_0, x)) + z^*(g(x))\} \leq y^*(f(x_0, b_0)) + z^*(g(b_0)) < 0,$$

which is a contradiction. Hence, $x_0 \in A$ is a solution of $(WVEP)$, and this completes the proof. □

With respect to assumption (ii) of Theorem 3.1 let us observe two facts: it is not trivial (in the sense that it does not hold for all functions) on one hand, and does not contradict the other hypotheses of this theorem, on the other hand. The next two examples are meant to show these claims.

Example 3.2. Take $S = [0,2]$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $Z = \mathbb{R}$, $K = \mathbb{R}_+$, $f(x,y) = y-x$ and $g(x) = -x$. It is easy to see that the set of all feasible solutions is $[0,2]$, and that 0 is the solution of the problem. Therefore,

$$\operatorname{cone}[E_0(S) + C \times K] = \{(x+c, -x+r) : x, c, r \in [0,\infty)\},$$

while

$$\operatorname{cone}[(f(0,S) + C) \times (g(S) + K)] = \mathbb{R}_+ \times \mathbb{R}.$$

Example 3.3. Take $S = [-1,1]$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $Z = \mathbb{R}$, $K = \mathbb{R}_+$, $f(x,y) = y - x$ and $g(x) = x$. It is easy to see that the set of all feasible solutions is $[-1,0]$ and that -1 is the solution of this problem. It is easy to check that

$$\operatorname{cone}[E_{-1}(S) + C \times K] = \operatorname{cone}[(f(-1,S) + C) \times (g(S) + K)] = \mathbb{R}_+ \times \mathbb{R}.$$

Thus, assumption (ii) of Theorem 3.1 holds. Moreover, this example fulfils the other two assumptions of this theorem.

The next theorem allows us to obtain the same conclusions by using a different approach.

Theorem 3.4. *Let Y and Z be real topological vector spaces. If x_0 is a solution of $(WVEP)$ such that the following conditions are satisfied:*

(i) the function E_{x_0} is $C \times K$-convexlike;

(ii) the set $E_{x_0}(S) + C \times K$ has nonempty interior;

(iii) the constraint qualification $0_Z \in \operatorname{int}(g(S) + K)$ holds;

then there exist $y^* \in C^* \setminus \{0^*_{Y^*}\}$ and $z^* \in K^*$ such that

$$\min_{x \in S}\{y^*(f(x_0,x)) + z^*(g(x))\} = 0 \text{ and } z^*(g(x_0)) = 0.$$

If $x_0 \in A$ and there exist $y^* \in C^* \setminus \{0^*_{Y^*}\}$ and $z^* \in K^*$ such that

$$\min_{x \in S}\{y^*(f(x_0,x)) + z^*(g(x))\} = 0\,,$$

then x_0 is a solution of $(WVEP)$.

Proof. Necessity. Let $x_0 \in A$ be a solution of $(WVEP)$, i.e.

$$f(x_0,x) \notin -\operatorname{int} C \text{ for all } x \in A.$$

We will prove that the set $M = E_{x_0}(S) + \operatorname{int} C \times K$ is convex and has nonempty interior. Let $(y_1,z_1), (y_2,z_2) \in M$ and $t \in [0,1]$. Hence, there exist $x_1, x_2 \in S$, $c_1, c_2 \in \operatorname{int} C$ and $k_1, k_2 \in K$ such that

$$t(y_1, z_1) + (1-t)(y_2, z_2) =$$
$$= t(f(x_0, x_1) + c_1, g(x_1) + k_1) + (1-t)(f(x_0, x_2) + c_2, g(x_2) + k_2)$$
$$= t(f(x_0, x_1) + \frac{c_1}{2}, g(x_1) + \frac{k_1}{2}) + (1-t)(f(x_0, x_2) + \frac{c_2}{2}, g(x_2) + \frac{k_2}{2}) +$$
$$+ t(\frac{c_1}{2}, \frac{k_1}{2}) + (1-t)(\frac{c_2}{2}, \frac{k_2}{2}).$$

By assumption (i) we get:

$$t(y_1, z_1) + (1-t)(y_2, z_2) \in E_{x_0}(S) + C \times K + \operatorname{int} C \times K \subseteq M.$$

Let $(y, z) \in \operatorname{int}(E_{x_0}(S) + C \times K)$. Thus, there exist $x \in S$, $c \in C$, $k \in K$, U, V open neighbourhoods of zero such that $y = f(x_0, x) + c$, $z = g(x) + k$ and

$$(f(x_0, x) + c + U) \times (g(x) + k + V) \subseteq E_{x_0}(S) + C \times K.$$

Moreover, this implies that

$$[f(x_0, x) + c + U \cap (\operatorname{int} C)] \times (g(x) + k + V) \subseteq E_{x_0}(S) + C \times K.$$

So, for any $u \in U \cap (\operatorname{int} C)$, $v \in V$ we have the existence of $s \in S$, $c_1 \in C$ and $k_1 \in K$ such that

$$f(x_0, x) + c + 2u = f(x_0, s) + c_1 + u \in f(x_0, s) + \operatorname{int} C, \tag{11}$$

and,

$$g(x) + k + v = g(s) + k_1 \in g(S) + K. \tag{12}$$

By relations (11) and (12) we obtain

$$(f(x_0, x) + c + 2u, g(x) + k + v) \in M.$$

Since u and v were arbitrarily chosen, by the above relation, and the openness of $U \cap \operatorname{int} C$, we conclude that M has nonempty interior.

Next, we show that $(0_Y, 0_Z) \notin M$. If this fails, then there is $x \in S$, such that $f(x_0, x) \in -\operatorname{int} C$ and $g(x) \in -K$, and this contradicts the fact that x_0 is a solution.

By Eidelheit's separation theorem (Theorem 2.8) we obtain the existence of a nonzero functional $(y^*, z^*) \in Y^* \times Z^*$ such that

$$y^*(f(x_0, x) + c) + z^*(g(x) + k) \geq 0 \text{ for all } x \in S, c \in \operatorname{int} C \text{ and } k \in K. \tag{13}$$

For any $t > 0$, $(f(x_0, x) + tc, g(x)) \in M$ for all $c \in \operatorname{int} C$, and by (13) we get $y^* \in C^*$. Using a similar argument we deduce that $z^* \in K^*$.

Suppose that $y^* = 0^*_{Y^*}$. Thus, $z^* \neq 0^*_{Z^*}$ and (13) becomes

$$z^*(g(x) + k) \geq 0, \text{ for all } x \in S \text{ and } k \in K. \tag{14}$$

Assumption (iii) assures the existence of an open neighbourhood U of zero such that $U \subseteq g(S) + K$. Let $z \in Z$ be arbitrarily chosen. Then there exists $t > 0$ such that $tz \in U$. Hence, by (14), $z^*(tz) \geq 0$, which is equivalent to $z^*(z) \geq 0$. Since z was arbitrarily chosen, we obtain $z^* = 0^*_{Z^*}$, which is a contradiction.

For every $x \in S$, $c \in \operatorname{int} C$, $k \in K$ and $t > 0$ we have

$$(f(x_0, x), g(x)) + t(c, k) \in M,$$

and by (7) it follows that

$$y^*(f(x_0, x)) + z^*(g(x)) + ty^*(c) + tz^*(z) \geq 0.$$

Taking the limit as t goes to 0 we get

$$0 = y^*(f(x_0, x_0)) + z^*(g(x_0)) = \min_{x \in S}\{y^*(f(x_0, x)) + z^*(g(x))\}.$$

The sufficient part follows the same idea as in the proof of Theorem 3.1. □

As we mentioned above, the approaches used in Theorem 3.1 and Theorem 3.4 are different, and they allow to obtain necessary and sufficient conditions under divers hypotheses. For instance, the functions from Example 3.2 satisfy all assumptions of Theorem 3.4, while the conditions of Theorem 3.1 fail. Example 3.3 meets the assumptions of both theorems. The next example illustrates a case where all assumptions of Theorem 3.1 are satisfied, while Theorem 3.4 cannot be applied.

Example 3.5. Let $S = [-1, 1]$, $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $Z = \mathbb{R}^2$, $K = \{(k, k) : k \geq 0\}$, $f(x, y) = y - x$ and $g(x) = (x, x)$. In this framework the set of feasible solutions is $[-1, 0]$, while -1 is the solution of the problem. Also,

$$\operatorname{cone}[E_{-1}(S)+C\times K] = \operatorname{cone}[(f(-1, S)+C)\times(g(S)+K)] = \mathbb{R}_+\times\{(r, r) : r \in \mathbb{R}\}.$$

Thus, assumption *(ii)* of Theorem 3.1 holds, and the other two assumptions of this theorem are also satisfied. On the other hand, Theorem 3.4 cannot be applied because $0_Z \notin \operatorname{int}(g(S) + K) = \emptyset$.

The next statement deals with a stronger constraint qualification.

Corollary 3.6. *Let Y and Z be real topological vector spaces. If x_0 is a solution of $(WVEP)$ such that the following conditions are satisfied:*

(i) the function E_{x_0} is $C \times K$-convexlike;

(ii) the set $E_{x_0}(S) + C \times K$ has nonempty interior;

(iii) the constraint qualification $0_Z \in \operatorname{int}(\operatorname{co} g(S))$ holds;

then there exist $y^* \in C^* \setminus \{0^*_{Y^*}\}$ and $z^* \in K^*$ such that

$$\min_{x \in S}\{y^*(f(x_0, x)) + z^*(g(x))\} = 0 \text{ and } z^*(g(x_0)) = 0.$$

If $x_0 \in A$ and there exist $y^* \in C^* \setminus \{0^*_{Y^*}\}$ and $z^* \in K^*$ such that

$$\min_{x \in S}\{y^*(f(x_0, x)) + z^*(g(x))\} = 0,$$

then x_0 is a solution of $(WVEP)$.

Proof. In order to apply Theorem 3.4 we still have to show that $0_Z \in \operatorname{int}(g(S)+K)$. Since $0_Z \in K$ and $0_Z \in \operatorname{int}(\operatorname{co} g(S))$, we get $0_Z \in \operatorname{int}(\operatorname{co} g(S)+K)$ (see [50]). Due to assumption (i), the set $g(S)+K$ is convex, and hence $0_Z \in \operatorname{int}(g(S)+K)$. □

The next result is established in [29, Theorem 3.1] and represents a particular case of Theorem 3.4.

Corollary 3.7. *Let Y and Z be real topological vector spaces, and let the following conditions be satisfied:*

(i) for each $x \in S$, the function $f(x,\cdot)$ is C-convex;

(ii) g is K-convex;

(iii) the Slater constraint qualification, $0_Z \in g(S)+\operatorname{int} K$, holds.

A point $x_0 \in A$ is solution of $(WVEP)$ if and only if there exist $y^* \in C^* \setminus \{0^*_{Y^*}\}$ and $z^* \in K^*$ such that

$$\min_{x\in S}\{y^*(f(x_0,x))+z^*(g(x))\}=0 \text{ and } z^*(g(x_0))=0.$$

Proof. It is an easy exercise to show that assumptions (i) and (ii) imply that the function E_{x_0} is $C\times K$-convex, and hence it is $C\times K$-convexlike. The interior assumptions on the cones C and K, provide that assumption (ii) of Theorem 3.4 is satisfied. Since every K-convexlike function is also K-subconvexlike, we obtain that the set $g(S)+\operatorname{int} K$ is convex. Thus, by [50], the constraint qualification can be written as

$$0_Z \in \operatorname{co} g(S)+\operatorname{int} K=\operatorname{int}(g(S)+K),$$

therefore, assumption (iii) of Theorem 3.4 is verified, and the conclusion follows now from this theorem. □

The necessary conditions we obtained above are not comparable with those established in [28], [30] and [52], while our sufficient conditions are milder than the ones existent in [30, Theorem 3.2], and [52, Theorem 3.4], respectively. In that results, the authors provide some conditions which imply our assumptions from the sufficiency part of Theorems 3.1 and 3.4.

4. Application: the generalized Nash equilibrium problem

The classical *Nash equlibrium problem* (abbreviated NEP) related to n-person noncooperative games was formally introduced by Nash [41, 42] and has antecedents in the works of von Neumann [43] and von Neumann and Morgenstern [44] on zero-sum two-person games. Extending the NEP in such a way that the strategy set of each player may depend on the strategies of the other players was necessary for a better description of practical problems arising from economics. This problem is widespreadly known as *generalized Nash equilibrium problem* (shortly GNEP) and was introduced by Debreu [20] (termed *social equilibrium*).

The latter can be seen as a preparation of the famous paper by Arrow and Debreu [3], where the authors studied economic equilibria. They termed GNEP as *abstract economy*. For an exhaustive survey on GNEP see Facchinei and Kanzow [24], where three specific examples have been described: the *Arrow-Debreu abstract economy model*, *the power control problem in telecommunication* and a GNEP arising from *Kyoto protocol*.

In this section we deduce an existence result for a two-person vector-valued GNEP by means of Theorem 3.1. To do this, let us recall the two-player noncooperative vector valued-game as mentioned in Section 1. More precisely, let X_1 and X_2 be nonempty sets and suppose that the strategy pair allowed for the two players is restricted to a subset $A \subseteq X_1 \times X_2$.

In order to introduce our GNEP, let $S := X_1 \times X_2$ and $A \subseteq S$ as before, i.e., $A = \{x \in S \mid g(x) \in -K\}$, where $g : S \to Z$ is a given function. Fix an arbitrary strategy pair $(x_1, x_2) \in X_1 \times X_2$ and consider the (strategy) sets $A_1(x_2) := \{x_1 \in X_1 \mid (x_1, x_2) \in A\}$ and $A_2(x_1) := \{x_2 \in X_2 \mid (x_1, x_2) \in A\}$, respectively. Let $F : X_1 \times X_2 \to Y$ be the payoff function of player 1 and consider the associated zero-sum game. For $U_i \subseteq X_i$ $(i = 1, 2)$, let us consider the sets:

$$F(U_1, x_2) = \{F(x_1, x_2) \mid \ x_1 \in U_1\} \text{ and } F(x_1, U_2) = \{F(x_1, x_2) \mid \ x_2 \in U_2\}.$$

In a similar way as in Section 1 we introduce now the concept of *restricted weak C-saddle point.*

Definition 4.1. A pair $(\bar{x}_1, \bar{x}_2) \in A$ is said to be a *restricted weak C-saddle point* for F if

$$F(\bar{x}_1, \bar{x}_2) \in \operatorname{Max}_{\mathrm{w}}(F(A_1(\bar{x}_2), \bar{x}_2), C) \cap \operatorname{Min}_{\mathrm{w}}(F(\bar{x}_1, A_2(\bar{x}_1)), C).$$

By means of Theorem 3.1 we obtain the following result.

Corollary 4.2. *Let $(\bar{x}_1, \bar{x}_2) \in A$. If there exists $(y^*, z^*) \in (C^* \setminus \{0^*_{Y^*}\}) \times K^*$ such that*

$$\min_{(x_1,x_2)\in X_1\times X_2}\{y^*(F(\bar{x}_1, x_2)) - y^*(F(x_1, \bar{x}_2)) + z^*(g(x_1, x_2))\} = 0,$$

then $(\bar{x}_1, \bar{x}_2)$ is a restricted weak C-saddle point of F.

Proof. Take $S = X_1 \times X_2$, and define the vector-valued function $f : S \times S \to Y$ as:

$$f(x, y) = F(x_1, y_2) - F(y_1, x_2) \text{ for all } x = (x_1, x_2),\ y = (y_1, y_2) \in S.$$

By the sufficiency of Theorem 3.1 we have that the feasible point $(\bar{x}_1, \bar{x}_2) \in A$ is a solution of problem $(WVEP)$, with f taken as above. Thus

$$F(\bar{x}_1, y_2) - F(y_1, \bar{x}_2) \notin -\operatorname{int} C \text{ for all } y \in A.$$

Set $y = (\bar{x}_1, y_2)$ with $y_2 \in A_2(\bar{x}_1)$, and $y = (y_1, \bar{x}_2)$ with $y_1 \in A_1(\bar{x}_2)$ in the above relation, and get

$$F(\bar{x}_1, y_2) \notin F(\bar{x}_1, \bar{x}_2) - \operatorname{int} C \text{ for all } y_2 \in A_2(\bar{x}_1), \tag{15}$$

and respectively,

$$F(y_1, \bar{x}_2) \notin F(\bar{x}_1, \bar{x}_2) + \operatorname{int} C \text{ for all } y_1 \in A_1(\bar{x}_2). \tag{16}$$

By (15) and (16) we deduce that

$$F(\bar{x}_1, \bar{x}_2) \in \operatorname{Max_w}(F(A_1(\bar{x}_2), \bar{x}_2), C) \cap \operatorname{Min_w}(F(\bar{x}_1, A_2(\bar{x}_1)), C),$$

which means that $(\bar{x}_1, \bar{x}_2)$ is a restricted weak C-saddle point, and the proof is completed. □

Remark 4.3. Note that Corollary 4.2 does not guarantee that $(\bar{x}_1, \bar{x}_2)$ is (an unrestricted) weak C-saddle point of F. Indeed, let $X_1 = X_2 = Y = Z = \mathbb{R}$, $C = K = \mathbb{R}_+$, $F(x_1, x_2) = x_2 - x_1$ and $g(x_1, x_2) = -\min\{x_1, x_2\}$. Then $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$ and take $(\bar{x}_1, \bar{x}_2) = (0, 0)$. For $y^* = 1$ and $z^* = 2$ it is obvious that

$$F(0, x_2) - F(x_1, 0) + 2g(x_1, x_2) = x_2 + x_1 - 2\min\{x_1, x_2\} \geq 0, \quad \forall x_1, x_2 \in \mathbb{R},$$

while $F(0, 0) - F(0, 0) - 2\min\{0, 0\} = 0$, thus the assumption of Corollary 4.2 is satisfied. However, it is easy to see that F does not have any saddle point (on $\mathbb{R} \times \mathbb{R}$), while its restricted saddle point is $(0, 0)$.

A natural question which comes in discussion is if any (restricted) weak C-saddle point of F is a solution of $(WVEP)$. The answer is negative, as the following examples illustrates.

Example 4.4. Let $X_1 = X_2 = [-1, 1]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and let $F : [-1, 1] \times [-1, 1] \to \mathbb{R}^2$ be defined by:

$$F(x_1, x_2) := \begin{cases} (x_1, x_2) & \text{if } x_1 \geq 0 \text{ and } x_2 \leq 0 \text{ or, } x_1 \leq 0 \text{ and } x_2 \geq 0 \\ (0, 0) & \text{otherwise.} \end{cases}$$

It is easy to check that $(0, 0)$ is a weak $\mathbb{R}^2_+$-saddle point of F. In order to verify whether the point $\bar{x} := (0, 0)$ is a solution for $(WVEP)$, let the set $S := [-1, 1] \times [-1, 1]$. So, we have to check whether

$$f(\bar{x}, y) = F(0, v) - F(u, 0) \notin -\operatorname{int} \mathbb{R}^2_+ \text{ for all } y := (u, v) \in S.$$

Taking $y := (1, -1) \in S$, we have

$$f(\bar{x}, y) = F(0, -1) - F(1, 0) = (-1, -1) \in -\operatorname{int} \mathbb{R}^2_+.$$

Hence, $(0, 0)$ is a weak $\mathbb{R}^2_+$-saddle point for F, but it is not a solution of $(WVEP)$.

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