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Cyclical Contractive Mappings in Hyperbolic Spaces

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We consider a complete metric space of cyclical nonexpansive mappings acting on a union of two sets in a complete hyperbolic space. Using the porosity notion we show that most cyclical nonexpansive mappings are contractive. In the case when the intersection of the sets is empty we show that the distance between iterates of a contractive mapping converges to the distance between these two sets.

Keywords: Complete metric space, cyclical mapping, generic element, iterate, porous set.

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1. Introduction

During more than fifty-five years now, there has been a lot of activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [4, 6, 12, 15, 16, 18, 19, 21, 22, 24, 28, 29, 30, 31, 35, 36, 38, 42, 43, 44, 52, 53] and the references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also covers the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility and common fixed point problems, which find important applications in engineering and medical sciences [5, 7, 8, 9, 14, 23, 45, 46, 52, 53].

In this paper we consider a complete metric space of cyclical nonexpansive mappings acting on a union of two sets in a complete hyperbolic space. The study of this class of mappings is an important topic in the fixed point theory [13, 20, 49, 50, 51]. Using the porosity notion we show that most cyclical nonexpansive mappings are contractive. In the case when the intersection of the sets is empty we show that the distance between iterates of a contractive mapping converges to the distance between these two sets. Namely, our paper has three main results: Theorem 2.4, Theorem 4.2 and Theorem 5.1.

In Theorem 2.4 we show that most cyclical nonexpansive mappings are contractive when the intersection of our two sets is not empty. In Theorem 4.2 we demonstrate that an analogous result is true in the case when the intersection is empty. Our final main result (Theorem 5.1) shows that for every complete metric space the distance between iterates of a contractive mappings converges to the distance between these two sets when the intersection of these two sets is empty.

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Since Theorems 2.4 and 4.2 hold for hyperbolic spaces we we briefly review this concept.

Let (X, ρ) be a metric space and let R^1 denote the real line. We say that a mapping $c : R^1 \to X$ is a *metric embedding* of R^1 into X if $\rho(c(s), c(t)) = |s - t|$ for all real s and t. The image of R^1 under a metric embedding will be called a *metric line*. The image of a real interval $[a, b] = \{t \in R^1 : a \le t \le b\}$ under such a mapping will be called a *metric segment*.

Assume that (X, ρ) contains a family M of metric lines such that for each pair of distinct points x and y in X there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this segment by $[x, y]$. For each $0 \le t \le 1$ there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \text{ and } \rho(z, y) = (1 - t)\rho(x, y).$$

This point will be denoted by $(1 - t)x \oplus ty$. We will say that X, or more precisely (X, ρ, M), is a *hyperbolic space* if

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \le \frac{1}{2}\rho(y, z)$$

for all x, y and z in X. An equivalent requirement is that

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z) \le \frac{1}{2}(\rho(x, w) + \rho(y, z))$$

for all x, y, z and w in X. A set $K \subset X$ is called ρ-convex if $[x, y] \subset K$ for all x and y in K.

In the sequel we will repeatedly use the following fact (cf. pp. 77 and 104 of [16], and [37]): If (X, ρ, M) is a hyperbolic space, then

$$\rho((1 - t)x \oplus tz, (1 - t)y \oplus tw) \le (1 - t)\rho(x, y) + t\rho(z, w) \tag{1.1}$$

for all x, y, z and w in X and $0 \le t \le 1$.

It is clear that all normed linear spaces are hyperbolic. A discussion of more examples of hyperbolic spaces and in particular of the Hilbert ball can be found, for example, in [3, 16, 37].

In this paper we use the Baire category approach which we describe now.

Let X be a complete metric space. According to Baire's theorem, the intersection of every countable collection of open dense subsets of X is dense in X. This rather simple, yet powerful result has found many applications. In particular, given a property which elements of X may have, it is of interest to determine whether this property is generic, that is, whether the set of elements which do enjoy this property contains a countable intersection of open dense sets. Such an approach, when a certain property is investigated for the whole space X and not just for a single point in X, has already been successfully applied in many areas of Analysis (see, for example, [2, 10, 11, 17, 25, 26, 27, 32, 33, 43, 47, 48] and the references mentioned there). According to the generic approach we say that a property holds for a generic (typical) element of a complete metric space (or the property holds generically) if the set of all elements of the metric space possessing this property contains a G_δ everywhere dense subset of the metric space which is a countable intersection of open everywhere dense sets.

Our result involves the notion of porosity which we now recall [11, 26, 27, 39, 40, 42, 43].

Let (Y, d) be a complete metric space. We denote by $B(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called *porous* (with respect to the metric d) if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists $z \in Y$ for which

$$B(z, \alpha r) \subset B(y, r) \setminus E.$$

A subset of the space Y is called *σ-porous* (with respect to d) if it is a countable union of porous subsets of Y.

Since porous sets are nowhere dense, all σ-porous sets are of the first Baire category. If Y is a finite-dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have Lebesgue measure 0 and are of the Baire first category. Also, every Banach space contains a set of the first Baire category which is not σ-porous.

To point out the difference between porous and nowhere dense sets, note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there is a point $z \in Y$ and a number $s > 0$ such that $B(z, s) \subset B(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which depends only on E.

2. The first main result

Let (X, ρ, M) be a complete hyperbolic space and A and B are nonempty, bounded and closed subsets of X such that $A \cap B \neq \emptyset$.

For each $x \in K$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \rho(x, y) \leq r\}.$$

Denote by $\mathcal{M}$ the set of all mappings $T : A \cup B \to A \cup B$ such that

$$T(A) \subset B, \ T(B) \subset A, \tag{2.1}$$

$$\rho(T(x), T(y)) \leq \rho(x, y) \text{ for each } x \in A \text{ and each } y \in B. \tag{2.2}$$

The following result was obtained in [20].

Theorem 2.1. *Assume that* $T \in \mathcal{M}$, $\alpha \in (0, 1)$ *and that*

$$\rho(T(x), T(y)) \leq \alpha \rho(x, y) \textit{ for each } x \in A \textit{ and each } y \in B.$$

Then T *has a unique fixed point in* $A \cap B$.

For each $T_1, T_2 \in \mathcal{M}$ define

$$(T_1, T_2) = \sup\{\|T_1(x) - T_2(x)\| : \ x \in A \cup B\}.$$

Clearly, the metric space $(\mathcal{M}, d)$ is complete. Set

$$D = \sup\{\rho(z_1, z_2) : \ z_1, z_2 \in A \cup B\}.$$

In [56] we proved the following result.

Theorem 2.2. *Assume that there exists* $x_* \in A \cap B$ *such that for each* $a \in A$, *each* $b \in B$ *and each* $\gamma \in (0,1)$,

$$\gamma x_* \oplus (1-\gamma)a \in A \ \text{ and } \ \gamma x_* \oplus (1-\gamma)b \in B.$$

Then there exists a set $\mathcal{F}_0 \subset \mathcal{M}$ *which is a countable intersection of open everywhere dense sets in* $(\mathcal{M}, d)$ *such that for each* $T \in \mathcal{F}_0$ *the following assertions hold.*

1. *There exists a unique point* $x_T \in A \cap B$ *such that* $T(x_T) = x_T$.

2. *For each* $\epsilon > 0$ *there exist an integer* $n_\epsilon \geq 1$ *and a neighborhood* $\mathcal{U}$ *of* T *in* $(\mathcal{M}, d)$ *such that for each* $S \in \mathcal{U}$, *each* $x \in A \cup B$ *and each integer* $n \geq n_\epsilon$,

$$\rho(S^n(x), x_T) \leq \epsilon.$$

A mapping $T : A \cup B \to A \cup B$ is called *contractive* (in the sense of Rakotch) [34, 43] if there exists a decreasing function $\phi : [0, \infty) \to [0, 1]$ such that

$$\phi(t) < 1, \ t \in (0, \infty)$$

and for any point $a \in A$ and any point $b \in B$, we have

$$\rho(T(a), T(b)) \leq \phi(\rho(a, b))\rho(a, b).$$

The results obtained in [54] imply the following theorem.

Theorem 2.3. *Assume that a mapping* $T : A \cup B \to A \cup B$ *is contractive. Then there exists* $x_* \in A \cap B$ *such that* $T(x_*) = x_*$ *and* x_* *is a unique fixed point of* T. *Moreover, for each* $\epsilon > 0$ *there exists* $\delta > 0$ *and a natural number* k *such that for every integer* $n > k$ *and any sequence* $\{x_i\}_{i=0}^{n-1} \subset A \cup B$ *which satisfies*

$$\rho(x_{i+1}, T(x_i)) \leq \delta, \ i = 0, \dots, n-1$$

the inequality $\rho(x_i, x_T) \leq \epsilon$ *is valid for all* $i \in \{k, \dots, n\}$.

Denote by $\mathcal{F}$ the set of all contractive mappings $T \in \mathcal{M}$. Now we state our first main result which is proved in the next section.

Theorem 2.4. *Assume that there exists* $x_* \in A \cap B$ *such that for each* $a \in A$, *each* $b \in B$ *and each* $\gamma \in (0,1)$,

$$(1-\gamma)a \oplus \gamma x_* \in A \text{ and } (1-\gamma)b \oplus \gamma x_* \in B.$$

Then the set $\mathcal{M} \setminus \mathcal{F}$ *is* σ*-porous in* $(\mathcal{M}, d)$.

An analogous result for noncyclic nonexpansive mappings was obtained in [42].

3. Proof of Theorem 2.4

Let $\gamma \in (0,1)$ and $T \in \mathcal{M}$. For each $a \in A$ and each $b \in B$ set

$$T_\gamma(a) = (1-\gamma)T(a) \oplus \gamma x_* \in A,\ T_\gamma(b) = (1-\gamma)T(b) \oplus \gamma x_* \in B. \tag{3.1}$$

By (2.2) and (3.1), for each $a \in A$ and each $b \in B$,

$$\begin{aligned}\rho(T_\gamma(a), T_\gamma(b)) &= \rho((1-\gamma)T(a) \oplus \gamma x_*, (1-\gamma)T(b) \oplus \gamma x_*)\\ &\le (1-\gamma)\rho(T(a), T(b)) \le (1-\gamma)\rho(a,b)\end{aligned} \tag{3.2}$$

and

$$\rho(T_\gamma(a), T(a)) = \rho((1-\gamma)T(a) \oplus \gamma x_*, T(a)) \le \gamma\rho(T(a), x_*) \le \gamma D, \tag{3.3}$$

$$\rho(T_\gamma(b), T(b)) = \rho((1-\gamma)T(b) \oplus \gamma x_*, T(b)) \le \gamma\rho(T(b), x_*) \le \gamma D. \tag{3.4}$$

Fix

$$\theta_1 \in A,\ \theta_2 \in B.$$

For each integer $n \ge 1$ denote by $\mathcal{F}_n$ the set of all $T \in \mathcal{M}$ such that

$$\sup\{\rho(T(a), T(b))\rho(a,b)^{-1} :$$

$$a \in A,\ b \in B \text{ and } \rho(a,b) \ge \min\{D/n,\ \rho(\theta_1,\theta_2)\}\} < 1. \tag{3.5}$$

Define

$$\mathcal{F} = \cap_{n=1}^{\infty} \mathcal{F}_n. \tag{3.6}$$

It is not difficult to see that every mapping $T \in \mathcal{F}_n$ is contractive. Indeed, if $T \in \mathcal{F}$, then we put $\phi(0) = 1$, for every $t > 0$ define

$$\phi(t) = \sup\{\rho(T(a), T(b))\rho(a,b)^{-1} :$$

$$a \in A,\ b \in B \textit{ and } \rho(a,b) \ge \min\{t,\ \rho(\theta_1,\theta_2)\}\} < 1$$

in view of (3.5) and now it is not difficult to see that for every $a \in A$ and every $b \in B$, we have

$$\rho(T(a), T(b)) \le \phi(\rho(a,b))\rho(a,b).$$

In order to complete the proof it is sufficient to show that for every integer $n \ge 1$, the set $\mathcal{M} \setminus \mathcal{F}_n$ is porous in $(\mathcal{M}, d)$.

Let $n \ge 1$ be an integer. Define

$$\alpha = 16^{-1}(D+1)^{-1}\min\{D/n, \rho(\theta_1,\theta_2)\}. \tag{3.7}$$

Let $T \in \mathcal{M}$ and $r \in (0,1]$. Set

$$\gamma = 4^{-1}r(D+1)^{-1} \tag{3.8}$$

and consider the mapping T_γ defined by (3.1). By (3.3), (3.4) and (3.8),

$$d(T_\gamma, T) \le \gamma D \le r/4. \tag{3.9}$$

Assume that $S \in \mathcal{M}$ satisfies

$$d(S, T_\gamma) \le \alpha r. \tag{3.10}$$

In view of (3.7), (3.9) and (3.10),

$$d(S,T) \le d(S,T_\gamma) + d(T_\gamma,T) \le \alpha r + r/4 \le r. \tag{3.11}$$

It follows from (3.2), (3.7), (3.8) and (3.10) that for each $a \in A$ and each $b \in B$ satisfying

$$\rho(a,b) \ge \min\{D/n,\ \rho(\theta_1,\theta_2)\}$$

we have

$$\begin{aligned}
&\rho(S(a),S(b))\rho(a,b)^{-1} \\
&\le (\rho(S(a),T_\gamma(a)) + \rho(T_\gamma(a),T_\gamma(b)) + \rho(T_\gamma(b),S(b)))\rho(a,b)^{-1} \\
&\le (2\alpha r + (1-\gamma)\rho(a,b))\rho(a,b)^{-1} \le 2\alpha r\rho(a,b)^{-1} + (1-\gamma) \\
&\le (1-\gamma) + 2\alpha r \min\{D/n,\ \rho(\theta_1,\theta_2)\}^{-1} \\
&\le 1 - 4^{-1}r(D+1)^{-1} + 8^{-1}r(D+1)^{-1} \le 1 - 8^{-1}r(D+1)^{-1}.
\end{aligned}$$

Hence $S \in \mathcal{F}_n$. Together with (3.11) this implies that

$$\{S \in \mathcal{M} :\ d(S,T_\gamma) \le \alpha r\} \subset \{S \in \mathcal{M} :\ d(S,T) \le r\} \cap \mathcal{F}_n.$$

Therefore the set $\mathcal{M} \setminus \mathcal{F}_n$ is porous. This completes the proof of Theorem 2.4.

4. The second main result

Let (X,ρ,M) be a complete hyperbolic space, A and B be nonempty, bounded and closed subsets of X such that

$$A \cap B = \emptyset \tag{4.1}$$

and let

$$x_A \in A,\ x_B \in B \tag{4.2}$$

satisfy

$$\rho(x_A,x_B) = \rho(A,B) := \inf\{\rho(a,b) :\ a \in A,\ b \in B\}. \tag{4.3}$$

Assume that for each $a \in A$, each $b \in B$ and each $\gamma \in (0,1)$,

$$\gamma x_A \oplus (1-\gamma)a \in A \tag{4.4}$$

$$\gamma x_B \oplus (1-\gamma)b \in B. \tag{4.5}$$

We continue to consider the space $\mathcal{M}$ of all mappings

$$T : A \cup B \to A \cup B \tag{4.6}$$

such that

$$T(A) \subset B,\ T(B) \subset A \tag{4.7}$$

and

$$\rho(T(x),T(y)) \le \rho(x,y) \text{ for all } x \in A \text{ and all } y \in B \tag{4.8}$$

equipped with the metric

$$d(T_1,T_2) = \sup\{\rho(T_1(x),T_2(x)) :\ x \in A \cup B\},\ T_1,T_2 \in \mathcal{M}. \tag{4.9}$$

Note that $(\mathcal{M},d)$ is a complete metric space and recall that

$$D = \sup\{\rho(a,b) :\ a,b \in A \cup B\}. \tag{4.10}$$

In [55] we proved the following result.

Theorem 4.1. *There exists a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense sets in $(\mathcal{M}, d)$ such that the following assertion holds.*

For each $S \in \mathcal{F}$ and each $\epsilon > 0$ there exist a natural number n_0 and a positive number δ such that for each sequence $\{x_i\}_{i=0}^{\infty} \subset A \cup B$ satisfying for each integer $i \geq 0$,

$$\text{if } x_i \in A, \text{ then } x_{i+1} \in B \text{ and } \text{ if } x_i \in B \text{ then } x_{i+1} \in A$$

and

$$\rho(S(x_i), x_{i+1}) \leq \delta$$

the inequality

$$\rho(x_n, x_{n+1}) \leq \rho(A, B) + \epsilon$$

holds for all integers $n \geq n_0$.

A mapping $T : A \cup B \to A \cup B$ is called *contractive* (in the sense of Rakotch) [34, 43] if there exists a decreasing function $\phi : [0, \infty) \to [0, 1]$ such that

$$\phi(t) < 1, \ t \in (0, \infty) \tag{4.11}$$

and for any point $a \in A$ and any point $b \in B$, we have

$$\rho(T(a), T(b)) - \rho(A, B) \leq \phi(\rho(a, b) - \rho(A, B))(\rho(a, b) - \rho(A, B)). \tag{4.12}$$

Here we use the notion of a contraction which differs from the contraction notion in Section 2 because in this section the intersection of the sets A and B is empty while in Section 2 it is nonempty (see Theorems 2.2–2.4).

Denote by $\tilde{\mathcal{F}}$ the set of all contractive mappings $T \in \mathcal{M}$. Now we prove our second main result.

Theorem 4.2. *The set $\mathcal{M} \setminus \tilde{\mathcal{F}}$ is σ-porous in $(\mathcal{M}, d)$.*

Proof. We may assume without loss of generality that

$$\tilde{D} := \sup\{\rho(a, b) : \ a \in A, \ b \in B\} - \rho(A, B) > 0.$$

Let $T \in \mathcal{M}$ and $\gamma \in (0, 1)$. Define

$$T_\gamma(a) = (1 - \gamma)T(a) \oplus \gamma x_B, \ a \in A, \tag{4.13}$$

$$T_\gamma(b) = (1 - \gamma)T(b) \oplus \gamma x_A, \ b \in B. \tag{4.14}$$

By (4.4), (4.5), (4.7), (4.13) and (4.14),

$$T_\gamma(A) \subset B, \ \ T_\gamma(B) \subset A. \tag{4.15}$$

It follows from (4.10), (4.13) and (4.14) that each $a \in A$ and each $b \in B$,

$$\rho(T_\gamma(a), T(a)) = \rho((1 - \gamma)T(a) \oplus \gamma x_B, T(a)) \leq \gamma\rho(x_B, T(a)) \leq \gamma D, \tag{4.16}$$

$$\rho(T_\gamma(b), T(b)) = \rho((1 - \gamma)T(b) \oplus \gamma x_A, T(b)) \leq \gamma\rho(x_A, T(b)) \leq \gamma D. \tag{4.17}$$

In view of (4.8), (4.13) and (4.14), for each $a \in A$ and each $b \in B$,

$$\begin{aligned}\rho(T_\gamma(a), T_\gamma(b)) &= \rho((1-\gamma)T(a) \oplus \gamma x_B,\ (1-\gamma)T(b) \oplus \gamma x_A) \\ &\le (1-\gamma)\rho(T(a), T(b)) + \gamma\rho(x_A, x_B) \\ &\le (1-\gamma)\rho(a, b) + \gamma\rho(x_A, x_B)\end{aligned}$$

and

$$\rho(a, b) - \rho(T_\gamma(a), T_\gamma(b)) \ge \gamma(\rho(a, b) - \rho(x_A, x_B)). \tag{4.18}$$

Let $n \ge 2$ be an integer. Define

$$\Omega_n = \{(a, b) \in A \times B :\ \rho(a, b) - \rho(x_A, x_B) \ge \tilde{D}/n\}. \tag{4.19}$$

Clearly,

$$\Omega_n \ne \emptyset.$$

It follows from (4.18) and (4.19) that for each $(a, b) \in \Omega_n$,

$$\rho(a, b) - \rho(T_\gamma(a), T_\gamma(b)) \ge \gamma(\rho(a, b) - \rho(x_A, x_B)) \ge \gamma\tilde{D}/n. \tag{4.20}$$

Denote by $\mathcal{M}_n$ the set of all $S \in \mathcal{M}$ such that

$$\inf\{\rho(a, b) - \rho(S(a), S(b))(\rho(a, b) - \rho(A, B))^{-1} :\ (a, b) \in \Omega_n\} > 0. \tag{4.21}$$

We show that the set $\mathcal{M} \setminus \mathcal{M}_n$ is porous in $(\mathcal{M}, d)$. Set

$$\alpha = 8^{-1}(D+1)^{-1}\tilde{D}/n. \tag{4.22}$$

Let $T \in \mathcal{M}$ and $r \in (0, 1]$. Set

$$\gamma = 2^{-1}r(D+1)^{-1} \tag{4.23}$$

and consider the mapping T_γ defined by (4.13) and (4.14). By (4.16), (4.17) and (4.23),

$$d(T_\gamma, T) \le \gamma D \le 2^{-1}r. \tag{4.24}$$

It follows from (4.18) and (4.23) that for each $(a, b) \in \Omega_n$,

$$\begin{aligned}(\rho(a, b) - \rho(T_\gamma(a), T_\gamma(b))) &\ge \gamma(\rho(a, b) - \rho(A, B)) \\ &= 2^{-1}r(D+1)^{-1}(\rho(a, b) - \rho(A, B)).\end{aligned} \tag{4.25}$$

Assume that $S \in \mathcal{M}$ satisfies

$$d(S, T_\gamma) \le \alpha r. \tag{4.26}$$

In view of (4.22), (4.24) and (4.26),

$$d(S, T) \le \alpha r + r/2 \le 3r/4. \tag{4.27}$$

Let

$$(a, b) \in \Omega_n. \tag{4.28}$$

It follows from (4.19), (4.22), (4.25), (4.26) and (4.28)

$$\begin{aligned}
&\rho(a,b) - \rho(S(a),S(b)) \\
&\geq \rho(a,b) - \rho(T_\gamma(a),T_\gamma(b)) - \rho(T_\gamma(a),S(a)) - \rho(T_\gamma(b),S(b)) \\
&\geq 2^{-1}r(D+1)^{-1}(\rho(a,b) - \rho(A,B)) - 2\alpha r \\
&\geq 2^{-1}r(D+1)^{-1}(\rho(a,b) - \rho(A,B) - 4(D+1)\alpha) \\
&\geq 2^{-1}r(D+1)^{-1}(\rho(a,b) - \rho(A,B) - 2^{-1}\tilde{D}/n) \\
&\geq 4^{-1}r(D+1)^{-1}(\rho(a,b) - \rho(A,B)). \qquad (4.29)
\end{aligned}$$

In view of (4.29), $S \in \mathcal{M}_n$.

Together with (4.27) this implies that

$$\{S \in \mathcal{M} :\ d(S,T_\gamma) \leq \alpha r\} \subset \{S \in \mathcal{M} :\ d(S,T) \leq r\} \cap \mathcal{M}_n.$$

Therefore the set $\mathcal{M} \setminus \mathcal{M}_n$ is porous. Set

$$\mathcal{F} = \cap_{n=2}^{\infty} \mathcal{M}_n. \qquad (4.30)$$

Clearly, $\mathcal{M} \setminus \mathcal{F}$ is σ-porous in $(\mathcal{M}, d)$. Assume that

$$S \in \mathcal{F}. \qquad (4.31)$$

Equations (4.21), (4.30) and (4.31) imply that for every integer $n \geq 2$, $S \in \mathcal{M}_n$ and there exists $\gamma_n \in (0,1)$ such that

$$(\rho(a,b) - \rho(S(a),S(b)))(\rho(a,b) - \rho(A,B))^{-1} > \gamma_n,\ (a,b) \in \Omega_n. \qquad (4.32)$$

In view of (4.32), for every integer $n \geq 2$ and every $(a,b) \in \Omega_n$ we have

$$\begin{aligned}
\rho(S(a),S(b)) &< \rho(a,b) - \gamma_n(\rho(a,b) - \rho(A,B)), \\
\rho(S(a),S(b)) - \rho(A,B) &< \rho(a,b) - \rho(A,B) - \gamma_n(\rho(a,b) - \rho(A,B)) \\
&< (1-\gamma_n)(\rho(a,b) - \rho(A,B))
\end{aligned}$$

and

$$\sup\{(\rho(S(a),S(b)) - \rho(A,B))(\rho(a,b) - \rho(A,B))^{-1} :\ (a,b) \in \Omega_n\} < 1. \qquad (4.33)$$

Since n is an arbitrary integer satisfying $n \geq 2$ it follows from (4.19) and (4.33) that for every $\epsilon > 0$ we have

$$\phi(\epsilon) := \sup\{(\rho(S(a),S(b)) - \rho(A,B))(\rho(a,b) - \rho(A,B))^{-1} :$$
$$(a,b) \in A \times B,\ \rho(a,b) - \rho(A,B) \geq \epsilon\} < 1.$$

(Here the supremum over an empty set is zero.) Set $\phi(0) = 1$. It is easy to see that the function $\phi : [0,\infty) \to [0,1]$ is decreasing and that for each $(a,b) \in A \times B$ we have,

$$\rho(S(a),S(b)) - \rho(A,B) \leq \phi(\rho(a,b) - \rho(A,B))(\rho(a,b) - \rho(A,B)).$$

Thus the mapping S is contractive. Theorem 4.2 is proved. □

5. The third main result

Let (X,ρ) be a complete metric space, A and B be nonempty and closed subsets of X such that

$$A \cap B = \emptyset, \tag{5.1}$$

$$\rho(A,B) := \inf\{\rho(a,b):\ a \in A,\ b \in B\} \tag{5.2}$$

and let a mapping $T : A \cup B \to A \cup B$ satisfies

$$T(A) \subset B,\ T(B) \subset A. \tag{5.3}$$

Assume that there exists a decreasing function $\phi : [0,\infty) \to [0,1]$ such that

$$\phi(t) < 1,\ t \in (0,\infty)$$

and that for any point $a \in A$ and any point $b \in B$, we have

$$\rho(T(a),T(b)) - \rho(A,B) \le \phi(\rho(a,b) - \rho(A,B))(\rho(a,b) - \rho(A,B)). \tag{5.4}$$

In other words, the mapping T is contractive (in the sense of Rakotch) [34, 43]. Here we use the same notion of a contraction as in Section 4.

Fix $\theta \in A \cup B$. We prove the following theorem which is our final main result.

Theorem 5.1. *Let $M > 0$ and $\epsilon \in (0,1)$. Then there exist a natural number n_0 and $\delta > 0$ such that for each sequence $\{x_i\}_{i=0}^{\infty}$ which satisfies*

$$\rho(x_0,\theta) \le M$$

and for each integer $i \ge 0$,

$$\text{if } x_i \in A,\ \text{then } x_{i+1} \in B \text{ and if } x_i \in B,\ \text{then } x_{i+1} \in A$$

and

$$\rho(x_{i+1},T(x_i)) \le \delta$$

the inequality $\rho(x_i,x_{i+1}) \le \rho(A,B) + \epsilon$ holds for all integers $n \ge n_0$.

Proof. Fix

$$x_A \in A,\ x_B \in B \tag{5.5}$$

satisfying

$$\rho(x_A,x_B) \le \rho(A,B) + 1. \tag{5.6}$$

Choose

$$\delta \in (0, 8^{-1}(1-\phi(\epsilon/2))\epsilon). \tag{5.7}$$

Assume that

$$z_0 \in A \cup B,\ \rho(z_0,\theta) \le M,\ z_1 \in X,\ \rho(z_1,T(z_0)) \le 1. \tag{5.8}$$

In view of (5.8),

$$\rho(z_0,x_A) \le \rho(z_0,\theta) + \rho(\theta,x_A) \le M + \rho(\theta,x_A) \tag{5.9}$$

and

$$\rho(z_0,x_B) \le \rho(z_0,\theta) + \rho(\theta,x_B) \le M + \rho(\theta,x_B). \tag{5.10}$$

If $z_0 \in A$, then by (5.4), (5.8) and (5.10),

$$\begin{aligned}\rho(z_0, T(z_0)) &\le \rho(z_0, x_B) + \rho(x_B, T(x_B)) + \rho(T(x_B), T(z_0)) \\ &\le 2\rho(z_0, x_B) + \rho(x_B, T(x_B)) \\ &\le 2M + 2\rho(\theta, x_B) + \rho(x_B, T(x_B))\end{aligned}$$

and

$$\rho(z_0, z_1) \le 1 + 2M + 2\rho(\theta, x_B) + \rho(x_B, T(x_B)). \tag{5.11}$$

If $z_0 \in B$, then by (5.4), (5.8) and (5.9),

$$\begin{aligned}\rho(z_0, T(z_0)) &\le \rho(z_0, x_A) + \rho(x_A, T(x_A)) + \rho(T(x_A), T(z_0)) \\ &\le 2M + 2\rho(\theta, x_A) + \rho(x_A, T(x_A))\end{aligned}$$

and

$$\rho(z_0, z_1) \le 1 + 2M + 2\rho(\theta, x_A) + \rho(x_A, T(x_A)).$$

Thus (see (5.11)) in both cases

$$\rho(z_0, z_1) \le c_1 := 1+2M+2\rho(\theta, x_A)+2\rho(\theta, x_B)+\rho(x_A, T(x_A))+\rho(x_B, T(x_B)). \tag{5.12}$$

Hence we showed that the following property holds:

(i) equations (5.8) imply (5.12).

Choose an integer

$$n_0 > 4c_1\epsilon^{-1}(1 - \phi(\epsilon/2))^{-1} + 4. \tag{5.13}$$

Assume that $\{x_i\}_{i=0}^{\infty} \subset A \cup B$ satisfies for each integer $i \ge 0$,

$$\text{if } x_i \in A, \text{ then } x_{i+1} \in B \text{ and if } x_i \in B, \text{ then } x_{i+1} \in A, \tag{5.14}$$

$$\rho(x_0, \theta) \le M \tag{5.15}$$

and for each integer $i \ge 0$,

$$\rho(x_{i+1}, T(x_i)) \le \delta. \tag{5.16}$$

We show that there exists an integer $j \in [0, n_0]$ such that

$$\rho(x_j, x_{j+1}) \le \rho(A, B) + \epsilon/2.$$

Assume the contrary. Then

$$\rho(x_i, x_{i+1}) > \rho(A, B) + \epsilon/2, \ i = 0, \dots, n_0. \tag{5.17}$$

By (5.4), (5.14) and (5.17) and monotonicity of ϕ,

$$\begin{aligned}&\rho(T(x_i), T(x_{i+1})) - \rho(A, B) \\ &\quad \le (\rho(x_i, x_{i+1}) - \rho(A, B))\phi(\rho(x_i, x_{i+1}) - \rho(A, B)) \\ &\quad \le \phi(\epsilon/2)(\rho(x_i, x_{i+1}) - \rho(A, B)),\end{aligned}$$

$$\begin{aligned}&\rho(x_i, x_{i+1}) - \rho(T(x_i), T(x_{i+1})) \\ &\quad = \rho(x_i, x_{i+1}) - \rho(A, B) - (\rho(T(x_i), T(x_{i+1})) - \rho(A, B)) \\ &\quad \ge (1 - \phi(\epsilon/2))(\rho(x_i, x_{i+1}) - \rho(A, B)) \ge (1 - \phi(\epsilon/2))\epsilon/2\end{aligned}$$

and together with (5.7) and (5.16) this implies that

$$\begin{aligned}&\rho(x_i, x_{i+1}) - \rho(x_{i+1}, x_{i+2})\\&\quad \geq \rho(x_i, x_{i+1}) - \rho(T(x_i), T(x_{i+1})) - \rho(T(x_i), x_{i+1}) - \rho(T(x_{i+1}), x_{i+2})\\&\quad \geq (1-\phi(\epsilon/2))\epsilon/2 - 2\delta \geq (1-\phi(\epsilon/2))\epsilon/4. \qquad (5.18)\end{aligned}$$

Property (i) and equations (5.15), (5.16) and (5.18) imply that

$$\begin{aligned}c_1 &\geq \rho(x_0, x_1) \geq \rho(x_0, x_1) - \rho(x_{n_0}, x_{n_0+1})\\&= \sum_{i=0}^{n_0-1} (\rho(x_i, x_{i+1}) - \rho(x_{i+1}, x_{i+2})) \geq 4^{-1}(1-\phi(\epsilon/2))\epsilon n_0\end{aligned}$$

and

$$n_0 \leq 4c_1(1-\phi(\epsilon/2))^{-1}\epsilon^{-1}.$$

This contradicts (5.13). The contradiction we have reached proves that there exists an integer

$$j \in [0, n_0] \qquad (5.19)$$

such that

$$\rho(x_j, x_{j+1}) \leq \rho(A, B) + \epsilon/2. \qquad (5.20)$$

In order to complete the proof of the theorem it is sufficient to show that

$$\rho(x_i, x_{i+1}) \leq \rho(A, B) + \epsilon$$

for all integers $i \geq j$.

Assume the contrary. Then there is an integer $k > j$ such that

$$\rho(x_k, x_{k+1}) > \rho(A, B) + \epsilon. \qquad (5.21)$$

We may assume without loss of generality that for all integers $i \in \{j, \dots k-1\}$,

$$\rho(x_i, x_{i+1}) \leq \rho(A, B) + \epsilon$$

and in particular

$$\rho(x_{k-1}, x_k) \leq \rho(A, B) + \epsilon. \qquad (5.22)$$

There are two cases:

$$\rho(x_{k-1}, x_k) \leq \rho(A, B) + \epsilon/2; \qquad (5.23)$$

$$\rho(x_{k-1}, x_k) > \rho(A, B) + \epsilon/2. \qquad (5.24)$$

Assume that (5.23) holds. Then by (5.4), (5.7), (5.16) and (5.23),

$$\begin{aligned}\rho(x_k, x_{k+1}) &\leq \rho(x_k, T(x_{k-1})) + \rho(T(x_{k-1}), T(x_k)) + \rho(T(x_k), x_{k+1})\\&\leq 2\delta + \rho(T(x_{k-1}), T(x_k)) \leq 2\delta + \rho(x_{k-1}, x_k)\\&\leq \rho(A, B) + \epsilon/2 + 2\delta \leq \rho(A, B) + \epsilon.\end{aligned}$$

This contradicts (5.21). The contradiction we have reached proves that (5.23) holds.

By (5.4), (5.24) and the monotonicity of ϕ we obtain

$$\begin{aligned}
&\rho(T(x_{k-1}), T(x_k)) - \rho(A, B) \\
&\qquad \le (\rho(x_{k-1}, x_k) - \rho(A, B))\phi(\rho(x_{k-1}, x_k) - \rho(A, B)), \\
&\rho(x_k, x_{k-1}) - \rho(T(x_{k-1}), T(x_k)) \\
&\qquad = \rho(x_{k-1}, x_k) - \rho(A, B) - (\rho(T(x_{k-1}), T(x_k)) - \rho(A, B)) \\
&\qquad \ge (\rho(x_{k-1}, x_k) - \rho(A, B))(1 - \phi(\rho(x_{k-1}, x_k) - \rho(A, B))) \\
&\qquad \ge (\rho(x_{k-1}, x_k) - \rho(A, B))(1 - \phi(\epsilon/2)) \\
&\qquad \ge (1 - \phi(\epsilon/2))\epsilon/2. \qquad (5.25)
\end{aligned}$$

It follows from (5.7), (5.16) and (5.25) that

$$\begin{aligned}
\rho(x_k, x_{k+1}) &\le \rho(x_k), T(x_{k-1})) + \rho(T(x_{k-1}), T(x_k)) + \rho(T(x_k), x_{k+1}) \\
&\le 2\delta + (1 - \phi(\epsilon/2))\epsilon/2 + \rho(A, B) < \rho(A, B) + \epsilon.
\end{aligned}$$

This contradicts (5.21). The contradiction we have reached completes the proof of Theorem 5.1. □

References

[1] S. Banach: *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math. 3 (1922) 133–181.

[2] C. Bargetz, M. Dymond, E. Medjic, S. Reich: *On the existence of fixed points for typical nonexpansive mappings on spaces with positive curvature*, Topol. Methods Nonlinear Analysis 57 (2021) 621–634.

[3] C. Bargetz, M. Dymond, S. Reich: *Porosity results for sets of strict contractions on geodesic metric spaces*, Topol. Methods Nonlinear Analysis 50 (2017) 89–124.

[4] A. Betiuk-Pilarska, T. Domínguez Benavides: *Fixed points for nonexpansive mappings and generalized nonexpansive mappings on Banach lattices*, Pure Appl. Funct. Analysis 1 (2016) 343–359.

[5] D. Butnariu, R. Davidi, G. T. Herman, I. G. Kazantsev: *Stable convergence behavior under summable perturbations of a class of projection methods for convex feasibility and optimization problems*, IEEE J. Selected Topics Signal Proc. 1 (2007) 540–547.

[6] D. Butnariu, S. Reich, A. J. Zaslavski: *Convergence to fixed points of inexact orbits of Bregman-monotone and of nonexpansive operators in Banach spaces*, in: Proc. 7th Int. Conf. Fixed-Point Theory Appl., H. Fetter Nathansky et al. (eds.), Guanajuato 2005, Yokohama Publishers, Yokohama (2006) 11–32.

[7] Y. Censor, R. Davidi, G. T. Herman: *Perturbation resilience and superiorization of iterative algorithms*, Inverse Problems 26 (2010), art. no. 6, 12 pp.

[8] Y. Censor, M. Zaknoon: *Algorithms and convergence results of projection methods for inconsistent feasibility problems: a review*, Pure Appl. Func. Analysis 3 (2018) 565–586.

[9] Y. Censor, A. J. Zaslavski: *Convergence and perturbation resilience of dynamic string-averaging projection methods*, Comp. Optimization Appl. 54 (2013) 65–76.

[10] F. S. De Blasi, J. Myjak: *Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach*, C. R. Acad. Sci. Paris 283 (1976) 185–187.

[11] F. S. De Blasi, J. Myjak: *On the porosity of the set of contractions without fixed points*, C. R. Acad. Sci. Paris 308 (1989) 51–54.

[12] F. S. De Blasi, J. Myjak, S. Reich, A. J. Zaslavski: *Generic existence and approximation of fixed points for nonexpansive set-valued maps*, Set-Valued Var. Analysis 17 (2009) 97–112.

[13] A. A. Eldred, P. Veeramani: *Existence and convergence of best proximity points*, J. Math. Analysis Appl. 323 (2006) 1001–-1006.

[14] A. Gibali: *A new split inverse problem and an application to least intensity feasible solutions*, Pure Appl. Funct. Analysis 2 (2017) 243–258.

[15] K. Goebel, W. A. Kirk: *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge (1990).

[16] K. Goebel, S. Reich: *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York (1984).

[17] A. D. Ioffe, A. J. Zaslavski: *Variational principles and well-posedness in optimization and calculus of variations*, SIAM J. Control Optimization 38 (2000) 566–581.

[18] J. Jachymski: *Extensions of the Dugundji-Granas and Nadler's theorems on the continuity of fixed points*, Pure Appl. Funct. Analysis 2 (2017) 657–666.

[19] W. A. Kirk, N. Shahzad: *Hyperbolic spaces and directional contractions*, Bull. Math. Sci. 9 (2019), art. no. 3, 49 pp.

[20] W. A. Kirk, P. S. Srinivasan, P. Veeramani: *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory 4 (2003) 79–89.

[21] U. Kohlenbach: *Some logical metatheorems with applications in functional analysis*, Trans. Amer. Math. Soc. 357 (2005) 89–128.

[22] U. Kohlenbach, L. Leustean: *Asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces*, J. Eur. Math. Soc. 12 (2010) 71–92.

[23] V. I. Kolobov, S. Reich, R. Zalas: *Finitely convergent deterministic and stochastic iterative methods for solving convex feasibility problems*, Math. Programming Ser. A 194 (2022) 1163–1183.

[24] R. Kubota, W. Takahashi, Y. Takeuchi: *Extensions of Browder's demiclosedness principle and Reich's lemma and their applications*, Pure Appl. Func. Analysis 1 (2016) 63–84.

[25] C. Li: *On well posed generalized best approximation problems*, J. Approx. Theory 107 (2000) 96–108.

[26] L. Peng, C. Li: *Porosity and fixed points of nonexpansive set-valued maps*, Set-Valued Var. Analysis 22 (2014) 333–348.

[27] L. Peng, C. Li, J.-C. Yao: *Porosity results on fixed points for nonexpansive set-valued maps in hyperbolic spaces*, J. Math. Analysis Appl. 428 (2015) 989–1004.

[28] A. Petrusel, G. Petrusel, J. C. Yao: *Coupled fixed point theorems for symmetric contractions in b-metric spaces with applications to operator equation systems*, Fixed Point Theory 17 (2016) 457–475.

[29] A. Petrusel, G. Petrusel, J. C. Yao: *Perov type theorems for orbital contractions*, J. Nonlinear Convex Analysis 21 (2020) 759–769.

[30] A. Petrusel, G. Petrusel, J. C. Yao: *Multi-valued graph contraction principle with applications*, Optimization 69 (2020) 1541–1556.

[31] A. Petrusel, G. Petrusel, J. C. Yao: *Graph contractions in vector-valued metric spaces and applications*, Optimization 70 (2021) 763–775.

[32] C. Planiden, X. Wang: *Most convex functions have unique minimizers*, J. Convex Analysis 23 (2016) 877–892.

[33] C. Planiden, X. Wang: *Strongly convex functions, Moreau envelopes, and the generic nature of convex functions with strong minimizers*, SIAM J. Optimization 26 (2016) 1341–1364.

[34] E. Rakotch: *A note on contractive mappings*, Proc. Amer. Math. Soc. 13 (1962) 459–465.

[35] S. Reich: *The fixed point property for nonexpansive mappings I*, Amer. Math. Monthly 83 (1976) 266–268.

[36] S. Reich: *The fixed point property for nonexpansive mappings II*, Amer. Math. Monthly 87 (1980) 292–294.

[37] S. Reich, I. Shafrir: *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Analysis 15 (1990) 537–558.

[38] S. Reich, A. J. Zaslavski: *Well-posedness of fixed point problems*, Far East J. Math. Sci., Special Volume, Part III (Functional Analysis and Its Applications), (2001) 393–401.

[39] S. Reich, A. J. Zaslavski: *The set of divergent descent methods in a Banach space is sigma-porous*, SIAM J. Optimization 11 (2001) 1003–1018.

[40] S. Reich, A. J. Zaslavski: *Well-posedness and porosity in best approximation problems*, Topol. Methods Nonlinear Analysis 18 (2001) 395–408.

[41] S. Reich, A. J. Zaslavski: *Generic aspects of metric fixed point theory*, in: *Handbook of Metric Fixed Point Theory*, W. A. Kirk et al. (eds.), Kluwer Academic Publishers, Dordrecht (2001) 557–575.

[42] S. S. Reich, A. J. Zaslavski: *The set of noncontractive mappings is sigma-porous in the space of all nonexpansive mappings*, C. R. Acad. Sci. Paris 333 (2001) 539–544.

[43] S. Reich, A. J. Zaslavski: *Genericity in Nonlinear Analysis*, Springer, New York (2014).

[44] S. Rezapour, J.-C. Yao, S. H. Zakeri: *A strong convergence theorem for quasi-contractive mappings and inverse strongly monotone mappings*, Pure Appl. Funct. Analysis 5 (2020) 733–745.

[45] W. Takahashi: *The split common fixed point problem and the shrinking projection method for new nonlinear mappings in two Banach spaces*, Pure Appl. Funct. Analysis 2 (2017) 685–699.

[46] W. Takahashi: *A general iterative method for split common fixed point problems in Hilbert spaces and applications*, Pure Appl. Funct. Analysis 3 (2018) 349–369.

[47] J. Vanderwerff: *On the residuality of certain classes of convex functions*, Pure Appl. Funct. Analysis 5 (2020) 791–806.

[48] X. Wang: *Most maximally monotone operators have a unique zero and a super-regular resolvent*, Nonlinear Analysis 87 (2013) 69–82.

[49] K. Wlodarczyk, R. Plebaniak, A. Banach: *Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces*, Nonlinear Analysis 70 (2009) 3332–3341.

[50] K. Wlodarczyk, R. Plebaniak, A. Banach: *Erratum to: "Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces*, Nonlinear Analysis 71 (2009) 3585–3586.

[51] K. Wlodarczyk, R. Plebaniak, C. Obczynski: *Convergence theorems, best approximation and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces*, Nonlinear Analysis 72 (2010) 794–805.

[52] A. J. Zaslavski: *Approximate Solutions of Common Fixed Point Problems*, Springer Optimization and Its Applications, Springer, Cham (2016).

[53] A. J. Zaslavski: *Algorithms for Solving Common Fixed Point Problems*, Springer Optimization and Its Applications, Springer, Cham (2018).

[54] A. J. Zaslavski: *Existence of a fixed point for a cyclical contractive mapping*, PanAmer. Math. J. 32 (2022) 57–64.

[55] A. J. Zaslavski: *Asymptotic behavior of iterates of a generic cyclical nonexpansive mapping*, Numer. Funct. Analysis Optimization 43/1 (2022) 116–125.

[56] A. J. Zaslavski: *Generic existence of fixed points for a class of cyclical nonlinear mappings*, Appl. Set-Valued Analysis Optimization, accepted.