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Local Mountain Pass for a Class of Elliptic Systems without Homogeneity on the Nonlinearity

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We consider the gradient elliptic system given by

$$\begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) + \lambda K_u(u,v) \ \text{ in } \ \mathbb{R}^N, \\ -\varepsilon^2 \mathrm{div}(b(x)\nabla v) + v = Q_v(u,v) + \lambda K_v(u,v) \ \text{ in } \ \mathbb{R}^N, \end{cases}$$

where the potentials a, b are continuous, the nonlinearity $Q+\lambda K$ is not homogeneous. We study the subcritical, critical and supercritical cases. For $\varepsilon > 0$ small we show existence and concentration results using the penalization method.

Keywords: Gradient elliptic systems, Schrödinger equation.

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1. Introduction

In a seminal paper [11], Del Pino and Felmer introduced the penalization method and showed existence of solution for a nonlinear Schrödinger equation given by

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = f(u) \ \text{ in } \mathbb{R}^N, \\ u > 0 \ \text{ in } \mathbb{R}^N, \end{cases}$$

where V is a continuous potential satisfying:

There exists an open and bounded domain $\Omega \subset \mathbb{R}^N$ such that

$$0 < \inf_{x \in \Omega} V(x) < \min_{x \in \partial\Omega} V(x).$$

Afterwards, many authors used this method to show results of existence, concentration and multiplicity of solutions in different cases. For example, see [2] and [12] for the cases with critical polynomial and critical exponential growths, respectively. For the case with subcritical growth and p-Laplacian operator see [3].

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In 2007, Alves [1], following Del Pino and Felmer's ideas, introduced a new penalization method for gradients systems-types and shows existence and concentration of solutions for a system given by

$$\begin{cases} -\varepsilon^2 \mathrm{div}(\nabla u) + W(x)u = Q_u(u,v) \ \text{ in } \ \mathbb{R}^N, \\ -\varepsilon^2 \mathrm{div}(\nabla v) + V(x)v = Q_v(u,v) \ \text{ in } \ \mathbb{R}^N, \end{cases}$$

where Q is a function of class C^1, homogeneous of degree p with $2 < p < 2^* = \frac{2N}{N-2}$ and $N \geq 3$. More precisely, the main hypothesis on the nonlinearity Q was

$(\widetilde{Q}_0)$ There exists $2 < p < 2^* := 2N/(N-2)$ such that

$$Q(tu,tv) = t^p Q(u,v) \quad \text{for each } t > 0, \ (u,v) \in \mathbb{R}^2_+.$$

Multiplicity results were obtained in [4] and in [5] for the subcritical and critical growths cases, respectively. For completeness, we also mention [6], [7] and [8] for some interesting results for fractional Laplacian systems with subcritical and critical growth.

The same method introduced by Alves [1] allowed the authors of this article to show results of existence, concentration and multiplicity of gradient systems of the type

$$\begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) \ \text{ in } \ \mathbb{R}^N, \\ -\varepsilon^2 \mathrm{div}(b(x)\nabla v) + v = Q_v(u,v) \ \text{ in } \ \mathbb{R}^N. \end{cases}$$

The subcritical case was studied in [13] and the critical case was studied in [14].

The main purpose of this article is to show that the penalization method introduced by Alves [1] is still true when the nonlinearity Q is not homogeneous.

In this paper, we study the existence and concentration of solutions for the following system

$$\begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) + \lambda K_u(u,v) \ \text{ in } \ \mathbb{R}^N, \\ -\varepsilon^2 \mathrm{div}(b(x)\nabla v) + v = Q_v(u,v) + \lambda K_v(u,v) \ \text{ in } \ \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$, $N \geq 3$, $\lambda \geq 0$, a, b are continuous potentials, and Q, K are functions with Q having subcritical growth. We consider the subcritical case with $\lambda = 0$ and the critical and supercritical cases with $\lambda \neq 0$.

The hypotheses on the functions a and b are the following:

(ab_1) $a, b \in C(\mathbb{R}^N, \mathbb{R})$, $0 < a_m \leq a(x) \leq a_M$ and $0 < b_m \leq b(x) \leq b_M$ for all $x \in \mathbb{R}^N$.

(ab_2) There exists a bounded domain $\Lambda \subset \mathbb{R}^N$ such that

$$a_m = \inf_{x \in \Lambda} a(x) < \inf_{x \in \partial\Lambda} a(x) \quad \text{and} \quad b_m = \inf_{x \in \Lambda} b(x) < \inf_{x \in \partial\Lambda} b(x).$$

Let us state the hypotheses on the nonlinearity Q:

(Q_0) $Q \in C^1(\mathbb{R}^2, \mathbb{R})$ such that $Q(s,t) > 0$ if $(s,t) \neq (0,0)$, $Q(0,0) = 0$, $Q_s(s,t) = 0$ if $s \leq 0$ and $Q_t(s,t) = 0$ if $t \leq 0$.

(Q_1) There exist $p_1, p_2 \in (2, 2^*)$ and $c_1 > 0$ such that

$$|Q_s(s,t)| + |Q_t(s,t)| \leq c_1(|s|^{p_1-1} + |t|^{p_2-1}) \quad \text{for all} \quad (s,t) \in \mathbb{R}^2.$$

(Q_2) There exists $2 < \mu < p_1, p_2$ such that

$$0 < \mu Q(s,t) \leq sQ_s(s,t) + tQ_t(s,t) \quad \text{for all} \quad (s,t) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

(Q_3) $\Upsilon \to \dfrac{sQ_s(\Upsilon s, \Upsilon t) + tQ_t(\Upsilon s, \Upsilon t)}{\Upsilon}$ is an increasing functions with $s, t, \Upsilon > 0$.

(Q_4) There exists $\sigma^* > 0$ such that $Q(s,t) \geq \frac{\sigma}{p_5} s^\beta t^\nu$ for all $s, t \geq 0$, $\beta, \nu \geq 1$, $p_5 \in (2, 2^*)$ with $\beta + \nu = p_5$, for all $\sigma > \sigma^*$ and σ^* to be fixed later.

$(\widetilde{Q}_4)$ There exists $\sigma > 0$ such that $Q(s,t) \geq \frac{\sigma}{p_5} s^\beta t^\nu$ for all $s, t \geq 0$, $\beta, \nu \geq 1$, $p_5 \in (2, 2^*)$ with $\beta + \nu = p_5$.

We will consider $K \in C^1(\mathbb{R}^2_+, \mathbb{R})$ where $\mathbb{R}^2_+ := [0, \infty) \times [0, \infty)$. In addition, the nonlinearity K satisfies the following properties

(K_0) K is 2^*-homogemeous; that is

$$K(\gamma s, \gamma t) = \gamma^{2^*} K(s,t), \quad \text{for each} \quad \gamma > 0, \ (s,t) \in \mathbb{R}^2_+.$$

(K_1) There exists $c_1 > 0$ such that

$$|K_s(s,t)| + |K_t(s,t)| \leq c_1 \left(s^{2^*-1} + t^{2^*-1} \right), \quad \text{for each } (s,t) \in \mathbb{R}^2_+.$$

(K_2) $K_s(0,1) = 0$, $K_t(1,0) = 0$.

(K_3) $K_s(1,0) = 0$, $K_t(0,1) = 0$.

(K_4) $K(s,t) > 0$ for each $s, t > 0$.

(K_5) $K_s(s,t), K_t(s,t) \geq 0$ for each $(s,t) \in \mathbb{R}^2_+$.

(K_6) The 1-homogeneous function $G : \mathbb{R}^2_+ \to \mathbb{R}$ given by $G(s^{2^*}, t^{2^*}) := K(s,t)$ is concave.

We also introduce the following set:

$$M := \{x \in \mathbb{R}^N : a(x) = a_m \ \text{ and } \ b(x) = b_m\}.$$

In the first part of the paper we will study the following problem

$$(S_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \operatorname{div}(a(x)\nabla u) + u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \operatorname{div}(b(x)\nabla v) + v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) \geq 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

In our first result we obtain, for $\varepsilon > 0$ small enough, the existence of a solution of (S_ε).

Theorem 1.1. *Assume that $(ab_1) - (ab_2)$ and $(Q_0) - (Q_3)$ hold and that $M \neq \emptyset$. There exists $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$, the system (S_ε) has a non-negative weak solution. Moreover, if $(u_\varepsilon, v_\varepsilon)$ is a solution for (S_ε) and if $\Pi_{\varepsilon,a}$ and $\Pi_{\varepsilon,b}$ are the maximum points of u_ε and v_ε respectively, then*

$$\Pi_{\varepsilon,a}, \Pi_{\varepsilon,b} \in \Lambda, \ \lim_{\varepsilon \to 0^+} a(\Pi_{\varepsilon,a}) = a_m \ \text{ and } \ \lim_{\varepsilon \to 0^+} b(\Pi_{\varepsilon,b}) = b_m.$$

In the second part of the paper we deal with a critical version of (S_ε) with $\lambda = \frac{1}{2^*}$, namely the problem

$$(C_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) + \frac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \mathrm{div}(b(x)\nabla v) + v = Q_v(u,v) + \frac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) \geq 0 \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

where $2^* := \frac{2N}{N-2}$.

The critical version of Theorem 1.1 can be stated as follows.

Theorem 1.2. *Assume that $(ab_1) - (ab_2)$, $(Q_0) - (Q_4)$ and $(K_0) - (K_6)$ hold and that $M \neq \emptyset$. There exists $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$, the system (C_ε) has a non-negative weak solution. Moreover, if $(u_\varepsilon, v_\varepsilon)$ is a solution for (C_ε) and if $\Pi_{\varepsilon,a}$ and $\Pi_{\varepsilon,b}$ are the maximum points of u_ε and v_ε respectively, then*

$$\Pi_{\varepsilon,a}, \Pi_{\varepsilon,b} \in \Lambda, \quad \lim_{\varepsilon \to 0^+} a(\Pi_{\varepsilon,a}) = a_m \quad \text{and} \quad \lim_{\varepsilon \to 0^+} b(\Pi_{\varepsilon,b}) = b_m.$$

In the last part of the paper, we study a supercritical system $(SC_{\varepsilon,\lambda})$.

$$(SC_{\varepsilon,\lambda}) \qquad \begin{cases} -\varepsilon^2 \mathrm{div}(a(x)\nabla u) + u = Q_u(u,v) + \lambda |u|^{q_1-2}u \text{ in } \mathbb{R}^N, \\ -\varepsilon^2 \mathrm{div}(b(x)\nabla v) + v = Q_v(u,v) + \lambda |v|^{q_2-2}v \text{ in } \mathbb{R}^N, \end{cases}$$

where $q_1, q_2 > 2^*$.

Our main result is as follows.

Theorem 1.3. *Assume that $(ab_1) - (ab_2)$, $(Q_0) - (Q_3)$ and $(\widetilde{Q}_4)$ hold and that $M \neq \emptyset$. Then there exists $\lambda_0 > 0$ with the following property: for any $\lambda \in (0, \lambda_0)$ and $\delta > 0$ given, there exists $\varepsilon_{\lambda,\delta} > 0$ such that, for any $\varepsilon \in (0, \varepsilon_{\lambda,\delta})$, the system $(SC_{\varepsilon,\lambda})$ has a non-negative weak solution. Moreover, if $(u_\varepsilon, v_\varepsilon)$ is a solution for $(SC_{\varepsilon,\lambda})$ and if $\Pi_{\varepsilon,a}$ and $\Pi_{\varepsilon,b}$ are the maximum points of u_ε and v_ε respectively, then*

$$\Pi_{\varepsilon,a}, \Pi_{\varepsilon,b} \in \Lambda, \quad \lim_{\varepsilon \to 0^+} a(\Pi_{\varepsilon,a}) = a_m \quad \text{and} \quad \lim_{\varepsilon \to 0^+} b(\Pi_{\varepsilon,b}) = b_m.$$

The present work is strongly influenced by the articles [1], [4], [5], [6], [13] and [14]. Below we list what we believe that are the main contributions of our paper.

(1) We show that the penalization method introduced by Alves [1] is still true when the nonlinearity Q is not homogeneous.

(2) Unlike [1], [4], [5], [6], [13] and [14], we show existence and concentration results for gradient systems type with nonlinearity Q not homogeneous.

(3) We complement the study that can be found in [1], [4], [5], [6], [13] and [14] because, in our results, we show existence and concentration solutions for the subcritical, critical and supercritical cases.

Concerning the class of nonlinearities we are dealing, we have the following examples that already appeared in [10]:

$$Q(s,t) = \sum_{\beta_i + \nu_i = p_1} a_i s^{\beta_i} t^{\nu_i} + \sum_{\upsilon_i + \iota_i = p_2} m_i s^{\upsilon_i} t^{\iota_i},$$

where $i \in \{1, \ldots, k\}$, $\beta_i, \nu_i, \upsilon_i, \iota_i \geq 1$ and $a_i, m_i \in \mathbb{R}$.

The following functions and its possible combinations, with appropriated choices of the coefficients a_i, m_i, satisfy our hypothesis on Q.

Condition (K_6) restricts the expression of the critical function K. However, it can have the polynomial form $K(s,t) = \sum_{\beta_i+\nu_i=2^*} a_i s^{\beta_i} t^{\nu_i}$.

This paper is organized as follows. In order to be able to deal variationally, in section 2 we show the penalization method for nonlinearity not homogeneous. The subcritical case was studied in section 3. The critical and supercritical cases were studied in section 4 and section 5, respectively.

2. Variational framework and modified system

In order to overcome the lack compactness originated by the unboundedness of $\mathbb{R}^N$ we use a penalization method.

Consider $\alpha > 0$ and $\eta : \mathbb{R} \to \mathbb{R}$ a non-increasing function of class C^2 such that

$$\eta \equiv 1 \quad \text{on} \quad (-\infty, \alpha], \eta \equiv 0 \quad \text{on} \quad [5\alpha, \infty) \quad \text{and} \quad |\eta'(s)| \leq \frac{c_2}{\alpha} \tag{1}$$

for each $s \in \mathbb{R}$ and for some constant $c_2 > 0$. Then, using the function η, we define $\widehat{Q} : \mathbb{R}^2 \to \mathbb{R}$ by

$$\widehat{Q}(s,t) := \eta(|(s,t)|)Q(s,t) + (1 - \eta(|(s,t)|))A(s^2+t^2),$$

where

$$A := \max\left\{\frac{Q(s,t)}{s^2+t^2} : (s,t) \in \mathbb{R}^2,\ \alpha \leq |(s,t)| \leq 5\alpha\right\}.$$

Notice that, since $A > 0$ tends to zero as $\alpha \to 0^+$, we may suppose that $A < 1$.

Denoting by χ_Λ the characteristic function of the set Λ, we define $H : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ by setting

$$H(x,s,t) := \chi_\Lambda(x)Q(s,t) + (1 - \chi_\Lambda(x))\widehat{Q}(s,t). \tag{2}$$

Lemma 2.1. *The function H satisfies the following estimates:*

(H_1) $\mu H(x,s,t) \leq sH_s(x,s,t) + tH_t(x,s,t)$, *for each* $x \in \Lambda$.

(H_2) $2H(x,s,t) \leq sH_s(x,s,t) + tH_t(x,s,t)$, *for each* $x \in \mathbb{R}^N \setminus \Lambda$.

(H_3) *For α small we have* $sH_s(x,s,t) + tH_t(x,s,t) \leq \frac{1}{4}(s^2+t^2)$ *for each* $x \in \mathbb{R}^N \setminus \Lambda$.

(H_4) *For α small we have* $\dfrac{|H_s(x,s,t)|}{\alpha}, \dfrac{|H_t(x,s,t)|}{\alpha} \leq \dfrac{1}{4}$ *for each* $x \in \mathbb{R}^N \setminus \Lambda$.

Proof. If $x \in \Lambda$ we have that $H(x,s,t) = Q(s,t)$. By using (Q_2) we get

$$sH_s(x,s,t) + tH_t(x,s,t) \geq \mu Q(s,t) = \mu H(x,s,t)$$

and the proof of (H_1) is finished.

From (2), we have that $H(x,s,t) = \widehat{Q}(s,t)$ for all $x \in \mathbb{R}^N \setminus \Lambda$. Then, from definition of $\widehat{Q}$, we have

$$\begin{aligned} \widehat{Q}_s(s,t) &= \eta'(|(s,t)|)\frac{s}{|(s,t)|}Q(s,t) + \eta(|(s,t)|)Q_s(s,t) \\ &\quad - \eta'(|(s,t)|)\frac{s}{|(s,t)|}A(s^2+t^2) + (1 - \eta(|(s,t)|))\,2As \end{aligned} \tag{3}$$

and
$$\widehat{Q}_t(s,t) = \eta'(|(s,t)|)\frac{t}{|(s,t)|}Q(s,t) + \eta(|(s,t)|)Q_t(s,t) - \eta'(|(s,t)|)\frac{t}{|(s,t)|}A(s^2+t^2) + (1-\eta(|(s,t)|))\,2At. \tag{4}$$

From (3) and (4) we get

$$s\widehat{Q}_s(s,t) + t\widehat{Q}_t(s,t) = \eta'(|(s,t)|)[Q(s,t) - A(s^2+t^2)]|(s,t)| + \eta(|(s,t)|)[sQ_s(s,t) + tQ_t(s,t)] + (1-\eta(|(s,t)|))2A(s^2+t^2). \tag{5}$$

Since η is non-increasing, we can use definition of A and (Q_2), and obtain

$$s\widehat{Q}_s(s,t) + t\widehat{Q}_t(s,t) \geq \eta(|(s,t)|)\mu Q(s,t) + 2(1-\eta(|(s,t)|))A(s^2+t^2) \geq 2\widehat{Q}(s,t),$$

and the proof of (H_2) is finished.

From (5) we get, using (1) and the definitions of A and (Q_1),

$$\frac{s\widehat{Q}_s(s,t) + t\widehat{Q}_t(s,t)}{s^2+t^2} \leq \frac{c_2}{\alpha}(5\alpha)(2A) + 4c_1\left((5\alpha)^{p_1-2} + (5\alpha)^{p_2-2}\right) + 4A.$$

Then, for α sufficiently small we get the proof of (H_3).

For proving (H_4), we can use (3), (1) and (Q_1) to get

$$|\widehat{Q}_s(s,t)| \leq \frac{c_2}{\alpha}A(5\alpha)^2 + c_1\left((5\alpha)^{p_1-1} + (5\alpha)^{p_2-1}\right) + \frac{c_2}{\alpha}A(5\alpha)^2 + 4A(5\alpha).$$

Then, for α sufficiently small we obtain $\frac{|H_s(x,s,t)|}{\alpha} \leq \frac{1}{4}$.

Using similar arguments, the second inequality is proved. □

3. Subcritical case

Changing variables by $x \mapsto \varepsilon x$, we can rewrite system (S_ε) into the following equivalent form

$$(\widehat{S}_\varepsilon) \qquad \begin{cases} -\operatorname{div}(a(\varepsilon x)\nabla u) + u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -\operatorname{div}(b(\varepsilon x)\nabla v) + v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) \geq 0 \ \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

If (u,v) is a solution of system $(\widehat{S}_\varepsilon)$, then $(\widehat{u}(x), \widehat{v}(x)) := (u(x/\varepsilon), v(x/\varepsilon))$ is a solution of system (S_ε). Thus, to study system (S_ε), it suffices to study system $(\widehat{S}_\varepsilon)$.

In view of definition (2), we deal in the sequel with the modified system

$$(S_{\varepsilon,aux}) \qquad \begin{cases} -\operatorname{div}(a(\varepsilon x)\nabla u) + u = H_u(\varepsilon x,u,v) \text{ in } \mathbb{R}^N, \\ -\operatorname{div}(b(\varepsilon x)\nabla v) + v = H_v(\varepsilon x,u,v) \text{ in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) \geq 0 \ \text{ for each } x \in \mathbb{R}^N, \end{cases}$$

and we will look for solutions $(u_\varepsilon, v_\varepsilon)$ verifying

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha, \quad \text{for each } \ x \in \mathbb{R}^N \setminus \Lambda_\varepsilon,$$

where $\Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$.

For each $\varepsilon > 0$ we denote by X_ε the Hilbert space

$$X_\varepsilon := \left\{(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|\nabla v|^2\right) dx < \infty\right\}$$

endowed with the norm

$$\|(u,v)\|_\varepsilon^2 := \int_{\mathbb{R}^N} \left(a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|\nabla v|^2 + |u|^2 + |v|^2\right) dx.$$

By using condition (ab_1) we can verify that

$$X_\varepsilon \hookrightarrow H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N) \ \text{ for all } \ 2 \le s \le 2^*.$$

Conditions (H_3) and (Q_1) imply that the critical points of the C^1-functional $I_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$I_\varepsilon(u,v) := \frac{1}{2}\int_{\mathbb{R}^N} \left(a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|\nabla v|^2 + |u|^2 + |v|^2\right) dx - \int_{\mathbb{R}^N} H(\varepsilon x, u, v) dx$$

are weak solution of $(S_{\varepsilon,aux})$.

We define the Palais-Smale compactness condition. A sequence $((u_n, v_n)) \subset X_\varepsilon$ is a *Palais-Smale sequence at level c for the functional I_ε* if

$$I_\varepsilon(u_n, v_n) \to c$$

and

$$I'_\varepsilon(u_n, v_n) \to 0 \ \text{ in } \ X'_\varepsilon,$$

where

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\varepsilon(\gamma(t)) > 0$$

and

$$\Gamma := \{\gamma \in C([0,1], X_\varepsilon) : \gamma(0) = (0,0), \ I_\varepsilon(\gamma(1)) < 0\}.$$

If every Palais-Smale sequence of I_ε has a strongly convergent subsequence then one says that I_ε satisfies the Palais-Smale condition ((PS) for short).

Lemma 3.1. *The functional I_ε satisfies the following conditions*

(i) *There exist positive numbers ρ and C such that*

$$I_\varepsilon(u,v) \ge C > 0, \quad \text{if } \ \|(u,v)\|_\varepsilon = \rho.$$

(ii) *There exists $(e_1, e_2) \in X_\varepsilon$ with $I_\varepsilon(e_1, e_2) < 0$ and $\|(e_1, e_2)\|_\varepsilon > \rho$.*

Proof. From $(H_1), (H_2), (H_3)$ and (Q_1) we have

$$\begin{aligned} I_\varepsilon(u,v) &= \frac{1}{2}\|(u,v)\|_\varepsilon^2 - \int_{\Lambda_\varepsilon} Q(u,v)dx - \int_{\mathbb{R}\setminus\Lambda_\varepsilon} H(\varepsilon x, u, v)dx \\ &\ge \frac{1}{2}\|(u,v)\|_\varepsilon^2 - \frac{1}{\mu}\int_{\Lambda_\varepsilon} [uQ_u(u,v) + vQ_v(u,v)]dx - \frac{1}{8}\int_{\mathbb{R}\setminus\Lambda_\varepsilon} (|u|^2 + |v|^2)dx \\ &\ge \frac{3}{8}\|(u,v)\|_\varepsilon^2 - \frac{2c_1}{\mu}\int_{\Lambda_\varepsilon} (|u|^{p_1} + |v|^{p_1})dx - \frac{2c_1}{\mu}\int_{\Lambda_\varepsilon} (|u|^{p_2} + |v|^{p_2})dx. \end{aligned}$$

Then, by Sobolev embeddings, there exists $c_3 > 0$ such that

$$I_\varepsilon(u,v) \geq \frac{3}{8}\|(u,v)\|_\varepsilon^2 - c_3\|(u,v)\|_\varepsilon^{p_1} - c_3\|(u,v)\|_\varepsilon^{p_2}$$

and the proof of item (i) follows by choosing $\rho > 0$ small enough.

Let $\phi \in C_0^\infty(\Lambda_\varepsilon)$. From (Q_2) there exist $c_4, c_5 > 0$ such that

$$\begin{aligned} I_\varepsilon(t(\phi,0)) &= \frac{1}{2}\|(t\phi,0)\|_\varepsilon^2 - \int_{\Lambda_\varepsilon} Q(t\phi,0)dx \\ &\leq \frac{t^2}{2}\|(\phi,0)\|_\varepsilon^2 - \int_{\Lambda_\varepsilon}(c_4|t\phi|^\mu - c_5)dx \\ &\leq \frac{t^2}{2}\|(\phi,0)\|_\varepsilon^2 - c_4 t^\mu \int_{\Lambda_\varepsilon}|\phi|^\mu dx + c_6 \end{aligned}$$

for all $t \geq 0$. The proof of item (ii) follow by considering $(e_1, e_2) = t^*(\phi, 0)$ for some $t^* > 0$ sufficiently large. □

From Lemma 3.1, I_ε has the mountain pass geometry. Hence, there exists a Palais-Smale sequence $((u_n, v_n)) \subset X_\varepsilon$ at level c_ε.

Lemma 3.2. *Let $((u_n, v_n))$ be a $(PS)_c$ sequence for I_ε. Then $((u_n, v_n))$ is bounded in X_ε.*

Proof. Since $((u_n, v_n))$ is a $(PS)_c$ sequence for I_ε,

$$I_\varepsilon(u_n, v_n) \to c \quad \text{and} \quad I'_\varepsilon(u_n, v_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, by using $(H_1), (H_2)$ and (H_3), we have

$$I_\varepsilon(u_n, v_n) - \frac{1}{\mu}I'_\varepsilon(u_n, v_n)(u_n, v_n)$$

$$\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|(u_n, v_n)\|_\varepsilon^2 - \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}\setminus\Lambda_\varepsilon}[u_n H_u + v_n H_v]dx \geq \frac{3}{4}\left(\frac{1}{2} - \frac{1}{\mu}\right)\|(u_n, v_n)\|_\varepsilon^2.$$

Then

$$\frac{3}{4}\left(\frac{1}{2} - \frac{1}{\mu}\right)\|(u_n, v_n)\|_\varepsilon^2 \leq I(u_n, v_n) - \frac{1}{\mu}I'_\varepsilon(u_n, v_n)(u_n, v_n) = c + o(\|(u_n, v_n)\|_\varepsilon),$$

where we conclude that $((u_n, v_n))$ is bounded in X_ε. □

The proof of the next two results is in the same spirit of [13, Lemma 3.2 and Lemma 3.3]. We omit the details.

Lemma 3.3. *Let $((u_n, v_n))$ be a $(PS)_c$ sequence for I_ε. Then for each $\xi > 0$, there exists $R = R(\xi)$ such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} \left(a(\varepsilon x)|\nabla u_n|^2 + b(\varepsilon x)|\nabla v_n|^2 + |u_n|^2 + |v_n|^2\right) dx < \xi.$$

Lemma 3.4. *The functional I_ε satisfies the Palais-Smale condition at any level c.*

Theorem 3.5. *Assume that conditions $(ab_1)-(ab_2)$ and $(Q_0)-(Q_2)$ are hold. Then, for all $\varepsilon > 0$, the system $(S_{\varepsilon,aux})$ has a non-negative weak solution.*

Proof. From Lemma 3.1, Lemma 3.4 and Mountain Pass Theorem due Ambrosetti Rabinowitz [9] it follows that I_ε has a critical point $(u,v) \in X_\varepsilon$ with $I_\varepsilon(u,v) = c_\varepsilon > 0$. Then $I'_\varepsilon(u,v)(\phi,\psi) = 0$ for all $(\phi,\psi) \in X_\varepsilon$, hence

$$\int_{\mathbb{R}^N} (a(\varepsilon x)\nabla u\nabla\phi + b(\varepsilon x)\nabla v\nabla\psi + u\phi + v\psi)\,dx - \int_{\mathbb{R}^N} (\phi H_u + \psi H_v)\,dx = 0.$$

Choosing $\phi = u^-$, $\psi = v^-$, where u^-, v^- are the non-positive parts of the functions u and v, respectively, we get

$$-\|(u^-,v^-)\|_\varepsilon^2 - \int_{\mathbb{R}^N} \left(u^- H_u(\varepsilon x,u,v) + v^- H_v(\varepsilon x,u,v)\right) dx = 0.$$

By using (3) and (4) we have

$$u^- H_u(\varepsilon x,u,v) + v^- H_v(\varepsilon x,u,v) = -\eta'(|(u,v)|)\frac{\left((u^-)^2 + (v^-)^2)\right)}{|(u,v)|}Q(u,v)$$
$$+\eta'(|(u,v)|)\frac{\left((u^-)^2 + (v^-)^2\right)}{|(u,v)|}A(u^2+v^2) - (1-\eta(|(u,v)|))2A\left((u^-)^2 + (v^-)^2\right)$$

for all $x \in \mathbb{R}^N \setminus \Lambda_\varepsilon$. Thus, by using (1) and definition of A, we have

$$|u^- H_u(\varepsilon x,u,v)+v^- H_v(\varepsilon x,u,v)| \le 2\frac{c_2}{\alpha}\left((u^-)^2+(v^-)^2\right)A(5\alpha) + 4A\left((u^-)^2+(v^-)^2\right)$$

for all $x \in \mathbb{R}^N \setminus \Lambda_\varepsilon$. Then, for α sufficiently small and (Q_0), we obtain

$$\|(u^-,v^-)\|_\varepsilon = 0.$$

Therefore $u \ge 0$ and $v \ge 0$ in $\mathbb{R}^N$. □

In order to study the concentration of solutions, we now consider the autonomous system

$$(S_0)\qquad \begin{cases} -a_m\Delta u + u = Q_u(u,v) \text{ in } \mathbb{R}^N, \\ -b_m\Delta v + v = Q_v(u,v) \text{ in } \mathbb{R}^N, \\ u,v \in H^1(\mathbb{R}^N), u(x),v(x) \ge 0 \text{ for each } x \in \mathbb{R}^N. \end{cases}$$

The corresponding functional is

$$I_0(u,v) := \frac{1}{2}\int_{\mathbb{R}^N} \left(a_m|\nabla u|^2 + b_m|\nabla v|^2 + |u|^2 + |v|^2\right) dx - \int_{\mathbb{R}^N} Q(u,v)dx$$

for $(u,v) \in X := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. We denote the norm in X by

$$\|(u,v)\|^2 := \int_{\mathbb{R}^N} \left(a_m|\nabla u|^2 + b_m|\nabla v|^2 + |u|^2 + |v|^2\right) dx.$$

Arguing as in Lemma 3.1, we can show that I_0 has the mountain pass geometry and therefore we can set the minimax level c_0 in the following way

$$c_0 := \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} I_0(\gamma(t)),$$

where $\Gamma_0 := \{\gamma \in C([0,1],X) : \gamma(0) = (0,0), I_0(\gamma(1)) < 0\}$.

On the other hand, by using (Q_3) we can argue as in [16] for to get

$$0 < c_0 = \inf_{(u,v)\in X\setminus\{(0,0)\}} \max_{t\geq 0} I_0(tu,tv) = \inf_{(u,v)\in\mathcal{M}} I_0(u,v) := m_0, \tag{6}$$

where $\mathcal{M} := \{(u,v) \in X \setminus \{(0,0)\} : I_0'(u,v)(u,v) = 0\}$.

Proposition 3.6. *Let $((u_n,v_n)) \subset \mathcal{M}$ be such that $I_0(u_n,v_n) \to m_0$. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $(w_n(x), z_n(x)) := (u_n(x+\widetilde{y}_n), v_n(x+\widetilde{y}_n)) \to (w,z) \in \mathcal{M}$ with $I_0(w,z) = m_0$.*

Proof. Arguing as in Lemma 3.2 $((u_n,v_n))$ is bounded in X. Then, up to a subsequence, we may suppose that $(u_n,v_n) \rightharpoonup (u,v)$ weakly in X. Using the Ekeland's Variational Principle [16, Theorem 8.5], we can assume that $((u_n,v_n))$ verifies the following limits

$$I_0(u_n,v_n) \to m_0 \quad \text{and} \quad I_0'(u_n,v_n) \to 0.$$

By using a density argument, we can conclude that (u,v) is a critical point of I_0. Now, we will divide our study in two cases.

Case 1. $(u,v) \not\equiv (0,0)$.

In this case $I_0'(u,v)(u,v) = 0$ and therefore $(u,v) \in \mathcal{M}$. Then, by using (Q_2) and Fatou's lemma we get

$$\begin{aligned}
m_0 &\leq I_0(u,v) - \frac{1}{\mu} I_0'(u,v)(u,v) \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right) \|(u,v)\|^2 + \int_{\mathbb{R}^N} \left[\frac{1}{\mu}\big(uQ_u(u,v) + vQ_v(u,v)\big) - Q(u,v)\right] dx \\
&\leq \liminf_{n\to\infty} \left(\frac{1}{2} - \frac{1}{\mu}\right) \|(u_n,v_n)\|^2 \\
&\quad + \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{\mu}\big(u_nQ_u(u_n,v_n) + v_nQ_v(u_n,v_n)\big) - Q(u_n,v_n)\right] dx \\
&\leq \liminf_{n\to\infty} \left(I_0(u_n,v_n) - \frac{1}{\mu} I_0'(u,v_n)(u_n,v_n)\right) = m_0
\end{aligned}$$

and thus $\lim_{n\to\infty} \|(u_n,v_n)\|^2 = \|(u,v)\|^2$.

Hence, $(u_n,v_n) \to (u,v)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Consequently $I_0(u,v) = m_0$. and $\widetilde{y}_n = 0$ for all $n \in \mathbb{N}$.

Case 2. $(u,v) = (0,0)$.

In this case, there exists $R, \eta > 0$ and $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_R(\widetilde{y}_n)} \left(|u_n|^2 + |v_n|^2\right) dx \geq \eta.$$

Because if this claim does not hold, we have by [15, Lemma I.1] that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^s dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^s dx = 0$$

for all $s \in (2, 2^*)$.

Thus, from (Q_1), we obtain

$$\int_{\mathbb{R}^N} (u_n Q_u(u_n, v_n) + v_n Q_v(u_n, v_n))\, dx \to 0 \quad \text{as} \quad n \to \infty.$$

Since $I_0'(u_n, v_n)(u_n, v_n) = 0$, we can use the above limit for to get

$$\|(u_n, v_n)\|^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently $I_0(u_n, v_n) \to 0$, which contradicts (6).

Now, we define $\widetilde{u}_n(x) := u_n(x + \widetilde{y}_n)$ and $\widetilde{v}_n(x) := v_n(x + \widetilde{y}_n)$.

Then, we can verify that

$$I_0(\widetilde{u}_n, \widetilde{v}_n) \to m_0 \quad \text{and} \quad I_0'(\widetilde{u}_n, \widetilde{v}_n) \to 0.$$

It is clear that $((\widetilde{u}_n, \widetilde{v}_n))$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and there exists $(\widetilde{u}, \widetilde{v}) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ with $(\widetilde{u}, \widetilde{v}) \neq (0, 0)$ such that

$$(\widetilde{u}_n, \widetilde{v}_n) \rightharpoonup (\widetilde{u}, \widetilde{v}).$$

Repeating the same arguments used in Case 1, it follows that $(\widetilde{u}_n, \widetilde{v}_n) \to (\widetilde{u}, \widetilde{v})$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. □

In similar way as in (6), we can show that

$$0 < c_\varepsilon = \inf_{(u,v) \in X_\varepsilon \setminus \{(0,0)\}} \max_{t \geq 0} I_\varepsilon(tu, tv) = \inf_{(u,v) \in \mathcal{N}_\varepsilon} I_\varepsilon(u, v) := m_\varepsilon, \tag{7}$$

where $\mathcal{N}_\varepsilon := \{(u, v) \in X_\varepsilon \setminus \{(0, 0)\} : I_\varepsilon'(u, v)(u, v) = 0\}$.

Proposition 3.7. *Let $(\varepsilon_n) \subset \mathbb{R}^+$ be such that $\varepsilon_n \to 0^+$ and (u_n, v_n) be a solution of $(S_{\varepsilon_n, aux})$ satisfying $I_{\varepsilon_n}(u_n, v_n) \to m_0$. Then there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that the sequence $(w_n, z_n) := (u_n(\cdot + \widetilde{y}_n), v_n(\cdot + \widetilde{y}_n))$ has a convergent subsequence in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Moreover, up to a subsequence, $(y_n) := (\varepsilon_n \widetilde{y}_n)$ is such that $y_n \to y \in M$.*

Proof. We star by proving that there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ and constants $R, \eta > 0$ such that

$$\liminf_{n \to \infty} \int_{B_R(\widetilde{y}_n)} \left(|u_n|^2 + |v_n|^2\right) dx \geq \eta. \tag{8}$$

Indeed, suppose that (8) is not satisfied. Since $((u_n, v_n))$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, then, from [15, Lemma I.1], we get

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^s dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |v_n|^s dx = 0$$

for all $s \in (2, 2^*)$. This and (Q_1) implies that

$$\lim_{n \to \infty} \int_{\Lambda_\varepsilon} (u_n Q_u(u_n, v_n) + v_n Q_v(u_n, v_n))\, dx = 0.$$

Since $I'_{\varepsilon_n}(u_n,v_n)(u_n,v_n)=0$, we can use (H_3) for to get

$$\|(u_n,v_n)\|_{\varepsilon_n}\to 0 \quad \text{as} \quad n\to\infty.$$

Then, by using $(H_1),(H_2),(H_3)$ and the above limit, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} H(\varepsilon_n x,u_n,v_n)dx=0.$$

Hence, $I_{\varepsilon_n}(u_n,v_n)\to 0$, contradicting $m_0>0$. Thus, (8) holds and, along a subsequence

$$(w_n,z_n):=(u_n(\cdot+\widetilde{y}_n),v_n(\cdot+\widetilde{y}_n))\rightharpoonup(w,z)\neq(0,0) \;\; \text{weakly in } H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N).$$

Let $t_n>0$ be such that $(\widetilde{w}_n,\widetilde{z}_n):=t_n(w_n,z_n)\in\mathcal{M}$.

Defining $y_n:=\varepsilon_n\widetilde{y}_n$, using again the fact that $I'_{\varepsilon_n}(u_n,v_n)(u_n,v_n)=0$, the change of variables $x\mapsto x+\widetilde{y}_n$ and (ab_1) we get

$$\begin{aligned}
m_0 &\leq I_0(\widetilde{w}_n,\widetilde{z}_n)\\
&\leq \frac{t_n^2}{2}\int_{\mathbb{R}^N}\left(a(\varepsilon_n x+y_n)|\nabla w_n|^2+b(\varepsilon_n x+y_n)|\nabla z_n|^2+|w_n|^2+|z_n|^2\right)dx\\
&\quad -\int_{\mathbb{R}^N}H(\varepsilon_n x+y_n,t_nw_n,t_nz_n)dx\\
&= \frac{t_n^2}{2}\int_{\mathbb{R}^N}\left(a(\varepsilon_n x)|\nabla u_n|^2+b(\varepsilon_n x)|\nabla v_n|^2+|u_n|^2+|v_n|^2\right)dx\\
&\quad -\int_{\mathbb{R}^N}H(\varepsilon_n x,t_nu_n,t_nv_n)dx\\
&= I_{\varepsilon_n}(t_nu_n,t_nv_n)\leq I_{\varepsilon_n}(u_n,v_n).
\end{aligned}$$

Hence $\lim_{n\to\infty} I_0(\widetilde{w}_n,\widetilde{z}_n)=m_0$, from which it follows that $(\widetilde{w}_n,\widetilde{z}_n)\not\to(0,0)$ in the space $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.

Since $((\widetilde{w}_n,\widetilde{z}_n))$ and $((w_n,z_n))$ are bounded in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, the sequence (t_n) is bounded. Thus, up to a subsequence, $t_n\to t_0\geq 0$. If $t_0=0$ then $\|(\widetilde{w}_n,\widetilde{z}_n)\|\to 0$, which does not make sense. Hence, $t_0>0$, and therefore the sequence $((\widetilde{w}_n,\widetilde{z}_n))$ satisfies

$$(\widetilde{w}_n,\widetilde{z}_n)\rightharpoonup(\widetilde{w},\widetilde{z}):=(t_0w,t_0z)\neq(0,0) \;\; \text{weakly in} \;\; H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N).$$

From Proposition 3.6, we conclude that $(\widetilde{w}_n,\widetilde{z}_n)\to(\widetilde{w},\widetilde{z})$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ and, as a consequence, $(w_n,z_n)\to(w,z)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. This proves the first part of the proposition.

To conclude the proof of the proposition, we show that (y_n) has a subsequence, still denoted by (y_n), satisfying $y_n\to y$ for $y\in M$. First of all, we claim that (y_n) is bounded. Indeed, if this is not the case, there exists a subsequence, still denoted by (y_n), verifying $|y_n|\to\infty$. Consider $r>0$ such that $\Lambda\subset B_r(0)$. Since we may suppose that $|y_n|>2r$, for any $x\in B_{r/\varepsilon_n}(0)$ we have

$$|\varepsilon_n x+y_n|\geq|y_n|-|\varepsilon_n x|>r.$$

Since $I'_{\varepsilon_n}(u_n,v_n)(u_n,v_n)=0$, we can use (ab_1), the change of variables $x\mapsto x+\widetilde{y}_n$ and (H_3) to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(a_m|\nabla w_n|^2+b_m|\nabla z_n|^2+|w_n|^2+|z_n|^2\right)dx\\
\le\ &\int_{\mathbb{R}^N}\left(a(\varepsilon_n x)|\nabla u_n|^2+b(\varepsilon_n x)|\nabla v_n|^2+|u_n|^2+|v_n|^2\right)dx\\
=\ &\int_{\mathbb{R}^N}\left(u_nH_u(\varepsilon_n x,u_n,v_n)+v_nH_v(\varepsilon_n x,u_n,v_n)\right)dx\\
=\ &\int_{\mathbb{R}^N}\left(w_nH_w(\varepsilon_n x+y_n,w_n,z_n)+z_nH_z(\varepsilon_n x+y_n,w_n,z_n)\right)dx\\
\le\ &\frac{1}{4}\int_{B_{r/\varepsilon_n}(0)}\left(|w_n|^2+|z_n|^2\right)dx+o_n(1).
\end{aligned}$$

It follows that $(w_n,z_n)\to(0,0)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, which is a contradiction because $m_0>0$. Thus, up to a subsequence, $y_n\to y\in\mathbb{R}^N$.

It remains to check that $y\in M$. It is sufficient to show that $a(y)=a_m$ and $b(y)=b_m$. Arguing by contradiction again, we suppose that $a(y)>a_m$ or $b(y)>b_m$. Then, recalling $(\widetilde{w}_n,\widetilde{z}_n)\to(\widetilde{w},\widetilde{z})$, we can use Fatou's lemma to obtain

$$\begin{aligned}
m_0 &= I_0(\widetilde{w},\widetilde{z})\\
&< \frac{1}{2}\int_{\mathbb{R}^N}\left(a(y)|\nabla\widetilde{w}|^2+b(y)|\nabla\widetilde{z}|^2+|\widetilde{w}|^2+|\widetilde{z}|^2\right)dx-\int_{\mathbb{R}^N}Q(\widetilde{w},\widetilde{z})dx\\
&\le \liminf_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}^N}\left(a(\varepsilon_n x+y_n)|\nabla\widetilde{w}_n|^2+b(\varepsilon_n x+y_n)|\nabla\widetilde{z}_n|^2+|\widetilde{w}_n|^2+|\widetilde{z}_n|^2\right)dx\\
&\quad -\lim_{n\to\infty}\int_{\mathbb{R}^N}Q(\widetilde{w}_n,\widetilde{z}_n)dx\\
&\le \liminf_{n\to\infty}\left[\frac{t_n^2}{2}\int_{\mathbb{R}^N}\left(a(\varepsilon_n x+y_n)|\nabla w_n|^2+b(\varepsilon_n x+y_n)|\nabla z_n|^2+|w_n|^2+|z_n|^2\right)dx\right.\\
&\quad\left.-\int_{\mathbb{R}^N}H(\varepsilon_n x+y_n,t_nw_n,t_nz_n)dx\right]\\
&= \liminf_{n\to\infty}I_{\varepsilon_n}(t_nu_n,t_nv_n)\le\liminf_{n\to\infty}I_{\varepsilon_n}(u_n,v_n)=m_0
\end{aligned}$$

which does not make sense. The proof is finished. □

From Proposition 3.6, there exists $(w_1,w_2)\in X$ solution of the system (S_0). For any $y\in M$, we define the function $\Psi_{i,\varepsilon,y}\in X_\varepsilon$ by setting

$$\Psi_{i,\varepsilon,y}(x):=\psi(\varepsilon x)w_i\left(\frac{\varepsilon x-y}{\varepsilon}\right),\quad i=1,2,$$

where ψ is a smoth function with compact support and with $\psi\equiv 1$ in a neighborhood of y. We denote by $t_\varepsilon>0$ the unique positive number verifying

$$I_\varepsilon(t_\varepsilon(\Psi_{1,\varepsilon,y},\Psi_{2,\varepsilon,y}))=\max_{t\ge 0}I_\varepsilon(t(\Psi_{1,\varepsilon,y},\Psi_{2,\varepsilon,y})).$$

In view of the above remarks the function $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ given by

$$\Phi_\varepsilon(y) := t_\varepsilon(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y})$$

is well defined. By direct calculations and the change of variables $z = (\varepsilon x - y)/\varepsilon$, it easy to check that

$$I_\varepsilon(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y}) = I_0(w_1, w_2) + o_\varepsilon(1).$$

Thus, $\max_{t\geq 0} I_\varepsilon(t(\Psi_{1,\varepsilon,y}, \Psi_{2,\varepsilon,y})) = m_0 + o_\varepsilon(1)$, which implies that

$$\lim_{\varepsilon\to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = m_0, \quad \text{uniformly for} \quad y \in M. \tag{9}$$

Now, we take the function $h : [0,\infty) \to [0,\infty)$ sucth that $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and set

$$\Sigma_\varepsilon := \{(u,v) \in \mathcal{N}_\varepsilon : I_\varepsilon(u,v) \leq m_0 + h(\varepsilon)\}.$$

Given $y \in M$, we can use (9) to conclude that $h(\varepsilon) = |I_\varepsilon(\Phi_\varepsilon(y)) - m_0|$ is such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, $\Phi_\varepsilon(y) \in \Sigma_\varepsilon$ and we have that $\Sigma_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$.

Lemma 3.8. *Let (ε_n) be a sequence such that $\varepsilon_n \to 0^+$ and for each $n \in \mathbb{N}$, let $(u_n, v_n) \in \Sigma_{\varepsilon_n}$ be a solution of system $(S_{\varepsilon_n,aux})$. Then $I_{\varepsilon_n}(u_n, v_n) \to m_0$ and $(u_n, v_n) \in L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$. Moreover, given $\epsilon > 0$, there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\|(w_n, z_n)\|_{L^\infty(\mathbb{R}^N\setminus B_R(0))\times L^\infty(\mathbb{R}^N\setminus B_R(0))} < \epsilon,$$

where $w_n(x) = u_n(x + \widetilde{y}_n)$, $z_n(x) = v_n(x + \widetilde{y}_n)$ and $(\widetilde{y}_n)$ is the sequence of Proposition 3.7.

Proof. The proof is similar to that presented in [13, Lemma 5.1]. □

We are now ready to present the proof of our existence and concentration theorem.

Proof of Theorem 1.1. From Theorem 3.5, for all $\varepsilon > 0$, there exists $(u_\varepsilon, v_\varepsilon) \in X_\varepsilon$ non-negative weak solution of the system $(S_{\varepsilon,aux})$. We first claim is that there exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, there holds

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \;\text{ for each }\; x \in \mathbb{R}^N \setminus \Lambda_\varepsilon.$$

In order to prove the claim, we argue by contradiction. Suppose that for some sequence $\varepsilon_n \to 0$ we can obtain (u_n, v_n) solution of $(S_{\varepsilon_n,aux})$ such that $I_{\varepsilon_n}(u_n, v_n) \to m_0$ and

$$\|(u_n, v_n)\|_{L^\infty(\mathbb{R}^N\setminus\Lambda_{\varepsilon_n})\times L^\infty(\mathbb{R}^N\setminus\Lambda_{\varepsilon_n})} > \alpha. \tag{10}$$

Then, we can use Proposition 3.7 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that we have $\varepsilon_n\widetilde{y}_n \to y_0 \in M$.

If we take $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Lambda$ we have that

$$B_{r/\varepsilon_n}(y_0/\varepsilon_n) = \frac{1}{\varepsilon_n} B_r(y_0) \subset \Lambda_\varepsilon.$$

Moreover, for any $z \in B_{r/\varepsilon}(\widetilde{y}_n)$, we have

$$\left|z - \frac{y_0}{\varepsilon_n}\right| < \frac{2r}{\varepsilon_n}$$

for n large. For these values of n we have that $B_{r/\varepsilon_n}(\widetilde{y}_n) \subset \Lambda_{\varepsilon_n}$. Thus, by applying Lemma 3.8 with $\epsilon = \alpha$ that, for any $n \geq n_0$ such that $r/\varepsilon_n > R$, we have

$$\|(u_n, v_n)\|_{L^\infty(\mathbb{R}^N \setminus \Lambda_\varepsilon) \times L^\infty(\mathbb{R}^N \setminus \Lambda_\varepsilon)} \leq \|(u_n, v_n)\|_{L^\infty(\mathbb{R}^N \setminus B_R(\widetilde{y}_n)) \times L^\infty(\mathbb{R}^N \setminus B_R(\widetilde{y}_n))} < \alpha$$

which contradicts (10). Thus, for $\varepsilon \in (0, \varepsilon_0)$, $H(\cdot, u_\varepsilon, v_\varepsilon) \equiv Q(u_\varepsilon, v_\varepsilon)$. Hence, $(u_\varepsilon, v_\varepsilon)$ is also a solution of system $(\widehat{S}_\varepsilon)$. An easy calculation shows that $(u_\varepsilon(\cdot/\varepsilon), v_\varepsilon(\cdot/\varepsilon))$ is a solution of the original system (S_ε).

To finish the proof, we consider $\varepsilon_n \to 0^+$ and we take $(u_n, v_n) \in X_{\varepsilon_n}$ be a non-negative solution of $(\widehat{S}_{\varepsilon_n})$ as above. By applying Lemma 3.8 we obtain $R > 0$ and $(\widetilde{y}_n) \subset \mathbb{R}^N$ such that

$$\|(u_n, v_n)\|_{L^\infty(\mathbb{R}^N \setminus B_R(\widetilde{y}_n)) \times L^\infty(\mathbb{R}^N \setminus B_R(\widetilde{y}_n))} < \epsilon.$$

Then, up to a subsequence, we can also assume that

$$\|(u_n, v_n)\|_{L^\infty(B_R(\widetilde{y}_n)) \times L^\infty(B_R(\widetilde{y}_n))} > \epsilon. \tag{11}$$

In the contrary case we have $\|(u_n, v_n)\|_{L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)} < \epsilon$.

Then, since (u_n, v_n) of solution of $(\widehat{S}_{\varepsilon_n})$ we can use (Q_1) to obtain

$$\begin{aligned}
&\int_{B_R(\widetilde{y}_n)} \left(|u_n|^2 + |v_n|^2\right) dx \leq \|(u_n, v_n)\|^2_{\varepsilon_n} \\
= \;& \int_{\mathbb{R}^N} \big(u_n Q_u(u_n, v_n) + v_n Q_v(u_n, v_n)\big) dx \\
\leq \;& 2c_1 \int_{\mathbb{R}^N} (|u_n|^{p_1} + |v_n|^{p_1})\, dx + 2c_1 \int_{\mathbb{R}^N} (|u_n|^{p_2} + |v_n|^{p_2})\, dx \\
\leq \;& 2c_1 \|(u_n, v_n)\|^{p_1 - 2}_{L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \left(|u_n|^2 + |v_n|^2\right) dx \\
& + 2c_1 \|(u_n, v_n)\|^{p_2 - 2}_{L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \left(|u_n|^2 + |v_n|^2\right) dx
\end{aligned}$$

which is a contradiction because of (8), for $n \geq n_0$. Thus, (11) holds.

From (11) we conclude that the maximum point $\pi_{n,a} \in \mathbb{R}^N$ of u_n and the maximum point $\pi_{n,b} \in \mathbb{R}^N$ of v_n belong to $B_R(\widetilde{y}_n)$. Hence $\pi_{n,a} = \widetilde{y}_n + q_{n,a}$, for some $q_{n,a} \in B_R(0)$ and $\pi_{n,b} = \widetilde{y}_n + q_{n,b}$, for some $q_{n,b} \in B_R(0)$. Recalling that the associated solution of (S_{ε_n}) is of the form $(\widehat{u}_n(x), \widehat{v}_n(x)) = (u_n(x/\varepsilon_n), v_n(x/\varepsilon_n))$, we conclude that the maximun point $\Pi_{\varepsilon_n,a}$ of $\widehat{u}_n$ and the maximum point $\Pi_{\varepsilon_n,b}$ of $\widehat{v}_n$ are

$$\Pi_{\varepsilon_n,a} := \varepsilon_n \widetilde{y}_n + \varepsilon_n q_{n,a} \quad \text{and} \quad \Pi_{\varepsilon_n,b} := \varepsilon_n \widetilde{y}_n + \varepsilon_n q_{n,b}.$$

Since $(q_{n,a}), (q_{n,b}) \subset B_R(0)$ are bounded and $\varepsilon_n \widetilde{y}_n \to y_0 \in M$, we obtain

$$\lim_{n \to \infty} a(\Pi_{\varepsilon_n,a}) = a(y_0) = a_m \quad \text{and} \quad \lim_{n \to \infty} b(\Pi_{\varepsilon_n,b}) = b(y_0) = b_m.$$

The last limits imply that the maximum points of $\widehat{u}_n$ and $\widehat{v}_n$ are concentrated around the set M. □

4. Critical case

In this section we present the proof of Theorem 1.2. Since many calculations are adaptions to that present in the two early sections, we will emphasis only the differences between the subcritical and critical case.

Since we are interested in non-negative solutions we extend the function K to the whole $\mathbb{R}^2$ by setting $K(u,v)=0$ if $u\le 0$ or $v\le 0$.

Hereafter, we will work with the following system equivalent to (C_ε).

$$(\widehat{C}_\varepsilon)\qquad \begin{cases} -\mathrm{div}(a(\varepsilon x)\nabla u)+u=Q_u(u,v)+\frac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N,\\ -\mathrm{div}(b(\varepsilon x)\nabla v)+v=Q_v(u,v)+\frac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N,\\ u,v\in H^1(\mathbb{R}^N), u(x),v(x)\ge 0 \text{ for each } x\in\mathbb{R}^N.\end{cases}$$

From (K_0) we have that the function K verifies the following identity

$$2^*K(s,t)=sK_s(s,t)+tK_t(s,t). \tag{12}$$

By using the function η given in (1), we define $\widehat{K}:\mathbb{R}^2\to\mathbb{R}$ by

$$\widehat{K}(s,t):=\eta(|(s,t)|)\Big(Q(s,t)+\frac{1}{2^*}K(s,t)\Big)+(1-\eta(|(s,t)|))\widetilde{A}(s^2+t^2),$$

where

$$\widetilde{A}:=\max\left\{\frac{Q(s,t)+\frac{1}{2^*}K(s,t)}{s^2+t^2}:(s,t)\in\mathbb{R}^2,\ \alpha\le|(s,t)|\le 5\alpha\right\}.$$

We define $\widetilde{H}:\mathbb{R}^N\times\mathbb{R}^2\to\mathbb{R}$ by setting

$$\widetilde{H}(x,s,t):=\chi_\Lambda(x)\Big(Q(s,t)+\frac{1}{2^*}K(u,v)\Big)+(1-\chi_\Lambda(x))\widehat{K}(s,t). \tag{13}$$

Using (Q_1), (Q_2), (K_1) and (12) we can argue as the proof of Lemma 2.1 to get

Lemma 4.1. *The function $\widetilde{H}$ satisfies the following estimates:*

$(\widetilde{H}_1)$ $\mu\widetilde{H}(x,s,t)\le s\widetilde{H}_s(x,s,t)+t\widetilde{H}_t(x,s,t)$, *for each* $x\in\Lambda$.

$(\widetilde{H}_2)$ $2\widetilde{H}(x,s,t)\le s\widetilde{H}_s(x,s,t)+t\widetilde{H}_t(x,s,t)$, *for each* $x\in\mathbb{R}^N\setminus\Lambda$.

$(\widetilde{H}_3)$ *For α small we have* $s\widetilde{H}_s(x,s,t)+t\widetilde{H}_t(x,s,t)\le\frac{1}{4}(s^2+t^2)$ *for each* $x\in\mathbb{R}^N\setminus\Lambda$.

$(\widetilde{H}_4)$ *For α small we have* $\frac{|\widetilde{H}_s(x,s,t)|}{\alpha},\frac{|\widetilde{H}_t(x,s,t)|}{\alpha}\le\frac{1}{4}$ *for each* $x\in\mathbb{R}^N\setminus\Lambda$.

In view of definition (13), we deal in the sequel with the modified system

$$(C_{\varepsilon,aux})\qquad \begin{cases} -\mathrm{div}(a(\varepsilon x)\nabla u)+u=\widetilde{H}_u(\varepsilon x,u,v) \text{ in } \mathbb{R}^N,\\ -\mathrm{div}(b(\varepsilon x)\nabla v)+v=\widetilde{H}_v(\varepsilon x,u,v) \text{ in } \mathbb{R}^N,\\ u,v\in H^1(\mathbb{R}^N), u(x),v(x)\ge 0 \text{ for each } x\in\mathbb{R}^N,\end{cases}$$

and we will look for solutions $(u_\varepsilon, v_\varepsilon)$ verifying

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \le \alpha, \quad \text{for each} \ \ x \in \mathbb{R}^N \setminus \Lambda_\varepsilon.$$

The conditions $(\widetilde{H}_3)$, (Q_1) and (K_1) imply that the critical points of the C^1-functional $\widetilde{I}_\varepsilon : X_\varepsilon \to \mathbb{R}$ given by

$$\widetilde{I}_\varepsilon(u,v) := \frac{1}{2}\int_{\mathbb{R}^N} \left(a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|\nabla v|^2 + |u|^2 + |v|^2\right) dx - \int_{\mathbb{R}^N} \widetilde{H}(\varepsilon x, u, v) dx$$

are weak solution of $(C_{\varepsilon,aux})$.

As in Lemma 3.1, the functional $\widetilde{I}_\varepsilon$ satisfies the mountain Pass Geometry. Hence there exists a Palais-Smale sequence $((u_n, v_n)) \subset X_\varepsilon$ at level $\widetilde{c}_\varepsilon$. The minimax level $\widetilde{c}_\varepsilon$ is given by

$$\widetilde{c}_\varepsilon := \inf_{\eta \in \widetilde{\Gamma}} \max_{t\in[0,1]} \widetilde{I}_\varepsilon(\eta(t)),$$

where $\widetilde{\Gamma} := \{\eta \in C([0,1], X_\varepsilon) : \eta(0) = (0,0), \widetilde{I}_\varepsilon(\eta(1)) < 0\}$.

Also, using (Q_3) we can show that

$$0 < \widetilde{c}_\varepsilon = \inf_{(u,v)\in X_\varepsilon\setminus\{(0,0)\}} \max_{t\ge 0} \widetilde{I}_\varepsilon(tu, tv) = \inf_{(u,v)\in \widetilde{\mathcal{N}}_\varepsilon} \widetilde{I}_\varepsilon(u,v) := \widetilde{m}_\varepsilon, \tag{14}$$

where $\widetilde{\mathcal{N}}_\varepsilon := \{(u,v) \in X_\varepsilon \setminus \{(0,0)\} : \widetilde{I}'_\varepsilon(u,v)(u,v) = 0\}$.

As usual, we denote by S the best constant of the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Now we consider

$$\widetilde{S}_K := \inf_{\substack{u,v\in D^{1,2}(\mathbb{R}^N)\\ u,v\neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx}{\left(\displaystyle\int_{\mathbb{R}^N} K(u,v) dx\right)^{2/2^*}}.$$

Lemma 4.2. *Any sequence $((u_n, v_n)) \subset X_\varepsilon$ such that*

$$\widetilde{I}_\varepsilon(u_n, v_n) \to \widetilde{c}_\varepsilon < \frac{1}{N}\left(\min\{a_m, b_m\}\widetilde{S}_K\right)^{N/2} \qquad \text{and} \qquad \widetilde{I}'_\varepsilon(u_n, v_n) \to 0$$

possesses a convergent subsequence.

Proof. Standart calculations show that $((u_n, v_n))$ is bounded in X_ε. Then, up to a subsequence, we may suppose that

$$\begin{aligned} &(u_n, v_n) \rightharpoonup (u,v) \ \ \text{weakly in} \ \ X_\varepsilon, \\ &u_n \to u, \ v_n \to v \ \ \text{strongly in} \ \ L^s_{\text{loc}}(\mathbb{R}^N), \ 2 \le s < 2^*, \\ &u_n(x) \to u(x), \ v_n(x) \to v(x) \ \ \text{for a.e.} \ \ x \in \mathbb{R}^N. \end{aligned} \tag{15}$$

Now, using density argument, we can show that (u,v) is a critical point of $\widetilde{I}_\varepsilon$. Hence

$$\|(u,v)\|^2_\varepsilon = \int_{\mathbb{R}^N} \left(u\widetilde{H}_u(\varepsilon x, u, v) + v\widetilde{H}_v(\varepsilon x, u, v)\right) dx. \tag{16}$$

Since $\widetilde{I}'_\varepsilon(u_n, v_n) \to 0$, we have

$$\|(u_n, v_n)\|_\varepsilon^2 = \int_{\mathbb{R}^N} \Big(u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)\Big)\, dx + o_n(1). \tag{17}$$

Claim 1. $\lim\limits_{n\to\infty} \displaystyle\int_{\Lambda_\varepsilon} K(u_n, v_n)dx = \int_{\Lambda_\varepsilon} K(u, v)dx.$

Since $((u_n, v_n))$ is bounded, we may suppose that

$$|\nabla u_n|^2 \rightharpoonup \widetilde{\mu}, \quad |\nabla v_n|^2 \rightharpoonup \widetilde{\sigma} \quad \text{and} \quad K(u_n, v_n) \rightharpoonup \widetilde{\nu} \;\; \text{(weak*-sense of measures).}$$

In the same way as in [10, Lemma 6], we obtain an at most countable index set Γ, sequences $(x_i) \in \mathbb{R}^N$, $(\widetilde{\mu}_i)$, $(\widetilde{\sigma}_i)$, $(\widetilde{\nu}_i) \subset (0, \infty)$ such that

$$\begin{aligned} \widetilde{\mu} \geq |\nabla u|^2 + \sum_{i\in\Gamma} \widetilde{\mu}_i \delta_{x_i}, \quad \widetilde{\sigma} \geq |\nabla v|^2 + \sum_{i\in\Gamma} \widetilde{\sigma}_i \delta_{x_i} \\ \widetilde{\nu} = K(u, v) + \sum_{i\in\Gamma} \widetilde{\nu}_i \delta_{x_i} \quad \text{and} \quad \widetilde{S}_K \widetilde{\nu}_i^{2/2^*} \leq \widetilde{\mu}_i + \widetilde{\sigma}_i \end{aligned} \tag{18}$$

for all $i \in \Gamma$, where δ_{x_i} is the Dirac mass at the point $x_i \in \mathbb{R}^N$.

Suppose that $\{x_i\}_{i\in\Gamma} \cap \Lambda_\varepsilon \neq \emptyset$, then exists $x_i \in \Lambda_\varepsilon$ for some $i \in \Gamma$. Define, for $\varrho > 0$, the function $\psi_\varrho(x) := \psi((x - x_i)/\varrho)$ where $\psi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ is such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$ and $|\nabla\psi|_\infty \leq 2$. We suppose that ϱ is chosen in such way that the support of ψ_ϱ is contained in Λ_ε. Since $((\psi_\varrho u_n, \psi_\varrho v_n))$ is bounded, $\widetilde{I}'_\varepsilon(u_n, v_n)(\psi_\varrho u_n, \psi_\varrho v_n) = o_n(1)$. Then

$$\begin{aligned} & \int_{\mathbb{R}^N} \big(a(\varepsilon x)\psi_\varrho |\nabla u_n|^2 + b(\varepsilon x)\psi_\varrho |\nabla v_n|^2\big)\, dx \\ & + \int_{\mathbb{R}^N} (a(\varepsilon x) u_n \nabla u_n \nabla \psi_\varrho + b(\varepsilon x) v_n \nabla v_n \nabla\psi_\varrho)\, dx + \int_{\mathbb{R}^N} \big(\psi_\varrho |u_n|^2 + \psi_\varrho |v_n|^2\big)\, dx \\ = & \int_{\mathbb{R}^N} \Big(u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)\Big)\, \psi_\varrho dx + o_n(1). \end{aligned}$$

Since $\operatorname{supp}(\psi_\varrho) \subset \Lambda_\varepsilon$, we can use definition of $\widetilde{H}$, (12) and (ab_1) to get

$$\begin{aligned} & \min\{a_m, b_m\} \int_{\mathbb{R}^N} \big(\psi_\varrho |\nabla u_n|^2 + \psi_\varrho |\nabla v_n|^2\big)\, dx \\ \leq & -\int_{\mathbb{R}^N} (a(\varepsilon x) u_n \nabla u_n \nabla\psi_\varrho + b(\varepsilon x) v_n \nabla v_n \nabla \psi_\varrho)\, dx \\ & + \int_{\mathbb{R}^N} (u_n Q_u(u_n, v_n) + v_n Q_v(u_n, v_n))\, \psi_\varrho dx + \int_{\mathbb{R}^N} K(u_n, v_n)\psi_\varrho dx + o_n(1). \end{aligned}$$

By using (Q_1) and since ψ_ϱ has compact support, we can let $n \to \infty$, $\varrho \to 0$ and use (18) to conclude that

$$\min\{a_m, b_m\}(\widetilde{\mu}_i + \widetilde{\sigma}_i) \leq \widetilde{\nu}_i.$$

As $\widetilde{S}_K \widetilde{\nu}_i^{2/2^*} \leq \widetilde{\mu}_i + \widetilde{\sigma}_i$, we get

$$\left(\min\{a_m, b_m\}\widetilde{S}_K\right)^{N/2} \leq \widetilde{\nu}_i.$$

On the other hand, by using $\widetilde{H}_3$, (Q_2) and (12) we get

$$\begin{aligned}
\widetilde{c}_\varepsilon &= \widetilde{I}_\varepsilon(u_n, v_n) - \frac{1}{2}\widetilde{I}'_\varepsilon(u_n, v_n)(u_n, v_n) + o_n(1) \\
&= \int_{\mathbb{R}\setminus\Lambda_\varepsilon} \left(\frac{1}{2}\left(u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)\right) - \widetilde{H}(\varepsilon x, u_n, v_n)\right) dx \\
&\quad + \int_{\Lambda_\varepsilon} \left(\frac{1}{2}\left(u_n Q_u(u_n, v_n) + v_n Q_v(u_n, v_n)\right) - Q(u_n, v_n)\right) dx \\
&\quad + \frac{1}{2^*}\int_{\Lambda_\varepsilon} \left(\frac{1}{2}\left(u_n K_u(u_n, v_n) + v_n K_v(u_n, v_n)\right) - K(u_n, v_n)\right) dx + o_n(1) \\
&\geq \frac{1}{N}\int_{\Lambda_\varepsilon} K(u_n, v_n) dx + o_n(1) \\
&\geq \frac{1}{N}\int_{\Lambda_\varepsilon} K(u_n, v_n)\psi_\varrho dx + o_n(1).
\end{aligned}$$

Thus, by taking the limit as $n \to \infty$ and using (18) we get

$$\widetilde{c}_\varepsilon \geq \frac{1}{N} \sum_{\{i\in\Gamma : x_i \in \Lambda_\varepsilon\}} \psi_\varrho(x_i)\widetilde{\nu}_i = \frac{1}{N} \sum_{\{i\in\Gamma : x_i \in \Lambda_\varepsilon\}} \widetilde{\nu}_i \geq \frac{1}{N}\left(\min\{a_m, b_m\}\widetilde{S}_K\right)^{N/2}$$

which does not make sense. Therefore $\{x_i\}_{i\in\Gamma} \cap \Lambda_\varepsilon = \emptyset$, concluding the proof of the claim.

Claim 2. The following limit holds as $n \to \infty$:

$$\int_{\mathbb{R}^N} [u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)] dx \to \int_{\mathbb{R}^N} [u \widetilde{H}_u(\varepsilon x, u, v) + v \widetilde{H}_v(\varepsilon x, u, v)] dx.$$

Arguing as in Lemma 3.3, for any $\xi > 0$ given, there exists $R > 0$ such that $\Lambda_\varepsilon \subset B_R(0)$ and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R(0)} \left(a(\varepsilon x)|\nabla u_n|^2 + b(\varepsilon x)|\nabla v_n|^2 + |u_n|^2 + |v_n|^2\right) dx < \xi.$$

This inequality and $(\widetilde{H}_3)$ imply that, for n large enough, we get

$$\int_{\mathbb{R}^N\setminus B_R(0)} \left(u_n \widetilde{H}_u(\varepsilon x, u_n, v_n) + v_n \widetilde{H}_v(\varepsilon x, u_n, v_n)\right) dx < \frac{1}{4}\xi. \tag{19}$$

On the other hand, taking R large enough, we can suppose that

$$\left|\int_{\mathbb{R}^N\setminus B_R(0)} \left(u \widetilde{H}_u(\varepsilon x, u, v) + v \widetilde{H}_v(\varepsilon x, u, v)\right) dx\right| < \xi. \tag{20}$$

Then, by (19) and (20), we obtain

$$\int_{\mathbb{R}^N\setminus B_R(0)} \Big(u_n\widetilde{H}_u(\varepsilon x,u_n,v_n)+v_n\widetilde{H}_v(\varepsilon x,u_n,v_n)\Big)\,dx$$
$$= \int_{\mathbb{R}^N\setminus B_R(0)} \Big(u\widetilde{H}_v(\varepsilon x,u,v)+v\widetilde{H}_v(\varepsilon x,u,v)\Big)\,dx+o_n(1). \tag{21}$$

Since the set $B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)$ is bounded, we can use $(\widetilde{H}_3)$, (15) and Lebesgue's theorem to conclude that

$$\lim_{n\to\infty}\int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)} \Big(u_n\widetilde{H}_u(\varepsilon x,u_n,v_n)+v_n\widetilde{H}_v(\varepsilon x,u_n,v_n)\Big)\,dx$$
$$= \int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)} \Big(u\widetilde{H}_v(\varepsilon x,u,v)+v\widetilde{H}_v(\varepsilon x,u,v)\Big)\,dx. \tag{22}$$

By using Claim 1, (Q_1), (K_1), (15) and Lebesgue's theorem again, we obtain

$$\lim_{n\to\infty}\int_{\Lambda_\varepsilon} \Big(u_n\widetilde{H}_u(\varepsilon x,u_n,v_n)+v_n\widetilde{H}_v(\varepsilon x,u_n,v_n)\Big)\,dx$$
$$= \int_{\Lambda_\varepsilon} \Big(u\widetilde{H}_v(\varepsilon x,u,v)+v\widetilde{H}_v(\varepsilon x,u,v)\Big)\,dx. \tag{23}$$

From (21), (22) and (23) Claim 2 is proved.

By using (16), Claim 2 and (17), we get $\|(u_n,v_n)\|_\varepsilon^2\to\|(u,v)\|_\varepsilon^2$. In conequence $(u_n,v_n)\to(u,v)$ in X_ε. □

In order to study the concentration of solutions, we first consider the critical version of the problem (S_0), namely

$$(CS_0)\qquad \begin{cases} -a_m\Delta u+u=Q_u(u,v)+\frac{1}{2^*}K_u(u,v) \text{ in } \mathbb{R}^N,\\ -b_m\Delta v+v=Q_v(u,v)+\frac{1}{2^*}K_v(u,v) \text{ in } \mathbb{R}^N,\\ u,v\in H^1(\mathbb{R}^N),\, u(x),v(x)\ge 0 \text{ for each } x\in\mathbb{R}^N,\end{cases}$$

whose solutions are related with the critical points of $\widetilde{I}_0 : X\to\mathbb{R}$ definided as

$$\widetilde{I}_0(u,v):=\frac{1}{2}\|(u,v)\|^2-\int_{\mathbb{R}^N}Q(u,v)dx-\frac{1}{2^*}\int_{\mathbb{R}^N}K(u,v)dx.$$

As in the subcritical case, we can show that $\widetilde{I}_0$ has the Mountain Pass geometry and therefore we can set the minimax level $\widetilde{c}_0$ in the following way

$$\widetilde{c}_0:=\inf_{\eta\in\widetilde{\Gamma}_0}\max_{t\in[0,1]}\widetilde{I}_0(\eta(t)),$$

where $\widetilde{\Gamma}_0:=\{\eta\in C([0,1],X):\eta(0)=(0,0),\widetilde{I}_0(\eta(1))<0\}$.

Using (Q_3) we can also verify that

$$0<\widetilde{c}_0=\inf_{(u,v)\in X\setminus\{(0,0)\}}\max_{t\ge 0}\widetilde{I}_0(tu,tv)=\inf_{(u,v)\in\widetilde{\mathcal{M}}}\widetilde{I}_0(u,v):=\widetilde{m}_0, \tag{24}$$

where $\widetilde{\mathcal{M}}:=\{(u,v)\in X\setminus\{(0,0)\}:\widetilde{I}_0'(u,v)(u,v)=0\}$.

Proposition 4.3. *There exists $\sigma^* > 0$ such that for all $\sigma > \sigma^*$*

$$\widetilde{c}_0 < \frac{1}{N}\left(\min\{a_m, b_m\}\widetilde{S}_K\right)^{N/2}.$$

Proof. From (24) it sufficies to obtain $(u, v) \in X \setminus \{(0, 0)\}$ such that

$$\sup_{t\geq 0} \widetilde{I}_0(tu, tv) < \frac{1}{N}\left(\min\{a_m, b_m\}\widetilde{S}_K\right)^{N/2}.$$

We first recall that, for any $\delta > 0$ the function

$$w_\delta(x) := [\delta N(N-2)]^{(N-2)/4}(\delta + |x|^2)^{(2-N)/2}$$

satisfies

$$\int_{\mathbb{R}^N} |\nabla w_\delta|^2 dx = \int_{\mathbb{R}^N} |w_\delta|^{2^*} dx = S^{N/2}.$$

Using (K_6) and by [10, Lemma 3], there exist $A, B \in \mathbb{R}$ such that $\widetilde{S}_K$ is attained by

$$\widetilde{S}_K = \frac{\int_{\mathbb{R}^N} \left(|\nabla(Aw_\delta)|^2 + |\nabla(Bw_\delta)|^2\right) dx}{\left(\int_{\mathbb{R}^N} K(Aw_\delta, Bw_\delta)dx\right)^{2/2^*}} = \frac{S^{N/2}(A^2 + B^2)}{\left(\int_{\mathbb{R}^N} K(Aw_\delta, Bw_\delta)dx\right)^{2/2^*}}.$$

Let $\eta \in C_0^\infty(\mathbb{R}^N, [0, 1])$ be such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ on $\mathbb{R}\setminus B_2(0)$. Consider

$$\psi_\delta(x) := \frac{\eta(x)w_\delta(x)}{|\eta w_\delta|_{2^*}}.$$

Then, by using (Q_4) we get

$$\begin{aligned}
&\widetilde{I}_0(tA\psi_\delta, tB\psi_\delta) \\
&= \frac{1}{2}\|(tA\psi_\delta, tB\psi_\delta)\|^2 - \int_{\mathbb{R}^N} Q(tA\psi_\delta, tB\psi_\delta)dx - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N} K(A\psi_\delta, B\psi_\delta)dx \\
&\leq \frac{t^2}{2}(A^2 + B^2)D_\delta - \frac{\sigma}{p_5}t^{p_5}A^\beta B^\nu \int_{\mathbb{R}^N} |\psi_\delta|^{p_5}dx,
\end{aligned}$$

where $p_5 \in (2, 2^*)$ is given by condition (Q_4) and

$$D_\delta = \int_{\mathbb{R}^N} \left(\max\{a_m, b_m\}|\nabla\psi_\delta|^2 + |\psi_\delta|^2\right) dx.$$

Thus $\widetilde{I}_0(tA\psi_\delta, tB\psi_\delta) \leq \max_{s\geq 0}\left\{\frac{s^2}{2}(A^2 + B^2)D_\delta - \frac{\sigma}{p_5}s^{p_5}A^\beta B^\nu\|\psi_\delta\|^{p_5}_{L^{p_5}(\mathbb{R}^N)}\right\}.$

Straightforward calculations show that

$$\begin{aligned}
\widetilde{I}_0(tA\psi_\delta, tB\psi_\delta) &\leq \frac{1}{\sigma^{2/(p_5-2)}}\left(\frac{1}{2} - \frac{1}{p_5}\right)\frac{(D_\delta(A^2 + B^2))^{p_5/(p_5-2)}}{\left(A^\beta B^\nu\|\psi_\delta\|^{p_5}_{L^{p_5}(\mathbb{R}^N)}\right)^{2/(p_5-2)}} \\
&= \frac{1}{\sigma^{2/(p_5-2)}}C_\delta.
\end{aligned}$$

Thus, $\max_{t\geq 0} \widetilde{I}_0(tA\psi_\delta, tB\psi_\delta) < \frac{1}{N}\left(\min\{a_m, b_m\}\widetilde{S}_K\right)^{N/2}$, for all $\sigma > \sigma^*$ where

$$\sigma^* := \left(\frac{C_\delta}{\frac{1}{N}\left(\min\{a_m, b_m\}\widetilde{S}_K\right)^{N/2}}\right)^{\frac{p_5-2}{2}}.$$

The proof is complete. □

The proof of the next result is in the same spirit of Proposition 3.6, but in this case, we need of the estimate given by Proposition 4.3.

Proposition 4.4. *Let* $((u_n, v_n)) \subset \widetilde{\mathcal{M}}$ *be such that* $\widetilde{I}_0(u_n, v_n) \to \widetilde{m}_0$. *Then there exists a sequence* $(\widetilde{y}_n) \subset \mathbb{R}^N$ *such that, up to a subsequence,* $(w_n(x), z_n(x)) := (u_n(x+\widetilde{y}_n), v_n(x+\widetilde{y}_n)) \to (w, z) \in \widetilde{\mathcal{M}}$ *with* $\widetilde{I}_0(w, z) = \widetilde{c}_0$.

From Proposition 4.4, there exists (w_1, w_2) solution of the system (CS_0). For any $y \in M$, we define the functions

$$u(x) = \psi(\varepsilon x) w_1\left(\frac{\varepsilon x - y}{\varepsilon}\right), \quad \text{and} \quad v(x) = \psi(\varepsilon x) w_2\left(\frac{\varepsilon x - y}{\varepsilon}\right),$$

where ψ is a smooth function with compact support and with $\psi \equiv 1$ in a neighborhood of y. By direct calculations and the change of variables $z = (\varepsilon x - y)/\varepsilon$, it easy to check that

$$\widetilde{I}_\varepsilon(u, v) = \widetilde{I}_0(w_1, w_2) + o_\varepsilon(1) = \widetilde{c}_0 + o_\varepsilon(1).$$

Thus, we can verify that

$$\widetilde{c}_\varepsilon \leq \sup_{t\geq 0} \widetilde{I}_\varepsilon(tu, tv) \leq \widetilde{c}_0 + o_\varepsilon(1). \tag{25}$$

We also have the following technical result. The proof is similar to that presented in Proposition 3.7 and it will be omitted.

Proposition 4.5. *Let* $(\varepsilon_n) \subset \mathbb{R}^+$ *be such that* $\varepsilon_n \to 0^+$ *and* (u_n, v_n) *be a solution of* $(C_{\varepsilon_n,aux})$ *satisfying* $\widetilde{I}_{\varepsilon_n}(u_n, v_n) \to m_0$. *Then there exists a sequence* $(\widetilde{y}_n) \subset \mathbb{R}^N$ *such that the sequence* $(w_n, z_n) := (u_n(\cdot + \widetilde{y}_n), v_n(\cdot + \widetilde{y}_n))$ *has a convergent subsequence in* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. *Moreover, up to a subsequence,* $(y_n) := (\varepsilon_n \widetilde{y}_n)$ *is such that* $y_n \to y \in M$.

As in the subcritical case, we can use Proposition 4.4 for to show that the set

$$\widetilde{\Sigma}_\varepsilon := \{(u, v) \in \widetilde{\mathcal{N}}_\varepsilon : \widetilde{I}_\varepsilon(u, v) \leq \widetilde{m}_0 + h(\varepsilon)\}$$

is not empty.

We are now ready to present the proof of Theorem 1.2.

Proof of Theorem 1.2. From (25) we obtain $\varepsilon_1 > 0$ such that $\widetilde{c}_\varepsilon < \widetilde{c}_0$ for any $\varepsilon \in (0, \varepsilon_1)$. For these values of ε, since $\widetilde{I}_\varepsilon$ has the mountain pass geometry, we can take a sequence $((u_n, v_n)) \subset X_\varepsilon$ such that

$$\widetilde{I}_\varepsilon(u_n, v_n) \to \widetilde{c}_\varepsilon \quad \text{and} \quad \widetilde{I}'_\varepsilon(u_n, v_n) \to 0.$$

By using Proposition 4.3, we garantee that

$$\widetilde{c}_\varepsilon < \frac{1}{N}\left(\min\{a_m, b_m\}\widetilde{S}_K\right)^{N/2}.$$

Thus, from Lemma 4.2 we get that, along a subsequence $(u_n, v_n) \to (u_\varepsilon, v_\varepsilon)$ with $(u_\varepsilon, v_\varepsilon)$ being critical point of $\widetilde{I}_\varepsilon$ and $\widetilde{I}_\varepsilon(u_\varepsilon, v_\varepsilon) = \widetilde{c}_\varepsilon$. The same calculations performed in Theorem 3.5 show that $u_\varepsilon \geq 0$ and $v_\varepsilon \geq 0$ in $\mathbb{R}^N$. Therefore $(u_\varepsilon, v_\varepsilon) \in X_\varepsilon$ is a non-negative weak solution of $(C_{\varepsilon,aux})$.

Now, by using Lemma 3.8 for the system $(C_{\varepsilon,aux})$ and following the same lines as in the proof of Theorem 1.1 we show that there exists $\varepsilon_2 > 0$ such that, for any $0 < \varepsilon < \varepsilon_2$, there holds

$$|(u_\varepsilon(\varepsilon x), v_\varepsilon(\varepsilon x))| \leq \alpha \quad \text{for each} \quad x \in \mathbb{R}^N \setminus \Lambda_\varepsilon.$$

Hence, $\widetilde{H}(\cdot, u_\varepsilon, v_\varepsilon) \equiv Q(u_\varepsilon, v_\varepsilon) + \frac{1}{2^*}K(u_\varepsilon, v_\varepsilon)$ for any $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2\})$, and therefore $(u_\varepsilon, v_\varepsilon)$ is also a non-negative weak solution of the system $(\widehat{C}_\varepsilon)$. An easy calculation shows that $(u_\varepsilon(\cdot/\varepsilon), v_\varepsilon(\cdot/\varepsilon))$ is a solution of the original system (C_ε).

The concentration follows along the same lines as in the proof of Theorem 1.1. $\square$

5. Supercritical case

To solve the system

$$(\widehat{SC}_{\varepsilon,\lambda}) \qquad \begin{cases} -\text{div}(a(\varepsilon x)\nabla u) + u = Q_u(u,v) + \lambda|u|^{q_1-2}u & \text{in } \mathbb{R}^N, \\ -\text{div}(b(\varepsilon x)\nabla v) + v = Q_v(u,v) + \lambda|v|^{q_2-2}v & \text{in } \mathbb{R}^N. \end{cases}$$

We first consider a truncated problem which involves only a subcritical growth. We show that any non-negative weak solution of truncated problem is a non-negative weak solution of $(\widehat{SC}_{\varepsilon,\lambda})$.

Since the growth of nonlinearity is supercritical, we can not use directly variational techniques because the functional correspondind to $(\widehat{SC}_{\varepsilon,\lambda})$ may not be well defined in X_ε. So to overcome this difficulty we construct a suitable truncation

$$(T_{\varepsilon,\lambda}) \qquad \begin{cases} -\text{div}(a(\varepsilon x)\nabla u) + u = \widetilde{K}_u(u,v) & \text{in } \mathbb{R}^N, \\ -\text{div}(b(\varepsilon x)\nabla v) + v = \widetilde{K}_v(u,v) & \text{in } \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), u(x), v(x) \geq 0 & \text{for each } x \in \mathbb{R}^N, \end{cases}$$

where $\widetilde{K}(s,t) = Q(s,t) + \lambda F_1(s) + \lambda F_2(t)$ with

$$F_i(t) = \int_0^t f_i(s)ds, \quad f_i(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ t^{q_i-1} & \text{if } 0 < t \leq \zeta, \\ \zeta^{q_i-p_i}t^{p_i-1} & \text{if } t \geq \zeta, \end{cases} \qquad i = 1, 2,$$

and the constant ζ will be determined later.

Note that $F_i(t) \leq \dfrac{\zeta^{q_i-p_i}}{p_i} t^{p_i}$. By using (Q_1), we can verify that

$$|\widetilde{K}_s(s,t)| + |\widetilde{K}_t(s,t)| \leq (c_1 + \lambda C_\zeta)(|s|^{p_1-1} + |t|^{p_2-1}), \tag{$\widetilde{K}_1$}$$

where $C_\zeta = \max\{\zeta^{q_1-p_1}, \zeta^{q_2-p_2}\}$.

Since $\mu < p_1, p_2 < 2^* < q_1, q_2$, we can use (Q_2) for to get

$$s\widetilde{K}_s(s,t) + t\widetilde{K}_t(s,t) \geq \mu Q(s,t) + \lambda p_1 F_1(s) + \lambda p_2 F_2(t) \geq \mu \widetilde{K}(s,t). \tag{$\widetilde{K}_2$}$$

Hence, $\widetilde{K}$ is a nonlinearity with subcritical growth and $\widetilde{K}$ verifies the conditions $(Q_0) - (Q_3)$. By directly applying Theorem 1.1 we obtain the following existence result for a truncated system $(T_{\varepsilon,\lambda})$

Proposition 5.1. *Let $\lambda > 0$ be fixed. Then, for any $\delta > 0$ given, there exists $\varepsilon_{\lambda,\delta} > 0$ such that, for any $\varepsilon \in (0, \varepsilon_{\lambda,\delta})$, the truncated system $(T_{\varepsilon,\lambda})$ has a nonnegative weak solution.*

Note that the solutions of $(T_{\varepsilon,\lambda})$ are critical points of the functional $I_{\varepsilon,\lambda} : X_\varepsilon \to \mathbb{R}$ given by

$$I_{\varepsilon,\lambda}(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} \left(a(\varepsilon x)|\nabla u|^2 + b(\varepsilon x)|\nabla v|^2 + |u|^2 + |v|^2\right) dx - \int_{\mathbb{R}^N} \widetilde{K}(u,v)dx.$$

Now, let $(u_{\varepsilon,\lambda}, v_{\varepsilon,\lambda})$ be a solution of $(T_{\varepsilon,\lambda})$ given by Proposition 5.1. Thus, by using $(\widetilde{K}_2)$ we have

$$\begin{aligned} c_{\varepsilon,\lambda} &= I_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}, v_{\varepsilon,\lambda}) - \frac{1}{\mu} I'_{\varepsilon,\lambda}(u_{\varepsilon,\lambda}, v_{\varepsilon,\lambda})(u_{\varepsilon,\lambda}, v_{\varepsilon,\lambda}) \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \|(u_{\varepsilon,\lambda}, v_{\varepsilon,\lambda})\|_\varepsilon^2. \end{aligned} \tag{26}$$

Since $I_{\varepsilon,\lambda}$ has the mountain pass geometry, given $(u,v) \in X_\varepsilon \setminus \{(0,0)\}$ there exists $t_0 > 0$ such that $I_{\varepsilon,\lambda}(t_0 u, t_0 v) < 0$. Define a path $\gamma : [0,1] \to tt_0(u,v)$. Clearly $\gamma \in \Gamma = \{\eta \in C([0,1], X_\varepsilon) : \eta(0) = (0,0),\ I_{\varepsilon,\lambda}(\eta(1)) < 0\}$ and as consequence of (Q_3) we have

$$c_{\varepsilon,\lambda} = \inf_{(u,v)\in X_\varepsilon\setminus\{(0,0)\}} \max_{t\geq 0} I_{\varepsilon,\lambda}(tu, tv).$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi \geq 0$. Thus, by using $(\widetilde{Q}_4)$ we have

$$\begin{aligned} c_{\varepsilon,\lambda} &\leq \max_{t\geq 0} I_{\varepsilon,\lambda}(t\varphi, t\varphi) \\ &\leq \max_{t\geq 0} \left[\frac{t^2}{2}\int_{\mathbb{R}^N} \left(a(\varepsilon x)|\nabla\varphi|^2 + b(\varepsilon x)|\nabla\varphi|^2 + 2|\varphi|^2\right) dx - \int_{\mathbb{R}^N} Q(t\varphi, t\varphi)dx\right] \\ &\leq \max_{t\geq 0} \left[t^2 \max\{a_M, b_M, 1\}\|\varphi\|^2_{H^1(\mathbb{R}^N)} - \frac{\sigma}{p_5} t^{p_5} \|\varphi\|^{p_5}_{L^{p_5}(\mathbb{R}^N)}\right] \\ &\leq 2^{2/(p_5-2)} \left(\frac{p_5 - 2}{p_5}\right) \frac{\left(\max\{a_M, b_M, 1\}\|\varphi\|^2_{H^1(\mathbb{R}^N)}\right)^{p_1/(p_5-2)}}{\sigma\|\varphi\|^{p_5}_{L^{p_5}(\mathbb{R}^N)}} =: \widetilde{D}. \end{aligned}$$

Thus, from (26) we get

$$\|(u_{\varepsilon,\lambda}, v_{\varepsilon,\lambda})\|_{\varepsilon}^2 \leq \left(\frac{2\mu}{\mu-2}\right)\widetilde{D} := D. \tag{27}$$

We are now ready to present the proof of Theorem 1.3.

Proof Theorem 1.3. For any $\lambda > 0$ and $\delta > 0$ we can apply Proposition 5.1 to obtain, for any $\varepsilon \in (0, \varepsilon_{\lambda,\delta})$ fixed, $(u_{\varepsilon,\lambda}, v_{\varepsilon,\lambda})$ non-negative solution of $(T_{\varepsilon,\lambda})$. To save notation, we will denote $(u, v) := (u_{\varepsilon,\lambda}, v_{\varepsilon,\lambda})$.

For each $L > 0$, we define

$$u_L(x) = \begin{cases} u(x) & \text{if } u(x) \leq L, \\ L & \text{if } u(x) > L, \end{cases}$$

and

$$v_L(x) = \begin{cases} v(x) & \text{if } v(x) \leq L, \\ L & \text{if } v(x) > L. \end{cases}$$

Consider $w := uu_L^{2(\tau-1)}$ and $z := vv_L^{2(\tau-1)}$, where $\tau > 1$ is a constant to be determined later. Taking $(w, 0)$ as a test function in $(T_{\varepsilon,\lambda})$ we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(a(\varepsilon x)u_L^{2(\tau-1)}|\nabla u|^2 + 2(\tau-1)a(\varepsilon x)uu_L^{2(\tau-1)-1}\nabla u\nabla u_L + u^2u_L^{2(\tau-1)}\right)dx \\ = &\int_{\mathbb{R}^N} uu_L^{2(\tau-1)}\widetilde{K}_u(u,v)dx.\end{aligned}$$

Since

$$\int_{\mathbb{R}^N} a(\varepsilon x)uu_L^{2(\tau-1)-1}\nabla u\nabla u_L dx = \int_{\{u\leq L\}} a(\varepsilon x)u^{2(\tau-1)}|\nabla u|^2 dx \geq 0$$

and $a(x) \geq a_m$ for all $x \in \mathbb{R}^N$ we have

$$a_m\int_{\mathbb{R}^N} u_L^{2(\tau-1)}|\nabla u|^2 dx \leq \int_{\mathbb{R}^N} uu_L^{2(\tau-1)}\widetilde{K}_u(u,v)dx.$$

From $(\widetilde{K}_1)$ we have

$$\lim_{s\to\infty}\frac{\widetilde{K}_s(s,t)}{s^{p_1-1}} \leq c_1 + \lambda C_\zeta \quad \text{and} \quad \lim_{s\to 0}\widetilde{K}_s(s,t) = 0.$$

Thus, there exists $C_1 > 0$ such that

$$a_m\int_{\mathbb{R}^N} u_L^{2(\tau-1)}|\nabla u|^2 dx \leq C_1(c_1 + \lambda C_\zeta)\int_{\mathbb{R}^N} u^{p_1}u_L^{2(\tau-1)}dx.$$

Now, we consider $\widehat{w} := uu_L^{\tau-1}$. By using Sobolev embedding we get

$$\begin{aligned}
\|\widehat{w}\|^2_{L^{2^*}(\mathbb{R}^N)} &\leq C_2 \int_{\mathbb{R}^N} |\nabla \widehat{w}|^2 dx = C_2 \int_{\mathbb{R}^N} |\nabla (uu_L^{\tau-1})|^2 dx \\
&\leq 2C_2 \left(\int_{\mathbb{R}^N} u_L^{2(\tau-1)} |\nabla u|^2 dx + (\tau-1)^2 \int_{\mathbb{R}^N} u^2 u_L^{2(\tau-2)} |\nabla u_L|^2 dx \right) \\
&= 2C_2 \left(\int_{\mathbb{R}^N} u_L^{2(\tau-1)} |\nabla u|^2 dx + (\tau-1)^2 \int_{\{u \leq L\}} u_L^{2(\tau-1)} |\nabla u|^2 dx \right) \\
&\leq 2C_2 \tau^2 \int_{\mathbb{R}^N} u_L^{2(\tau-1)} |\nabla u|^2 dx \\
&\leq 2C_2 a_m^{-1} C_1 (c_1 + \lambda C_\zeta) \tau^2 \int_{\mathbb{R}^N} u^{p_1} u_L^{2(\tau-1)} dx \\
&= C_3 (c_1 + \lambda C_\zeta) \tau^2 \int_{\mathbb{R}^N} u^{p_1-2} |\widehat{w}|^2 dx.
\end{aligned}$$

Applying now Hölder inequality with exponents $\frac{2^*}{p_1-2}$ and $\frac{2^*}{2^*-(p_1-2)} := \frac{\theta}{2}$, we have

$$\|\widehat{w}\|^2_{L^{2^*}(\mathbb{R}^N)} \leq C_3 (c_1 + \lambda C_\zeta) \tau^2 \|u\|^{p_1-2}_{L^{2^*}(\mathbb{R}^N)} \|\widehat{w}\|^2_{L^\theta(\mathbb{R}^N)}.$$

Since $u_L \leq u$, we can use (27) for to get

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} u^{2^*} u_L^{2^*(\tau-1)} dx \right)^{2/2^*} &\leq C_3 (c_1 + \lambda C_\zeta) \tau^2 \|u\|^{p_1-2}_{L^{2^*}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} u^{\tau\theta} dx \right)^{2/\theta} \\
&\leq C_4 (c_1 + \lambda C_\zeta) \tau^2 \|u\|^{2\tau}_{L^{\tau\theta}(\mathbb{R}^N)}.
\end{aligned}$$

By Fatou's lemma in the variable L, we get

$$\|u\|_{L^{\tau 2^*}(\mathbb{R}^N)} \leq C_4^{1/2\tau} (c_1 + \lambda C_\zeta)^{1/2\tau} \tau^{1/\tau} \|u\|_{L^{\tau\theta}(\mathbb{R}^N)} \tag{28}$$

whenever $u^{\tau\theta} \in L^1(\mathbb{R}^N)$.

We now set $\tau := \frac{2^*}{\theta} > 1$ and note that, since $u \in L^{2^*}(\mathbb{R}^N)$, the above inequality holds for this choice of τ. Thus, since $\tau^2\theta = \tau 2^*$, it follows that (28) also holds with τ replaced by τ^2. Hence,

$$\begin{aligned}
\|u\|_{L^{\tau^2 2^*}(\mathbb{R}^N)} &\leq [C_4(c_1 + \lambda C_\zeta)]^{\frac{1}{2\tau^2}} (\tau^2)^{\frac{1}{\tau^2}} \|u\|_{L^{\tau^2\theta}(\mathbb{R}^N)} \\
&\leq [C_4(c_1 + \lambda C_\zeta)]^{\frac{1}{2}\left(\frac{1}{\tau} + \frac{1}{\tau^2}\right)} \tau^{\frac{1}{\tau} + \frac{2}{\tau^2}} \|u\|_{L^{\tau\theta}(\mathbb{R}^N)}.
\end{aligned}$$

By iterating this process and using that $\tau\theta = 2^*$, we obtain

$$\|u\|_{L^{\tau^m 2^*}(\mathbb{R}^N)} \leq [C_4(c_1 + \lambda C_\zeta)]^{2^{-1} \sum_{i=1}^m \tau^{-i}} \tau^{\sum_{i=1}^m i\tau^{-i}} \|u\|_{L^{2^*}(\mathbb{R}^N)}.$$

Taking the limit as $m \to \infty$ and using (27) again, we get

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq [C_5(c_1 + \lambda C_\zeta)]^{\sigma_1} \tau^{\sigma_2},$$

where $$\sigma_1 = \frac{1}{2}\sum_{i=1}^{\infty}\tau^{-i} \quad \text{and} \quad \sigma_2 = \sum_{i=1}^{\infty} i\tau^{-i}.$$

We want to check that, for a suitable value of ζ and λ small enough, we have

$$[C_5(c_1+\lambda C_\zeta)]^{\sigma_1}\,\tau^{\sigma_2} \le \zeta$$

or equivalently $$c_1+\lambda C_\zeta \le C_5^{-1}\tau^{-\sigma_2/\sigma_1}\zeta^{1/\sigma_1} = C_6\zeta^{1/\sigma_1}.$$

So, we choose $\zeta>0$ such that $C_6\zeta^{1/\sigma_1}=2c_1$ and take $\lambda>0$ such that $\lambda\le\lambda_0:=\frac{c_1}{C_\zeta}$.

Thus, $$\|u\|_{L^\infty(\mathbb{R}^N)} \le \zeta.$$

Repeating the same reasoning with the test function $(0,z)$, we have

$$\|v\|_{L^\infty(\mathbb{R}^N)} \le \zeta.$$

Then, in view of the definition of $\widetilde{K}$, we have that $\widetilde{K}(u,v)=Q(u,v)+\frac{\lambda}{q_1}u^{q_1}+\frac{\lambda}{q_2}v^{q_2}$ and therefore $(u_{\varepsilon,\lambda},v_{\varepsilon,\lambda})$ is also a solution of the system $(\widehat{SC}_{\varepsilon,\lambda})$.

As a consequence of Theorem 1.1 we get that $(\widehat{u}_\varepsilon,\widehat{v}_\varepsilon)=(u_{\varepsilon,\lambda}(\cdot/\varepsilon),v_{\varepsilon,\lambda}(\cdot/\varepsilon))$ is a solution of the original system $(SC_{\varepsilon,\lambda})$ and the maximum points of $\widehat{u}_\varepsilon$ and $\widehat{v}_\varepsilon$ are concentrated around the set M. □

References

[1] C. O. Alves: *Local mountain pass for a class of elliptic system*, J. Math. Anal. Appl. 335/1 (2007) 135-50.

[2] C. O. Alves, J. M. B. do Ó, M. A. S. Souto: *Local mountain-pass for a class of elliptic problems in $\mathbb{R}^N$ involving critical growth*, Nonlinear Analysis, Ser. A: Theory Methods 46/4 (2001) 495–510.

[3] C. O. Alves, G. M. Figueiredo: *Multiplicity of positive solutions for a quasilinear problem in $\mathbb{R}^N$ via penalization method*, Adv. Nonlinear Stud. 5/4 (2005) 551–572.

[4] C. O. Alves, G. M. Figueiredo, M. F. Furtado: *Multiplicity of solutions for elliptic systems via local mountain pass method*, Commun. Pure Appl. Analysis 8/6 (2009) 1745–1758.

[5] C. O. Alves, G. M. Figueiredo, M. F. Furtado: *Multiple solutions for critical elliptic systems via penalization method*, Diff. Int. Equations 23/7-8 (2010) 703-723.

[6] V. Ambrosio: *Concentration phenomena for critical fractional Schrödinger systems*, Commun. Pure Appl. Analysis 17/5 (2018) 2085–2123.

[7] V. Ambrosio: *Multiplicity and concentration of solutions for fractional Schrödinger systems via penalization method*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 30/3 (2019) 543–581.

[8] V. Ambrosio: *Multiplicity of solutions for fractional Schrödinger systems in $\mathbb{R}^N$*, Complex Var. Elliptic Equations 65/5 (2020) 856–885.

[9] A. Ambrosetti, P. H. Rabinowitz: *Dual variational methods in critical point theory and applications*, J. Funct. Analysis 14 (1973) 349–381.

[10] D. C. de Morais Filho, M. A. S. Souto: *Systems of p-Laplacean equations involving homogeneous nonlinearities with critical Sobolev exponent degrees*, Comm. Partial Diff. Equations 24 (1999) 1537–1553.

[11] M. del Pino, P. L. Felmer: *Local Mountain Pass for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Diff. Equations 4 (1996) 121–137.

[12] J. M. B. do Ó, M. A. S. Souto: *On a class of nonlinear Schrödinger equations in* $\mathbb{R}^2$ *involving critical growth*, J. Diff. Equations 174/2 (2001) 289–311.

[13] G. M. Figueiredo, S. M. A. Salirrosas: *Concentration, existence of ground state and multiplicity of solutions for a subcritical elliptic system via penalization method*, SN Partial Differ. Equation Appl. 2/1 (2021), art. no. 6, 30 pp.

[14] G. M. Figueiredo, S. M. A. Salirrosas: *On multiplicity and concentration behavior of solutions for a critical system with equations in divergence form*, J. Math. Analysis Appl. 494/1 (2021), art. no. 124446, 30 pp.

[15] P. L. Lions: *The concentration-compacteness principle in te calculus of variations. The locally compact case. Part II*, Ann. Inst. H. Poincaré Anal. Non Linéare I 4 (1984) 223–283.

[16] M. Willem: *Minimax Theorems*, Birkhäuser, Boston (1996).