

Saddle Points of some Integral Functionals and Solutions of Elliptic Systems

Lucio Boccardo

*Istituto Lombardo, Milano, Italy
boccardo@mat.uniroma1.it*

Pasquale Imparato

pasquale.imparato1994@gmail.com

Luigi Orsina

*Sapienza Università di Roma, Italy
orsina@mat.uniroma1.it*

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We prove the existence of finite energy solutions u and ψ for two systems, one of which is

$$\begin{cases} u \in W_0^{1,2}(\Omega) : & -\operatorname{div}(a(x) \nabla u) = -\operatorname{div}(\psi E(x)), \\ \psi \in W_0^{1,p}(\Omega) : & -\operatorname{div}(a(x) |\nabla \psi|^{p-2} \nabla \psi) + E(x) \cdot \nabla u = f(x), \end{cases}$$

under some assumptions on p and on the vector field $E(x)$.

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1. Introduction

Let Ω be a bounded, open subset of \mathbb{R}^N , with $N > 2$. Define, for v and φ in $W_0^{1,2}(\Omega)$, the functional

$$\mathcal{J}(\varphi, v) = \frac{1}{2} \int_{\Omega} a(x) |\nabla \varphi|^2 - \frac{1}{2} \int_{\Omega} a(x) |\nabla v|^2 + \int_{\Omega} \varphi E(x) \cdot \nabla v - \int_{\Omega} f(x) \varphi. \quad (1)$$

where $a(x)$ is a measurable function on Ω such that

$$0 < \alpha \leq a(x) \leq \beta, \quad (2)$$

for almost every x in Ω and

$$f \in L^m(\Omega), \quad m \geq 2_* \stackrel{\text{def}}{=} \frac{2N}{N+2}. \quad (3)$$

In the paper [7] it is proved the existence of a saddle point of the functional \mathcal{J} if the vector field E is very singular. More precisely, it is first proved that if $E(x)$ belongs to $(L^N(\Omega))^N$, then there exist u and ψ in $W_0^{1,2}(\Omega)$ such that

$$\mathcal{J}(\psi, v) \leq \mathcal{J}(\psi, u) \leq \mathcal{J}(\varphi, u), \quad \forall v, \varphi \in W_0^{1,2}(\Omega).$$

Thus, using Calculus of Variations techniques it follows that u and ψ are weak solutions of the system

$$\begin{cases} u \in W_0^{1,2}(\Omega) : -\operatorname{div}(a(x) \nabla u) = -\operatorname{div}(\psi E(x)), \\ \psi \in W_0^{1,2}(\Omega) : -\operatorname{div}(a(x) \nabla \psi) + E(x) \cdot \nabla u = f(x), \end{cases} \quad (4)$$

that is

$$\begin{cases} \int_{\Omega} a(x) \nabla u \cdot \nabla v = \int_{\Omega} \psi E(x) \cdot \nabla v, \quad \forall v \in W_0^{1,2}(\Omega), \\ \int_{\Omega} a(x) \nabla \psi \cdot \nabla \varphi + \int_{\Omega} E(x) \cdot \nabla u \varphi = \int_{\Omega} f(x) \varphi, \quad \forall \varphi \in W_0^{1,2}(\Omega). \end{cases} \quad (5)$$

Then a *regularizing* property of the above system is proved: the existence of weak solutions u and ψ in $W_0^{1,2}(\Omega)$ of the system (4) under the weaker assumption that $E(x)$ belongs to $(L^2(\Omega))^N$.

Note that the existence of solutions in the “energy space” $W_0^{1,2}(\Omega)$ is *false* if one considers the equations separately, and is a consequence of the fact that we are dealing with a system of equations. Indeed, the term $\psi E(x)$ only belongs to $(L^{\frac{N}{N-1}}(\Omega))^N$, so that $-\operatorname{div}(\psi E(x))$ does not belong to the dual of $W_0^{1,2}(\Omega)$, and the term $E(x) \cdot \nabla u$ only belongs to $L^1(\Omega)$.

This regularizing effect is a consequence of a *natural* feature of the system: if we take $v = u$ as test function in the first identity of (5), and $\varphi = \psi$ in the second, and then add the results, the integrals containing $E(x)$ cancel out; therefore, the summability of $E(x)$ has no effect on the regularity properties of the solutions.

In this paper, we study the case of functionals (like J below) having nonlinear terms with nonquadratic growth, which differ from \mathcal{J} , where the nonlinear terms have quadratic growth.

In the first problem, we allow a growth of order p , $p \neq 2$, to one of the two integrals of the principal part, so that in the second equation of system (4) there is a p -laplacian operator. Such systems are loosely related to those studied in [4].

In the second problem, the term with nonlinear growth is the coupling term depending on ∇v and φ .

The common point between the problems studied in [7] and the problems studied in this paper is the “cancellation property” of the terms containing $E(x)$, which holds also for the Euler-Lagrange equation of the functional J below (see (8)).

2. Problems with nonlinear principal part

Let Ω be a bounded, open subset of \mathbb{R}^N , with $N > 2$, let

$$2_* = \frac{2N}{N+2} \leq p < N, \quad (6)$$

and let $r \geq 2$ be such that

$$\frac{1}{p} + \frac{1}{r} \leq \frac{1}{2} + \frac{1}{N}. \quad (7)$$

Let us define, for φ in $W_0^{1,p}(\Omega)$ and v in $W_0^{1,2}(\Omega)$,

$$J(\varphi, v) = \frac{1}{p} \int_{\Omega} a(x) |\nabla \varphi|^p - \frac{1}{2} \int_{\Omega} a(x) |\nabla v|^2 + \int_{\Omega} \varphi E(x) \cdot \nabla v - \int_{\Omega} f(x) \varphi. \quad (8)$$

where $a(x)$ satisfies (2),

$$f \in L^m(\Omega), \quad m \geq (p^*)' \stackrel{\text{def}}{=} p_*, \quad (9)$$

and

$$E \in (L^r(\Omega))^N, \quad (10)$$

with r such that (7) holds.

As in the paper [7], we look for saddle points of the functional J , that is functions ψ in $W_0^{1,p}(\Omega)$ and u in $W_0^{1,2}(\Omega)$ such that

$$J(\psi, v) \leq J(\psi, u) \leq J(\varphi, u), \quad \forall \varphi \in W_0^{1,p}(\Omega), \quad \forall v \in W_0^{1,2}(\Omega).$$

In order to prove the existence of a saddle point for J , we will work with the system of Euler-Lagrange equations for the critical points of J , which is

$$\begin{cases} u \in W_0^{1,2}(\Omega) : & -\operatorname{div}(a(x) \nabla u) = -\operatorname{div}(\psi E(x)), \\ \psi \in W_0^{1,p}(\Omega) : & -\operatorname{div}(a(x) |\nabla \psi|^{p-2} \nabla \psi) + E(x) \cdot \nabla u = f(x). \end{cases} \quad (11)$$

We will prove two results: the first one is the existence of weak solutions u and ψ of system (11) under the assumption that $E(x)$ only belongs to $(L^2(\Omega))^N$; however, under such assumption on $E(x)$ the functional J is not well defined, since the term $\varphi E(x) \cdot \nabla v$ may not belong to $L^1(\Omega)$; therefore, to prove that (ψ, u) is a saddle point of J we need the stronger assumption that $E(x)$ belongs to $(L^r(\Omega))^N$, with r such that (7) holds.

Our first result is the following.

Theorem 2.1. *Let $f(x)$ be such that (9) holds, and let $E(x)$ be a vector field in $(L^2(\Omega))^N$. Then there exist u in $W_0^{1,2}(\Omega)$ and ψ in $W_0^{1,p}(\Omega)$, solutions of system (11) that is*

$$\int_{\Omega} a(x) \nabla u \cdot \nabla v = \int_{\Omega} \psi E(x) \cdot \nabla v, \quad \forall v \in C_0^1(\Omega), \quad (12)$$

$$\int_{\Omega} a(x) |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \varphi + \int_{\Omega} E(x) \cdot \nabla u \varphi = \int_{\Omega} f(x) \varphi, \quad (13)$$

$$\forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega).$$

Note that with our assumptions on p and r we have that $\psi E(x)$ belongs to $(L^1(\Omega))^N$ (since $\frac{1}{p^*} + \frac{1}{2} \leq 1$), and that $E(x) \cdot \nabla u$ belongs to $L^1(\Omega)$ (since both ∇u and $E(x)$ belong to $(L^2(\Omega))^N$).

We begin with an existence theorem for an approximated system with bounded data; we will use the following function of one real variable, depending on a parameter $k \geq 0$:

$$T_k(s) = \max(-k, \min(s, k)).$$

Theorem 2.2. *Let n in \mathbb{N} , let $g(x)$ be a function in $L^\infty(\Omega)$, and let $G(x)$ be a vector field in $(L^\infty(\Omega))^N$. Then there exist v and φ , weak solutions of the system*

$$\begin{cases} v \in W_0^{1,2}(\Omega) : & -\operatorname{div}(a(x)\nabla v) = -\operatorname{div}(T_n(\varphi)G(x)), \\ \varphi \in W_0^{1,p}(\Omega) : & -\operatorname{div}(a(x)|\nabla\varphi|^{p-2}\nabla\varphi) + G(x) \cdot \nabla v = g(x), \end{cases} \quad (14)$$

that is

$$\begin{cases} \int_{\Omega} a(x)\nabla v \cdot \nabla z = \int_{\Omega} T_n(\varphi)G(x) \cdot \nabla z, & \forall z \in W_0^{1,2}(\Omega), \\ \int_{\Omega} a(x)|\nabla\varphi|^{p-2}\nabla\varphi \cdot \nabla\eta + \int_{\Omega} G(x) \cdot \nabla v \eta = \int_{\Omega} g(x)\eta, & \forall \eta \in W_0^{1,p}(\Omega). \end{cases} \quad (15)$$

Proof. Let ϕ belong to $L^p(\Omega)$, and let w in $W_0^{1,2}(\Omega)$ be the unique weak solution of the problem

$$w \in W_0^{1,2}(\Omega) : -\operatorname{div}(a(x)\nabla w) = -\operatorname{div}(T_n(\phi)G(x)), \quad (16)$$

that is
$$\int_{\Omega} a(x)\nabla w \cdot \nabla z = \int_{\Omega} T_n(\phi)G(x) \cdot \nabla z, \quad \forall z \in W_0^{1,2}(\Omega).$$

Such a solution exists (by Lax-Milgram theorem) since $T_n(\phi)G(x)$ is in $(L^\infty(\Omega))^N$. Let then Φ in $W_0^{1,p}(\Omega)$ be the unique solution of

$$\Phi \in W_0^{1,p}(\Omega) : -\operatorname{div}(a(x)|\nabla\Phi|^{p-2}\nabla\Phi) + G(x) \cdot \nabla w = g(x),$$

whose weak formulation is

$$\int_{\Omega} a(x)|\nabla\Phi|^{p-2}\nabla\Phi \cdot \nabla\eta + \int_{\Omega} G(x) \cdot \nabla w \eta = \int_{\Omega} g(x)\eta, \quad \forall \eta \in W_0^{1,p}(\Omega).$$

Observing that since $p \geq 2_*$ one has $p^* \geq 2$, so that $p_* \leq 2$, the existence of Φ follows from the results of [9] since both $g(x)$ and $G(x) \cdot \nabla w$ belong to $L^2(\Omega)$, hence to $L^{p^*}(\Omega)$.

From the equation solved by w , and thanks to Lax-Milgram theorem (for example) we have that there exists $C \geq 0$ (independent on n) such that

$$\|w\|_{W_0^{1,2}(\Omega)} \leq C \|T_n(\phi)G(x)\|_{(L^2(\Omega))^N} \leq C n \|G\|_{(L^\infty(\Omega))^N}. \quad (17)$$

From the equation solved by Φ , rewritten as

$$-\operatorname{div}(a(x)|\nabla\Phi|^{p-2}\nabla\Phi) = g(x) - G(x) \cdot \nabla w,$$

we have, thanks to standard estimates (choose Φ as test function, and use Sobolev embedding and Hölder inequality), that there exists $C \geq 0$ (independent on n) such that

$$\begin{aligned} \|\Phi\|_{W_0^{1,p}(\Omega)} &\leq C \|g\|_{L^{p^*}(\Omega)} + C \|G(x) \cdot \nabla w\|_{L^{p^*}(\Omega)} \\ &\leq C \|g\|_{L^\infty(\Omega)} + C \|G\|_{L^\infty(\Omega)} \|\nabla w\|_{(L^{p^*}(\Omega))^N} \\ &\leq C \|g\|_{L^\infty(\Omega)} + C \|G\|_{L^\infty(\Omega)} \|w\|_{W_0^{1,2}(\Omega)} \\ &\leq C \|g\|_{L^\infty(\Omega)} + C n \|G\|_{L^\infty(\Omega)}^2, \end{aligned}$$

where in the last passage we have used (17). Thus, also recalling Poincaré inequality, we have that

$$\|\Phi\|_{L^p(\Omega)} \leq C \|\Phi\|_{W_0^{1,p}(\Omega)} \leq C \|g\|_{L^\infty(\Omega)} + C n \|G\|_{L^\infty(\Omega)}^2 \stackrel{\text{def}}{=} R. \tag{18}$$

Thanks to the above estimate, we have that the ball $B_R(0)$ of radius R in $L^p(\Omega)$, is invariant for the map $S : L^p(\Omega) \rightarrow L^p(\Omega)$ defined by $\phi \mapsto \Phi$.

Thanks to the Rellich theorem, and to (18), the map S is compact. Furthermore, since the map $\phi \mapsto w$ is continuous from $L^p(\Omega)$ to $W_0^{1,2}(\Omega)$ (recall that in the divergence term appears $T_n(\phi)$, which belongs to $L^\infty(\Omega)$), and since the map $w \mapsto \Phi$ is continuous from $W_0^{1,2}(\Omega)$ to $W_0^{1,p}(\Omega)$ (thanks to the assumptions on $g(x)$ and $G(x)$), the map S is continuous from $L^p(\Omega)$ to $W_0^{1,p}(\Omega)$, hence from $L^p(\Omega)$ to $L^p(\Omega)$. Therefore, S satisfies the assumptions of Schauder’s theorem, so that there exists φ in $L^p(\Omega)$ such that $S(\varphi) = \varphi$. Defining v as the solution of equation (16) with $\phi = \varphi$, we have proved that for every n in \mathbb{N} there exist weak solutions v in $W_0^{1,2}(\Omega)$ and φ in $W_0^{1,p}(\Omega)$ of (14) (in the sense (15)). \square

We can now prove Theorem 2.1.

Proof of Theorem 2.1. Let n in \mathbb{N} , and let

$$f_n(x) = T_n(f(x)), \quad E_n(x) = \frac{E(x)}{1 + \frac{1}{n}|E(x)|}. \tag{19}$$

Since f_n belongs to $L^\infty(\Omega)$, and E_n belongs to $(L^\infty(\Omega))^N$, by Theorem 2.2, applied with $g(x) = f_n(x)$, and $G(x) = E_n(x)$, there exist u_n in $W_0^{1,2}(\Omega)$ and ψ_n in $W_0^{1,p}(\Omega)$, weak solutions of (14) in the sense (15); that is

$$\int_{\Omega} a(x) \nabla u_n \cdot \nabla z = \int_{\Omega} T_n(\psi_n) E_n(x) \cdot \nabla z, \quad \forall z \in W_0^{1,2}(\Omega), \tag{20}$$

and

$$\int_{\Omega} a(x) |\nabla \psi_n|^{p-2} \nabla \psi_n \cdot \nabla \eta + \int_{\Omega} E_n(x) \cdot \nabla u_n \eta = \int_{\Omega} f_n(x) \eta, \quad \forall \eta \in W_0^{1,p}(\Omega). \tag{21}$$

Choose $z = u_n$ in (20), and $\eta = T_n(\psi_n)$ in (21), to obtain

$$\int_{\Omega} a(x) |\nabla u_n|^2 = \int_{\Omega} T_n(\psi_n) E_n(x) \cdot \nabla u_n,$$

and

$$\int_{\Omega} a(x) |\nabla T_n(\psi_n)|^p + \int_{\Omega} T_n(\psi_n) E_n(x) \cdot \nabla u_n = \int_{\Omega} f_n(x) T_n(\psi_n).$$

Summing the two identities, and recalling (2), we have that

$$\alpha \int_{\Omega} |\nabla T_n(\psi_n)|^p + \alpha \int_{\Omega} |\nabla u_n|^2 \leq \int_{\Omega} f_n(x) T_n(\psi_n).$$

Recalling the definition of f_n and the assumptions on $f(x)$, and using Hölder inequality and Sobolev embedding, we have that

$$\begin{aligned} \alpha \int_{\Omega} |\nabla T_n(\psi_n)|^p + \alpha \int_{\Omega} |\nabla u_n|^2 &\leq \|f_n\|_{L^{p^*}(\Omega)} \|T_n(\psi_n)\|_{L^{p^*}(\Omega)} \\ &\leq C \|f\|_{L^{p^*}(\Omega)} \|T_n(\psi_n)\|_{W_0^{1,p}(\Omega)}. \end{aligned}$$

From this inequality it easily follows that

$$\int_{\Omega} |\nabla T_n(\psi_n)|^p + \int_{\Omega} |\nabla u_n|^2 \leq C \|f\|_{L^{p^*}(\Omega)}^p, \quad (22)$$

so that the sequences $\{u_n\}$ and $\{T_n(\psi_n)\}$ are bounded in $W_0^{1,2}(\Omega)$ and in $W_0^{1,p}(\Omega)$, respectively.

We now focus our attention on the equation solved by ψ_n , which we rewrite as

$$-\operatorname{div}(a(x) |\nabla \psi_n|^{p-2} \nabla \psi_n) = f_n(x) - E_n(x) \cdot \nabla u_n \stackrel{\text{def}}{=} g_n(x).$$

Since the sequences $\{E_n(x)\}$ and $\{\nabla u_n\}$ are bounded in $(L^2(\Omega))^N$, and since the sequence $\{f_n\}$ is bounded in $L^{p^*}(\Omega)$, the sequence $\{g_n(x)\}$ is bounded in $L^1(\Omega)$. From this fact it follows, using the results of [1] and [3], that

- (i) the sequence $\{T_k(\psi_n)\}$ is bounded in $W_0^{1,p}(\Omega)$ for every $k \geq 0$, and the sequence $\{|\nabla \psi_n|^{p-1}\}$ is bounded in $L^q(\Omega)$, for every $1 < q < \frac{N}{N-1}$;
- (ii) there exists a measurable function ψ , with the property that $T_k(\psi)$ belongs to $W_0^{1,p}(\Omega)$ for every $k \geq 0$, such that ψ_n almost everywhere converges to ψ ;
- (iii) the sequence $\{T_k(\psi_n)\}$ weakly converges to $T_k(\psi)$ in $W_0^{1,p}(\Omega)$;
- (iv) the sequence $\{\nabla \psi_n\}$ almost everywhere converges to $\nabla \psi$, where $\nabla \psi$ is the vector function z defined by $z = \nabla T_k(\psi)$ almost everywhere on $\{|\psi| \leq k\}$.

We now remark that from (22) it follows that, if $k \leq n$,

$$\int_{\Omega} |\nabla T_k(\psi_n)|^p \leq \int_{\Omega} |\nabla T_n(\psi_n)|^p \leq C \|f\|_{L^{p^*}(\Omega)}^p,$$

so that, using (iii) we have, by weak lower semicontinuity, that

$$\int_{\Omega} |\nabla T_k(\psi)|^p \leq C \|f\|_{L^{p^*}(\Omega)}^p, \quad \forall k \geq 0.$$

Letting k tend to infinity, we thus have that ψ belongs to $W_0^{1,p}(\Omega)$ (and its gradient is the distributional gradient of ψ , see [1]).

We are now able to pass to the limit in (20); indeed, from (ii) above it follows that $T_n(\psi_n)$ almost everywhere converges to ψ , so that $\{T_n(\psi_n)\}$ weakly converges to ψ in $L^{p^*}(\Omega)$; since $\{E_n(x)\}$ strongly converges to $E(x)$ in $(L^2(\Omega))^N$, the assumption $p^* \geq 2$ implies that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} T_n(\psi_n) E_n(x) \cdot \nabla v = \int_{\Omega} \psi E(x) \cdot \nabla v, \quad \forall v \in C_0^1(\Omega).$$

Therefore, if u is the weak limit in $W_0^{1,2}(\Omega)$ of the sequence $\{u_n\}$, we have that u in $W_0^{1,2}(\Omega)$ is such that

$$\int_{\Omega} a(x) \nabla u \cdot \nabla v = \int_{\Omega} \psi E(x) \cdot \nabla v, \quad \forall v \in C_0^1(\Omega),$$

which is (12).

As far as the second equation is concerned, we can pass to the limit using Minty’s method (see [6]). Let ϕ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, let $k \geq 0$ and choose $\eta = T_k(\psi_n - \phi)$ as test function in (21). We obtain

$$\int_{\Omega} a(x) |\nabla \psi_n|^{p-2} \nabla \psi_n \cdot \nabla T_k(\psi_n - \phi) + \int_{\Omega} E_n(x) \cdot \nabla u_n T_k(\psi_n - \phi) = \int_{\Omega} f_n(x) T_k(\psi_n - \phi).$$

Adding on both side of these identities the term

$$- \int_{\Omega} a(x) |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla T_k(\psi_n - \phi),$$

we have that

$$\begin{aligned} & \int_{\Omega} a(x) [|\nabla \psi_n|^{p-2} \nabla \psi_n - |\nabla \phi|^{p-2} \nabla \phi] \cdot \nabla T_k(\psi_n - \phi) + \int_{\Omega} E_n(x) \cdot \nabla u_n T_k(\psi_n - \phi) \\ &= \int_{\Omega} f_n(x) T_k(\psi_n - \phi) - \int_{\Omega} a(x) |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla T_k(\psi_n - \phi). \end{aligned}$$

Since the first term is positive, we have that

$$\int_{\Omega} a(x) |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla T_k(\psi_n - \phi) + \int_{\Omega} E_n(x) \cdot \nabla u_n T_k(\psi_n - \phi) \leq \int_{\Omega} f_n(x) T_k(\psi_n - \phi).$$

Recalling the $T_k(\psi_n - \phi)$ converges to $T_k(\psi - \phi)$ weakly in $W_0^{1,p}(\Omega)$, weakly* in $L^\infty(\Omega)$ and almost everywhere, that $E_n(x)$ strongly converges to $E(x)$ in $(L^2(\Omega))^N$, and that u_n weakly converges to u in $W_0^{1,2}(\Omega)$, we can pass to the limit in n in the previous inequality to have that

$$\int_{\Omega} a(x) |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla T_k(\psi - \phi) + \int_{\Omega} E(x) \cdot \nabla u T_k(\psi - \phi) \leq \int_{\Omega} f(x) T_k(\psi - \phi), \quad (23)$$

for every ϕ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We now choose $\phi = T_h(\psi) - t\varphi$, with $h \geq 0$, $0 < t < 1$, and φ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Since

$$T_k(\psi - \phi) = T_k(\psi - T_h(\psi) + t\varphi),$$

from (23) it follows that

$$\begin{aligned} & \int_{\Omega} a(x) |\nabla(T_h(\psi) - t\varphi)|^{p-2} \nabla(T_h(\psi) - t\varphi) \cdot \nabla T_k(\psi - T_h(\psi) + t\varphi) \\ & \quad + \int_{\Omega} E(x) \cdot \nabla u T_k(\psi - T_h(\psi) + t\varphi) \\ & \leq \int_{\Omega} f(x) T_k(\psi - T_h(\psi) + t\varphi). \end{aligned}$$

Passing to the limit as h tends to infinity, we obtain, using Lebesgue theorem and the fact that ψ belongs to $W_0^{1,p}(\Omega)$,

$$\int_{\Omega} a(x)|\nabla(\psi - t\varphi)|^{p-2}\nabla(\psi - t\varphi) \cdot \nabla T_k(t\varphi) + \int_{\Omega} E(x) \cdot \nabla u T_k(t\varphi) \leq \int_{\Omega} f(x) T_k(t\varphi).$$

Choosing $k \geq \|\varphi\|_{L^\infty(\Omega)}$, we have that $T_k(t\varphi) = t\varphi$ (since $0 < t < 1$); hence, the previous inequality becomes

$$t \int_{\Omega} a(x)|\nabla(\psi - t\varphi)|^{p-2}\nabla(\psi - t\varphi) \cdot \nabla \varphi + t \int_{\Omega} E(x) \cdot \nabla u \varphi \leq t \int_{\Omega} f(x) \varphi.$$

Dividing by t , and letting t tend to zero, we obtain

$$\int_{\Omega} a(x)|\nabla\psi|^{p-2}\nabla\psi \cdot \nabla\varphi + \int_{\Omega} E(x) \cdot \nabla u \varphi \leq \int_{\Omega} f(x) \varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Replacing φ with $-\varphi$ we obtain the reverse inequality, so that

$$\int_{\Omega} a(x)|\nabla\psi|^{p-2}\nabla\psi \cdot \nabla\varphi + \int_{\Omega} E(x) \cdot \nabla u \varphi = \int_{\Omega} f(x) \varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

which is (13). □

Remark 2.3. One may wonder why only one equation of system (11) has a non-linear principal part, and not both of them. Should both equations be nonlinear, we will have to consider the system

$$\begin{cases} u \in W_0^{1,p}(\Omega) : & -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = -\operatorname{div}(\psi E(x)), \\ \psi \in W_0^{1,p}(\Omega) : & -\operatorname{div}(a(x)|\nabla\psi|^{p-2}\nabla\psi) + E(x) \cdot \nabla u = f(x). \end{cases} \tag{24}$$

The main problem with this system is the divergence term in the first equation; indeed, while for the second equation the fact that the sequence $\{E_n(x) \cdot \nabla u_n\}$ is bounded in $L^1(\Omega)$ was enough to prove the almost everywhere convergence of ψ_n and $\nabla\psi_n$ (see the proof of Theorem 2.1) using the results of [5] and [1], similar results do not exist for nonlinear equations with data in divergence form.

Open Problem 2.4. *Prove an existence result for system (24).*

We now assume that p and r satisfy (7), and prove that (ψ, u) is a saddle point of the functional J .

Theorem 2.5. *Let $f(x)$ be such that (9) holds, and let $E(x)$ be a vector field in $(L^r(\Omega))^N$, with r such that (7) holds. Then u and ψ are weak solutions of system (11), that is*

$$\int_{\Omega} a(x) \nabla u \cdot \nabla v = \int_{\Omega} \psi E(x) \cdot \nabla v, \quad \forall v \in W_0^{1,2}(\Omega),$$

and
$$\int_{\Omega} a(x)|\nabla\psi|^{p-2}\nabla\psi \cdot \nabla\varphi + \int_{\Omega} E(x) \cdot \nabla u \varphi = \int_{\Omega} f(x) \varphi, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

Furthermore, (ψ, u) is a saddle point of J , that is

$$J(\psi, v) \leq J(\psi, u) \leq J(\varphi, u), \quad \forall v \in W_0^{1,2}(\Omega), \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{25}$$

Proof. If we assume that $p \geq 2_*$ and $r \geq 2$ are such that (7) holds, then

$$\frac{1}{p^*} + \frac{1}{r} \leq \frac{1}{2},$$

so that $\psi E(x)$ belongs to $(L^2(\Omega))^N$, and

$$\frac{1}{r} + \frac{1}{2} \leq \frac{1}{p_*},$$

so that $E(x) \cdot \nabla u$ belongs to $L^{p^*}(\Omega)$. Therefore, reasoning by density one may choose test functions v in $W_0^{1,2}(\Omega)$ in (20), and φ in $W_0^{1,p}(\Omega)$ in (21).

We now prove that $J(\psi, v) \leq J(\psi, u)$ for every v in $W_0^{1,2}(\Omega)$. Indeed, choosing $u - v$ as test function in (20), we have

$$\int_{\Omega} a(x) \nabla u \cdot \nabla(u - v) = \int_{\Omega} \psi E(x) \cdot \nabla(u - v). \tag{26}$$

Since the function $\xi \mapsto |\xi|^2$ is convex, we have that

$$|\xi|^2 \geq |\eta|^2 - 2\eta \cdot (\xi - \eta), \quad \forall \xi, \eta \in \mathbb{R}^N.$$

Rearranging terms, we thus have that

$$\eta \cdot (\eta - \xi) \geq \frac{1}{2} |\eta|^2 - \frac{1}{2} |\xi|^2.$$

Using this inequality in (26), with $\eta = \nabla u(x)$ and $\xi = \nabla v(x)$ we obtain (recall that $a(x) \geq 0$)

$$\frac{1}{2} \int_{\Omega} a(x) |\nabla u|^2 - \frac{1}{2} \int_{\Omega} a(x) |\nabla v|^2 \leq \int_{\Omega} \psi E(x) \cdot \nabla u - \int_{\Omega} \psi E(x) \cdot \nabla v,$$

Changing signs, rearranging terms and adding to both sides the term

$$\frac{1}{p} \int_{\Omega} a(x) |\nabla \psi|^p - \int_{\Omega} f(x) \psi,$$

we arrive at $J(\psi, v) \leq J(\psi, u)$ for every v in $W_0^{1,2}(\Omega)$.

As far as the inequality $J(\psi, u) \leq J(\varphi, u)$ is concerned, we choose $\psi - \varphi$ as test function in (21), with φ in $W_0^{1,p}(\Omega)$, to obtain

$$\int_{\Omega} a(x) |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla(\psi - \varphi) + \int_{\Omega} E(x) \cdot \nabla u (\psi - \varphi) = \int_{\Omega} f(x) (\psi - \varphi). \tag{27}$$

As before, we have that from the convexity of the function $\xi \mapsto |\xi|^p$ it follows that

$$|\xi|^p \geq |\eta|^p + p |\eta|^{p-2} \eta \cdot (\xi - \eta), \quad \forall \eta, \xi \in \mathbb{R}^N,$$

which can be rewritten as $|\eta|^{p-2} \eta (\eta - \xi) \geq \frac{1}{p} |\eta|^p - \frac{1}{p} |\xi|^p$.

Therefore, choosing $\eta = \nabla\psi(x)$ and $\xi = \nabla\varphi(x)$, we have that (recall that $a(x) \geq 0$)

$$\int_{\Omega} a(x) |\nabla\psi|^{p-2} \nabla\psi \cdot \nabla(\psi - \varphi) \geq \frac{1}{p} \int_{\Omega} a(x) |\nabla\psi|^p - \frac{1}{p} \int_{\Omega} a(x) |\nabla\varphi|^p.$$

Thus, from (27) it follows (after rearranging terms) that,

$$\begin{aligned} & \frac{1}{p} \int_{\Omega} a(x) |\nabla\psi|^p + \int_{\Omega} E(x) \cdot \nabla u \psi - \int_{\Omega} f(x) \psi \\ & \leq \frac{1}{p} \int_{\Omega} a(x) |\nabla\varphi|^p + \int_{\Omega} E(x) \cdot \nabla u \varphi - \int_{\Omega} f(x) \varphi. \end{aligned}$$

Subtracting from both sides the term $\frac{1}{2} \int_{\Omega} a(x) |\nabla u|^2$, we thus arrive at the inequality $J(\psi, u) \leq J(\varphi, u)$, as desired. □

Remark 2.6. Note that if p tends to 2_* in (7) then r tends to infinity, while if p tends to N then r tends to 2.

Remark 2.7. The main reason one needs to assume that r satisfies (10) in order to prove the existence of a saddle point for J is that the term $\varphi E(x) \cdot \nabla v$ may not belong to $L^1(\Omega)$. One can however replace the concept of minimum with the concept of *weak minimum* (see [8], [7] and also [2], where a similar definition is introduced) as follows. If $E(x)$ belongs to $(L^2(\Omega))^N$ we say that ψ in $W_0^{1,p}(\Omega)$ is a weak minimum of $J(\cdot, u)$, with u in $W_0^{1,2}(\Omega)$, if

$$\frac{1}{p} \int_{\Omega} a(x) |\nabla\psi|^p + \int_{\Omega} E(x) \cdot \nabla u \varphi - \int_{\Omega} f(x) \varphi \leq \frac{1}{p} \int_{\Omega} a(x) |\nabla(\psi - \varphi)|^p, \quad \forall \varphi \in C_0^1(\Omega).$$

This inequality can be formally obtained from the relation $J(\psi, u) \leq J(\varphi, u)$ after cancelling the terms

$$\int_{\Omega} E(x) \cdot \nabla u \psi,$$

which appear in both sides of the inequalities, and that the term

$$\int_{\Omega} E(x) \cdot \nabla u \varphi$$

is well defined since $E(x) \cdot \nabla u$ belongs to $L^1(\Omega)$ and φ is bounded. Analogously, u in $W_0^{1,2}(\Omega)$ is a weak minimum of $-J(\psi, \cdot)$, with ψ in $W_0^{1,p}(\Omega)$, if

$$\frac{1}{2} \int_{\Omega} a(x) |\nabla u|^2 + \int_{\Omega} E(x) \cdot \nabla v \psi \leq \frac{1}{2} \int_{\Omega} a(x) |\nabla(u - v)|^2, \quad \forall v \in C_0^1(\Omega).$$

Once again, this inequality can be formally obtained from $-J(\psi, u) \leq -J(\psi, v)$ after cancelling equal terms; remark that the term

$$\int_{\Omega} E(x) \cdot \nabla v \psi$$

is well defined since $\psi E(x)$ belongs to $(L^1(\Omega))^N$ (recall that $p^* \geq 2$ due to the assumption $p \geq \frac{2N}{N+2}$) and ∇v is bounded.

The fact that ψ and u (given by Theorem 2.1) are weak minima of $J(\cdot, u)$ and of $-J(\psi, u)$ respectively, follows from the fact that if u_n and ψ_n are the solutions of (20) and (21), then

$$\frac{1}{p} \int_{\Omega} a(x) |\nabla \psi_n|^p + \int_{\Omega} E_n(x) \cdot \nabla u_n \varphi - \int_{\Omega} f_n(x) \varphi \leq \frac{1}{p} \int_{\Omega} a(x) |\nabla(\psi_n - \varphi)|^p,$$

for every φ in $C_0^1(\Omega)$, and

$$\frac{1}{2} \int_{\Omega} a(x) |\nabla u_n|^2 + \int_{\Omega} T_n(\psi_n) E_n(x) \cdot \nabla v \leq \frac{1}{2} \int_{\Omega} a(x) |\nabla(u_n - v)|^2,$$

for every v in $C_0^1(\Omega)$. One can then obtain that ψ and u are weak minima passing to the limit in the two inequalities using the properties of $\{\psi_n\}$ and $\{u_n\}$. Note that one can use Lebesgue theorem pass to the limit in the *difference*

$$\frac{1}{p} \int_{\Omega} a(x) [|\nabla \psi_n|^p - |\nabla(\psi_n - \varphi)|^p],$$

since $||\nabla \psi_n|^p - |\nabla(\psi_n - \varphi)|^p| \leq C (|\nabla \psi_n|^{p-1} + |\nabla \varphi|^{p-1}) |\nabla \varphi|$,

and since the sequence $\{|\nabla \psi_n|^{p-1}\}$ is strongly convergent in $L^q(\Omega)$, for $1 < q < \frac{N}{N-1}$ (see (i) and (iv) in the proof of Theorem 2.1), and one can pass to the limit in the difference

$$\frac{1}{2} \int_{\Omega} a(x) [|\nabla u_n|^2 - |\nabla(u_n - v)|^2],$$

since it is equal to

$$\int_{\Omega} a(x) \nabla u_n \cdot \nabla v - \frac{1}{2} \int_{\Omega} a(x) |\nabla v|^2,$$

and the sequence $\{u_n\}$ weakly converges to u in $W_0^{1,2}(\Omega)$. After passing to the limit in the difference, one can than split the integrals since both terms are finite (recall that ψ is in $W_0^{1,p}(\Omega)$ and u is in $W_0^{1,2}(\Omega)$).

3. Problem with nonlinear first order terms

In this section, we are going to consider systems with nonlinear first order terms, formally obtained as Euler-Lagrange equations of saddle points of the functional

$$I(\varphi, v) = \frac{1}{2} \int_{\Omega} a(x) |\nabla \varphi|^2 - \frac{1}{2} \int_{\Omega} a(x) |\nabla v|^2 + \int_{\Omega} |\varphi|^{q-1} \varphi E(x) \cdot \nabla v - \int_{\Omega} f(x) \varphi,$$

with φ and v in $W_0^{1,2}(\Omega)$, and $q > 1$.

Our result is the following.

Theorem 3.1. *Let $q > 1$ and $r \geq 2$ be such that*

$$\frac{q}{2^*} + \frac{1}{r} < 1 - \frac{1}{N}. \quad (28)$$

Let $f(x)$ be a function in $L^{2^}(\Omega)$, and let $E(x)$ be a vector field in $(L^r(\Omega))^N$. Then there exist u and ψ in $W_0^{1,2}(\Omega)$, weak solutions of the system*

$$\begin{cases} u \in W_0^{1,2}(\Omega) : -\operatorname{div}(a(x) \nabla u) = -\operatorname{div}(|\psi|^{q-1} \psi E(x)), \\ \psi \in W_0^{1,2}(\Omega) : -\operatorname{div}(a(x) \nabla \psi) + q |\psi|^{q-1} E(x) \cdot \nabla u = f(x), \end{cases} \quad (29)$$

that is

$$\begin{cases} \int_{\Omega} a(x) \nabla u \cdot \nabla v = \int_{\Omega} |\psi|^{q-1} \psi E(x) \cdot \nabla v, \\ \int_{\Omega} a(x) \nabla \psi \cdot \nabla \varphi + q \int_{\Omega} |\psi|^{q-1} E(x) \cdot \nabla u \varphi = \int_{\Omega} f(x) \varphi, \end{cases} \quad (30)$$

for every v and φ in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Proof. Let $f(x)$ be a function in $L^{2^*}(\Omega)$, let $E(x)$ belong to $(L^r(\Omega))^N$, with r such that (28) holds, let n in \mathbb{N} and let f_n and E_n be defined as in (19) in the proof of Theorem 2.1. Consider the system

$$\begin{cases} u_n \in W_0^{1,2}(\Omega) : -\operatorname{div}(a(x) \nabla u_n) = -\operatorname{div}(|T_n(\psi_n)|^{q-1} T_n(\psi_n) E_n(x)), \\ \psi_n \in W_0^{1,2}(\Omega) : -\operatorname{div}(a(x) \nabla \psi_n) + q |T_n(\psi_n)|^{q-1} E_n(x) \cdot \nabla u_n = f_n(x). \end{cases} \quad (31)$$

Existence of solutions for system (31) can be proved using Schauder's theorem as in the proof of Theorem 2.2: let ϕ in $L^1(\Omega)$, let w be the unique solution of

$$w \in W_0^{1,2}(\Omega) : -\operatorname{div}(a(x) \nabla w) = -\operatorname{div}(|T_n(\phi)|^{q-1} T_n(\phi) E_n(x)), \quad (32)$$

which exists by Lax-Milgram theorem, and let Φ be the unique solution of

$$\Phi \in W_0^{1,2}(\Omega) : -\operatorname{div}(a(x) \nabla \Phi) + q |T_n(\phi)|^{q-1} E_n(x) \cdot \nabla v = f_n(x),$$

whose existence is guaranteed again by Lax-Milgram theorem. Using the fact that both $T_n(\phi)$ and $|E_n|$ belong to $L^\infty(\Omega)$, one can prove that, for a positive constant C ,

$$\|\Phi\|_{L^1(\Omega)} \leq C \|\Phi\|_{W_0^{1,2}(\Omega)} \leq C n^{2q+1} \stackrel{\text{def}}{=} R,$$

so that the ball of $L^1(\Omega)$ of radius R is invariant for the map $S : L^1(\Omega) \rightarrow L^1(\Omega)$ defined by $S(\phi) = \Phi$. Since S is both continuous and compact, there exist a function ψ_n in $L^1(\Omega)$ such that $S(\psi_n) = \psi_n$. Defining u_n as the solution of (32) with $\phi = \psi_n$ we have thus found solutions u_n and ψ_n of system (31).

Choosing u_n as test function in the first equation of system (31), and $\frac{T_n(\psi_n)}{q}$ as test function in the second one, we get

$$\begin{aligned} \int_{\Omega} a(x) |\nabla u_n|^2 &= \int_{\Omega} |T_n(\psi_n)|^{q-1} T_n(\psi_n) E_n(x) \cdot \nabla u_n, \quad \text{and} \\ \frac{1}{q} \int_{\Omega} a(x) |\nabla T_n(\psi_n)|^2 + \int_{\Omega} |T_n(\psi_n)|^{q-1} T_n(\psi_n) E_n(x) \cdot \nabla u_n &= \frac{1}{q} \int_{\Omega} f_n(x) T_n(\psi_n). \end{aligned}$$

Thus, summing the two identities, and recalling (2), we have

$$\alpha \int_{\Omega} |\nabla u_n|^2 + \frac{\alpha}{q} \int_{\Omega} |\nabla T_n(\psi_n)|^2 \leq \frac{1}{q} \int_{\Omega} f_n(x) T_n(\psi_n).$$

Using that the sequence $\{f_n\}$ is bounded in $L^{2^*}(\Omega)$, it is easy to prove that one has

$$\int_{\Omega} |\nabla u_n|^2 + \int_{\Omega} |\nabla T_n(\psi_n)|^2 \leq C \|f\|_{L^{2^*}(\Omega)}^2, \tag{33}$$

so that the sequences $\{u_n\}$ and $\{T_n(\psi_n)\}$ are bounded in $W_0^{1,2}(\Omega)$. This implies that the sequence $\{T_n(\psi_n)\}$ is bounded in $L^{2^*}(\Omega)$, so that the sequence $\{|T_n(\psi_n)|^{q-1}\}$ is bounded in $L^s(\Omega)$, with $s = 2^*/(p - 1)$. This fact, and the boundedness of the sequence $\{E_n\}$ in $(L^r(\Omega))^N$ and $\{\nabla u_n\}$ in $(L^2(\Omega))^N$ respectively, yield that the sequence $\{|T_n(\psi_n)|^{q-1} E_n(x) \cdot \nabla u_n\}$ is bounded in $L^m(\Omega)$, with

$$\frac{1}{m} = \frac{q - 1}{2^*} + \frac{1}{r} + \frac{1}{2}.$$

Using (28), we have that $m > 1$. Thus, since ψ_n is the solution of the equation

$$-\operatorname{div}(a(x) \nabla \psi_n) = f_n(x) - q |T_n(\psi_n)|^{q-1} E_n(x) \cdot \nabla u_n,$$

with a sequence of data bounded in $L^1(\Omega)$, the results of [3] imply that the sequence $\{\psi_n\}$ is bounded in $W_0^{1,s}(\Omega)$, for every $s < \frac{N}{N-1}$. Thus, up to subsequences, it converges, strongly in $L^1(\Omega)$, to some function ψ . On the other hand, we have that, up to subsequences, the sequence $\{T_n(\psi_n)\}$ weakly converges in $W_0^{1,2}(\Omega)$ to some function z belonging to $W_0^{1,2}(\Omega)$. Recalling the almost everywhere convergence of ψ_n to ψ , we have that $z = \psi$, so that ψ belongs to $W_0^{1,2}(\Omega)$. We now observe that the sequence $\{|T_n(\psi_n)|^{q-1} T_n(\psi_n)\}$ weakly converges to $|\psi|^{q-1} \psi$ in $L^\rho(\Omega)$, with $\rho = 2^*/q$, while the sequence $\{E_n\}$ strongly converges to $E(x)$ in $(L^r(\Omega))^N$. Since assumption (28) implies that

$$\frac{q}{2^*} + \frac{1}{r} < 1,$$

the sequence $\{|T_n(\psi_n)|^{q-1} T_n(\psi_n) E_n(x)\}$ converges in $(L^1(\Omega))^N$ to the vector field $|\psi|^{q-1} \psi E(x)$. Thus, one has

$$\int_{\Omega} a(x) \nabla u \cdot \nabla \varphi = \int_{\Omega} |\psi|^{q-1} \psi E(x) \cdot \nabla \varphi, \quad \forall \varphi \in C_0^1(\Omega).$$

Using the strong convergence of $|T_n(\psi_n)|^{q-1}$ to $|\psi|^{q-1}$ in the space $L^s(\Omega)$, for every $s < 2^*/(q - 1)$, as well as the strong convergence of $E_n(x)$ to $E(x)$ in $(L^r(\Omega))^N$, and the weak convergence of u_n to u in $W_0^{1,2}(\Omega)$, one also has that

$$\int_{\Omega} a(x) \nabla \psi \cdot \nabla v + q \int_{\Omega} |\psi|^{q-1} E(x) \cdot \nabla u v = \int_{\Omega} f(x) v, \quad \forall v \in C_0^1(\Omega). \quad \square$$

Remark 3.2. If q tends to 1, from (28) it follows that r can be chosen any number larger than 2 (as is in Section 2). If r tends to infinity, then q can be chosen any number smaller than $2 \frac{N-1}{N-2}$.

Remark 3.3. If one assumes that

$$\frac{q}{2^*} + \frac{1}{r} \leq \frac{1}{2}, \quad (34)$$

which also implies that

$$\frac{q-1}{2^*} + \frac{1}{r} + \frac{1}{2} \leq \frac{1}{2_*},$$

one can choose test functions in $W_0^{1,2}(\Omega)$ in both equations of system (29) and prove, using the convexity of the maps $\xi \mapsto |\xi|^2$ and $t \mapsto |t|^q$ as in the proof of Theorem 2.5, that (ψ, u) is a saddle point of the functional I .

Note that assumption (34) is stronger than (28) since $N > 2$. In this case, if q tends to 1 then r tends to N , while if r tends to infinity then q tends to $\frac{N}{N-2}$.

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