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Multiplicity Theorems for Biharmonic Kirchhoff-Type Elliptic Problems

Lingju Kong
Department of Mathematics, University of Tennessee at Chattanooga, U.S.A.
lingju-kong@utc.edu

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We study the existence of multiple weak solutions for the biharmonic Kirchhoff-type elliptic problem

$$\begin{cases} M\left(\int_\Omega (|\Delta u|^p + |\nabla u|^p)dx\right)\left(\Delta_p^2 u - \Delta_p u\right) = \sum_{i=1}^k \alpha_i(x) f_i(u) + \gamma(x) \text{ in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \quad \int_\Omega (|\Delta u|^p + |\nabla u|^p)\, dx < \rho. \end{cases}$$

We establish necessary and sufficient conditions on f_i, $i = 1, \dots, k$, under which there exists functions $\alpha_i, \gamma \in C(\overline{\Omega})$, $i = 1, \dots, k$, such that the above problem has at least two weak solutions. Our proof uses the variational approaches and relies on an existence result for crical points of functionals in Banach spaces recently obtained by Ricceri.

Keywords: Kirchhoff-type problems, p-Laplacian operator, p-biharmonic operator, weak solutions, critical points, contraction mapping theorem.

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1. Introduction

In this paper, we are concerned with the biharmonic Kirchhoff-type elliptic problem

$$\begin{cases} M\left(\int_\Omega (|\Delta u|^p + |\nabla u|^p)dx\right)\left(\Delta_p^2 u - \Delta_p u\right) = \sum_{i=1}^k \alpha_i(x) f_i(u) + \gamma(x) \text{ in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \quad \int_\Omega (|\Delta u|^p + |\nabla u|^p)\, dx < \rho, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $p > N \geq 1$, $k \geq 1$, $\rho > 0$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $\Delta_p^2 u = \Delta(|\Delta|^{p-2}\Delta u)$ is the p-biharmonic operator, and $M : [0, \rho) \to [0, \infty)$, $\alpha_i, \gamma : \Omega \to \mathbb{R}$, $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, k$, are continuous functions.

As is well known, fourth order differential equations have numerous applications in physics and engineering. For instance, they arise in the study of traveling waves in suspension bridges [4, 13]. In recent years, the existence of solutions of various fourth order biharmonic problems have been studied by many authors. See, for example, [3, 7, 8, 10, 11, 12, 14] and the references therein. Problem (1) is a nonlocal problem because of the presence of M in the equation. Nonlocal problems model many

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physical and biological phenomena. Such problems are often referred as Kirchhoff-type problems in the literature as, in 1883, Kirchhoff [9] proposed a second order nonlocal equation as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Below we briefly recall some early work on fourth order nonlocal problems. In 1950, Woinowsky-Krieger [22] studied the following equation when the space is one dimensional

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho}\frac{\partial^4 u}{\partial x^4} - \left(\frac{H}{\rho} + \frac{EA}{2\rho L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 ds\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

where L is the length of the string, A is the cross-sectional area, E is the Young modulus of the material, ρ is the mass density, I is the cross-sectional moment of inertia, and H is the tension in the rest position. In 1955, Berger [2] introduced the following model when the space is two dimensional

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + \left(Q + \int_\Omega |\nabla u|^2 ds\right)\Delta u = f(u, u_t, x) \tag{3}$$

to describes large deflection of a plate, where Q describes in-plane forces applied to the plate and f denotes transverse loads, which may depend on the displacement u and the velocity u_t. We comment that problem (1) can be regarded as a generalization of the stationary problems associated with equations (2) and (3).

Recently, many authors have investigated the existence of weak solutions for various biharmonic Kirchhoff-type problems. We refer the reader to [1, 5, 6, 19, 20] and the references therein for a small sample of recent work on on the subject. For instance, using variational methods, paper [20] studied the existence of a unique weak solution of the problem

$$\begin{cases} \Delta_p^2 u - (a\int_\Omega |\nabla u|^2 dx + b)\Delta u + cu = \alpha(x)|u|^{-\gamma} - \lambda|u|^{p-2}u \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $0 < \gamma < 1$, $\lambda > 0$, and α belongs to a given Lebesgue space. Paper [6] applied the mountain pass theorem and the concentration compactness principle to derive conditions for the existence of at least one weak solution to the problem

$$\begin{cases} M\left(\int_\Omega |\Delta u|^p dx\right)\Delta_p^2 u = \lambda f(x,u) + |u|^{p^{**}-2}u \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\frac{\partial u}{\partial \nu}$ is the outer normal derivative, $\lambda > 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, p^{**} is the critical exponent (see [6, equation (1.2)] for the definition).

In this paper, we study existence of at least two weak solutions of problem (1). This work is inspired by a recent paper of Ricceri [18], where the author first proved a general existence theorem in a reflexive real Banach space (see Lemma 2.5 below), and then presented its applications to some second order Kirchhoff-type problem on $\mathbb{R}$. Here, we explore further applications of Ricceri's theorem to establish new existence theorem for problem (1). In particular, with the help of Ricceri's theorem, we are able to derive a necessary and sufficient condition on f_i, $i = 1, \dots, k$, under which there exists $\alpha_i, \gamma \in C(\overline{\Omega})$, $i = 1, \ldots, k$, such that problem (1) has at least two weak solutions. The condition imposed on f_i, $i = 1, \ldots, k$, is simple and easy to verify in applications. See Theorem 2.2 in Section 2 for details. Moreover, as a consequence of our Theorem 2.2, we also study the problem

$$\begin{cases} M\left(\int_\Omega(|\Delta u|^p + |\nabla u|^p)dx\right)\left(\Delta_p^2 u - \Delta_p u\right) = \alpha(x)f(u) + \gamma(x) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \quad \int_\Omega (|\Delta u|^p + |\nabla u|^p)\, dx < \rho, \end{cases} \tag{4}$$

and find a necessary and sufficient condition on f under which there exists some $(\alpha, \gamma) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ such that the problem has at least two weak solutions. See Corollary 2.3 below. To the best of our knowledge, no result of the same kind has been established in the literature for biharmonic Kirchhoff-type problems. Finally, we point out that Lemma 2.1 below is an application of a minimax theorem given in [16, Theorem 1.1], and thus Theorem 2.1 itself can be regarded as a significant application of that minimax theorem.

In the next section, we present our theorem and its proof.

2. Main results

Let $X = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$. Then, equipped with the norm

$$\|u\| = \left(\int_\Omega (|\Delta u|^p + |\nabla u|^p)\, dx\right)^{1/p},$$

X is a reflexive and separable Banach space.

Definition 2.1. By a *weak solution* of problem (1), we mean a function $u \in X$ such that $\int_\Omega (|\Delta u|^p + |\nabla u|^p)\, dx < \rho$ and

$$\begin{aligned} &M\left(\int_\Omega(|\Delta u|^p + |\nabla u|^p)dx\right)\int_\Omega \left(|\Delta u|^{p-2}\Delta u\Delta v + |\nabla u|^{p-2}\nabla u\nabla v\right) dx \\ &= \int_\Omega \left(\sum_{i=1}^k \alpha_i(x)f_i(u) + \gamma(x)\right) v dx \quad \text{in } \Omega \text{ for all } v \in X. \end{aligned}$$

We introduce a constant c as follows: $c := \sup_{u \in X\setminus\{0\}} \frac{\sup_{x\in\Omega}|u(x)|}{\|u\|}$.

Since $p > N$, $W_0^{1,p}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Then, we have $c < \infty$ and

$$\sup_{x\in\Omega}|u(x)| \le c\|u\| \quad \text{for all } u \in X, \tag{5}$$

Moreover, in view of [21, Theorem 2.E.], it is easy to see that

$$c \le N^{-1/p}\omega_N^{-1/N}\left(\frac{p-1}{p-N}\right)^{(p-1)/p}|\Omega|^{1/N-1/p},$$

where $\omega_N = \frac{\pi^{N/2}}{\Gamma(N/2+1)}$ is the volume of the unit ball in $\mathbb{R}^N$ and $|\Omega|$ is the volume of Ω.

We now state the main theorem of the paper.

Theorem 2.2. *Assume that M is increasing on $[0,\rho)$ and*

$$\lim_{t\to\rho^-}\int_0^t M(s)ds=\infty. \tag{6}$$

Let
$$E=\underbrace{C(\overline{\Omega})\times\ldots\times C(\overline{\Omega})}_{k+1\ \ times} \tag{7}$$

be endowed with the norm $\|(w_1,\ldots,w_{k+1})\|_E=\displaystyle\int_\Omega\sum_{i=1}^{k+1}|w_i(x)|\,dx.$

Then, the following assertions are equivalent:

(a) *At least one of the restrictions of $f_1,\ldots,f_k$ to $\left(-c\rho^{1/p},c\rho^{1/p}\right)$ is not constant.*

(b) *For every convex set $S\subset E$ dense in E, there exists $(\alpha_1,\ldots\alpha_k,\gamma)\in S$ such that problem (1) has at least two weak solutions.*

The following result is a direct consequence of Theorem 2.2 by setting $k=1$, $\alpha_1=\alpha$, and $f_1=f$ in (1).

Corollary 2.3. *Assume that M is increasing on $[0,\rho)$ and (6) holds. Consider the space $E_1=C(\overline{\Omega})\times C(\overline{\Omega})$ endowed with the norm $\|(\alpha,\gamma)\|_E=\int_\Omega(|\alpha(x)|+|\gamma(x)|)\,dx$. Then, the following assertions are equivalent:*

(a) *The restriction of f to $\left(-c\rho^{1/p},c\rho^{1/p}\right)$ is not constant.*

(b) *For every convex set $S_1\subset E_1$ dense in E_1, there exists $(\alpha,\gamma)\in S_1$ such that problem (4) has at least two weak solutions.*

Remark 2.4. There are many increasing functions M satisfying (6). For instance, here are two examples of such functions:

$$M(t)=\frac{1}{(\rho-t)^\mu}\ \text{ with } \mu>1, \ \text{ and } \ M(t)=e^{1/(\rho-t)}.$$

Next, we prove Theorem 2.2. Lemma 2.5 below plays a key role in the proof and is taken from [18, Theorem 2.7]. For some other related results, see [15, 16, 17] the references therein.

Lemma 2.5. *Let V be a reflexive real Banach space with the norm $\|\cdot\|$, let E be a real normed space with topological dual E^*, let $u_0\in V$ and $r>0$, let B_r be the open ball in V of radius r, centered at u_0, let $\chi:[0,r)\to\mathbb{R}$ be such that $\lim_{t\to r^-}\chi(t)=\infty$, let $I:B_r\to\mathbb{R}$ and $\psi:B_r\to E$ be two Gâteaux differentiable functions. Moreover, assume that I is sequentially weakly lower semicontinuous, that ψ is sequentially weakly continuous, that $\psi(B_r)$ is bounded and non-convex, and that*

$$\chi(\|u-u_0\|)\le I(u)\quad \textit{for all } u\in B_r. \tag{8}$$

Then, for every convex set $S\subseteq E^*$ weakly-star dense in E^*, there exists $\eta\in S$ such that the equation

$$I'(u)+(\eta\circ\psi)'(u)=0$$

has at least two solutions in B_r.

Proof of Theorem 2.2. Let the functionals $I, J_{\alpha_1,\dots,\alpha_k,\gamma} : X \to \mathbb{R}$ be defined by

$$I(u) = \frac{1}{p}\widetilde{M}\left(\int_\Omega (|\Delta u|^p + |\nabla u|^p)\, dx\right), \tag{9}$$

$$J_{\alpha_1,\dots,\alpha_k,\gamma}(u) = \int_\Omega \left(\sum_{i=1}^k \alpha_i(x)F_i(u(x)) + \gamma(x)u(x)\right) dx, \tag{10}$$

where $\widetilde{M}(t) = \int_0^t M(s)ds$ and $F_i(t) = \int_0^t f_i(s)ds,\ i = 1,\dots,k.$ (11)

Then, we have $I, J_{\alpha_1,\dots,\alpha_k,\gamma} \in C^1(X,\mathbb{R})$ and their Gâteaux derivatives $I', J'_{\alpha_1,\dots,\alpha_k,\gamma} : X \to X^*$ are given by

$$\langle I'(u), v\rangle = M\left(\int_\Omega (|\Delta u|^p + |\nabla u|^p)dx\right)\int_\Omega \left(|\Delta u|^{p-2}\Delta u\Delta v + |\nabla u|^{p-2}\nabla u\nabla v\right) dx$$

and
$$\langle J'_{\alpha_1,\dots,\alpha_k,\gamma}(u), v\rangle = \int_\Omega \left(\sum_{i=1}^k \alpha_i(x)f_i(u) + \gamma(x)\right) v dx$$

for all $(u, v) \in X$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and its dual space X^*. Let $B_{\rho^{1/p}}$ be the open ball in X centered at 0 with radius $\rho^{1/p}$. Then, it is obvious that $u \in X$ is a weak solution of problem (1) if and only if u is critical point in $B_{\rho^{1/p}}$ of the functional $I - J_{\alpha_1,\dots,\alpha_k,\gamma}$.

We first prove that (a) $\Rightarrow$ (b). To this end, we will apply Lemma 2.5 where $V = X$, $u_0 = 0$, $r = \rho^{1/p}$, I is defined by (9), $\chi(t) = \frac{1}{p}\widetilde{M}(t^p)$, E is given by (7), and $\psi : B_{\rho^{1/p}} \to E$ is defined by

$$\psi(u)(\cdot) = (F_1(u(\cdot)),\dots,F_k(u(\cdot)), u(\cdot)) \quad \text{for all } u \in B_{\rho^{1/p}}.$$

It is easy to check that I is sequentially weakly lower semicontinuous and ψ is sequentially weakly continuous. In view of (5), we have $\psi(B_{\rho^{1/p}})$ is bounded, and from (6) and the definition of $\widetilde{M}$ in (11), we see that $\lim_{t\to r^-}\chi(t) = \infty$. Below, we show that $\psi(B_{\rho^{1/p}})$ is non-convex. Assume, to the contrary, that $\psi(B_{\rho^{1/p}})$ is convex. For any $u_1, u_2 \in B_{\rho^{1/p}}$, there exist $w_1, w_2 \in \psi(B_{\rho^{1/p}})$ such that

$$w_1 = (F_1(u_1(\cdot)),\dots,F_k(u_1(\cdot)), u_1(\cdot)) \quad \text{and} \quad w_2 = (F_1(u_2(\cdot)),\dots,F_k(u_2(\cdot)), u_2(\cdot)).$$

Since $\psi(B_{\rho^{1/p}})$ is convex, we have $\tau w_1 + (1-\tau)w_2 \in \psi(B_{\rho^{1/p}})$ for all $\tau \in [0,1]$. Then, there exists $u_3 \in B_{\rho^{1/p}}$ such that

$$\tau w_1 + (1-\tau)w_2 = (F_1(u_3(\cdot)),\dots,F_k(u_3(\cdot)), u_3(\cdot)).$$

Thus,
$$F_i(u_3(\cdot)) = \tau F_i(u_1(\cdot)) + (1-\tau)F_i(u_2(\cdot)), \quad i = 1,\dots,k,$$

and
$$u_3(\cdot) = \tau u_1(\cdot) + (1-\tau)u_2(\cdot).$$

Then, from the definitions of F_i in (11), it follows that for all $x \in \Omega$

$$\int_0^{\tau u_1(x)+(1-\tau)u_2(x)} f_i(s)ds = \tau\int_0^{u_1(x)} f_i(s)ds + (1-\tau)\int_0^{u_2(x)} f_i(s)ds,\ i = 1,\dots,k. \tag{12}$$

Differentiating both side of (12) with respect to τ yields that

$$f_i(\tau u_1(x)+(1-\tau)u_2(x))(u_1(x)-u_2(x)) = \int_0^{u_1(x)} f_i(s)ds - \int_0^{u_2(x)} f_i(s)ds, \quad (13)$$

for $i = 1,\dots,k$. In assertion (a), we first assume that the restriction of f_1 to $\left(-c\rho^{1/p}, c\rho^{1/p}\right)$ is not constant. Now, let $t_1, t_2 \in \left(-c\rho^{1/p}, c\rho^{1/p}\right)$ be such that $t_1 < t_2$ and $f_1(t_1) \neq f_1(t_2)$. Then, fix $\epsilon > 0$ so that $-c\rho^{1/p}+\epsilon < t_1 < t_2 < c\rho^{1/p}-\epsilon$.

Choose $v \in X \setminus \{0\}$ so that $\dfrac{\sup_{x\in\Omega}|v(x)|}{\|v\|} > c - \dfrac{\epsilon}{\rho^{1/p}}$.

Finally, fix $x_0 \in \Omega$ and $\lambda > 0$ such that $\dfrac{c\rho^{1/p}-\epsilon}{|v(x_0)|} < \lambda < \dfrac{\rho^{1/p}}{\|v\|}$.

Consequently, if we choose $u = \lambda v$, we have $\|u\| < \rho^{1/p}$ and $|u(x_0)| > c\rho^{1/p}-\epsilon$. Then, choosing $u_1(x) = u(x)$ and $u_2(x) = -u(x)$, we have either $u_2(x_0) < t_1 < t_2 < u_1(x_0)$ or $u_1(x_0) < t_1 < t_2 < u_2(x_0)$. Hence, from (13) with $i = 1$ and $x = x_0$, it follows that

$$2u(x_0)f_1\left(2\tau u(x_0)-u(x_0)\right) = \int_{-u(x_0)}^{u(x_0)} f_1(s)ds = 2u(x_0)f_1(\theta)\ ,$$

where θ is a suitable point of the open interval whose extremes are $u_1(x_0)$ and $u_2(x_0)$. Thus, we have $f_1\left(2\tau u(x_0)-u(x_0)\right) = f_1(\theta)$. Since $\tau \in [0,1]$ is arbitrary, f is constant on the interval whose extremes are $u_1(x_0)$ and $u_2(x_0)$. Thus, $f_1(t_1) = f_1(t_2)$, contradicting to the assumption that $f_1(t_1) \neq f_1(t_2)$. When the restriction of other f_i to $\left(-c\rho^{1/p}, c\rho^{1/p}\right)$ is not constant, a similar contradiction can be derived by using (13). Therefore, $\psi(B_{\rho^{1/p}})$ is non-convex. We have shown that all the assumptions of Lemma 2.5 are satisfied. Now, define the operator $T : E \to E^*$ by

$$T(v_1,\dots,v_{k+1})(w_1,\dots,w_{k+1}) = \int_\Omega \sum_{i=1}^{k+1} v_i(x)w_i(x)\,dx \quad (14)$$

for all $(v_1,\dots,v_{k+1}),(w_1,\dots,w_{k+1}) \in E$. Clearly, T is linear and the linear subspace $T(E)$ is total over E. Then, $T(E)$ is weakly-star dense in E^*. We claim that T is continuous with respect to the weak-star topology of E^*. To see this, let $\{(v_{1n},\dots,v_{(k+1)n})\}$ be a sequence in E converging to some $(\bar{v}_1,\dots,\bar{v}_{k+1})) \in E$. Then, we have

$$\lim_{n\to\infty}\int_\Omega \sum_{i=1}^{k+1} |v_{in}(x)-\bar{v}_i(x)|\,dx = 0. \quad (15)$$

For any $(w_1,\dots,w_{k+1}) \in E$, (14) and (15) imply that

$$\begin{aligned}&|T(v_{1n},\dots,v_{(k+1)n})(w_1,\dots,w_{k+1}) - T(\bar{v}_1,\dots,\bar{v}_{k+1}))(w_1,\dots,w_{k+1})|\\ &\le \int_\Omega \sum_{i=1}^{k+1} |v_{in}(x)-\bar{v}(x)|\,|w_i(x)|\,dx\\ &\le \max\Big\{\max_{x\in\overline{\Omega}}|w_i(x)|,\ i=1,\dots,k+1\Big\}\lim_{n\to\infty}\int_\Omega\sum_{i=1}^{k+1}|v_{in}(x)-\bar{v}_i(x)|\,dx \to 0\end{aligned}$$

as $n \to \infty$. Thus, the claim is true.

Finally, for every convex set $S \subset E$ dense in E, as above, we can show that $T(-S)$ is weakly-star dense in E^*. Note from (10) and (14) that $J_{\alpha_1,\dots,\alpha_k,\gamma} = \eta \circ \psi$ for any $\eta \in T(S)$. Then, by Lemma (2.5), we see that there exists $(\hat{\alpha}_1, \dots, \hat{\alpha}_k, \hat{\gamma}) \in -S$ such that if we set

$$\alpha = -\hat{\alpha}_1, \dots, \alpha_k = -\hat{\alpha}_k, \quad \text{and} \quad \gamma = -\hat{\gamma},$$

the functional $I - J_{\alpha_1,\dots,\alpha_k,\gamma}$ has at least two critical points which are weak solutions of problem (1). This proves assertion (b).

Next, we prove that (b) $\Rightarrow$ (a). Assume that the restrictions of all the functions f_i, $i = 1, \dots, k$, to $\left(-c\rho^{1/p}, c\rho^{1/p}\right)$ are constant. Let d_i, $i = 1, \dots, k$, be the values of f and g, respectively, on $\left[-c\rho^{1/p}, c\rho^{1/p}\right]$. Then, the weak solutions of the problem

$$\begin{cases} M\left(\int_\Omega (|\Delta u|^p + |\nabla u|^p) dx\right)\left(\Delta_p^2 u - \Delta_p u\right) = \sum_{i=1}^k d_i \alpha_i(x) + \gamma(x) \text{ in } \Omega, \\ u = \Delta u = 0 \text{ on } \partial\Omega, \quad \int_\Omega \left(|\Delta u|^p + |\nabla u|^p\right) dx < \rho, \end{cases}$$

are the critical points in $B_{\rho^{1/p}}$ of the functional $\Phi : X \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{p}\widetilde{M}\left(\int_\Omega \left(|\Delta u|^p + |\nabla u|^p\right) dx\right) - \int_\Omega \left(\sum_{i=1}^k d_i \alpha_i(x) + \gamma(x)\right) u(x) dx.$$

It can be easily verified that $\Phi' : X \to X^*$ is strictly monotone. Hence, Φ has a unique critical point (see, for example, [23, Theorem 26.A (c)]). Thus, (b) $\Rightarrow$ (a). This completes the proof of the theorem. □

We end this paper with the following proposition and remark to show that the density requirement of S in E in assertion (b) of Theorem 2.2 (respectively, S_1 in E_1 in Corollary 2.3) is necessary for the conclusion to be true.

Proposition 2.6. *Let $\rho > 1$. Consider the problem*

$$\begin{cases} \left(\rho - \int_0^1 (|u''|^2 + |u'|^2) dx\right)^{-2} \left(u^{(4)} - u''\right) = \alpha(x) u + \gamma(x) \quad \text{in } (0,1), \\ u(0) = u(1) = 0, \quad u''(0) = u''(1) = 0, \quad \int_0^1 \left(|u''|^2 + |u'|^2\right) dx < \rho. \end{cases} \tag{16}$$

Then, for any $(\alpha, \gamma) \in E_1 = C(\overline{\Omega}) \times C(\overline{\Omega})$ satisfying

$$\begin{cases} \left(\rho^{5/2} \int_0^1 s(1-s)|\alpha(s)| ds + \rho^2 \int_0^1 s(1-s)|\gamma(s)| ds + \frac{1}{\sqrt{30}}\rho^{1/2}\right)^2 < \frac{4}{13}\rho, \\ \kappa = \int_0^1 s(1-s)\left(9\rho^2|\alpha(s)| + 8\rho^{3/2}|\gamma(s)| + 1\right) ds < 1, \end{cases} \tag{17}$$

the problem (16) *has a unique solution.*

Remark 2.7. It is trivial to see that problem (16) is a special case of of problem (4) with $p = 2$, $N = 1$, $\Omega = (0,1)$, $M(t) = (\rho - t)^{-2}$, and $f(t) = t$. Obviously, M is increasing on $[0, \rho)$ and satisfies condition (6). Moreover, assertion (a) of Corollary 2.3 holds. Thus, Proposition 2.6 implies that, assertion (b) of Corollary 2.3 is no longer true without the density requirement of S_1 in E_1.

Proof of Proposition 2.6. We first notice that a weak solution of problem (16) coincides with a classical solution of the problem and such a solution can be sought in the Banach space

$$Y = \left\{ u \in C^2[0,1] \;\middle|\; \begin{array}{l} u(0) = u(1) = 0,\ u''(0) = u''(1) = 0 \\ \text{and } \int_0^1 (|u''|^2 + |u'|^2)\,dx < \rho \end{array} \right\}$$

equipped with the norm $\|u\|_Y = \max_{x\in[0,1]} |u''(x)|$.

Let
$$G(x,s) = \begin{cases} x(s-1), & 0 \le x \le s \le 1, \\ s(x-1), & 0 \le s \le x \le 1, \end{cases}$$

and
$$H(x,s) = \int_0^1 G(x,\tau)G(\tau,s)d\tau.$$

Then, $H(x,s)$ is the Green's function for the problem

$$\begin{cases} u^{(4)} = 0 \quad \text{in } (0,1), \\ u(0) = u(1) = 0,\ u''(0) = u''(1) = 0. \end{cases}$$

Rewrite the equation in (16) as

$$\begin{aligned} u^{(4)} &= \big(\alpha(x)u + \gamma(x)\big)\left(\rho - \int_0^1 (|u''|^2 + |u'|^2)dx\right)^2 + u'' \\ &= \big(\alpha(x)u + \gamma(x)\big)K(u) + u'' \quad \text{in } (0,1), \end{aligned}$$

where
$$K(u) = \left(\rho - \int_0^1 (|u''|^2 + |u'|^2)dx\right)^2.$$

Define an operator $S : Y \to C^2[0,1]$ by

$$Su(x) = \int_0^1 H(x,s)\left(\big(\alpha(s)u(s) + \gamma(s)\big)K(u) + u''(s)\right) ds.$$

Then, $u \in Y$ is a solution of problem (17) if and only if u is a fixed point of S in Y. Now, we show that $S : Y \to Y$. For any $u \in Y$ and $x \in [0,1]$, we have

$$\begin{aligned} (Su(x))' &= \int_0^1 \tau \int_0^1 G(\tau,s)\left(\big(\alpha(s)u(s) + \gamma(s)\big)K(u) + u''(s)\right) ds d\tau \\ &\quad - \int_x^1 \int_0^1 G(\tau,s)\left(\big(\alpha(s)u(s) + \gamma(s)\big)K(u) + u''(s)\right) ds d\tau \end{aligned}$$

and

$$(Su(x))'' = \int_0^1 G(x,s)\left(\big(\alpha(s)u(s) + \gamma(s)\big)K(u) + u''(s)\right) ds.$$

Then, since $|G(t,s)| \le s(1-s)$ on $[0,1]\times[0,1]$, we obtain that

$$\begin{aligned} |(Su(x))'| \ \le\ & \int_0^1 \tau \int_0^1 s(1-s)\left(\big(|\alpha(s)|\,|u(s)|+|\gamma(s)|\big)K(u)+|u''(s)|\right) dsd\tau \\ & + \int_x^1 \int_0^1 s(1-s)\left(\big(|\alpha(s)|\,|u(s)|+|\gamma(s)|\big)K(u)+|u''(s)|\right) dsd\tau \end{aligned}$$

and

$$|(Su(x))''| = \int_0^1 s(1-s)\left(\big(|\alpha(s)|\,|u(s)|+|\gamma(s)|\big)K(u)+|u''(s)|\right) ds.$$

By Hölder's inequality, we see that

$$|u(s)| = \left|\int_0^s u'(s)ds\right| \le \int_0^1 |u'(s)|ds \le \left(\int_0^1 |u'(s)|^2 ds\right)^{1/2} \le \rho^{1/2}. \tag{18}$$

Then, from the fact that $K(u)\le\rho^2$ and Hölder's inequality, it follows that

$$\begin{aligned} |(Su(x))'| &\le \left(\frac{3}{2}-x\right)\rho^{5/2}\int_0^1 s(1-s)|\alpha(s)|ds + \left(\frac{3}{2}-x\right)\rho^2\int_0^1 s(1-s)|\gamma(s)|ds \\ &\quad + \left(\frac{3}{2}-x\right)\int_0^1 s(1-s)|u''(s)|ds \\ &\le \frac{3}{2}\rho^{5/2}\int_0^1 s(1-s)|\alpha(s)|ds + \frac{3}{2}\rho^2\int_0^1 s(1-s)|\gamma(s)|ds \\ &\quad + \frac{3}{2}\left(\int_0^1 s^2(1-s)^2 ds\right)^{1/2}\left(\int_0^1 |u''(s)|^2 ds\right)^{1/2} \\ &\le \frac{3}{2}\rho^{5/2}\int_0^1 s(1-s)|\alpha(s)|ds + \frac{3}{2}\rho^2\int_0^1 s(1-s)|\gamma(s)|ds + \frac{3}{2\sqrt{30}}\rho^{1/2} \end{aligned}$$

and

$$\begin{aligned} |(Su(x))''| &\le \rho^{5/2}\int_0^1 s(1-s)|\alpha(s)|ds + \rho^2\int_0^1 s(1-s)|\gamma(s)|ds \\ &\quad + \int_0^1 s(1-s)|u''(s)|ds \\ &\le \rho^{5/2}\int_0^1 s(1-s)|\alpha(s)|ds + \rho^2\int_0^1 s(1-s)|\gamma(s)|ds \\ &\quad + \left(\int_0^1 s^2(1-s)^2 ds\right)^{1/2}\left(\int_0^1 |u''(s)|^2 ds\right)^{1/2} \\ &\le \rho^{5/2}\int_0^1 s(1-s)|\alpha(s)|ds + \rho^2\int_0^1 s(1-s)|\gamma(s)|ds + \frac{1}{\sqrt{30}}\rho^{1/2}. \end{aligned}$$

Thus, by the first inequality in (17), we have

$$\begin{aligned}
&\int_0^1 \left(|(Su(x))'|^2 + |(Su(x))''|^2\right) dx \\
&\leq \frac{13}{4}\left(\rho^{5/2}\int_0^1 s(1-s)|\alpha(s)|ds + \rho^2\int_0^1 s(1-s)|\gamma(s)|ds + \frac{1}{\sqrt{30}}\rho^{1/2}\right)^2 \\
&\leq \rho.
\end{aligned}$$

Hence, $S(Y) \subset Y$.

Next, we prove that S is a contraction mapping. For any $u_1, u_2 \in Y$ and $x \in [0,1]$, we have

$$\begin{aligned}
|(Su_1(x) - Su_2(x))''| \ \leq\ & \int_0^1 |G(x,s)|\big(|\alpha(s)|\,|u_1(s)K(u_1) - u_2(s)K(u_2)| \\
&+|\gamma(s)|\,|K(u_1) - K(u_2)| + |u_1''(s) - u_2''(s)|\big)ds.
\end{aligned}$$

Since $|G(t,s)| \leq s(1-s)$ on $[0,1]\times[0,1]$ and

$$\begin{aligned}
|u_1(s)K(u_1) - u_2(s)K(u_2)| \ &\leq\ K(u_1)|u_1(s) - u_2(s)| + |u_2(s)|\,|K(u_2) - K(u_1)| \\
&\leq\ \rho^2|u_1(s) - u_2(s)| + |u_2(s)|\,|K(u_2) - K(u_1)|,
\end{aligned}$$

then from (18) with $u = u_2$, we reach that

$$\begin{aligned}
&|(Su_1(x) - Su_2(x))''| \\
&\leq \int_0^1 s(1-s)\big[|\alpha(s)|\left(\rho^2|u_1(s) - u_2(s)| + |u_2(s)|\,|K(u_2) - K(u_1)|\right) \\
&\quad + |\gamma(s)|\,|K(u_1) - K(u_2)| + |u_1''(s) - u_2''(s)|\big]ds \\
&\leq \int_0^1 s(1-s)\big[|\alpha(s)|\left(\rho^2|u_1(s) - u_2(s)| + \rho^{1/2}|K(u_2) - K(u_1)|\right) \\
&\quad + |\gamma(s)|\,|K(u_1) - K(u_2)| + |u_1''(s) - u_2''(s)|\big]ds.
\end{aligned} \tag{19}$$

Note that

$$\begin{aligned}
&|K(u_2) - K(u_1)| \\
&= \left|\left(\rho - \int_0^1 (|u_2''|^2 + |u_2'|^2)dx\right)^2 - \left(\rho - \int_0^1 (|u_1''|^2 + |u_1'|^2)dx\right)^2\right| \\
&= \left|2\rho - \int_0^1 (|u_2''|^2 + |u_2'|^2)dx - \int_0^1 (|u_1''|^2 + |u_1'|^2)dx\right| \\
&\quad \times \left|\int_0^1 (|u_2''|^2 + |u_2'|^2)dx - \int_0^1 (|u_1''|^2 + |u_1'|)dx\right| \\
&\leq 2\rho\int_0^1 \left((|u_2''| + |u_1''|)|u_2'' - u_1''| + (|u_2'| + |u_1'|)|u_2' - u_1'|\right) dx.
\end{aligned}$$

Then, in view of $|u_2'(x) - u_1'(x)| \leq \|u_2 - u_1\|_Y$ and from Hölder's inequality,

$$\begin{aligned}
&|K(u_2) - K(u_1)| \\
&\leq 2\rho\|u_2 - u_1\|_Y \int_0^1 (|u_2''(s)| + |u_2'(s)| + |u_1''(s)| + |u_1'(s)|)\, dx \\
&\leq 2\rho\|u_2 - u_1\|_Y \Bigg(\left(\int_0^1 |u_2''(s)|^2 ds\right)^{1/2} + \left(\int_0^1 |u_2'(s)|^2 ds\right)^{1/2} \\
&\quad + \left(\int_0^1 |u_1''(s)|^2 ds\right)^{1/2} + \left(\int_0^1 |u_1'(s)|^2 ds\right)^{1/2} \Bigg) \\
&\leq 8\rho^{3/2}\|u_2 - u_1\|_Y. \qquad (20)
\end{aligned}$$

It is easy to check that $|u_1(s) - u_2(s)| \leq \|u_1 - u_2\|_Y$. Then, from (19) and (20), it follows that

$$\begin{aligned}
|(Su_1(x) - Su_2(x))''| &\leq \|u_2 - u_1\|_Y \int_0^1 s(1-s)\big(9\rho^2|\alpha(s)| + 8\rho^{3/2}|\gamma(s)| + 1\big) ds \\
&= \kappa\|u_2 - u_1\|_Y,
\end{aligned}$$

where $\kappa < 1$ is given by the second equation on (17). In conseqence we obtain $\|Su_1 - Su_2\|_Y \leq \kappa\|u_2 - u_1\|_Y$. Thus, S is a contraction mapping. Hence, by the contraction mapping theorem, S has a unique fixed point in Y. This show that if $(\alpha, \gamma) \in E_1$ satisfies (17), problem (16) has a unique solution. This completes the proof of the proposition. □

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