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Ground State Homoclinic Solutions for Fourth Order Differential Equations

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We are interested in the existence of ground state homoclinic orbits for a class of periodic fourth-order differential equations under superquadratic conditions weaker than the ones known in the literature. To the best of our knowledge, there has been no work focussed in this case.

Keywords: Fourth-order differential equations, ground state homoclinic solutions, critical points, superquadratic growth, concentration compactness principle.

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1. Introduction

In this paper, we consider a class of periodic fourth-order differential equations

$$(\mathcal{F}) \qquad u^{(4)}(x) + \omega u''(x) + a(x)u(x) = f(x, u(x)), \ \forall x \in \mathbb{R}$$

where ω is a constant, $a \in C(\mathbb{R}, \mathbb{R})$ and $f \in C(\mathbb{R}^2, \mathbb{R})$ are two real functions periodic in x. Here as usual, we say that a solution u of $(\mathcal{F})$ is homoclinic (to 0) if we have $u \in C^4(\mathbb{R}, \mathbb{R})$, $u(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$ and $u \neq 0$. A ground state homoclinic solution is an homoclinic solution that minimizes the energy among all homoclinic solutions or on the Nehari-Pankov manifold.

Many problems arising in science and engineering call for the solution of nonlinear ordinary differential equations or systems. In particular, the fourth-order differential equation $(\mathcal{F})$ has been put forward as the mathematical model for the study of pattern formation in physics and mechanics, see for example [3, 6, 8, 17]. These equations are difficult to solve and there are very few general techniques that can be applied to solve them. However, it was noticed that in the two last decades, the existence and multiplicity of homoclinic orbits for equation $(\mathcal{F})$ have been studied in many papers via critical point theory and variational methods, see [1, 2, 9–13, 16, 19–21, 25–30]. Many resolvability conditions are given, such as the superquadratic condition [9, 26, 28, 31], the subquadratic condition [12, 29], the asymptotically quadratic condition [13, 21, 30] and the local condition [27]. For the superquadratic case, most of the results on the existence and multiplicity of homoclinic solutions for $(\mathcal{F})$ were obtained under the following well-known Ambrosetti-Rabinowitz condition

$$(\mathcal{AR}) \qquad \begin{array}{l} \text{There exists a constant } \mu > 2 \text{ such that} \\ 0 < \mu F(x, u) \leq f(x, u), \ \forall (x, u) \in \mathbb{R}^2 \setminus \{0\}), \end{array}$$

where $F(x, u) = \int_0^u f(x, v)dv$.

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The condition $(\mathcal{AR})$ is very useful in studying the mountain pass geometry and verifying the Palais-Smale condition for the associated energy functional. Recently, there are many papers devoted to replacing condition $(\mathcal{AR})$ with weaker ones, see [11, 25, 26, 28]. In particular, in [25, 26, 28] the author studied equation $(\mathcal{F})$ replacing the $(\mathcal{AR})$ condition respectively by the following conditions

$$\text{(C1)} \quad \begin{cases} \text{There exist constants } c, r > 0 \text{ and } \sigma > 1 \text{ such that} \\ \left(\dfrac{|f(x,u)|}{|u|}\right)^{\sigma} \leq c\widetilde{F}(x,u), \ \forall x \in \mathbb{R}, \ \forall\, |u| \geq r, \end{cases}$$

where $\widetilde{F}(x,u) = \frac{1}{2} f(x,u)u - F(x,u)$.

$$\text{(C2)} \quad \begin{cases} \text{There exists a constant } \sigma \geq 1 \text{ such that} \\ \sigma\widetilde{F}(x,u) \geq \widetilde{F}(x,su), \ \forall (s,x,u) \in [0,1] \times \mathbb{R}^2. \end{cases}$$

$$\text{(C3)} \quad \begin{cases} \text{There exist } g \in L^1(\mathbb{R}) \text{ and constants } b_0,\, c_0 > 0 \text{ and } \nu \in]0,2[\text{ such that} \\ \widetilde{F}(x,u) \geq \begin{cases} g(u), \ \forall x \in \mathbb{R}, \ |u| \leq R_0 \\ b_0\,|u|^{\nu}, \ \forall x \in \mathbb{R}, \ |u| \geq R_0, \end{cases} \\ |F(x,u)| \leq c_0\,|u|^{2-\nu}\,\widetilde{F}(x,u), \ \forall (x,u) \in \mathbb{R}^2 \ \textit{with} \ |u| \geq R_0. \end{cases}$$

However, there are many superquadratic functions satisfying none of $(\mathcal{AR})$, (C1), (C2), (C3) as we will see in the examples below.

In the present paper, motivated by the above papers, we are concerned with the existence of ground state homoclinic solutions for $(\mathcal{F})$ under some kind of superquadratic conditions weaker than the above mentioned conditions. To state our main first result, we still need the following conditions

$(\mathcal{A})$ $a \in C(\mathbb{R},\mathbb{R})$ is T-periodic $(T > 0)$,

(χ) $\sup\left(\sigma\left(\frac{d^4}{dx^4}+\omega\frac{d^2}{dx^2}+a(x)\right)\cap]-\infty,0[\right)<0<\inf\left(\sigma\left(\frac{d^4}{dx^4}+\omega\frac{d^2}{dx^2}+a(x)\right)\cap]0,\infty[\right)$,

(F_1) $F(x,u) \in C^1(\mathbb{R}^2,\mathbb{R})$ is T-periodic in x and $F(x,u) \geq 0$ for all $(x,u) \in \mathbb{R}^2$,

(F_2) $F(x,0) = 0$ and $f(x,u) = o(|u|)$ uniformly for $x \in \mathbb{R}$,

(F_3) $\lim\limits_{|u|\to\infty} \frac{F(x,u)}{|u|^2} = +\infty$, a.e. $x \in \mathbb{R}$,

(F_4) $\widetilde{F}(x,u) \geq 0$, $\forall (x,u) \in \mathbb{R}^2$, and there exists $g \in C(\mathbb{R}_+^*,\mathbb{R}_+^*)$ with $\lim\limits_{s\to\infty} g(s) = +\infty$ such that $\quad \frac{|f(x,u)|}{|u|} \geq \frac{1}{4\eta_2^2} \implies |f(x,u)| \leq \frac{|u|}{g(|u|)}\widetilde{F}(x,u)$,

where η_2 is the Sobolev embedding constant given in (2.5).

Theorem 1.1. *Assume that* $(\mathcal{A})$, (χ) *and* $(F_1)-(F_4)$ *are satisfied. Then the fourth-order differential equation* $(\mathcal{F})$ *possesses a ground state homoclinic solution, i.e a homoclinic solution that minimizes the energy among all homoclinic solutions.*

Note that the ground state homoclinic solution for $(\mathcal{F})$ in Theorem 1.1 is in fact a nontrivial homoclinic solution u which satisfies $\Phi(u) = \inf_{\mathcal{M}} \Phi$, where

$$\mathcal{M} = \{v \in E \setminus \{0\} \,/ \Phi'(v) = 0\},$$

where $E = E^{-} \oplus E^{+}$ is the working space on which the energy functional Φ associated with $(\mathcal{F})$ is defined. In 2005, Pankov [18] introduced the following set

$$\mathcal{N}^{-} = \left\{ u \in E \setminus E^{-} / \Phi'(u)u = \Phi'(u)v, \forall v \in E^{-} \right\},$$

which is a subset of the Nehari manifold

$$\mathcal{N} = \left\{ u \in E \setminus \{0\} / \Phi'(u)v = 0, \forall v \in E^{-} \right\}.$$

In general, $\mathcal{M}$ is a very small subset of $\mathcal{N}^{-}$. By definition, we say that $u \in E \setminus \{0\}$ is a ground state homoclinic solution of $(\mathcal{F})$ if it satisfies $\Phi(u) = \inf_{\mathcal{N}^{-}} \Phi$ and $\Phi'(u) = 0$. Based on variational methods and critical point theory, many authors study the existence of ground state homoclinic solutions of $(\mathcal{F})$ which minimizes the energy on the Nehari-Pankov manifold $\mathcal{N}^{-}$ for the Schrödinger equation, see [5, 22–24] and the references therein. However, to the best of our knowledge there is no similar results on the existence of a ground state homoclinic solution for fourth order differential equation $(\mathcal{F})$. Note that it is much more difficult to find a solution u of $(\mathcal{F})$ which satisfies $\Phi(u) = \inf_{\mathcal{N}^{-}} \Phi$ than one satisfying $\Phi(u) = \inf_{\mathcal{M}} \Phi$. Motivated by the above papers, in the following, we will study the existence of ground state homoclinic solution for $(\mathcal{F})$ which minimizes the associated energy on the Nehari-Pankov manifold $\mathcal{N}^{-}$. More precisely, we obtain the following result

Theorem 1.2. *Assume that $(\mathcal{A})$, (χ), $(F_1) - (F_3)$ and the following condition are satisfied*

(F_5) $\quad \dfrac{1-\theta^2}{2} f(x,u)u - \theta f(x,u)v + F(x,\theta u + v) - F(x,u) \geq 0, \ \forall \theta \geq 0, \ u, v \in \mathbb{R}.$

Then the fourth-order differential equation $(\mathcal{F})$ possesses a homoclinic solution satisfying $\Phi(u) = \inf_{\mathcal{N}^{-}} \Phi$.

2. Preliminaries

In the following, we shall use $\|.\|_{L^s}$ to denote the norm of $L^s(\mathbb{R})$ for any $s \in [2, \infty]$. Let $H^2(\mathbb{R})$ be the Sobolev space with inner product and norm given respectively by

$$< u, v >_{H^2} = \int_{\mathbb{R}} \left[u''(x)v''(x) + u'(x)v'(x) + u(x)v(x) \right] dx, \quad \|u\|_{H^2} = < u, u >_{H^2}^{\frac{1}{2}}$$

for all $u, v \in H^2(\mathbb{R})$. In what follows it will always be assumed that a satisfies conditions $(\mathcal{A})$ and (χ). We denote by χ the selfadjoint extension of the operator $\frac{d^4}{dx^4} + \omega \frac{d^2}{dx^2} + a(x)$ with the domain $\mathcal{D}(\chi) \subset L^2(\mathbb{R})$. Let $\{\mathcal{E}(\lambda) / -\infty < \lambda < \infty\}$ and $|\chi|$ be the spectral family and the absolute value of χ, respectively, and $|\chi|^{\frac{1}{2}}$ be the square root of $|\chi|$. Set $U = id - \mathcal{E}(0) - \mathcal{E}(0^{-})$. Then U commutes with χ, $|\chi|$ and $|\chi|^{\frac{1}{2}}$, and $\chi = U|\chi|$ is the polar decomposition of χ (see [4]). Let

$$E = \mathcal{D}(|\chi|^{\frac{1}{2}}), \ E^{-} = \mathcal{E}(0^{-})E, \ E^{0} = (\mathcal{E}(0) - \mathcal{E}(0^{-}))E, \ E^{+} = (id - \mathcal{E}(0))E. \quad (2.1)$$

By (χ), one has $E^0 = \{0\}$. Hence for any $u \in E$, we have $u = u^{-} + u^{+}$ and

$$\chi u^{-} = -|\chi| u, \chi u^{+} = |\chi| u, \ \forall u \in E \cap \mathcal{D}(\chi), \quad (2.2)$$

where $u^{-} = \mathcal{E}(0^{-})u \in E^{-}, \ u^{+} = (id - \mathcal{E}(0))u \in E^{+}$.

We can define an inner product

$$\prec u, v \succ = \prec |\chi|^{\frac{1}{2}} u, |\chi|^{\frac{1}{2}} v \succ_{L^2}, \ \forall u, v \in E \tag{2.3}$$

and the corresponding norm

$$\|u\| = \prec u, u \succ^{\frac{1}{2}}, \ \forall u \in E. \tag{2.4}$$

Since $E = H^2(\mathbb{R})$ with equivalent norms under condition $(\mathcal{A})$, E is continuously embedded in $L^s(\mathbb{R})$ for all $2 \leq s \leq \infty$ and compactly embedded in $L^s(\mathbb{R})$ for all $2 \leq s < \infty$. Hence, for all $2 \leq s \leq \infty$, there exists a constant $\eta_s > 0$ such that

$$\|u\|_{L^s} \leq \eta_s \|u\|, \ \forall u \in E. \tag{2.5}$$

In addition, one has the decomposition $E = E^- \oplus E^+$ orthogonal with respect to both inner product $\prec .,. \succ$ and $\prec .,. \succ_{L^2}$. In view of (2.2) and (2.4), we have for all $u \in E$

$$\int_{\mathbb{R}} [u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2]dx = \left\|u^+\right\|^2 - \left\|u^-\right\|^2. \tag{2.6}$$

Now, in order to prove our main results, the following critical point theorem will be needed.

Definition 2.1. Let E be a real Hilbert space with $E = E^- \oplus E^+$ and $E^- \perp E^+$. A functional $\Phi \in C^1(E, \mathbb{R})$ is said to be *weakly sequentially lower semi-continuous* if for any $u_n \rightharpoonup u$ in E, one has $\Phi(u) \leq \liminf_{n\longrightarrow\infty} \Phi(u_n)$, and Φ' is said to be *weakly sequentially continuous* if $\lim_{n\longrightarrow\infty} \Phi'(u_n)v = \Phi'(u)v$ for each $v \in E$.

Lemma 2.1. [7, 14] *Let $(E, \|.\|)$ be a Hilbert space with $E = E^- \oplus E^+$ and $E^- \perp E^+$, and let $\Phi \in C^1(E, \mathbb{R})$ be of the form*

$$\Phi(u) = \frac{1}{2}\left(\left\|u^+\right\|^2 - \left\|u^-\right\|^2 \right) - g(u), \ u = u^- + u^+ \in E^- \oplus E^+.$$

Assume that the following conditions are satisfied

(1) *$g \in C^1(E, \mathbb{R})$ is bounded from below and weakly lower sequentially semi-continuous,*

(2) *g' is weakly sequentially continuous,*

(3) *there exist $r > \rho > 0$ and $e \in E^+$ with $\|e\| = 1$ such that*

$$\alpha = \inf \Phi(S_\rho^+) > \sup \Phi(\partial\Lambda),$$

where $S_\rho^+ = \{u \in E^+ / \|u\| = \rho\}$ and $\Lambda = \{v + se / v \in E^-, s \geq 0, \|v + se\| \leq r\}$. Then there exist a constant $c \in [\alpha, \sup \Phi(\Lambda)]$ and a sequence $(u_n) \subset E$ satisfying

$$\Phi(u_n) \longrightarrow c, \ \|\Phi'(u_n)\| (1 + \|u_n\|) \longrightarrow 0.$$

Definition 2.2. Let (u_n) be a bounded sequence in a Banach space. We say that (u_n) is *vanishing* if, for each $R > 0$,

$$\lim_{n\longrightarrow\infty} \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} |u_n|^2 \, dx = 0$$

and (u_n) is *nonvanishing* if there exist $\sigma > 0$, $R > 0$ and $(y_n) \subset \mathbb{R}$ such that

$$\liminf_{n \longrightarrow \infty} \int_{y_n - R}^{y_n + R} |u_n|^2 \, dx \geq \sigma.$$

In the vanishing case, we have the following result, which is a special case of the Lion's concentration compactness principle.

Lemma 2.2. [15] *Let (u_n) be a bounded sequence, if for any $R > 0$*

$$\lim_{n \longrightarrow \infty} \sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} |u_n|^2 \, dx = 0,$$

then $u_n \longrightarrow 0$ in $L^s(\mathbb{R})$ for $2 < s < \infty$.

3. Proof of Theorem 1.1.

Now we are going to establish the corresponding variational framework to obtain the existence of ground state homoclinic solution of $(\mathcal{F})$. For this end, define the energy functional Φ associated to equation $(\mathcal{F})$

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}} \left[u^{''}(x)^2 - \omega u^{'}(x)^2 + a(x) u(x)^2 \right] dx - \int_{\mathbb{R}} F(x, u(x)) dx$$

defined on the Hilbert space E introduced in Section 2. By (2.6), Φ can be rewritten as

$$\Phi(u) = \frac{1}{2} \Big(\left\| u^+ \right\|^2 - \left\| u^- \right\|^2 \Big) - \int_{\mathbb{R}} F(x, u(x)) dx$$

for $u = u^- + u^+ \in E = E^- \oplus E^+$. It is well known that Φ is continuously differentiable on E and

$$\begin{aligned}
\Phi'(u)v &= \int_{\mathbb{R}} \left[u''(x)v''(x) - \omega u'(x)v'(x) + a(x)u(x)v(x) \right] dx - \int_{\mathbb{R}} f(x, u(x))v(x) dx \\
&= < u, v > - \int_{\mathbb{R}} f(x, u(x))v(x) dx
\end{aligned}$$

for all $u, v \in E$. Moreover, the nontrivial critical points of Φ on E are homoclinic solutions of $(\mathcal{F})$. In the following, we will proceed by successive lemmas.

Lemma 3.1. *Assume that $(\mathcal{A})$, (χ), (F_1) and (F_2) are satisfied. Then the functional*

$$g(u) = \int_{\mathbb{R}} F(x, u(x)) dx, \;\; u \in E$$

is nonnegative, weakly sequentially lower semi-continuous and g' is weakly sequentially continuous.

Proof. Let $u_n \rightharpoonup u$ in E. Taking a subsequence if necessary, we have $u_n \longrightarrow u$ a.e. on $\mathbb{R}$.

By (F_2), Fatou's lemma, we have

$$\liminf_{n\longrightarrow\infty} g(u_n) \leq g(u),$$

which means that g is weakly sequentially lower semi-continuous. g' is weakly sequentially continuous on E is due to [25]. □

Lemma 3.2. *Assume that $(\mathcal{A})$, (χ), (F_1) and (F_2) are satisfied. Then there exists a constant $\rho > 0$ such that $\alpha = \inf \Phi(S_\rho^+) > 0$, where*

$$S_\rho^+ = \left\{u \in E^+ / \left\|u\right\| = \rho\right\}.$$

Proof. By (F_2), we have

$$\forall \epsilon > 0,\ \exists r > 0 / \forall x \in \mathbb{R},\ |u| \leq r,\ |f(x,u)| \leq \epsilon\,|u|\,,$$

which by the Mean Value Theorem implies

$$0 \leq F(x,u) = \int_0^1 f(x,su)uds \leq \frac{\epsilon}{2}\,|u|^2\,. \tag{3.1}$$

Let $u \in E^+$ be such that $\|u\| \leq \frac{r}{\eta_\infty}$, then by (2.5), one has $\|u\|_{L^\infty} \leq r$. Hence

$$\Phi(u) = \frac{1}{2}\left\|u\right\|^2 - \int_{\mathbb{R}} F(x,u(x))dx \geq \frac{1}{2}\left\|u\right\|^2 - \int_{\mathbb{R}} \frac{\epsilon}{2}\,|u(x)|^2\,dx \geq \frac{1}{2}(1-\epsilon\eta_2)\left\|u\right\|^2.$$

Take $\epsilon = \frac{\eta_2}{2}$. We obtain $\Phi(u) \geq \dfrac{1}{4}\left\|u\right\|^2$ for all $\|u\| \leq \dfrac{r}{\eta_\infty},\ u \in E^+$.

It suffices to take $\rho = \dfrac{r}{\eta_\infty}$ and $\alpha = \dfrac{\rho^2}{4}$. The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Assume that $(\mathcal{A})$, (χ), (F_1) and (F_2) are satisfied. Let $e \in E^+$ with $\|e\| = 1$. Then there is $r_0 > \rho$ such that $\sup \Phi(\partial\Lambda) \leq 0$ for $r \geq r_0$, where*

$$\Lambda = \left\{w + se \mid w \in E^-, s \geq 0, \|w + se\| \leq r\right\}.$$

Proof. We have

$$\partial\Lambda = \left\{w \mid w \in E^-, \|w\| \leq r\right\} \cup \left\{w + se \mid w \in E^-, s \geq 0, \|w + se\| = r\right\}.$$

We have $\Phi(w) \leq 0$ for all $w \in E^-$. We claim that $\Phi(w+se) \to -\infty$ as $\|w + se\| \to \infty$. Arguing indirectly, assume that there exists a sequence $(w_n + s_n e) \subset E^- \oplus \mathbb{R}e$ with $\|w_n + s_n e\| \longrightarrow \infty$ such that $\Phi(w_n + s_n e) \geq -M$ for all $n \in \mathbb{N}$, where M is a constant. Set $\frac{w_n + s_n e}{\|w_n + s_n e\|} = v_n^- + \tau_n e$, then $\|v_n^- + \tau_n e\| = 1$. Passing to a subsequence if necessary, we may assume that $\tau_n \longrightarrow \bar{\tau}$, $v_n^- \rightharpoonup v^-$ and $v_n^- \longrightarrow v^-$ a.e. on $\mathbb{R}$. Hence

$$\frac{-M}{\|w_n + s_n e\|^2} \leq \frac{\Phi(w_n + s_n e)}{\|w_n + s_n e\|^2} = \frac{\tau_n^2}{2} - \frac{1}{2}\left\|v_n^-\right\|^2 - \int_{\mathbb{R}} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2}dx. \tag{3.2}$$

If $\bar{\tau} = 0$, then it follows from (3.2) that

$$0 \leq \frac{1}{2}\left\|v_n^-\right\|^2 + \int_{\mathbb{R}} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} dx \leq \frac{\tau_n^2}{2} + \frac{M}{\|w_n + s_n e\|^2}$$

which yields $\|v_n^-\| \longrightarrow 0$ and so $1 = \|v_n^- + \tau_n e\| \longrightarrow 0$, a contradiction.

If $\bar{\tau} \neq 0$, then it follows from (3.2), (F_3) and Fatou's lemma

$$\begin{aligned} 0 &\leq \limsup_{n\longrightarrow\infty}\left[\frac{\tau_n^2}{2} - \frac{1}{2}\left\|v_n^-\right\|^2 - \int_{\mathbb{R}} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} dx\right] \\ &\leq \frac{\bar{\tau}^2}{2} - \liminf_{n\longrightarrow\infty} \int_{\mathbb{R}} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} dx \leq \frac{\bar{\tau}^2}{2} - \int_{\mathbb{R}} \liminf_{n\longrightarrow\infty} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2} dx = -\infty, \end{aligned}$$

a contradiction again. Hence $\Phi(w + se) \longrightarrow -\infty$ as $\|w + se\| \longrightarrow \infty$, and then there exists $r_0 > \rho$ such that $\sup \Phi(\partial\Lambda) \leq 0$ for $r \geq r_0$. □

Lemma 3.4. *Assume that* $(\mathcal{A})$, (χ), (F_1) *and* (F_4) *are satisfied. Then any sequence* (u_n) *satisfying*

$$\Phi(u_n) \longrightarrow c > 0,\ \|\Phi'(u_n)\|\,(1 + \|u_n\|) \longrightarrow 0 \tag{3.3}$$

is bounded in E.

Proof. To prove the boundedness of (u_n), arguing by contradiction, suppose that $\|u_n\| \longrightarrow \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$ and $\|v_n\|_{L^s} \leq \eta_s \|v_n\| = \eta_s$ for $2 \leq s \leq \infty$. Observe that

$$c + o(1) = \Phi(u_n) - \frac{1}{2}\Phi'(u_n)u_n = \int_{\mathbb{R}} \widetilde{F}(x, u_n)dx. \tag{3.4}$$

Since $\lim_{s\longrightarrow\infty} g(s) = +\infty$, there exists $R > 0$ such that

$$g(s) \geq 8(c + 1)\eta_\infty^2,\ \forall s \geq R. \tag{3.5}$$

Let

$$A_n = \left\{x \in \mathbb{R} \mid \frac{|f(x, u_n|}{|u_n|} \leq \frac{1}{4\eta_2^2}\right\} \text{ and } B_n = \{x \in \mathbb{R}/ |u_n| \geq R\}. \tag{3.6}$$

Hence, it follows from (3.6) that

$$\int_{A_n} \frac{|f(x, u_n)|}{\|u_n\|} |v_n|\, dx = \int_{A_n} \frac{|f(x, u_n)|}{|u_n|} |v_n|^2\, dx \leq \frac{1}{4\eta_2^2} \int_{A_n} |v_n|^2 \leq \frac{1}{4}. \tag{3.7}$$

From (F_4), (3.4)–(3.6), one has

$$\begin{aligned} &\int_{(\mathbb{R}\backslash A_n)\cap B_n} \frac{|f(x, u_n)|}{\|u_n\|} |v_n|\, dx = \int_{(\mathbb{R}\backslash A_n)\cap B_n} \frac{|f(x, u_n)|}{|u_n|} |v_n|^2\, dx \\ &\leq \|v_n\|_{L^\infty}^2 \int_{(\mathbb{R}\backslash A_n)\cap B_n} \frac{|f(x, u_n)|}{|u_n|} dx \leq \eta_\infty^2 \int_{(\mathbb{R}\backslash A_n)\cap B_n} \frac{\widetilde{F}(x, u_n)}{g(|u_n|)} dx \\ &\leq \eta_\infty^2 \frac{1}{8(c + 1)\eta_\infty^2} \int_{(\mathbb{R}\backslash A_n)\cap B_n} \widetilde{F}(x, u_n)dx = \frac{1}{8(c + 1)})(c + o(1)) \leq \frac{1}{8} + o(1) \end{aligned} \tag{3.8}$$

and $$\int_{(\mathbb{R}\backslash A_n)\cap(\mathbb{R}\backslash B_n)} \frac{|f(x,u_n)|}{\|u_n\|}|v_n|\,dx \leq \frac{\|v_n\|_{L^\infty}}{\|u_n\|}\int_{(\mathbb{R}\backslash A_n)\cap(\mathbb{R}\backslash B_n)}|f(x,u_n)|\,dx$$
$$\leq \frac{\|v_n\|_{L^\infty}}{\|u_n\|}\int_{(\mathbb{R}\backslash A_n)\cap(\mathbb{R}\backslash B_n)}\frac{|u_n|}{g(|u_n|)}\widetilde{F}(x,u_n)\,dx \leq \frac{\eta_\infty}{\|u_n\|}\sup_{|x|\leq R}\frac{|x|}{g(|x|)}\int_{(\mathbb{R}\backslash A_n)\cap(\mathbb{R}\backslash B_n)}\widetilde{F}(x,u_n)\,dx$$
$$\leq \frac{c_1}{\|u_n\|}\int_{(\mathbb{R}\backslash A_n)\cap(\mathbb{R}\backslash B_n)}\widetilde{F}(x,u_n)\,dx = \frac{c_1(c+o(1))}{\|u_n\|} = o(1). \tag{3.9}$$

Now, combining (3.3) with (3.7)–(3.9) yields

$$\begin{aligned}1+o(1) &= \frac{\|u_n\|^2-\Phi'(u_n)u_n}{\|u_n\|^2} = \frac{1}{\|u_n\|^2}\int_{\mathbb{R}} f(x,u_n)u_n dx\\ &= \frac{1}{\|u_n\|}\int_{\mathbb{R}} f(x,u_n)v_n dx \leq \frac{1}{\|u_n\|}\int_{\mathbb{R}}|f(x,u_n)|\,|v_n|\,dx\\ &= \int_{A_n}\frac{|f(x,u_n)|}{\|u_n\|}|v_n|\,dx + \int_{(\mathbb{R}\backslash A_n)\cap B_n}\frac{|f(x,u_n)|}{\|u_n\|}|v_n|\,dx + \int_{(\mathbb{R}\backslash A_n)\cap(\mathbb{R}\backslash B_n)}\frac{|f(x,u_n)|}{\|u_n\|}|v_n|\,dx\\ &\leq \frac{3}{8}+o(1).\end{aligned} \tag{3.10}$$

This contradiction implies that (u_n) is bounded in E. □

Proof of Theorem 1.1. By Lemmas 2.1 and 3.1–3.3, there exist a onstant $c \geq \alpha$ and a sequence $(u_n) \subset E$ satisfying

$$\Phi(u_n) \longrightarrow c \text{ and } \|\Phi'(u_n)\|(1+\|u_n\|) \longrightarrow 0. \tag{3.11}$$

In view of Lemma 3.4, (u_n) is bounded. So, there exists a constant $R > 0$ such that

$$\|u_n\| \leq R,\ \forall n \in \mathbb{N}. \tag{3.12}$$

By virtue of (F_1) and (F_2), for all $\epsilon > 0$, there exists $r_\epsilon \in]0, R\eta_2[$ such that

$$|f(x,u)| \leq \epsilon|u|,\ \forall(x,u)\in\mathbb{R}^2,\ |u|\leq r_\epsilon.$$

Let $C_\epsilon = \max\left\{\frac{|f(x,u)|}{|u|^2}/x\in\mathbb{R},\ r_\epsilon\leq|u|\leq R\eta_2\right\}$. Then we have

$$|f(x,u)| \leq \epsilon|u| + C_\epsilon|u|^2,\ \forall(x,u)\in\mathbb{R}^2,\ |u|\leq R\eta_2. \tag{3.13}$$

Since E is continuously embedded in $L^s(\mathbb{R})$ for $s = 2, 3$, there exists a constant $c_2 > 0$ such that $\|u_n\|_{L^2}^2 + \|u_n\|_{L^3}^3 \leq c_2$. If

$$\delta = \limsup_{n\longrightarrow\infty}\sup_{y\in\mathbb{R}^*}\int_{I(y,2)}|u_n^+|^2\,dx = 0$$

where $I(y,2)$ is the interval of $\mathbb{R}$ centered at y with radius 2, then by Lemma 2.2, $u_n^+ \longrightarrow 0$ in $L^s(\mathbb{R})$ for $2 < s < \infty$. By (2.5) and (3.12), one has

$$\|u_n\|_{L^\infty} \leq \eta_\infty R,\ \forall n\in\mathbb{N}. \tag{3.14}$$

Combining (3.13), (3.14) and Hölder's inequality yields for a positive constant c_3

$$\begin{aligned}2c+o(1)=2\Phi(u_n)&=\left\|u_n^+\right\|^2-\left\|u_n^-\right\|^2-2\int_{\mathbb{R}}F(x,u_n)dx\\&\leq\left\|u_n^+\right\|^2=\Phi'(u_n)u_n^++\int_{\mathbb{R}}f(x,u_n)u_n^+dx\\&\leq\Phi'(u_n)u_n^++\int_{\mathbb{R}}\left[\epsilon\left|u_n\right|+C_\epsilon\left|u_n\right|^2\right]\left|u_n^+\right|dx\\&\leq\Phi'(u_n)u_n^++\epsilon\left\|u_n\right\|_{L^2}\left\|u_n^+\right\|_{L^2}+C_\epsilon\left\|u_n\right\|_{L^3}^2\left\|u_n^+\right\|_{L^3}\leq\epsilon c_3+o(1).\end{aligned}$$

This is a contradiction since $\epsilon>0$ is arbitrary. Thus $\delta>0$. Going to a subsequence if necessary, we can assume the existence of $(k_n)\subset\mathbb{Z}$ such that

$$\int_{I(k_n,2)}\left|u_n^+\right|^2dx\geq\frac{\delta}{2},\ \forall n\in\mathbb{N}.$$

Let us define $w_n(x)=u_n(x+k_nT)$ so that

$$\int_{I(0,2)}\left|w_n^+\right|^2dx\geq\frac{\delta}{2},\ \forall n\in\mathbb{N}. \tag{3.15}$$

Since $F(x,u)$ is T-periodic in x, we have $\|w_n\|=\|u_n\|$ and

$$\Phi(w_n)\longrightarrow c\in[\alpha,\sup\Phi(\Lambda)]\quad\text{and}\quad\|\Phi'(w_n)\|\left(1+\|w_n\|\right)\longrightarrow 0. \tag{3.16}$$

Passing to a subsequence, we get $w_n\rightharpoonup w_0$ in E, $w_n\longrightarrow w_0$ in $L^s_{loc}(\mathbb{R})$ for $2\leq s<\infty$ and $w_n\longrightarrow w_0$ a.e. on $\mathbb{R}$. Since Φ' is weakly sequentially continuous, one has

$$\Phi'(w_n)v\longrightarrow\Phi'(w_0)v,\ \forall v\in E,$$

and by (3.16), $\Phi'(w_n)\longrightarrow 0$ in E'. Then we have $\Phi'(w_0)=0$. By (3.15) and the fact $w_n\longrightarrow w_0$ in $L^s_{loc}(\mathbb{R})$, we have

$$\int_{I(0,2)}\left|w_0^+\right|^2dx\geq\lim_{n\longrightarrow\infty}\int_{I(0,2)}\left|w_n^+\right|^2dx\geq\frac{\delta}{2}.$$

Hence $w_0^+\neq 0$ and $w_0\in\mathcal{M}$. Therefore $\mathcal{M}$ is not empty. Now, let $c=\inf_{u\in\mathcal{M}}\Phi(u)$. For any critical point $u\in\mathcal{M}\setminus\{0\}$, assumption (F_4) implies

$$\Phi(u)=\Phi(u)-\frac{1}{2}\Phi'(u)u=\int_{\mathbb{R}}\Big(\frac{1}{2}f(x,u)u-F(x,u)\Big)dx\geq 0.$$

Therefore $c\geq 0$. We shall prove that $c>0$ and there is $u_0\in\mathcal{M}\setminus\{0\}$ such that $\Phi(u_0)=c$. Let $(u_n)\subset\mathcal{M}\setminus\{0\}$ be such that $\Phi(u_n)\longrightarrow c$. Then the proof of Lemma 3.4 shows that (u_n) is bounded. So, there exists a constant $R_0>0$ such that $\|u_n\|\leq R_0$ for all $n\in\mathbb{N}$. By (2.5), we get

$$\|u_n\|_{L^\infty}\leq\eta_\infty R_0=R,\ \forall n\in\mathbb{N}. \tag{3.17}$$

As in (3.13), for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|f(x,u)| \leq \epsilon |x| + C_\epsilon |x|^2, \ \forall x \in \mathbb{R}, \ |u| \leq R.$$

So
$$\left\|u_n^+\right\|^2 \leq \int_{\mathbb{R}} f(x,u_n)u_n^+ dx \leq \int_{\mathbb{R}} \left[\epsilon |u_n| + C_\epsilon |u_n|^2\right] \left|u_n^+\right| dx \tag{3.18}$$
$$\leq \epsilon \|u_n\|_{L^2} \left\|u_n^+\right\|_{L^2} + C_\epsilon \|u_n\|_{L^3}^2 \left\|u_n^+\right\|_{L^3} \leq \epsilon\eta_2^2 \|u_n\|^2 + C_\epsilon \eta_3^2 \|u_n\|^2 \|u_n\|_{L^3}.$$

Similarly, we have
$$\left\|u_n^-\right\|^2 \leq \epsilon\eta_2^2 \|u_n\|^2 + C_\epsilon \eta_3^2 \|u_n\|^2 \|u_n\|_{L^3}. \tag{3.19}$$

From (3.18) and (3.19), we get

$$\|u_n\|^2 \leq 2\epsilon\eta_2^2 \|u_n\|^2 + 2C_\epsilon \eta_3^2 \|u_n\|^2 \|u_n\|_{L^3},$$

which implies
$$\|u_n\|_{L^3} \geq d, \ \forall n \in \mathbb{N} \tag{3.20}$$

for a positive constant d. If

$$\delta = \lim_{n \longrightarrow \infty} \sup \sup_{y \in \mathbb{R}^*} \int_{I(y,2)} |u_n|^2 dx = 0,$$

then, by Lemma 2.2, $u_n \longrightarrow 0$ in $L^s(\mathbb{R})$ for $2 < s < \infty$. This is a contradiction with (3.20). Thus $\delta > 0$. Taking a subsequence if necessary, we can assume as above the existence of $(k_n) \subset \mathbb{Z}$ such that

$$\int_{I(k_n,2)} |u_n|^2 dx \geq \frac{\delta}{2}, \ \forall n \in \mathbb{N}.$$

Let us define $w_n(x) = u_n(x + k_n T)$ so that

$$\int_{I(0,2)} |w_n|^2 dx \geq \frac{\delta}{2}, \ \forall n \in \mathbb{N}. \tag{3.21}$$

Since $F(x,u)$ is T-periodic in x, we have $\|w_n\| = \|u_n\|$ and

$$\Phi(w_n) \longrightarrow c \ \text{ and } \ \|\Phi'(w_n)\| (1 + \|w_n\|) \longrightarrow 0. \tag{3.22}$$

Passing to a subsequence, we get $w_n \rightharpoonup w_0$ in E, $w_n \longrightarrow w_0$ in $L^s_{loc}(\mathbb{R})$ for $2 \leq s < \infty$ and $w_n \longrightarrow w_0$ a.e. on $\mathbb{R}$. Since Φ' is weakly sequentially continuous, then we have $\Phi'(w_0) = 0$. Since $w_n \longrightarrow w_0$ in $L^2_{loc}(\mathbb{R})$, then (3.21) implies

$$\int_{I(0,2)} |w_0|^2 dx = \lim_{n \longrightarrow \infty} \int_{I(0,2)} |w_n|^2 dx \geq \frac{\delta}{2}.$$

Hence $w_0 \neq 0$ and $c = \Phi(w_0) > 0$. The proof of Theorem 1.1 is complete. □

4. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following several lemmas:

Lemma 4.1. *Assume that $(\mathcal{A})$, (χ), (F_1) and (F_5) are satisfied. Then for all $\theta \geq 0$, $u \in E$ and $w \in E^-$*

$$\Phi(u) \geq \Phi(\theta u + w) + \frac{1}{2} \|w\|^2 + \frac{1 - \theta^2}{2} \Phi'(u)u - \theta \Phi'(u)w. \tag{4.1}$$

Proof. By (W_5), we have for $\theta \geq 0$, $u \in E$ and $w \in E^-$

$$\begin{aligned}
&\Phi(u) - \Phi(\theta u + w) \\
&= \frac{1}{2}\Big(\left\|u^+\right\|^2 - \left\|u^-\right\|^2 \Big) - \int_{\mathbb{R}} F(x, u(x))dx \\
&\quad - \frac{1}{2}\Big(\left\|\theta u^+\right\|^2 - \left\|\theta u^- + w\right\|^2 \Big) + \int_{\mathbb{R}} F(x, \theta u + w)dx \\
&= \frac{1}{2}\Big(\left\|u^+\right\|^2 - \left\|u^-\right\|^2 \Big) - \frac{1}{2}\Big(\theta^2 \left\|u^+\right\|^2 - \theta^2 \left\|u^-\right\|^2 - \left\|w\right\|^2 \\
&\quad - \theta \prec u^-, w \succ \Big) + \int_{\mathbb{R}} \big[F(x, \theta u + w) - F(x, u)\big]dx \\
&= \frac{1}{2}\left\|w\right\|^2 + \frac{1-\theta^2}{2}\Big(\left\|u^+\right\|^2 - \left\|u^-\right\|^2 - \int_{\mathbb{R}} f(x, u)udx\Big) \\
&\quad - \theta\Big[\prec u, w \succ - \int_{\mathbb{R}} f(x, u)wdx\Big] \\
&\quad + \int_{\mathbb{R}} \Big[\frac{1-\theta^2}{2} f(x, u)u - \theta f(x, u)w + F(x, \theta u + w) - F(x, u)\Big]dx \\
&= \frac{1}{2}\left\|w\right\|^2 + \frac{1-\theta^2}{2}\Phi'(u)u - \theta\Phi'(u)w \\
&\quad + \int_{\mathbb{R}} \Big[\frac{1-\theta^2}{2} f(x, u)u - \theta f(x, u)w + F(x, \theta u + w) - F(x, u)\Big]dx \\
&\geq \frac{1}{2}\left\|w\right\|^2 + \frac{1-\theta^2}{2}\Phi'(u)u - \theta\Phi'(u)w,
\end{aligned}$$

which completes the proof of Lemma 4.1. □

Corollary 4.2. *Assume that $(\mathcal{A})$, (χ), (F_1) and (F_5) are satisfied. Then for any $u \in \mathcal{N}^-$, one has*

$$\Phi(u) \geq \Phi(\theta u + w), \ \forall \theta \geq 0, \ w \in E^-. \tag{4.2}$$

Proof. This is evident since $u \in \mathcal{N}^-$ implies $\Phi'(u)u = \Phi'(u)w = 0$. □

Corollary 4.3. *Assume that $(\mathcal{A})$, (χ), (F_1), (F_2) and (F_5) are satisfied. Then, for all $u \in E$ and $\theta \geq 0$*

$$\Phi(u) \geq \frac{\theta^2}{2}\left\|u\right\|^2 + \frac{1-\theta^2}{2}\Phi'(u)u + \theta^2\Phi'(u)u^- - \int_{\mathbb{R}} F(x, \theta u^+)dx. \tag{4.3}$$

Proof. Take $w = -\theta u^-$ in Lemma 4.1, one has

$$\begin{aligned}
\Phi(u) &\geq \Phi(\theta u^+) + \frac{\theta^2}{2}\left\|u^-\right\|^2 + \frac{1-\theta^2}{2}\Phi'(u)u + \theta^2\Phi'(u)u^- \\
&= \frac{\theta^2}{2}\left\|u^+\right\|^2 - \int_{\mathbb{R}} F(x, \theta u^+)dx + \frac{\theta^2}{2}\left\|u^-\right\|^2 + \frac{1-\theta^2}{2}\Phi'(u)u + \theta^2\Phi'(u)u^- \\
&= \frac{\theta^2}{2}\left\|u\right\|^2 + \frac{1-\theta^2}{2}\Phi'(u)u + \theta^2\Phi'(u)u^- - \int_{\mathbb{R}} F(x, \theta u^+)dx,
\end{aligned}$$

which ends the proof of Corollary 4.3. □

Lemma 4.4. *Assume that* $(\mathcal{A})$, (χ), (F_1), (F_2) *and* (F_5) *are satisfied. Then*
(i) *there exists* $\rho > 0$ *such that*

$$m = \inf_{\mathcal{N}^-} \Phi \geq \alpha = \inf \left\{ \Phi(u)/u \in E^+,\ \|u\| = \rho \right\} > 0,$$

(ii)
$$\left\|u^+\right\| \geq \max \left\{ \left\|u^-\right\|, \sqrt{2m} \right\},\ for\ all\ u \in \mathcal{N}^-.$$

Proof. (i) Take $w = -\theta u^-$ in Corollary 4.3, one has

$$\Phi(u) \geq \Phi(\theta u^+),\ \forall \theta \geq 0,\ u \in \mathcal{N}^-.$$

Let ρ be defined in Lemma 3.2, we have

$$\alpha = \inf \left\{ \Phi(u)/u \in E^+,\ \|u\| = \rho \right\} > 0.$$

Then, we have for all $u \in \mathcal{N}^-$ (take $\theta = \frac{\rho}{\|u^+\|}$)

$$\Phi(u) \geq \Phi(\rho \frac{u^+}{\|u^+\|}) \geq \alpha = \inf \left\{ \Phi(v) \mid v \in E^+,\ \|v\| = \rho \right\}.$$

(ii) For $u \in \mathcal{N}^-$, we have

$$m \leq \frac{1}{2}\Big(\left\|u^+\right\|^2 - \left\|u^-\right\|^2 \Big) - \int_{\mathbb{R}} F(x, u(x))dx \leq \frac{1}{2}\Big(\left\|u^+\right\|^2 - \left\|u^-\right\|^2 \Big),$$

which implies $2m + \left\|u^-\right\|^2 \leq \left\|u^+\right\|^2$ and hence $\|u^+\| \geq \max\left\{\sqrt{2m}, \|u^-\|\right\}$.
The proof of Lemma 4.4 is complete. □

Lemma 4.5. *Assume that* $(\mathcal{A})$, (χ), $(F_1) - (F_3)$ *and* (F_5) *are satisfied. Then any sequence* $(u_n) \subset E$ *satisfying*

$$\Phi(u_n) \longrightarrow c > 0,\ \Phi'(u_n)u_n^{\pm} \longrightarrow 0 \tag{4.4}$$

is bounded.

Proof. To prove the boundedness of (u_n), arguing by contradiction, suppose that $\|u_n\| \longrightarrow \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. By (2.5), $\|v_n\|_{L^2} \leq \eta_2$. If

$$\delta = \lim_{n \longrightarrow \infty} \sup \sup_{s \in \mathbb{R}} \int_{s-T}^{s+T} \left|v_n^+\right|^2 dx = 0,$$

then by Lemma 2.2 [15], $v_n^+ \longrightarrow 0$ in $L^s(\mathbb{R})$ for $s \in]2, \infty[$. Fix $R > [2(1+c)]^{\frac{1}{2}}$. By virtue of (F_1) and (F_2), for $\epsilon = \frac{1}{4(R\eta_2)^2}$, there exists $r \in]0, R\eta_\infty[$ such that

$$F(x, u) \leq \frac{1}{4(R\eta_2)^2} |u|^2,\ \forall (x, u) \in \mathbb{R}^2,\ |u| \leq r. \tag{4.5}$$

Let
$$\beta = \max \left\{ \frac{F(x, u)}{|u|^3} / x \in \mathbb{R}, r \leq |u| \leq R\eta_\infty \right\}. \tag{4.6}$$

Then $0 \leq \beta < \infty$. Hence, it follows from (4.5) and (4.6) that

$$F(x, u) \leq \frac{1}{4(R\eta_2)^2} |u|^2 + \beta |u|^3,\ \forall (x, u) \in \mathbb{R}^2,\ |u| \leq R\eta_\infty. \tag{4.7}$$

Combining (4.7) with the fact $\|v_n^+\|_{L^\infty} \leq \eta_\infty$, one has

$$\limsup_{n\longrightarrow\infty}\int_{\mathbb{R}} F(x, Rv_n^+)dx \leq \frac{1}{4\eta_2^2}\limsup_{n\longrightarrow\infty}\int_{\mathbb{R}}\left|v_n^+\right|^2 dx + R^3\beta\limsup_{n\longrightarrow\infty}\int_{\mathbb{R}}\left|v_n^+\right|^3 dx \leq \frac{1}{4}. \quad (4.8)$$

Set $\theta_n = \frac{R}{\|u_n\|}$. Hence, by virtue of (4.3), (4.4) and (4.8), one has

$$\begin{aligned}
c + o(1) &= \Phi(u_n)\\
&\geq \frac{\theta_n^2}{2}\|u_n\|^2 - \int_{\mathbb{R}} F(x, \theta_n u_n^+)dx + \frac{1-\theta_n^2}{2}\Phi'(u_n)u_n + \theta_n^2\Phi'(u_n)u_n^-\\
&\geq \frac{R^2}{2} - \int_{\mathbb{R}} F(x, \frac{R}{\|u_n\|}u_n^+)dx + \frac{1}{2}\Big(1 - \frac{R^2}{\|u_n\|^2}\Big)\Phi'(u_n)u_n + \frac{R^2}{\|u_n\|^2}\Phi'(u_n)u_n^-\\
&\geq \frac{R^2}{2} - \int_{\mathbb{R}} F(x, \frac{R}{\|u_n\|}u_n^+)dx + o(1)\\
&\geq \frac{R^2}{2} - \frac{1}{4} + o(1) > c + 1 - \frac{1}{4} + o(1) = c + \frac{3}{4} + o(1).
\end{aligned}$$

This contradiction shows that $\delta > 0$. We may assume that there exists $(k_n) \subset \mathcal{Z}$ such that

$$\int_{(k_n-2)T}^{(k_n+2)T}\left|v_n^+\right|^2 dx > \frac{\delta}{2},\ \forall n \in \mathbb{N}.$$

Let $w_n(x) = v_n(x + k_nT)$. Then

$$\int_{-2T}^{2T}\left|w_n^+\right|^2 dx > \frac{\delta}{2},\ \forall n \in \mathbb{N}. \quad (4.9)$$

Now, we define $\widetilde{u}_n(x) = u_n(x + k_nT)$. Then $\frac{\widetilde{u}_n}{\|u_n\|} = w_n$ and $\|w_n\| = 1$. Up to a subsequence if necessary, we can assume $w_n \rightharpoonup w$ in E, $w_n \longrightarrow w$ in $L^2_{loc}(\mathbb{R})$ and $w_n \longrightarrow w$ a.e. on $\mathbb{R}$. Obviously, (4.9) implies that $w \neq 0$. Hence, it follows from (4.4), (F_3) and Fatou's lemma that

$$\begin{aligned}
0 = \lim_{n\longrightarrow\infty}\frac{c+o(1)}{\|u_n\|^2} &= \lim_{n\longrightarrow\infty}\frac{\Phi(u_n)}{\|u_n\|^2}\\
&= \lim_{n\longrightarrow\infty}\Big[\frac{1}{2}\big(\|w_n^+\|^2 - \|w_n^-\|^2\big) - \int_{\mathbb{R}}\frac{F(x,u_n)}{\|u_n\|^2}dx\\
&= \lim_{n\longrightarrow\infty}\Big[\frac{1}{2}\big(\|w_n^+\|^2 - \|w_n^-\|^2\big) - \int_{\mathbb{R}}\frac{F(x,u_n)}{|\widetilde{u}_n|^2}|w_n|^2 dx\\
&\leq \frac{1}{2} - \liminf_{n\longrightarrow\infty}\int_{\mathbb{R}}\frac{F(x,u_n)}{|\widetilde{u}_n|^2}|w_n|^2 dx = -\infty,
\end{aligned}$$

which is a contradiction. Thus (u_n) is bounded and the proof of Lemma 4.5 is complete. □

Lemma 4.6. *Assume that $(\mathcal{A})$, (χ), $(F_1) - (F_3)$ and (F_5) are satisfied. Then there exist a constant $c \in [\alpha, \sup\Phi(\Lambda)]$ and a sequence $(u_n) \subset E$ satisfying*

$$\Phi(u_n) \longrightarrow c \ \ \textit{and}\ \ \|\Phi'(u_n)\|(1 + \|u_n\|) \longrightarrow 0. \quad (4.10)$$

Proof. By Lemmas 3.1, 4.4(i), the functional Φ satisfies all the conditions of Lemma 2.1. Hence, by Lemma 2.1, there exist a constant $c \in [\alpha, \sup \Phi(\Lambda)]$ and a sequence $(u_n) \subset E$ satisfying (4.10). □

Lemma 4.7. *Assume that* $(\mathcal{A})$, (χ), $(F_1) - (F_3)$ *and* (F_5) *are satisfied. Then there exist a constant* $c \in [\alpha, m]$ *and a sequence* $(u_n) \subset E$ *satisfying*

$$\Phi(u_n) \longrightarrow c \quad \textit{and} \quad \|\Phi'(u_n)\| (1 + \|u_n\|) \longrightarrow 0. \tag{4.11}$$

Proof. Choose $(v_n) \subset \mathcal{N}^-$ such that

$$m \leq \Phi(v_n) < m + \frac{1}{n}, \ \forall n \in \mathbb{N}. \tag{4.12}$$

By Lemma 4.4, $\|v_n^+\| \geq \sqrt{2m}$. Set $e_n = \frac{v_n^+}{\|v_n^+\|}$. Then $e_n \in E^+$ and $\|e_n\| = 1$. In view of Lemma 3.3, there exists $r_n > \max\{\rho, \|v_n\|\}$ such that $\sup \Phi(\partial \Lambda_n) \leq 0$, where

$$\Lambda_n = \left\{ w + se_n / w \in E^-, \ s \geq 0, \ \|w + se_n\| \leq r_n \right\}, \ n \in \mathbb{N}. \tag{4.13}$$

Hence, applying Lemma 4.5 to the above set Λ_n, there exist a constant $c_n \in [\alpha, \sup \Phi(\Lambda_n)]$ and a sequence $(u_{n,k}) \subset E$ satisfying

$$\Phi(u_{n,k}) \longrightarrow c_n \ \textit{and} \ \|\Phi'(u_{n,k})\| (1 + \|u_{n,k}\|) \longrightarrow 0, \ \forall n \in \mathbb{N}. \tag{4.14}$$

By virtue of Corollary 4.3, we can get

$$\Phi(v_n) \geq \Phi(sv_n + w), \ \forall s \geq 0, \ w \in E^-. \tag{4.15}$$

Since $v_n \in \Lambda_n$, it follows from (4.13) and (4.15) that $\Phi(v_n) = \sup \Phi(\Lambda_n)$. Hence, by (4.12) and (4.14), one has

$$\Phi(u_{n,k}) \longrightarrow c_n < m + \frac{1}{n} \ \text{ and } \ \|\Phi'(u_{n,k})\| (1 + \|u_{n,k}\|) \longrightarrow 0, \ \forall n \in \mathbb{N}. \tag{4.16}$$

Now, we can choose a sequence $(k_n) \subset \mathbb{N}$ such that

$$\Phi(u_{n,k_n}) < m + \frac{1}{n} \ \text{ and } \ \|\Phi'(u_{n,k_n})\| (1 + \|u_{n,k_n}\|) < \frac{1}{n}, \ \forall n \in \mathbb{N}. \tag{4.17}$$

Let $u_n = u_{n,k_n}$ for $n \in \mathbb{N}$, then going to a subsequence if necessary, we obtain

$$\Phi(u_n) \longrightarrow c \in [\alpha, m] \ \text{ and } \ \|\Phi'(u_n)\| (1 + \|u_n\|) \longrightarrow 0.$$

The proof of Lemma 4.7 is completed. □

Proof of Theorem 1.2. In view of Lemma 4.7, there exist a constant $c \in [\alpha, m]$ and a sequence $(u_n) \subset E$ satisfying

$$\Phi(u_n) \longrightarrow c \ \text{ and } \ \|\Phi'(u_n)\| (1 + \|u_n\|) \longrightarrow 0. \tag{4.18}$$

By Lemma 4.5, (u_n) is bounded. Proceeding as in the proof of Theorem 1.1, there exist a constant $\delta > 0$ and a sequence $(w_n) \subset E$ satisfying

$$\int_{I(0,2)} \left|w_n^+\right|^2 dx \geq \frac{\delta}{2}, \ \forall n \in \mathbb{N} \tag{4.19}$$

and

$$\Phi(w_n) \longrightarrow c \ \text{ and } \ \|\Phi'(w_n)\| (1 + \|w_n\|) \longrightarrow 0. \tag{4.20}$$

By Lemma 4.5, (w_n) is bounded. Hence, passing to a subsequence, we may assume that $w_n \rightharpoonup w_0$ in E, $w_n \longrightarrow w_0$ in $L^2_{loc}(\mathbb{R})$ and $w_n \longrightarrow w_0$ a.e. on $\mathbb{R}$. As in the proof of Theorem 1.1, we get $\Phi'(w_0) = 0$ and

$$\int_{I(0,2)} \left|w_0^+\right|^2 dx \geq \frac{\delta}{2},$$

which implies that $w_0^+ \neq 0$ and $w_0 \in \mathcal{N}^-$. Hence $\Phi(w_0) \geq m$. On the other hand, by using (4.20), (F_5) and Fatou's lemma, we have

$$\begin{aligned} m \geq c &= \lim_{n\longrightarrow\infty} \left[\Phi(w_n) - \frac{1}{2}\Phi'(w_n)w_n\right] = \lim_{n\longrightarrow\infty} \int_{\mathbb{R}} \left[\frac{1}{2}f(x,w_n)w_n - F(x,w_n)\right] dx \\ &\geq \int_{\mathbb{R}} \lim_{n\longrightarrow\infty} \left[\frac{1}{2}f(x,w_n)w_n - F(x,w_n)\right] dx \geq \int_{\mathbb{R}} \left[\frac{1}{2}f(x,w_0)w_0 - F(x,w_0)\right] dx \\ &= \Phi(w_0) - \frac{1}{2}\Phi'(w_0)w_0 = \Phi(w_0). \end{aligned}$$

This shows that $m \geq \Phi(w_0)$ and so $\Phi(w_0) = m = \inf_{\mathcal{N}^-} f$. The proof of Theorem 1.2 is complete. □

5. Examples

In this Section, we give some examples for our results.

Example 5.1. Let $F(x,u) = \cos^2 x\, |u|^2 \ln(1+|u|), \forall (x,u) \in \mathbb{R}^2$.

It is easy to check assumptions $(F_1) - (F_3)$. It remains to prove (F_4). We have

$$\lim_{|u|\longrightarrow 0} \frac{|f(x,u)|}{|u|} \leq \lim_{|u|\longrightarrow 0} \left[2\ln(1+|u|) + \frac{|u|}{1+|u|}\right] = 0.$$

So, there exists a positive constant r_0 such that

$$|u| \leq r_0 \implies \frac{|f(x,u)|}{|u|} < \frac{1}{4\eta_2^2}, \ \forall x \in \mathbb{R}.$$

By contraposition, we get for all $x \in \mathbb{R}$

$$\frac{|f(x,u)|}{|u|} \geq \frac{1}{4\eta_2^2} \implies |u| \geq r_0.$$

On the other hand, we have $\widetilde{F}(x,u) = \frac{1}{2}\cos^2 x \frac{|u|^3}{1+|u|}$ and then

$$\frac{|f(x,u)|}{\widetilde{F}(x,u)\,|u|} = 2\,\frac{2(1+|u|)ln(1+|u|) + |u|}{|u|^3}.$$

Setting

$$g_0(t) = \frac{t^2}{4ln(1+t)}, \quad t \geq r_0,$$

we get

$$\lim_{|x|\longrightarrow\infty} \frac{|f(x,u)|}{\widetilde{F}(x,u)\,|x|} g_0(|x|) = 1$$

hence $(x,u) \longmapsto \frac{|f(x,u)|}{\widetilde{F}(x,u)|u|} g_0(|u|)$ is bounded on $\mathbb{R} \times \big(\mathbb{R}\backslash]-r_0, r_0[\big)$ and achieves its maximum.

Let
$$M_0=\max_{x\in\mathbb{R},|u|\geq r_0}\frac{|f(x,u)|}{\widetilde{F}(x,u)\,|u|}g_0(|u|)\ \text{ and }\ g(s)=\frac{g_0(s)}{M_0}.$$

Then, we have
$$|f(x,u)|\leq\frac{|u|}{g(|u|)}\widetilde{F}(x,u).$$

Therefore $F(x,u)$ satisfies the condition (F_4). However, F does not satisfy any of the conditions (C1)-(C3). In fact, we have

(a) For any $\mu>2$
$$f(x,u)u-\mu F(x,u)=u^2\cos^2(x)\Big[(2-\mu)\ln(1+|u|)+\frac{u}{1+|u|}\Big]\rightarrow-\infty\ \text{ as }\ |u|\rightarrow\infty,$$
for almost every $x\in\mathbb{R}$, which means that F does not satisfy the condition (C1).

(b) Assume that there exists a constant $\sigma\geq 1$ such that
$$\sigma\widetilde{F}(x,u)\geq\widetilde{F}(x,su),\ \forall(s,x,u)\in[0,1]\times\mathbb{R}^2,$$
then we obtain $\sigma\dfrac{u^3}{1+|u|}\geq\dfrac{s^3u^3}{1+s\,|u|}$ for all $(s,x,u)\in[0,1]\times\mathbb{R}^2$.

Taking $u=-1$ and $s=0$, we get $\sigma\leq 0$, witch is a contradiction. Hence, F does not satisfy the condition (C2).

(c) Since $\widetilde{F}(x,u)=\dfrac{1}{2}\cos^2 x\,\dfrac{u^3}{1+|u|}$ and $x\longmapsto\cos^2 x$ cannot be lower bounded by a positive constant, then F does not satisfy the condition (C3).

Example 5.2. Let $F(x,u)=\theta(x)\Big[\,|u|^{\frac{13}{4}}-|u|^{\frac{11}{4}}+|u|^{\frac{9}{4}}\Big]$ where $\theta\in C(\mathbb{R},\mathbb{R}^+)$ is periodic. It is easy to see that W satisfies $(F_1)-(F_3)$. Let us prove condition (F_4).

We have
$$f(x,u)=\frac{\theta(x)}{4}\Big[13\,|u|-11\,|u|^{\frac{1}{2}}+1\Big]u$$
and
$$\widetilde{F}(x,u)=\frac{\theta(x)}{8}\Big[5\,|u|-3\,|u|^{\frac{1}{2}}+1\Big]\,|u|^{\frac{9}{4}}$$
so
$$\frac{|f(x,u)|}{\widetilde{F}(x,u)}=2\,\frac{13\,|u|-11\,|u|^{\frac{1}{2}}+1}{5\,|u|-3\,|u|^{\frac{1}{2}}+1}\,|u|^{-1}.$$

Set $M_0=2\sup\limits_{t\in\mathbb{R}^+}\dfrac{13t^2-11t+1}{5t^2-3t+1}$, then we get $\dfrac{|f(x,u)|}{\widetilde{F}(x,u)}\leq\dfrac{|u|}{g(|u|)}$, where $g(t)=\dfrac{t^2}{M_0}$, $t\in\mathbb{R}^+$.

Therefore
$$|f(x,u)|\leq\frac{|u|}{g(|u|)}\widetilde{F}(x,u)\ \forall\ u\neq 0$$
with $g\in C(\mathbb{R}^*_+,\mathbb{R}^*_+)$. Therefore W satisfies (F_4).

Example 5.3. Let $F(x,u)=a(x)\Big[\,|u|^p+(p-2)\,|u|^{p-\epsilon}\sin^2\big(\frac{|u|^\epsilon}{\epsilon}\big)\Big]$, where $p>2$, $a\in C(\mathbb{R},\mathbb{R}^*_+)$ is periodic and $0<\epsilon<p-2$. It is easy to check that F satisfies $(F_1)-(F_3)$ and (F_5). However, let $u_n=[\epsilon(n\pi+\frac{3\pi}{4})]^{\frac{1}{\epsilon}}$.

Then for any $\eta > 2$, one has

$$
\begin{aligned}
&f(x,u_n)u_n - \eta F(x,u_n)\\
&= a(x)\Big[(\gamma-\eta)\,|u_n|^{\gamma} + (\gamma-2)(\gamma-\eta-\epsilon)\,|u_n|^{\gamma-\epsilon}\sin^2\Big(\frac{|u_n|^{\epsilon}}{\epsilon}\Big) + (\gamma-2)\,|u_n|^{\gamma}\sin\Big(2\frac{|u_n|^{\epsilon}}{\epsilon}\Big)\Big]\\
&= a(x)\,|u_n|^{\gamma}\left[\gamma-\eta+\frac{(\gamma-2)(\gamma-\eta-\epsilon)}{|u_n|^{\gamma}}\sin^2\Big(\frac{|u_n|^{\epsilon}}{\epsilon}\Big) + (\gamma-2)\,sin\Big(2\frac{|u_n|^{\epsilon}}{\epsilon}\Big)\right]\\
&= a(x)\,|u_n|^{\gamma}\left[2-\eta+\frac{(\gamma-2)(\gamma-\eta-\epsilon)}{|u_n|^{\gamma}}\Big(\frac{\sqrt{2}}{2}\Big)^2\right] \longrightarrow -\infty \quad \text{as} \quad n \longrightarrow \infty.
\end{aligned}
$$

That is the condition (C1) can not be satisfied for any $\eta > 2$. Similarly, we prove that F does not satisfy any of the conditions (C2) and (C3).

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