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Convergence to Consensus for a Hegselmann-Krause-Type Model with Distributed Time Delay

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We study a Hegselmann-Krause opinion formation model with distributed time delay and positive influence functions. Through a Lyapunov functional approach, we provide a consensus result under a smallness assumption on the initial delay. Furthermore, we analyze a transport equation, obtained as mean-field limit of the particle one. We prove global existence and uniqueness of the measure-valued solution for the delayed transport equation and its convergence to consensus under a smallness assumption on the delay, using a priori estimates which are uniform with respect to the number of agents.

Keywords: Hegselmann-Krause model, opinion formation, delay, consensus.

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1. Introduction

In recent years, many researchers have focused their attention to multi-agent systems. One aspect of these models is the natural self-aggregation, which has been studied in different fields such as biology [1], robotics [12], sociology, economics [19], computer science, control theory [21, 22, 28], social sciences [26, 27] and many other areas. In these last decades a large number of mathematical models has been proposed to study the consensus behavior. First order models, such as the Hegselmann-Krause model [16], have been proposed to study opinion formation. We mention also [17], in which bounded confidence yields the so-called clustering phenomenon. Second order models, in particular Cucker-Smale model [11], have been studied by many authors [13, 14, 23], in order to describe, for example, flocking of birds, swarming of bacteria, or schooling of fishes.

In addition, it is reasonable to introduce a delay in the model as a reaction time or simply as a time to receive the information from outside, in order to let the dynamics more realistic. For first order models, we refer to [5, 8, 10], while for delayed Cucker-Smale-type models we mention [6, 7, 15, 25]. In particular, in very recent papers (see [9, 18, 24]), the authors analyzed modified Cucker-Smale models with distributed time delay, thanks to which agents are influenced by the other ones on a time interval $[t-\tau(t), t]$.

Furthermore, delayed and non-delayed kinetic and transport equations associated to the particle multi-agent systems have been studied in [2, 3, 4, 6, 8, 9].

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In this paper, we are interested in the evolution of opinions among N agents, with $N \in \mathbb{N}$. Let $x_i \in \mathbb{R}^d$ be the opinion of the i-th agent, for any $i = 1, ..., N$. Then, the dynamics is given by the following Hegselmann-Krause-type model:

$$\begin{cases} \dfrac{dx_i(t)}{dt} = \dfrac{1}{Nh(t)} \displaystyle\sum_{j \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ij}(t;s)(x_j(s) - x_i(t))ds, \quad t > 0 \\ x_i(s) = x_{i,0}(s), \quad s \in [-\tau(0), 0], \end{cases} \tag{1}$$

where $\tau : [0, +\infty) \to (0, +\infty)$ is the time delay. It is a function in $W^{1,\infty}([0,T])$, for any $T > 0$ and we assume that $\tau(t) \geq \tau_*$ for some $\tau_* > 0$, and

$$\tau'(t) \leq 0, \qquad \forall\, t \geq 0. \tag{2}$$

This implies that $\tau(t) \leq \tau(0)$, for any $t \geq 0$. We stress the fact that constant delays $\tau(t) \equiv \bar{\tau} > 0$ are allowed. Motivated by [11, 17, 20], we take the communication rates $a_{ij}(t;s)$ either of the form

$$a_{ij}(t;s) = \psi(|x_j(s) - x_i(t)|), \tag{3}$$

for any $i, j \in \{1, ..., N\}$, where $\psi : [0, +\infty) \to (0, +\infty)$ is a non-increasing function, or

$$a_{ij}(t;s) = \frac{N\psi(|x_j(s) - x_i(t)|)}{\sum_{k=1}^{N} \psi(|x_k(s) - x_i(t)|)}, \quad \forall\, t \geq 0. \tag{4}$$

Without loss of generality, we can assume that $\psi(0) = 1$. We notice that in both cases we have that

$$\frac{1}{N} \sum_{j=1}^{N} a_{ij}(t;s) \leq 1, \quad \forall\, t \geq 0. \tag{5}$$

Moreover, $\alpha : [0, \tau(0)] \to [0, +\infty)$ is a weight function which satisfies

$$\underline{A} := \int_0^{\tau_*} \alpha(s)ds > 0.$$

Furthermore, we define for any $t \geq 0$: $\quad h(t) := \displaystyle\int_0^{\tau(t)} \alpha(s)ds. \qquad (6)$

Remark 1.1. We notice that if $\alpha(s) = \delta_{\tau(t)}(s)$, then system (1) can be rewritten as

$$\frac{dx_i(t)}{dt} = \frac{1}{N} \sum_{j \neq i} a_{ij}(t; t-\tau(t))(x_j(t-\tau(t)) - x_i(t)), \quad x_i(s) = x_{i,0}(s), \quad s \in [-\tau(0), 0],$$

which is already analyzed in [8]. □

We define, now, the following quantity: $d_X(t) := \max_{1 \leq i,j \leq N} |x_i(t) - x_j(t)|$.

Definition 1.2. A solution $\{x_i(t)\}_{i=1,...,N}$ to (1) converges to *consensus* if

$$\lim_{t \to +\infty} d_X(t) = 0.$$

We will prove the following consensus result.

Theorem 1.3. *Let* $\{x_i(t)\}_{i=1}^N$ *be the solution to* (1). *Suppose that*

$$\left(e^{\tau(0)} - 1\right) h(0) \leq \frac{A\psi(2R)^3}{2 + \psi(2R)^2}. \tag{7}$$

Then, there exist two positive constants C, K *such that*

$$d_X(t) \leq Ce^{-Kt}, \qquad \forall\ t \geq 0. \tag{8}$$

Remark 1.4. Here, we stress the fact that the quantity

$$\left(e^{\tau(0)} - 1\right) \int_0^{\tau(0)} \alpha(s) ds$$

is increasing with respect to $\tau(0)$. Then, (7) represents a smallness assumption on $\tau(0)$. Moreover, the right-hand side of (7) is increasing with respect to $\psi(2R)$. Therefore, we observe that if R is small enough and/or the decay of ψ is not too fast, then the quantity

$$\frac{\psi(2R)^3}{2 + \psi(2R)^2}$$

becomes large and consensus occurs for more values of $\tau(0)$. □

The transport equation associated to (1) can be obtained as mean-field limit of the particle system (1) when $N \to +\infty$. Let $\mathcal{M}(\mathbb{R}^d)$ be the set of probability measures on the space $\mathbb{R}^d$. Then, the transport equation associated to (1) reads as

$$\begin{cases} \partial_t \mu_t + \operatorname{div} \left(\dfrac{1}{h(t)} \displaystyle\int_{t-\tau(t)}^t \alpha(t-s) F[\mu_s] ds\ \mu_t \right) = 0, \\ x \in \mathbb{R}^d, \quad t \geq 0, \quad \mu_s = g_s, \quad s \in [-\tau(0), 0], \end{cases} \tag{9}$$

where F is given by either

$$F[\mu_s](x) = \int_{\mathbb{R}^d} \psi(|x-y|)(y-x) d\mu_s(y), \tag{10}$$

or

$$F[\mu_s](x) = \frac{\int_{\mathbb{R}^d} \psi(|x-y|)(y-x) d\mu_s(y)}{\int_{\mathbb{R}^d} \psi(|x-y|) d\mu_s(y)}, \tag{11}$$

according to the choice of (3) and (4).

Furthermore, we take $g_s \in \mathcal{C}([-\tau(0), 0]; \mathcal{M}(\mathbb{R}^d))$.

Definition 1.5. Let $T > 0$. We say that $\mu_t \in \mathcal{C}([0,T); \mathcal{M}(\mathbb{R}^d))$ is a *weak solution* to (9) on the time interval $[0, T)$ if for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times [0,T))$ we have

$$\int_0^T \int_{\mathbb{R}^d} \left(\partial_t \varphi + \frac{1}{h(t)} \int_{t-\tau(t)}^t \alpha(t-s) F[\mu_s](x) ds \cdot \nabla_x \varphi \right) d\mu_t(x) dt \\ + \int_{\mathbb{R}^d} \varphi(x, 0) dg_0(x) = 0, \tag{12}$$

where $F[\mu_s]$ is defined as in (10) or (11).

We will prove the following theorem.

Theorem 1.6. *Let $\mu_t \in \mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d))$ be a weak solution to* (9)*, with compactly supported initial measure $g_s \in \mathcal{C}([-\tau(0),0];\mathcal{P}_1(\mathbb{R}^d))$ and let F as in* (10) *or* (11)*.*

Suppose that
$$\left(e^{\tau(0)}-1\right)h(0) \le \frac{\underline{A}\psi(2R)^3}{2+\psi(2R)^2}. \tag{13}$$

Then, there exists a constant $C>0$ independent of t such that
$$d_X(\mu_t) \le \left(\max_{s\in[-\tau(0),0]} d_X(g_s)\right) e^{-Ct}, \tag{14}$$

for all $t \ge 0$, where $d_X(\mu_t) :=$ diam supp μ_t.

The paper is organized as follows. In Section 2 we study the consensus behavior of solution to (1), after assuming an upper-bound on the initial delay $\tau(0)$, namely we will prove Theorem 1.3. In Section 3 we focus our attention on system (9) and we study the existence and uniqueness of the solution and its convergence to consensus.

2. Consensus results

We notice that d_X may be not differentiable at some $t \ge 0$. Then, we will use a suitable generalized derivative. We define the upper Dini derivative of a continuous function F as follows:
$$D^+F(t) := \limsup_{h\to 0^+} \frac{F(t+h)-F(t)}{h}.$$

Before studying the convergence to consensus of the solution to (1), we state the following lemma.

Lemma 2.1. *Let $\{x_i(t)\}_{i=1}^N$ be a solution to* (1)*. Suppose that the initial functions $x_{i,0}(s)$ are continuous on the time interval $[-\tau(0),0]$ for all $i=1,\dots,N$. Set*
$$R := \max_{s\in[-\tau(0),0]} \max_{1\le i\le N} |x_i(s)|.$$

Then, for all $t \ge 0$,
$$\max_{1\le i\le N} |x_i(t)| \le R. \tag{15}$$

Proof. Let $\epsilon > 0$ and define $R_\epsilon := R+\epsilon$. Set
$$S^\epsilon = \left\{ t>0 \ : \ \max_{1\le i\le N} |x_i(s)| < R_\epsilon, \quad \forall\ s\in[0,t) \right\}.$$

By continuity, $S^\epsilon \ne \emptyset$. Denote $T^\epsilon := \sup S^\epsilon$ and assume by contradiction $T^\epsilon < +\infty$. Then,
$$\lim_{t\to T^{\epsilon-}} \max_{1\le i\le N} |x_i(t)| = R^\epsilon. \tag{16}$$

On the other hand, we have that for any $t \le T^\epsilon$,
$$\begin{aligned}\frac{1}{2}D^+|x_i(t)|^2 &\le \left\langle x_i(t), \frac{dx_i(t)}{dt}\right\rangle \\ &= \left\langle x_i(t), \frac{1}{Nh(t)}\sum_{j\ne i}\int_{t-\tau(t)}^t \alpha(t-s)a_{ij}(t;s)(x_j(s)-x_i(t))ds \right\rangle\end{aligned}$$

$$= \frac{1}{Nh(t)} \sum_{j \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ij}(t;s) \langle x_i(t), x_j(s) - x_i(t) \rangle ds$$

$$= \frac{1}{Nh(t)} \sum_{j \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ij}(t;s) \left(\langle x_i(t), x_j(s) \rangle - |x_i(t)|^2 \right) ds$$

$$\leq \frac{1}{Nh(t)} \sum_{j \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ij}(t;s) |x_i(t)| \left(|x_j(s)| - |x_i(t)| \right) ds.$$

Using (5) and the fact that $t \leq T^\epsilon$ yield

$$\frac{1}{2} D^+ |x_i(t)|^2 \leq \frac{1}{h(t)} \int_{t-\tau(t)}^{t} \alpha(t-s) ds \, |x_i(t)| (R^\epsilon - |x_i(t)|) = |x_i(t)| (R^\epsilon - |x_i(t)|).$$

Hence, we have that $D^+ |x_i(t)| \leq R^\epsilon - |x_i(t)|$.

By Gronwall inequality, we obtain $|x_i(t)| \leq e^{-t} \left(|x_i(0)| - R^\epsilon \right) + R^\epsilon < R^\epsilon$.

Therefore, $\lim_{t \to T^{\epsilon-}} \max_{1 \leq i \leq N} |x_i(t)| < R^\epsilon$, which is in contradiction with (16). Moreover, since ϵ is arbitary, we obtain (15). □

Remark 2.2. Thanks to the previous lemma, we can find a control on $a_{ij}(t;s)$ from below. Indeed, for any $i, j \in \{1, \dots, N\}$, for any $t \geq 0$ and $s \in [t - \tau(t), t]$, we have

$$|x_j(s) - x_i(t)| \leq |x_j(s)| + |x_i(t)| \leq 2R.$$

Hence, from (3) and (4), we can deduce that

$$a_{ij}(t;s) \geq \psi(2R), \quad \forall \, t \geq 0. \qquad \square \tag{17}$$

Lemma 2.3. *Let* $\{x_i(t)\}_{i=1}^N$ *be the solution to* (1)*. Moreover, define*

$$\gamma(t) := \frac{1}{h(t)} \int_{t-\tau(t)}^{t} \alpha(t-s) \int_{s}^{t} \max_{1 \leq k \leq N} \left| \frac{dx_k(z)}{dz} \right| dz ds, \quad \forall \, t \geq 0. \tag{18}$$

Then,

$$D^+ d_X(t) \leq \frac{2}{\psi(2R)} \gamma(t) - \psi(2R) d_X(t), \quad \forall \, t \geq 0. \tag{19}$$

Proof. Due to continuity of $x_i(t)$, for any $i \in \{1, \dots, N\}$, there exists a sequence of times $\{t_k\}_{k \in \mathbb{N}}$ such that

$$\bigcup_{k \in \mathbb{N}} [t_k, t_{k+1}) = [0, +\infty),$$

and for each $k \in \mathbb{N}$ and for any $t \in (t_k, t_{k+1})$ there exist $i, j \in \{1, \dots, N\}$ such that

$$d_X(t) = |x_i(t) - x_j(t)|.$$

Hence, we obtain

$$\frac{1}{2} D^+ d_X^2(t) \leq \left\langle x_i(t) - x_j(t), \frac{dx_i(t)}{dt} - \frac{dx_j(t)}{dt} \right\rangle$$

$$= \frac{1}{Nh(t)} \left\langle x_i(t) - x_j(t), \sum_{k \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ik}(t;s)(x_k(s) - x_i(t)) ds \right\rangle$$
$$- \frac{1}{Nh(t)} \left\langle x_i(t) - x_j(t), \sum_{k \neq j} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{jk}(t;s)(x_k(s) - x_j(t)) ds \right\rangle$$
$$=: I_1 + I_2. \tag{20}$$

Now, I_1 and I_2 can be rewritten in the following way:

$$I_1 = \frac{1}{Nh(t)} \sum_{k \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ik}(t;s) \langle x_i(t) - x_j(t), x_k(s) - x_k(t) \rangle ds$$
$$+ \frac{1}{Nh(t)} \sum_{k \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ik}(t;s) \langle x_i(t) - x_j(t), x_k(t) - x_i(t) \rangle ds \tag{21}$$

and

$$I_2 = -\frac{1}{Nh(t)} \sum_{k \neq j} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{jk}(t;s) \langle x_i(t) - x_j(t), x_k(s) - x_k(t) \rangle ds$$
$$- \frac{1}{Nh(t)} \sum_{k \neq j} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{jk}(t;s) \langle x_i(t) - x_j(t), x_k(t) - x_j(t) \rangle ds.$$

We observe (as in [8]) that for any $t \geq 0$,

$$\langle x_i(t) - x_j(t), x_k(t) - x_i(t) \rangle \leq 0, \quad \forall k \in \{1, \dots, N\}.$$

Moreover, we notice that for any $i, j \in \{1, \dots, N\}$

$$a_{ij}(t;s) \leq \frac{1}{\psi(2R)} \tag{22}$$

in both cases (3) and (4). Indeed, if a_{ij} are as in (4), for any $i, j = 1, ..., N$, then we obtain (22), using (17) and the fact that ψ is a non-increasing function with $\psi(0) = 1$. Moreover, if we take a_{ij} as in (3), then (22) immediately follows, using the fact that $a_{ij}(t;s) \leq 1$, for any $i, j = 1, ..., N$, and $\psi(2R) \leq 1$. Therefore, using (17) and (22) in (21) yield

$$I_1 \leq \frac{1}{Nh(t)} \frac{d_X(t)}{\psi(2R)} \sum_{k \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) |x_k(s) - x_k(t)| ds$$
$$+ \frac{\psi(2R)}{N} \sum_{k=1}^{N} \langle x_i(t) - x_j(t), x_k(t) - x_i(t) \rangle. \tag{23}$$

As before, we observe that for any $t \geq 0$

$$-\langle x_i(t) - x_j(t), x_k(t) - x_j(t) \rangle \leq 0, \quad \forall k \in \{1, \dots, N\}.$$

Hence, using again (17) and (22), we can obtain a similar estimate for I_2, namely

$$I_2 \leq \frac{1}{Nh(t)} \frac{d_X(t)}{\psi(2R)} \sum_{k \neq j} \int_{t-\tau(t)}^{t} \alpha(t-s)|x_k(s) - x_k(t)|ds + \frac{\psi(2R)}{N} \sum_{k=1}^{N} \langle x_i(t) - x_j(t), x_j(t) - x_k(t) \rangle. \tag{24}$$

Using (23) and (24) in (20), we have that

$$\frac{1}{2} D^+ d_X(t)^2 \leq \frac{2}{Nh(t)} \frac{d_X(t)}{\psi(2R)} \sum_{k=1}^{N} \int_{t-\tau(t)}^{t} \alpha(t-s)|x_k(s) - x_k(t)|ds - \psi(2R) d_X(t)^2. \tag{25}$$

Moreover, we notice that, for $s < t$,

$$\sum_{k=1}^{N} |x_k(s) - x_k(t)| \leq \sum_{k=1}^{N} \int_s^t \left| \frac{dx_k(z)}{dz} \right| dz \leq N \int_s^t \max_{1 \leq k \leq N} \left| \frac{dx_k(z)}{dz} \right| dz.$$

Substituting this estimate in (25), we obtain

$$\frac{1}{2} D^+ d_X(t)^2 \leq \frac{2 d_X(t)}{\psi(2R)} \gamma(t) - \psi(2R) d_X(t)^2,$$

which yields (19). □

Lemma 2.4. *Let $\{x_i(t)\}_{i=1}^N$ be the solution to* (1)*. Then, for any $t \geq 0$*

$$\max_{1 \leq i \leq N} \left| \frac{dx_i(t)}{dt} \right| \leq \frac{1}{\psi(2R)} \gamma(t) + \frac{1}{\psi(2R)} d_X(t). \tag{26}$$

Proof. We have that for any $i \in \{1, \dots, N\}$,

$$\left| \frac{dx_i(t)}{dt} \right| \leq \frac{1}{Nh(t)} \sum_{k \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ik}(t;s) |x_k(s) - x_k(t)| ds + \frac{1}{Nh(t)} \sum_{k \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ik}(t;s) |x_k(t) - x_i(t)| ds.$$

Using (22) yields

$$\left| \frac{dx_i(t)}{dt} \right| \leq \frac{1}{\psi(2R)} \gamma(t) + \frac{1}{\psi(2R)} d_X(t).$$

Taking the maximum, we obtain (26). □

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Define the following Lyapunov functional:

$$\mathcal{L}(t) := d_X(t) + \beta \int_0^{\tau(t)} \alpha(s) \int_{t-s}^{t} e^{-(t-\sigma)} \int_\sigma^t \max_{1\le k\le N} \left|\frac{dx_k(\rho)}{d\rho}\right| d\rho d\sigma ds,$$

with $\beta > 0$. Then,

$$\begin{aligned}D^+\mathcal{L}(t) = D^+ d_X(t) + \beta\tau'(t)\alpha(\tau(t)) \int_{t-\tau(t)}^{t} e^{-(t-\sigma)} \int_\sigma^t \max_{1\le k\le N} \left|\frac{dx_k(\rho)}{d\rho}\right| d\rho d\sigma \\ -\beta \int_0^{\tau(t)} \alpha(s) e^{-s} \int_{t-s}^t \max_{1\le k\le N} \left|\frac{dx_k(\rho)}{d\rho}\right| d\rho ds \\ -\beta \int_0^{\tau(t)} \alpha(s) \int_{t-s}^{t} e^{-(t-\sigma)} \int_\sigma^t \max_{1\le k\le N} \left|\frac{dx_k(\rho)}{d\rho}\right| d\rho d\sigma ds \\ +\beta \max_{1\le k\le N} \left|\frac{dx_k(t)}{dt}\right| \int_0^{\tau(t)} \alpha(s) \int_{t-s}^t e^{-(t-\sigma)} d\sigma ds.\end{aligned}$$

Using $\underline{A} \le h(t) \le h(0)$ and $\tau'(t) \le 0$, we deduce

$$\begin{aligned}D^+\mathcal{L}(t) \le D^+ d_X(t) - \beta e^{-\tau(0)} \underline{A}\gamma(t) - \beta \int_0^{\tau(t)} \alpha(s) \int_{t-s}^{t} e^{-(t-\sigma)} \int_\sigma^t \max_{1\le k\le N} \left|\frac{dx_k(\rho)}{d\rho}\right| d\rho d\sigma ds \\ + \beta h(0)(1-e^{-\tau(0)}) \max_{1\le k\le N} \left|\frac{dx_k(t)}{dt}\right|.\end{aligned}$$

Now, since (19) and (26) hold, we have

$$\begin{aligned}D^+\mathcal{L}(t) \le \left(\frac{2}{\psi(2R)} - \beta e^{-\tau(0)}\underline{A} + \beta h(0)(1-e^{-\tau(0)})\frac{1}{\psi(2R)}\right)\gamma(t) \\ + \left(-\psi(2R) + \beta h(0)\frac{1-e^{-\tau(0)}}{\psi(2R)}\right) d_X(t) \\ - \beta \int_0^{\tau(t)} \alpha(s) \int_{t-s}^{t} e^{-(t-\sigma)} \int_\sigma^t \max_{1\le k\le N} \left|\frac{dx_k(\rho)}{d\rho}\right| d\rho d\sigma ds.\end{aligned}$$

We want to show that for $\tau(0)$ sufficiently small we obtain the existence of $K > 0$ such that

$$D^+\mathcal{L}(t) \le -K\mathcal{L}(t), \quad \forall\, t \ge 0. \tag{27}$$

This is true if the following two conditions hold:

$$\frac{2}{\psi(2R)} - \beta e^{-\tau(0)}\underline{A} + \beta h(0)(1-e^{-\tau(0)})\frac{1}{\psi(2R)} \le 0, \tag{28}$$

$$-\psi(2R) + \beta h(0)\frac{1-e^{-\tau(0)}}{\psi(2R)} < 0. \tag{29}$$

The inequality (29) is satisfied for

$$\beta < \frac{\psi(2R)^2}{h(0)(1-e^{-\tau(0)})}. \tag{30}$$

Now, in order to have (28), we need $h(0)\left(e^{\tau(0)}-1\right) < \underline{A}\psi(2R)$.

Hence, (28) is satisfied if

$$\beta \geq \frac{2}{e^{-\tau(0)}\underline{A}\psi(2R) - h(0)(1-e^{-\tau(0)})}. \tag{31}$$

Then, in order to have the existence of the parameter $\beta > 0$ such that (30) and (31) hold, we need

$$\frac{2}{e^{-\tau(0)}\underline{A}\psi(2R) - h(0)(1-e^{-\tau(0)})} < \frac{\psi(2R)^2}{h(0)(1-e^{-\tau(0)})},$$

which is true for any $\tau(0)$ satisfying (7). Choosing

$$K = \min\left\{\beta, \psi(2R) - \beta h(0)\frac{1-e^{-\tau(0)}}{\psi(2R)}\right\},$$

we obtain (27). We notice that since β satisfies (30), then $K > 0$. This implies immediately (8). Hence, the theorem is proved. □

3. Consensus of solution to (9)

In this section we want to analyse the transport equation (9) associated to (1), obtained as mean-field limit of the particle system when $N \to +\infty$. To do so, we consider ψ Lipschitz continuous and we denote by L its Lipschitz constant.

Before proving the existence and uniqueness of solutions to (9), we first recall some tools on probability spaces and measures.

Definition 3.1. Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ be two probability measures on $\mathbb{R}^d$. We define the *1-Wasserstein distance between μ and ν* as

$$d_1(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y| d\pi(x,y),$$

where $\Pi(\mu,\nu)$ is the space of all couplings for μ and ν, namely all those probability measures on $\mathbb{R}^{2d}$ having as marginals μ and ν:

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x)d\pi(x,y) = \int_{\mathbb{R}^d} \varphi(x)d\mu(x), \quad \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(y)d\pi(x,y) = \int_{\mathbb{R}^d} \varphi(y)d\nu(y),$$

for all $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$.

It is well-known that $(\mathcal{P}_1(\mathbb{R}^d), d_1)$ (where $\mathcal{P}_1$ is the space of all probability measures with finite first-order moment) is a complete metric space. Moreover, in order to prove the existence of solution to (9), we need the following definition.

Definition 3.2. Let μ be a Borel measure on $\mathbb{R}^d$ and let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable map. We define the *push-forward of μ via T* as the measure

$$T\#\mu(A) := \mu(T^{-1}(A)),$$

for all Borel sets $A \subset \mathbb{R}^d$.

Then, we have the following theorem.

Theorem 3.3. *Consider the system* (9) *with* $g_s \in \mathcal{C}([-\tau(0), 0]; \mathcal{P}_1(\mathbb{R}^d))$. *Suppose that there exists a constant* $R > 0$ *such that*

$$supp\ g_t \in B^d(0, R),$$

for all $t \in [-\tau(0), 0]$, *where* $B^d(0, R)$ *denotes the ball of radius* R *in* $\mathbb{R}^d$ *centered at the origin. Then, for any* $T > 0$ *there exists a unique weak solution* $\mu_t \in \mathcal{C}([0, T); \mathcal{P}_1(\mathbb{R}^d))$ *of* (9) *in the sense of* (12). *Moreover,* μ_t *is uniformly compactly supported and*

$$\mu_t = X(t; \cdot)\#\mu_0, \tag{32}$$

where $X(t; \cdot)$ *is the solution of the characteristic system associated to* (9) *for* $t \in [0, T)$.

Proof. First of all we claim that for any $t \in [0, T]$, there exist two positive constants $C, K > 0$ such that

$$\left| \frac{1}{h(t)} \int_{t-\tau(t)}^{t} \alpha(t-s) F[\mu_s](x) ds - \frac{1}{h(t)} \int_{t-\tau(t)}^{t} \alpha(t-s) F[\mu_s](\tilde{x}) ds \right| \leq C|x - \tilde{x}|,$$

for any $x, \tilde{x} \in B^d(0, R)$, and

$$\left| \frac{1}{h(t)} \int_{t-\tau(t)}^{t} \alpha(t-s) F[\mu_s](x) ds \right| \leq K,$$

for all $x \in B^d(0, R)$, with F as in (10) or in (11). The proof of this claim is very similar to [8, Lemma 3.4]. Then, from [2, Theorem 3.10], we deduce that there exists a unique weak solution to (9) in the sense of (12) and it exists as long as μ_t is compactly supported. Hence, we need to estimate the growth of support. To do so, we set

$$R_X[\mu_t] := \max_{x \in \overline{supp\ \mu_t}} |x|,$$

for $t \in [0, T]$ and we define

$$R_X(t) := \max_{-\tau(0) \leq s \leq t} R_X[\mu_s].$$

Now, we proceed by steps. We consider $t \in [0, \tau_*]$ and we construct the system of characteristics $X(t; x) : [0, \tau_*] \times \mathbb{R}^d \to \mathbb{R}^d$ associated to (9):

$$\begin{cases} \dfrac{dX(t;x)}{dt} = \dfrac{1}{h(t)} \displaystyle\int_{t-\tau(t)}^{t} \alpha(t-s) F[\mu_s](X(s;x)) ds, \\ X(0; x) = x, \quad x \in \mathbb{R}^d. \end{cases} \tag{33}$$

We notice that the system (33) is well-defined, since the velocity field

$$\frac{1}{h(t)} \int_{t-\tau(t)}^{t} \alpha(t-s) F[\mu_s] ds$$

is locally Lipschitz and locally bounded. Then, arguing as in Lemma 2.1, we have that

$$\frac{d|X(t;x)|}{dt} \leq R_X(t) - |X(t;x)|,$$

which yields $R_X(t) < R_X(0)$, for any $t \in [0, \tau_*]$.

Thus, we obtain a unique solution μ_t to (9) on the time interval $[0, \tau_*]$. We can iterate this process on all the intervals of the type $[k\tau_*, (k+1)\tau_*]$, with $k = 1, 2, \dots$, until we reach the final time T. Moreover, following [2], it's possible to find a measure μ_t which satisfies (32) and this is equivalent to the definiton of weak solution (12). □

3.1. Consensus behavior

In this subsection we will prove the consensus behavior of the solution to (9), with F as in (10) or (11). To do so, we firstly need the following stability result.

Lemma 3.4. *Let $\mu_t^1, \mu_t^2 \in \mathcal{C}([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ be two weak solutions to* (9)*, with compactly supported initial data $g_s^1, g_s^2 \in \mathcal{C}([-\tau(0), 0]; \mathcal{P}_1(\mathbb{R}^d))$ respectively. Then, there exists a constant $C > 0$ depending only on T such that, for any $t \in [0,T]$,*

$$d_1(\mu_t^1, \mu_t^2) \leq C \max_{s \in [-\tau(0), 0]} d_1(g_s^1, g_s^2). \tag{34}$$

Proof. For $i = 1, 2$ let $X^i(t;x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be the characteristics associated to (9), which obey to

$$\begin{cases} \dfrac{dX^i(t;x)}{dt} = \dfrac{1}{h(t)} \displaystyle\int_{t-\tau(t)}^{t} \alpha(t-s) F[\mu_s](X^i(s;x)) ds, \\ X^i(0;x) = x, \end{cases}$$

for any $x \in \mathbb{R}^d$. We remember that the characteristics X^i are well-defined in $[0,T]$ since, by Theorem 3.3, μ_t^i have uniformly compact support on such interval. Then, we have that

$$\mu_t^i = X^i(t;\cdot) \# \mu_s^i, \quad \forall t, s \in [0,T].$$

Moreover, as before, we define $R_{i;X}^T := \max_{s \in [-\tau(0), T]} R_X[\mu_s^i]$.

Then, we choose an optimal transport map between μ_0^1 and μ_0^2 with respect to d_1 (call it $S_0(x)$) such that $\mu_0^2 = S_0 \# \mu_0^1$ and

$$d_1(\mu_0^1, \mu_0^2) = \int_{\mathbb{R}^d} |x - S_0(x)| d\mu_0^1(x).$$

Moreover, we define the map T^t for any $t \in [0,T]$ as

$$T^t := X^2(t;\cdot) \circ S_0 \circ X^1(t;\cdot)^{-1}. \tag{35}$$

Therefore, we can write $T^t \# \mu_t^1 = \mu_t^2$, $\forall t \in [0,T]$ and

$$d_1(\mu_t^1, \mu_t^2) \leq \int_{\mathbb{R}^d} |x - T^t(x)| d\mu_t^1(x) := u(t).$$

Using (35) yields

$$u(t) = \int_{\mathbb{R}^d} \left| X^1(t;x) - X^2(t;S_0(x)) \right| d\mu_0^1(x).$$

Moreover, we extend the definition of T^t on the interval $[-\tau(0), 0]$ and we define $u(t)$ for $t \in [-\tau(0), 0]$ as

$$u(t) := d_1(g_t^1, g_t^2) = \int_{\mathbb{R}^d} \left| x - T^t(x) \right| dg_t^1(x).$$

Now, differentiating $u(t)$ and using (35), we obtain

$$\frac{du(t)}{dt} \leq \frac{1}{h(t)} \int_{\mathbb{R}^d} \int_{t-\tau(t)}^{t} \alpha(t-s) \left| F[\mu_s^1](x) - F[\mu_s^2](T^t(x)) \right| ds d\mu_t^1(x) =: J.$$

We consider now the case of F as in (10). Then,

$$\begin{aligned}
&|F[\mu_s^1](x) - F[\mu_s^2](T^t(x))| \\
&\qquad \leq \int_{\mathbb{R}^d} \left| \psi(|x-y|)(y-x) - \psi(|T^t(x) - T^s(y)|)(T^s(y) - T^t(x)) \right| d\mu_s^1(y) \\
&\qquad \leq \int_{\mathbb{R}^d} \left| \psi(|x-y|) - \psi(|T^t(x) - T^s(y)|) \right| \cdot |y-x| d\mu_s^1(y) \\
&\qquad\qquad + \int_{\mathbb{R}^d} \psi(|T^t(x) - T^s(y)|) \cdot \left| y - x - (T^s(y) - T^t(x)) \right| d\mu_s^1(y) \\
&\qquad = (1) \ + \ (2).
\end{aligned}$$

Now,

$$\begin{aligned}
(1) &\leq L \int_{\mathbb{R}^d} \left| x - y - T^t(x) + T^s(y) \right| \cdot |y-x| d\mu_s^1(y) \\
&\leq L(|x| + R_{1;X}^T) \left[|x - T^t(x)| + \int_{\mathbb{R}^d} |y - T^s(y)| d\mu_s^1(y) \right],
\end{aligned}$$

and

$$(2) \leq |x - T^t(x)| + \int_{\mathbb{R}^d} |y - T^s(y)| d\mu_s^1(y).$$

Therefore, there exists a constant $C > 0$ depending only on T such that

$$J \leq C \left(u(t) + \frac{1}{h(t)} \int_{t-\tau(t)}^{t} \alpha(t-s) u(s) ds \right).$$

Now, if we take F as in (11), we obtain

$$\begin{aligned}
&\left| F[\mu_s^1](x) - F[\mu_s^2](T^t(x)) \right| \\
&= \left| \frac{\int_{\mathbb{R}^d} \psi(|x-y|)(y-x) d\mu_s^1(y)}{\int_{\mathbb{R}^d} \psi(|x-y|) d\mu_s^1(y)} - \frac{\int_{\mathbb{R}^d} \psi(|T^t(x) - y|)(y - T^t(x)) d\mu_s^2(y)}{\int_{\mathbb{R}^d} \psi(|T^t(x) - y|) d\mu_s^2(y)} \right| \\
&\leq \frac{1}{\psi(R_{1;X}^T)} \left| \int_{\mathbb{R}^d} \psi(|x-y|)(y-x) d\mu_s^1(y) - \int_{\mathbb{R}^d} \psi(|T^t(x) - y|)(y - T^t(x)) d\mu_s^2(y) \right|
\end{aligned}$$

$$+\frac{1}{\psi(R^T_{1;X})\psi(R^T_{2;X})}\left|\int_{\mathbb{R}^d}\psi(|T^t(x)-y|)(y-T^t(x))d\mu^2_s(y)\right|$$
$$\times\left|\int_{\mathbb{R}^d}\psi(|x-y|)d\mu^1_s(y)-\int_{\mathbb{R}^d}\psi(|T^t(x)-y|)d\mu^2_s(y)\right|.$$

As before we have that

$$\left|\int_{\mathbb{R}^d}\psi(|x-y|)(y-x)d\mu^1_s(y)-\int_{\mathbb{R}^d}\psi(|T^t(x)-y|)(y-T^t(x))d\mu^2_s(y)\right|$$
$$\leq \left[(|x|+R^T_{1;X})L+1\right](|x-T^t(x)|+u(s)).$$

Furthermore,

$$\left|\int_{\mathbb{R}^d}\psi(|T^t(x)-y|)(y-T^t(x))d\mu^2_s(y)\right|\leq R^T_{2;X}+|T^t(x)|,$$

and

$$\left|\int_{\mathbb{R}^d}\psi(|x-y|)d\mu^1_s(y)-\int_{\mathbb{R}^d}\psi(|T^t(x)-y|)d\mu^2_s(y)\right|\leq L(|x-T^t(x)|+u(s)).$$

Hence, we obtain again the existence of a constant $C>0$ depending only on L and T such that

$$\frac{du(t)}{dt}\leq C\left(u(t)+\frac{1}{h(t)}\int_{t-\tau(t)}^t\alpha(t-s)u(s)ds\right).$$

Denote
$$\overline{u}=\max_{s\in[-\tau(0),0]}u(s)=\max_{s\in[-\tau(0),0]}d_1(g^1_s,g^2_s),$$

and define $w(t):=e^{-Ct}u(t)$. Then, we have that

$$\frac{dw(t)}{dt}\leq\frac{C}{h(t)}\int_{-\tau(0)}^t\alpha(t-s)w(s)ds. \tag{36}$$

Thus, we can rewrite (36) as

$$\frac{dw(t)}{dt}\leq K\tau(0)\overline{u}+K\int_0^t w(s)ds,$$

for some $K>0$. This gives us the following estimate:

$$w(t)\leq\tilde{K}\overline{u},\qquad\forall t\in[0,T],$$

for some $\tilde{K}>0$. Then, by definition of w we have

$$d_1(\mu^1_t,\mu^2_t)\leq u(t)\leq\tilde{K}e^{CT}\overline{u},\quad\forall\ t\in[0,T],$$

which gives us the thesis of this lemma. □

We are finally ready to prove Theorem 1.6.

Proof of Theorem 1.6. Fixed $g_s \in \mathcal{C}([-\tau(0),0];\mathcal{P}_1(\mathbb{R}^d))$, we construct the familiy of N- particle approximations of g_s, which is a family $\{g_s^N\}_{N\in\mathbb{N}}$ such that

$$g_s^N = \sum_{i=1}^{N} \delta(x - x_i^0(s)),$$

where $x_i^0 \in \mathcal{C}([-\tau(0),0];\mathbb{R}^d)$ satisfy

$$\max_{s\in[-\tau(0),0]} d_1(g_s^N, g_s) \to 0, \quad \text{as } N \to +\infty.$$

Moreover, let $\{x_i^N\}$ be the solution to (1), with initial data $x_i(s) = x_i^0(s)$ for any $s \in [-\tau(0),0]$ and we denote

$$\mu_t^N := \sum_{i=1}^{N} \delta(x - x_i^N(t)),$$

for any $t \in [0,T]$, which is a weak solution to (9). Now, since (13) holds, then we know that there exists a constant $C > 0$ such that

$$d_X(t) \le d_X(0)e^{-Ct} \le \left(\max_{s\in[-\tau(0),0]} d_X(s)\right) e^{-Ct},$$

for any $t \ge 0$. Fixing $T \ge 0$, by Lemma 3.4 we have that there exists a constant $K > 0$ independent of N such that

$$d_1(\mu_t, \mu_t^N) \le K \max_{s\in[-\tau(0),0]} d_1(g_s, g_s^N),$$

for any $t \in [0,T]$, where μ_t is the weak solution to (9) with initial measure g_s. Sending $N \to +\infty$ we have that $d_X(t) \to d_X(\mu_t)$ and for any $s \in [-\tau(0),0]$, $d_X(g_s) = d_X(s)$. This gives (14) for any $t \in [0,T]$. Since T can be chosen arbitrarly, then the theorem is proved. □

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