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Mixed Variational Approach to Finding Guaranteed Estimates for Solutions and Right-Hand Sides of the Second-Order Linear Elliptic Equations under Incomplete Data

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We investigate the problem of guaranteed estimation of values of linear continuous functionals defined on solutions to mixed variational equations generated by linear elliptic problems from indirect noisy observations of these solutions. We assume that right-hand sides of the equations, as well as the second moments of noises in observations are not known; the only available information is that they belong to given bounded sets in the appropriate functional spaces. We are looking for linear with respect to observations optimal estimates of solutions of aforementioned equations called minimax or guaranteed estimates. We develop constructive methods for finding these estimates and estimation errors which are expressed in terms of solutions to special mixed variational equations and prove that Galerkin approximations of the obtained variational equations converge to their exact solutions. Finally we study the problem of guaranteed estimation of right-hand sides of mixed variational equations.

Introduction

Estimation theory for systems with lumped and distributed parameters under uncertainty conditions was developed intensively during the last 30 years. That was motivated by the fact that the realistic setting of boundary value problems describing physical processes often contains perturbations of unknown (or partially unknown) nature. In such cases the minimax estimation method proved to be useful, making it possible to obtain optimal estimates both for the unknown solutions (or right-hand sides of equations appearing in the boundary value problems) and for linear functionals from them; that is, estimates looked for in the class of linear estimates with respect to observations for which the maximal mean square error taken over all the realizations of perturbations from certain given sets takes its minimal value. Here we understand observations of unknown solutions as the functions that are linear transformations of the same solutions distorted by additive random noises.

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The above estimation method was investigated in the works by N. N. Krasovsky, A. B. Kurzhansky, A. G. Nakonechny, and others (see [7], [8], [11]–[13], [15], [10]). This approach makes it possible to find optimal estimates of parameters of boundary value problems reckoning on the "worst" realizations of perturbations.

A. G. Nakonechny [10] used traditional variational formulations of boundary value problems (their solvability is based on the Lax-Milgram lemma), to obtain systems of variational equations whose solutions generate the minimax mean square estimates.

At the same time many physical processes of the real world are described by mixed variational problems. Among such processes, there are flows of viscous fluids, propagation of electromagnetic and acoustical waves. In addition, many classical boundary value problems admit mixed variational formulations. The mixed method consists of simultaneous finding, from systems of variational equations, both solutions and certain expression generated by solutions taken as new auxiliary unknowns. As a rule, these unknowns are related to derivatives of the solutions and have important physical meaning (such as flux, bending moment etc), and their calculation or estimation often has even greater practical significance.

The theory of mixed variational methods of solving boundary value problems and their numerical implementation, the mixed finite element methods, was developed by Babuška, Brezzi, Fortin, Raviard, Glowinski and others (see [2]–[4]). Brezzi and Fortin proved solvability theorems for a wide class of mixed variational problems and their discrete analogs.

In this paper we show that mixed variational formulations of boundary value problems can be used also for a guaranteed estimation of linear functionals from unknown solutions and their gradients, as well as functionals from unknown right-hand sides of second order linear elliptic equations. It is proved that guaranteed estimates of these functionals and estimation errors are expressed explicitly from the solutions of special systems of mixed variational equations, for which the unique solvability is established. We develop, on the basis of the Galerkin method, numerical methods of finding these solutions and prove the convergence of the approximate solutions to exact ones.

The estimation methods proposed here yield, for example, in stationary and non-stationary heat conduction problems, estimates of heat flux from temperature observations, or conversely, estimates of temperature from heat flux observations, as well as estimates of the unknown distribution of density of sources from heat flux observations. The theory of guaranteed estimation developed in this work provides an essential generalization of well-known results in this direction by the authors mentioned above.

Note that the available estimation methods do not provide solution of such estimation problems, so that the methods developed here are essentially new.

1. Preliminaries and auxiliary results

We denote matrices and vectors by bold letters; $x = (x_1, \ldots, x_n)$ denotes a spatial variable in an open domain $D \subset \mathbb{R}^n$ with Lipschitzian boundary Γ; $dx = dx_1 \cdots dx_n$ is Lebesgue measure in $\mathbb{R}^n$; $H^1(D)$, $H_0^1(D)$, and $H(\mathrm{div}\,; D)$ are standard Sobolev spaces of the first order in the domain D with corresponding norm.

If X is a Hilbert space over $\mathbb{R}$ with inner product $(\cdot, \cdot)_X$ and norm $\| \cdot \|_X$, then $J_X \in \mathcal{L}(X, X')$ denotes the Riesz operator acting from X to its adjoint X' and determined by the equality $(v, u)_X = < v, J_X u >_{X \times X'}$ $\forall u, v \in X$ (we note that this operator exists according to the Riesz theorem), where $< x, f >_{X \times X'} := f(x)$ for $x \in X$, $f \in X'$.

By $L^2(\Omega, X)$ we denote the Bochner space composed of random variables $\xi = \xi(\omega)$ defined on a certain probability space $(\Omega, \mathcal{B}, P)$ with values in a separable Hilbert space X such that

$$\|\xi\|^2_{L^2(\Omega,X)} = \int_\Omega \|\xi(\omega)\|^2_X dP(\omega) < \infty\,, \tag{1}$$

where a random variable ξ with values in X is considered as a function $\xi : \Omega \to X$ mapping random events $E \in \mathcal{B}$ to Borel sets in H (Borel σ-algebra in X is generated by open sets in X). In this case there exists the Bochner integral

$$\mathbb{E}\xi := \int_\Omega \xi(\omega)\, dP(\omega) \in X \tag{2}$$

called the mathematical expectation or the mean value of random variable $\xi(\omega)$ which satisfies the condition

$$(h, \mathbb{E}\xi)_X = \int_\Omega (h, \xi(\omega))_X\, dP(\omega) \quad \forall h \in X. \tag{3}$$

Being applied to a random variable ξ with values in $\mathbb{R}$ this expression leads to a usual definition of its mathematical expectation because the Bochner integral (2) reduces to a Lebesgue integral with probability measure $dP(\omega)$.

In $L^2(\Omega, X)$ one can introduce the inner product

$$(\xi, \eta)_{L^2(\Omega,X)} := \int_\Omega (\xi(\omega), \eta(\omega))_X\, dP(\omega) \quad \forall \xi, \eta \in L^2(\Omega, X). \tag{4}$$

Applying the sign of mathematical expectation one can write the relationships (1)–(4) as

$$\|\xi\|^2_{L^2(\Omega,X)} = \mathbb{E}\|\xi(\omega)\|^2_X, \tag{5}$$

$$(h, \mathbb{E}\xi)_X = \mathbb{E}(h, \xi(\omega))_X \quad \forall h \in X, \tag{6}$$

$$(\xi, \eta)_{L^2(\Omega,X)} := \mathbb{E}(\xi(\omega), \eta(\omega))_X \quad \forall \xi, \eta \in L^2(\Omega, X). \tag{7}$$

$L^2(\Omega, X)$ equipped with norm (5) and inner product (7) is a Hilbert space.

2. Statement of the estimation problem of linear functionals from solutions to mixed variational equations

Let the state of a system be characterized by the function $\varphi(x)$ which is defined as a solution of the Dirichlet boundary value problem:

$$-\mathrm{div}\,(\mathbf{A}\,\mathbf{grad}\,\varphi) + c\varphi = f \quad \text{in} \quad D, \tag{8}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma. \tag{9}$$

Introducing the additional unknown $\mathbf{j} = -\mathbf{A}\,\mathbf{grad}\,\varphi$ in D we rewrite this problem as the first-order system

$$\mathbf{A}^{-1}\,\mathbf{j} = -\mathbf{grad}\,\varphi \quad \text{in} \quad D, \tag{10}$$

$$\mathrm{div}\,\mathbf{j} + c\varphi = f \qquad D, \quad \varphi = 0 \quad \text{on} \quad \Gamma, \tag{11}$$

where $\mathbf{A} = \mathbf{A}(x) = (a_{ij}(x))$ is an $n \times n$ matrix with entries $a_{ij} \in L^\infty(D)$ for which there exists a positive number μ such that

$$\mu \sum_{i=1}^{n} \xi_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \forall x \in D \quad \forall \xi = (\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n,$$

$\mathbf{A}^{-1}$ is the inverse matrix of $\mathbf{A}$, and c is a piecewise continuous function satisfying for $x \in D$ the inequality $c_0 \le c(x) \le c_1$, $c_0, c_1 = \text{const}$, $0 \le c_0 \le c_1$.

According to [2] and [5], by a solution of problem (10)–(11) we will mean a pair of functions $(\mathbf{j}, \varphi) \in H(\mathrm{div}; D) \times L^2(\Omega)$ such that

$$\int_D ((\mathbf{A}(x))^{-1}\mathbf{j}(x), \mathbf{q}(x))_{\mathbb{R}^n} dx - \int_D \varphi(x)\mathrm{div}\,\mathbf{q}(x)dx = 0 \quad \forall \mathbf{q} \in H(\mathrm{div}; D) \tag{12}$$

$$\int_D v(x)\mathrm{div}\,\mathbf{j}(x)dx + \int_D c(x)\varphi(x)v(x)dx = \int_D f(x)v(x)\,dx \quad \forall v \in L^2(D). \tag{13}$$

Note that from equations (12)–(13) it follows that $\varphi \in H_0^1(D)$, i.e. the boundary condition $\varphi|_\Gamma = 0$ is implicitly contained in these equations. sdsProblem (12)–(13) is commonly referred to as the mixed formulation of (10)–(11). From a physical point of view problem (10)–(11) simulates a stationary process of the propagation of heat in the domain D, and the functions $\varphi(x)$, $\mathbf{j}(x)$, and $f(x)$ have the sense of temperature, heat flux, and volume density of heat sources, respectively, at the point x.

We introduce the bilinear forms a, b, c and the functional l as

$$a(\mathbf{q}_1, \mathbf{q}_2) := \int_D ((\mathbf{A}(x))^{-1}\mathbf{q}_1, \mathbf{q}_2)_{\mathbb{R}^n} dx \quad \forall \mathbf{q}_1, \mathbf{q}_2 \in H(\mathrm{div}; \mathrm{D}), \tag{14}$$

$$b(\mathbf{q}, v) := -\int_D v\,\mathrm{div}\,\mathbf{q}\,dx \quad \forall \mathbf{q} \in H(\mathrm{div}; \mathrm{D}), \mathrm{v} \in \mathrm{L}^2(\mathrm{D}), \tag{15}$$

$$c(v,v) := \int_D c(x)v_1(x)v_2(x)\,dx \quad \forall v_1, v_2 \in L^2(D), \tag{16}$$

$$l(v) := -\int_D fv\,dx \quad \forall v \in L^2(D). \tag{17}$$

Then the problem under study may be stated as follows:

Find $(\mathbf{j}, \varphi) \in H(\text{div}; D) \times L^2(\Omega)$ such that

$$a(\mathbf{j}, \mathbf{q}) + b(\mathbf{q}, \varphi) = 0 \quad \forall \mathbf{q} \in H(\text{div}; D), \text{ and} \tag{18}$$

$$b(\mathbf{j}, v) - c(\varphi, v) = l(v) \quad \forall v \in L^2(D). \tag{19}$$

Denote by $B : H(\text{div}; \mathrm{D}) \to \mathrm{L}^2(\mathrm{D})$ the operator associated with the bilinear form b. It is easy to see that a, b and c are continuous bilinear forms with a being coercive on $\mathrm{Ker\,B}$, c being symmetric, positive semidefinite and b satisfying the standard inf-sup condition (Brezzi condition). Since $\operatorname{Im} B = L^2(D)$, we have $\operatorname{Ker} B^t = \{\emptyset\}$, where $B^t : L^2(D) \to H(\text{div}; D)'$ is the transpose operator of B defined by

$$< v, B^t q >_{H(\text{div},D)\times H(\text{div},D)'} = b(v,q) \quad \forall v \in H(\text{div}, D), \quad \forall q \in L^2(D).$$

Consequently, it follows from Theorem 1.2 of [2] that problem (18)–(19) has a unique solution and the following *a priori* estimate is valid

$$\|\mathbf{j}\|_{H(\text{div};D)} + \|\varphi\|_{L^2(D)} \leq C\|f\|_{L^2(D)} \quad (C = \text{const}).$$

Further we assume that the function $f(x)$ in equations (11) and (13) is unknown and belongs to the set

$$G_0 := \Big\{\tilde{f} \in L^2(D) : \Big(Q(\tilde{f} - f_0), \tilde{f} - f_0\Big)_{L^2(D)} \leq \epsilon_1\Big\}, \tag{20}$$

where $f_0 \in L^2(D)$ is a given function, $\epsilon_1 > 0$ is a given constant, and $Q : L^2(D) \to L^2(D)$ is a bounded positive selfadjoint operator for which there exists the inverse bounded operator Q^{-1}. It is known that the operator Q^{-1} is positive and selfadjoint.

In this paper we focus on the following estimation problem:

From observations of the random variables

$$y_1 = C_1\mathbf{j} + \eta_1, \quad y_2 = C_2\varphi + \eta_2, \tag{21}$$

with values in separable Hilbert spaces H_1 and H_2 over $\mathbb{R}$, respectively, it is necessary to estimate the value of the linear functional

$$l(\mathbf{j}, \varphi) := \int_D (\mathbf{l}_1(x), \mathbf{j}(x))_{\mathbb{R}^n}\,dx + \int_D l_2(x)\varphi(x)\,dx \tag{22}$$

in the class of the estimates linear with respect to observations, which have the form

$$\widehat{l(\mathbf{j},\varphi)} := (y_1,u_1)_{H_1} + (y_2,u_2)_{H_2} + c, \tag{23}$$

where $(\mathbf{j},\varphi)$ is a solution of problem (12)–(13), $\mathbf{l}_1$ and l_2 are given functions from $L^2(D)^n$ and $L^2(D)$, $u_1 \in H_1$, $u_2 \in H_2$, $c \in \mathbb{R}$, $C_1 \in \mathcal{L}(L^2(D)^n, H_1)$, and $C_2 \in \mathcal{L}(L^2(D), H_2)$ are linear bounded operators,

$$\eta := (\eta_1,\eta_2) \in G_1; \tag{24}$$

by G_1 we denote the set of pairs $\{(\tilde\eta_1,\tilde\eta_2)\}$ of uncorrelated random variables $\tilde\eta_1 \in L^2(\Omega,H_1)$ and $\tilde\eta_2 \in L^2(\Omega,H_2)$ with zero expectations satisfying the condition

$$\mathbb{E}(\tilde Q_1\tilde\eta_1,\tilde\eta_1)_{H_1} \le \epsilon_2, \quad \mathbb{E}(\tilde Q_2\tilde\eta_2,\tilde\eta_2)_{H_2} \le \epsilon_3, \tag{25}$$

where $\tilde Q_1$ and $\tilde Q_2$ are bounded positive-definite selfadjoint operators in H_1 and H_2, respectively, for which there exist the inverse bounded operators $\tilde Q_1^{-1}$ and $\tilde Q_2^{-1}$. We note that the random variables $\xi_1 \in H_1$ and $\xi_2 \in H_2$ are called *uncorrelated* if

$$\mathbb{E}(\xi_1,u_1)_{H_1}(\xi_2,u_2)_{H_2} = 0 \quad \forall u_1 \in H_1, u_2 \in H_2 \tag{26}$$

(see, for example, [18], p. 146). It is known that the operators $\tilde Q_1^{-1}$ and $\tilde Q_2^{-1}$ are positive definite and selfadjoint, that is, there exists a positive number α such that

$$(\tilde Q_1^{-1}u_1,u_1)_{H_1} \ge \alpha\|u_1\|^2_{H_1}\ \forall u_1 \in H_1,\ (\tilde Q_2^{-1}u_2,u_2)_{H_2} \ge \alpha\|u_2\|^2_{H_2}\ \forall u_2 \in H_2. \tag{27}$$

Set $u := (u_1,u_2) \in H := H_1 \times H_2$.

Definition 2.1. An estimate

$$\widehat{\widehat{l(\mathbf{j},\varphi)}} = (y_1,\hat u_1)_{H_1} + (y_2,\hat u_2)_{H_2} + \hat c$$

is called a *guaranteed* (or *minimax*) *estimate of* $l(\mathbf{j},\varphi)$, if elements $\hat u_1 \in H_1$, $\hat u_2 \in H_2$ and a number $\hat c$ are determined from the condition

$$\inf_{u\in H,\, c\in\mathbb{R}} \sigma(u,c) = \sigma(\hat u,\hat c),$$

where

$$\sigma(u,c) := \sup_{\tilde f\in G_0,(\tilde\eta_1,\tilde\eta_2)\in G_1} \mathbb{E}|l(\tilde{\mathbf{j}},\tilde\varphi) - \widehat{l(\tilde{\mathbf{j}},\tilde\varphi)}|^2,$$

and $(\tilde{\mathbf{j}},\tilde\varphi)$ is a solution of problem (10)–(11) at $f(x) = \tilde f(x)$,

$$\widehat{l(\tilde{\mathbf{j}},\tilde\varphi)} := (\tilde y_1,u_1)_{H_1} + (\tilde y_2,u_2)_{H_2} + c, \quad \tilde y_1 = C_1\tilde{\mathbf{j}} + \tilde\eta_1, \quad \tilde y_2 = C_2\tilde\varphi + \tilde\eta_2\,.$$

The quantity

$$\sigma := [\sigma(\hat u,\hat c)]^{1/2} \tag{28}$$

is called the *error of the guaranteed estimation of* $l(\mathbf{j},\varphi)$.

Thus, the guaranteed estimate is an estimate minimizing the maximal mean-square estimation error calculated for the "worst" implementation of perturbations. Further, without loss of generality, we may set $\epsilon_k = 1$, $k = 1, 2, 3$, in (20) and (25).

3. Reduction of the estimation problem to the optimal control problem of a system governed by mixed variational equations

Introduce a pair of functions $(\mathbf{z}_1(\cdot; u), z_2(\cdot; u)) \in H(\mathrm{div}; D) \times L^2(D)$ as a solution of the problem:

$$\int_D (((\mathbf{A}(x))^{-1})^T \mathbf{z}_1(x; u), \mathbf{q}(x))_{\mathbb{R}^n} dx - \int_D z_2(x; u) \mathrm{div}\, \mathbf{q}(x)\, dx$$

$$= \int_D (\mathbf{l}_1(x) - (C_1^t J_{H_1} u_1)(x), \mathbf{q}(x))_{\mathbb{R}^n} dx \quad \forall \mathbf{q} \in H(\mathrm{div}, D), \quad (29)$$

$$-\int_D v(x) \mathrm{div}\, \mathbf{z}_1(x; u)\, dx - \int_D c(x) z_2(\cdot; u) v(x)\, dx$$

$$= \int_D (l_2(x) - (C_2^t J_{H_2} u_2)(x)) v(x)\, dx \quad \forall v \in L^2(D), \quad (30)$$

where $u \in H$, $C_1^t : H_1' \to L^2(D)^n$ and $C_2^t : H_2' \to L^2(D)$ are the transpose operators of C_1 and C_2, respectively, defined by

$$\int_D (v(x), C_1^t w(x))_{\mathbb{R}^n} dx = < Cv, w >_{H_1 \times H_1'} \quad \forall v \in L^2(D)^n,\ w \in H_1', \text{ and}$$

$$\int_D v(x) C_2^t w(x)\, dx = < Cv, w >_{H_2 \times H_2'} \quad \forall v \in L^2(D),\ w \in H_2'.$$

From the theory of mixed variational problems it is known that the pair $(\mathbf{z}_1(x; u)$, $z_2(x; u))$ is uniquely determined. [1]

Lemma 3.1. *The problem of guaranteed estimation of the functional $l(\mathbf{j}, \varphi)$ (i.e. the determination of $\hat{u} = (\hat{u}_1, \hat{u}_2)$ and $\hat{c}$) is equivalent to the problem of optimal*

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[1] In fact, note that problem (29)–(30) can be rewritten in the form

$$a^*(\mathbf{z}_1, \mathbf{q}) + b(\mathbf{q}, z_2) = l_1(\mathbf{q}) \quad \forall \mathbf{q} \in H(\mathrm{div}; \mathrm{D}),$$

$$b(\mathbf{z}_1; v) - c(z_2; v) = l_2(v) \quad \forall v \in L^2(D),$$

where $a^*(\mathbf{z}_1, \mathbf{q}) = a(\mathbf{q}, \mathbf{z}_1)$, the bilinear forms a, b, and c, are defined by (14), (15), and (16), respectively, $l_1(\mathbf{q}) = (\mathbf{l}_1 - C_1^t J_{H_1} u_1, \mathbf{q})_{L^2(D)^n}$, $l_2(v) = (l_2 - C_2^t J_{H_2} u_2, v)_{L^2(D)}$. Since $a^*(\mathbf{q}, \mathbf{q}) = a(\mathbf{q}, \mathbf{q})$ then the bilinear form $a^*(\mathbf{z}_1, \mathbf{q})$ is also coercive on $\mathrm{Ker\,B}$ and, hence, by Theorem 1.2 from [2] problem (29), (30) is uniquely solvable. Moreover we have:

$$\|\mathbf{z}_1\|_{H(\mathrm{div}; D)} + \|z_2\|_{L^2(D)} \le C(\|l_1\|_{H(\mathrm{div}; D)'} + \|l_2\|_{L^2(D)}) \le$$

$$\le C(\mathbf{l}_1 - C_1^t J_{H_1} u_1)_{L^2(D)^n} + (l_2 - C_2^t J_{H_2} u_2)_{L^2(D)}), \quad (31)$$

where C =const.

control of the system described by the mixed variational problem (29)–(30) with the cost function

$$I(u) = (Q^{-1}z_2(\cdot;u), z_2(\cdot;u))_{L^2(D)} + (\tilde{Q}_1^{-1}u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2} \to \inf_{u\in H}. \quad (32)$$

Proof. From the relations (21)–(23) at $\mathbf{j} = \tilde{\mathbf{j}}$, $\varphi = \tilde{\varphi}$, $\eta_1 = \tilde{\eta}_1$, $\eta_2 = \tilde{\eta}_2$, we have

$$l(\tilde{\mathbf{j}}, \tilde{\varphi}) - \widehat{l(\tilde{\mathbf{j}}, \tilde{\varphi})} =$$

$$= (\mathbf{l}_1, \tilde{\mathbf{j}})_{L^2(D)^n} + (l_2, \tilde{\varphi})_{L^2(D)} - (\tilde{y}_1, u_1)_{H_1} - (\tilde{y}_2, u_2)_{H_2} - c =$$

$$= (\mathbf{l}_1, \tilde{\mathbf{j}})_{L^2(D)^n} + (l_2, \tilde{\varphi})_{L^2(D)} - (u_1, C_1\tilde{\mathbf{j}} + \tilde{\eta}_1)_{H_1} - (u_2, C_2\tilde{\varphi} + \tilde{\eta}_2)_{H_2} - c =$$

$$= (\mathbf{l}_1, \tilde{\mathbf{j}})_{L^2(D)^n} + (l_2, \tilde{\varphi})_{L^2(D)} - < J_{H_1}u_1, C_1\tilde{\mathbf{j}} >_{H_1'\times H_1} -$$
$$- < J_{H_2}u_2, C_2\tilde{\varphi} >_{H_2'\times H_2} - (u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c =$$

$$= (\mathbf{l}_1, \tilde{\mathbf{j}})_{L^2(D)^n} + (l_2, \tilde{\varphi})_{L^2(D)} - (C_1^t J_{H_1}u_1, \tilde{\mathbf{j}})_{L^2(D)^n} -$$
$$- (C_2^t J_{H_2}u_2, \tilde{\varphi})_{L^2(D)} - (u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c =$$

$$= (\mathbf{l}_1 - C_1^t J_{H_1}u_1, \tilde{\mathbf{j}})_{L^2(D)^n} + (l_2 - C_2^t J_{H_2}u_2, \tilde{\varphi})_{L^2(D)} - (u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c. \quad (33)$$

Further, taking into account the system of variational equations

$$\int_D ((\mathbf{A}(x))^{-1}\tilde{\mathbf{j}}(x), \mathbf{q}(x))_{\mathbb{R}^n} dx - \int_D \tilde{\varphi}(x) \operatorname{div} \mathbf{q}(x)\, dx = 0 \quad \forall \mathbf{q} \in H(\mathrm{div}; D), \quad (34)$$

$$\int_D v(x) \operatorname{div} \tilde{\mathbf{j}}(x)\, dx + \int_D c(x)\tilde{\varphi}(x)v(x)\, dx = \int_D \tilde{f}(x)v(x)\, dx \quad \forall v \in L^2(D), \quad (35)$$

which follows from (12)–(13) if we set $\mathbf{f} = \tilde{\mathbf{f}}$, and (29)–(30), we transform the third and fourth summands in (33). By setting $\mathbf{q} = \tilde{\mathbf{j}}$ in (29) and $v = \tilde{\varphi}$ in (30), we receive

$$\int_D ((\mathbf{A}(x))^{-1}\tilde{\mathbf{j}}(x), \mathbf{z}_1(x;u))_{\mathbb{R}^n} dx - \int_D z_2(x;u) \operatorname{div} \tilde{\mathbf{j}}(x)\, dx =$$
$$= \int_D (\mathbf{l}_1(x) - (C_1^t J_{H_1}u_1)(x), \tilde{\mathbf{j}}(x))_{\mathbb{R}^n} dx, \quad (36)$$

$$- \int_D \tilde{\varphi}(x) \operatorname{div} \mathbf{z}_1(x;u)\, dx - \int_D c(x) z_2(x;u)\tilde{\varphi}(x)\, dx =$$
$$= \int_D (l_2(x) - (C_2^t J_{H_2}u_2)(x))\tilde{\varphi}(x)\, dx. \quad (37)$$

On the other hand, putting $\mathbf{q} = \mathbf{z}_1(\cdot;u)$ in (34) and $v = z_2(\cdot;u)$ in (35), we find

$$\int_D ((\mathbf{A}(x))^{-1}\tilde{\mathbf{j}}(x), \mathbf{z}_1(x;u))_{\mathbb{R}^n} dx - \int_D \tilde{\varphi}(x) \operatorname{div} \mathbf{z}_1(x;u)\, dx = 0, \quad (38)$$

$$\int_D z_2(x;u)\,\mathrm{div}\,\tilde{\mathbf{j}}(x)\,dx+\int_D c(x)\tilde{\varphi}(x)z_2(x;u)\,dx=\int_D \tilde{f}(x)z_2(x;u)\,dx. \tag{39}$$

From (36)–(39), we get

$$(\mathbf{l}_1-C_1^tJ_{H_1}u_1,\tilde{\mathbf{j}})_{L^2(D)^n}+(l_2-C_2^tJ_{H_2}u_2,\tilde{\varphi})_{L^2(D)}=$$

$$=\int_D((\mathbf{A}(x))^{-1}\tilde{\mathbf{j}}(x),\mathbf{z}_1(x;u))_{\mathbb{R}^n}dx-\int_D z_2(x;u)\,\mathrm{div}\,\tilde{\mathbf{j}}(x)\,dx-$$

$$-\int_D \tilde{\varphi}(x)\,\mathrm{div}\,\mathbf{z}_1(x;u)\,dx-\int_D c(x)z_2(x;u)\tilde{\varphi}(x)\,dx=$$

$$=\int_D((\mathbf{A}(x))^{-1}\tilde{\mathbf{j}}(x),\mathbf{z}_1(x;u)_{\mathbb{R}^n}dx-\int_D \tilde{\varphi}(x)\,\mathrm{div}\,\mathbf{z}_1(x;u)\,dx-$$

$$-\int_D z_2(x;u)\,\mathrm{div}\,\tilde{\mathbf{j}}(x)\,dx-\int_D c(x)z_2(x;u)\tilde{\varphi}(x)\,dx=$$

$$=0-(\tilde{f},z_2(\cdot;u)_{L^2(D)}=-(\tilde{f},z_2(\cdot;u)_{L^2(D)}. \tag{40}$$

Equalities (40) and (33) imply

$$l(\tilde{\mathbf{j}},\tilde{\varphi})-\widehat{l(\tilde{\mathbf{j}},\tilde{\varphi})}=-(\tilde{f},z_2(\cdot\,;u)_{L^2(D)}-(u_1,\tilde{\eta}_1)_{H_1}-(u_2,\tilde{\eta}_2)_{H_2}-c=$$

$$=-(\tilde{f}-f_0,z_2(\cdot\,;u))_{L^2(D)}-(f_0,z_2(\cdot\,;u))_{L^2(D)}-$$

$$-(u_1,\tilde{\eta}_1)_{H_1}-(u_2,\tilde{\eta}_2)_{H_2}-c=:\xi, \tag{41}$$

where by ξ we denote the random variable defined by the right-hand side of the latter equality. It is obvious that

$$\mathbb{E}\xi=-(\tilde{f}-f_0,z_2(\cdot\,;u))_{L^2(D)}-(f_0,z_2(\cdot\,;u))_{L^2(D)}-c\,,$$

$$\xi-\mathbb{E}\xi=-(u_1,\tilde{\eta}_1)_{H_1}-(u_2,\tilde{\eta}_2)_{H_2}\,.$$

Taking into consideration the relationship

$$\mathbf{D}\xi=\mathbb{E}(\xi-\mathbb{E}\xi)^2=\mathbb{E}\xi^2-(\mathbb{E}\xi)^2 \tag{42}$$

that couples the dispersion $\mathbf{D}\xi$ of the random variable ξ and its expectation $\mathbb{E}\xi$, we obtain from (41)

$$\mathbb{E}\left|l(\tilde{\mathbf{j}},\tilde{\varphi})-(\widehat{l(\tilde{\mathbf{j}},\tilde{\varphi})}\right|^2=$$

$$=\left|(\tilde{f}_2-f_0,z_2(\cdot;u))_{L^2(D)}+(f_0,z_2(\cdot;u))_{L^2(D)}+c\right|^2+\mathbb{E}[(u_1,\tilde{\eta}_1)_{H_1}+(u_2,\tilde{\eta}_2)_{H_2}]^2,$$

whence we get

$$\inf_{c\in\mathbb{R}}\sup_{\tilde{f}\in G_0,(\tilde{\eta}_1,\tilde{\eta})\in G_1}\mathbb{E}|l(\tilde{\mathbf{j}},\tilde{\varphi})-\widehat{l(\tilde{\mathbf{j}},\tilde{\varphi})}|^2=$$

$$=\inf_{c\in\mathbb{R}}\sup_{\tilde{f}\in G_0}\Big[(\tilde{f}-f_0,z_2(\cdot;u))_{L^2(D)}\;+(f_0,z_2(\cdot;u))_{L^2(D)}+c\Big]^2+$$

$$+\sup_{(\tilde\eta_1,\tilde\eta_2)\in G_1}\mathbb{E}[(\tilde\eta_1,u_1)_{H_1}+(\tilde\eta_2,u_2)_{H_2}]^2=$$

$$=\sup_{\tilde f\in G_0}\left[(\tilde f_2-f_2^{(0)},z_2(\cdot;u))_{L^2(D)}\right]^2+\sup_{(\tilde\eta_1,\tilde\eta_2)\in G_1}\mathbb{E}[(\tilde\eta_1,u_1)_{H_1}+(\tilde\eta_2,u_2)_{H_2}]^2, \qquad (43)$$

with

$$c=-(f_0,z_2(\cdot;u))_{L^2(D)}.$$

In order to calculate the first term on the right-hand side of (43) we make use of the Cauchy-Bunyakovsky inequality (see [6], p. 186) and (20). We have

$$|(\tilde f-f_0,z_2(\cdot;u))_{L^2(D)}|^2\le$$

$$\le(Q^{-1}z_2(\cdot;u),z_2(\cdot;u))_{L^2(D)}(Q(\tilde f-f_0),\tilde f-f_0)_{L^2(D)}\le(Q^{-1}z_2(\cdot;u),z_2(\cdot;u))_{L^2(D)}.$$

The direct substitution shows that last inequality is transformed to an equality on the element

$$\tilde f=f_0+\frac{Q^{-1}z_2(\cdot;u)}{(Q^{-1}z_2(\cdot;u),z(\cdot;u))_{L^2(D)}^{1/2}}\,.$$

Hence,

$$\sup_{\tilde f\in G_0}\left[(\tilde f_2-f_2^{(0)},z_2(\cdot;u))_{L^2(D)}\right]^2=(Q^{-1}z_2(\cdot;u),z_2(\cdot;u))_{L^2(D)}\,. \qquad (44)$$

In order to calculate the second term on the right-hand side of (43), note that the Cauchy−Bunyakovsky inequality, (25), (6), and (26) yields

$$\sup_{(\tilde\eta_1,\tilde\eta_2)\in G_1}\mathbb{E}[(\tilde\eta_1,u_1)_{H_1}+(\tilde\eta_2,u_2)_{H_2}]^2\le$$

$$\le\sup_{(\tilde\eta_1,\tilde\eta_2)\in G_1}[(\tilde Q_1^{-1}u_1,u_1)_{H_1}\mathbb{E}(\tilde Q_1\tilde\eta_1,\tilde\eta_1)_{H_1}+(\tilde Q_2^{-1}u_2,u_2)_{H_2}\mathbb{E}(\tilde Q_2\tilde\eta_2,\tilde\eta_2)_{H_2}]\le$$

$$\le(\tilde Q_1^{-1}u_1,u_1)_{H_1}+(\tilde Q_2^{-1}u_2,u_2)_{H_2}\,. \qquad (45)$$

It is easy to see that (45) takes the form (44) at

$$\tilde\eta_1=\nu_1\tilde Q_1^{-1}u_1/(\tilde Q_1^{-1}u_1,u_1)^{1/2},\quad \tilde\eta_2=\nu_2\tilde Q_2^{-1}u_1/(\tilde Q_2^{-1}u_1,u_1)^{1/2},$$

where ν_1 and ν_2 are uncorrelated random variables with $\mathbb{E}\nu_1=\mathbb{E}\nu_2=0$ and $\mathbb{E}\nu_1^2=\mathbb{E}\nu_2^2=1$. Therefore,

$$\sup_{(\tilde\eta_1,\tilde\eta_2)\in G_1}\mathbb{E}[(\tilde\eta_1,u_1)_{H_1}+(\tilde\eta_2,u_2)_{H_2}]^2=(\tilde Q_1^{-1}u_1,u_1)_{H_1}+(\tilde Q_2^{-1}u_2,u_2)_{H_2}. \qquad (46)$$

From (46), (44), and (43), we find

$$\inf_{c\in\mathbb{R}}\sup_{\tilde f\in G_0,(\tilde\eta_1,\tilde\eta)\in G_1}\mathbb{E}|l(\tilde{\mathbf{j}},\tilde\varphi)-\widehat{l(\tilde{\mathbf{j}},\tilde\varphi)}|^2=I(u),$$

at $c=-(z_2(\cdot;u),f_0)_{L^2(D)}$, where $I(u)$ is determined by (32). This proves the required assertion. □

4. Representation for guaranteed estimates and errors of estimation via solutions of mixed variational equations

Solving the optimal control problem (29)–(32), we come to the following result.

Theorem 4.1. *There exists a unique guaranteed eatimate of* $l(\mathbf{j},\varphi)$ *which has the form*

$$\widehat{\widehat{l(\mathbf{j},\varphi)}} = (y_1,\hat{u}_1)_{H_1} + (y_2,\hat{u}_2)_{H_2} + \hat{c}, \tag{47}$$

where

$$\hat{c} = -\int_D \hat{z}_2(x) f_0(x)\,dx, \quad \hat{u}_1 = \tilde{Q}_1 C_1 \mathbf{p}_1, \quad \hat{u}_2 = \tilde{Q}_2 C_2 p_2, \tag{48}$$

and the functions $\mathbf{p}_1 \in H(\mathrm{div}, D)$ *and* $\hat{z}_2, p_2 \in L^2(D)$ *are determined as a solution of the following uniquely solvable problem:*

$$\int_D (((\mathbf{A}(x))^{-1})^T\,\hat{\mathbf{z}}_1(x), \mathbf{q}_1(x))_{\mathbb{R}^n} dx - \int_D \hat{z}_2(x)\,\mathrm{div}\,\mathbf{q}_1(x)\,dx =$$
$$= \int_D (\mathbf{l}_1(x) - C_1^t J_{H_1}\tilde{Q}_1 C_1 \mathbf{p}_1(x), \mathbf{q}_1(x))_{\mathbb{R}^n}\,dx \quad \forall \mathbf{q}_1 \in H(\mathrm{div}, D), \tag{49}$$

$$-\int_D v_1(x)\,\mathrm{div}\,\hat{\mathbf{z}}_1(x)\,dx - \int_D c(x)\hat{z}_2(x) v_1(x)\,dx =$$
$$= \int_D (l_2(x) - C_2^t J_{H_2}\tilde{Q}_2 C_2 p_2(x)) v_1(x)\,dx \quad \forall v_1 \in L^2(D), \tag{50}$$

$$\int_D ((\mathbf{A}(x))^{-1}\mathbf{p}_1(x), \mathbf{q}_2(x))_{\mathbb{R}^n} dx - \int_D p_2(x)\mathrm{div}\,\mathbf{q}_2(x)\,dx = 0 \quad \forall \mathbf{q}_2 \in H(\mathrm{div}, D), \tag{51}$$

$$-\int_D v_2(x)\,\mathrm{div}\,\mathbf{p}_1(x)\,dx - \int_D c(x) p_2(x) v_2(x) dx =$$
$$= \int_D v_2(x) Q^{-1}\hat{z}_2(x)\,dx \quad \forall v_2 \in L^2(D), \tag{52}$$

where $\hat{\mathbf{z}}_1 \in H(\mathrm{div}, D)$*. The error of estimation* σ *is given by an expression*

$$\sigma = l(\mathbf{p}_1, p_2)^{1/2}. \tag{53}$$

Proof. We prove first that the solution to the optimal control problem (29)–(32) can be reduced to the solution of system (49)–(52). Note first that the functional $I(u)$, where $u \in H$, can be represented in the form

$$I(u) = \tilde{I}(u) + L(u) + \int_D Q^{-1}\tilde{z}_2^{(0)}(x)\tilde{z}_2^{(0)}(x)\,dx,$$

where

$$\tilde{I}(u) = \int_D Q^{-1}\tilde{z}_2(x;u)\tilde{z}_2(x;u)\,dx + (\tilde{Q}_1^{-1}u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2},$$

$$L(u) = 2\int_D Q^{-1}\tilde{z}_2(x;u)\tilde{z}_2^{(0)}(x)\,dx,$$

$\tilde z_2(x;u)$ is the second component of the pair $(\tilde{\mathbf{z}}_1(x;u), \tilde z_2(x;u))$ which is the unique solution to problem (29)–(30) at $\mathbf{l}_0^{(1)}(x)=0$, $l_0^{(2)}(x)=0$, and $\tilde z_2^{(0)}(x)$ is the second component of the pair $(\tilde{\mathbf{z}}_1^{(0)}(x), \tilde z_2^{(0)}(x))$ which is the unique solution to the same problem at $u=0$. We show that $\tilde I(u)$ is a quadratic form corresponding to a symmetric continuous bilinear form

$$\pi(u,v) := \int_D Q^{-1}\tilde z_2(x;u)\tilde z_2(x;v)\,dx + (\tilde Q_1^{-1}u_1, v_1)_{H_1} + (\tilde Q_2^{-1}u_2, v_2)_{H_2} \tag{54}$$

on $H\times H$ and $L(u)$ is a linear continuous functional defined on H. The continuity of the form $\pi(u,v)$ on $H\times H$ means that for all $u,v\in H$ the inequality

$$|\pi(u,v)| \le C\|u\|_H\|v\|_H \tag{55}$$

must be valid, where $C=\text{const}$. To prove (55), we use the estimate

$$\int_D \tilde z_2^2(x;u)\,dx \le c_1\Big(\big\|C_1^t J_{H_1}u_1\big\|^2_{L^2(D)^n} + \big\|C_2^t J_{H_2}u_2\big\|^2_{L^2(D)}\Big), \quad c_1=\text{const.}, \tag{56}$$

which follows from the inequality (31) at $\mathbf{l}_1=0$ and $l_2=0$. For the first term in the right-hand side of (54), due to the Cauchy–Bunyakovsky inequality and (56) we have

$$\left|\int_D Q^{-1}\tilde z_2(x;u)\tilde z_2(x;v)\,dx\right| \le c_2\left(\int_D \tilde z_2^2(x;u)\,dx\right)^{1/2}\left(\int_D \tilde z_2^2(x;v)\,dx\right)^{1/2} \le$$

$$\le c_2c_3\Big(\big\|C_1^t J_{H_1}u_1\big\|^2_{L^2(D)^n} + \big\|C_2^t J_{H_2}u_2\big\|^2_{L^2(D)}\Big)^{1/2}\times$$

$$\times\, c_3\Big(\big\|C_1^t J_{H_1}v_1\big\|^2_{L^2(D)^n} + \big\|C_2^t J_{H_2}v_2\big\|^2_{L^2(D)}\Big)^{1/2} \le$$

$$\le c_4\Big(\|u_1\|^2_{H_1} + \|u_2\|^2_{H_2}\Big)^{1/2}\Big(\|v_1\|^2_{H_1} + \|v_2\|^2_{H_2}\Big)^{1/2} = c_4\|u\|_H\|v\|_H, \tag{57}$$

where $c_2, c_3, c_4 = \text{const}$. Analogously,

$$(\tilde Q_1^{-1}u_1, v_1)_{H_1} + (\tilde Q_2^{-1}u_2, v_2)_{H_2} \le c_5\|u\|_H\|v\|_H, \quad c_5=\text{const}.$$

From this estimate and (57) the validity of inequality (55) follows.

The continuity of the linear functional $L(u)$ on H can be proved similarly. It is obvious that

$$\tilde I(u) = \pi(u,u) \ge (Q_1^{-1}u_1, u_1)_{H_1} + (\tilde Q_2^{-1}u_2, u_2)_{H_2} \ge \alpha\|u\|_H^2 \quad \forall u\in H,$$

where α is a constant from (27). In line with Theorem 1.1 proved in [9], p. 11, the latter statements imply the existence of the unique element $\hat u := (\hat u_1, \hat u_2)\in H$ such that

$$I(\hat u) = \inf_{u\in H} I(u).$$

Therefore, for any fixed $w \in H$ and $\tau \in \mathbb{R}$ the function $s(\tau) := I(\hat{u} + \tau w)$ reaches its minimum at a unique point $\tau = 0$, so that,

$$\frac{d}{d\tau} I(\hat{u} + \tau w) \,|_{\tau=0} = 0\,. \tag{58}$$

Since $z_2(x; \hat{u} + \tau w) = z_2(x; \hat{u}) + \tau \tilde{z}_2(x; w)$, relation (58) yields

$$\frac{1}{2}\frac{d}{dt} I(\hat{u} + \tau w)\Big|_{\tau=0} =$$

$$= (Q^{-1} z_2(\cdot; \hat{u}), \tilde{z}_2(\cdot; w))_{L^2(D)} + (\tilde{Q}_1^{-1} \hat{u}_1, w_1)_{H_1} + (\tilde{Q}_2^{-1} \hat{u}_2, w_2)_{H_2} = 0. \tag{59}$$

We introduce a pair of functions $(\mathbf{p}_1, p_2) \in H(\mathrm{div}, D) \times L^2(D)$ as the unique solution of the problem

$$\int_D ((\mathbf{A}(x))^{-1} \mathbf{p}_1(x), \mathbf{q}_2(x))_{\mathbb{R}^n} dx - \int_D p_2(x) \mathrm{div}\, \mathbf{q}_2(x)\, dx = 0 \tag{60}$$

for all $\mathbf{q}_2 \in H(\mathrm{div}, D)$; and for all $v_2 \in L^2(D)$

$$-\int_D v_2(x) \mathrm{div}\, \mathbf{p}_1(x)\, dx - \int_D c(x) p_2(x) v_2(x) dx = \int_D v_2(x) Q^{-1} z_2(x; \hat{u})\, dx\,. \tag{61}$$

Setting in (60) $\mathbf{q}_2 = \tilde{\mathbf{z}}_1(\cdot\,; w)$ and in (61) $v_2 = \tilde{z}_2(\cdot\,; w)$, we obtain

$$\int_D ((\mathbf{A}(x))^{-1} \mathbf{p}_1(x), \tilde{\mathbf{z}}_1(x; w))_{\mathbb{R}^n} dx - \int_D p_2(x) \mathrm{div}\, \tilde{\mathbf{z}}_1(x; w)\, dx = 0, \text{ and} \tag{62}$$

$$-\int_D \tilde{z}_2(x; w) \mathrm{div}\, \mathbf{p}_1(x)\, dx - \int_D c(x) p_2(x) \tilde{z}_2(x; w) dx = \int_D v_2(x) Q^{-1} z_2(x; \hat{u})\, dx\,. \tag{63}$$

From (62) and (63), we find

$$(Q^{-1} z_2(\cdot; \hat{u}), \tilde{z}_2(\cdot; w))_{L^2(D)} =$$

$$= -\int_D \tilde{z}_2(x; w) \mathrm{div}\, \mathbf{p}_1(x)\, dx - \int_D c(x) p_2(x) \tilde{z}_2(x; w) dx +$$

$$+ \int_D ((\mathbf{A}(x))^{-1} \mathbf{p}_1(x), \tilde{\mathbf{z}}_1(x; w))_{\mathbb{R}^n} dx - \int_D p_2(x)\, \mathrm{div}\, \tilde{\mathbf{z}}_1(x; w)\, dx =$$

$$= \int_D (((\mathbf{A}(x))^{-1})^T \tilde{\mathbf{z}}_1(x; w), \mathbf{p}_1(x))_{\mathbb{R}^n} dx - \int_D \tilde{z}_2(x; w)\, \mathrm{div}\, \mathbf{p}_1(x)\, dx -$$

$$- \int_D p_2(x) \mathrm{div}\, \tilde{\mathbf{z}}_1(x; w)\, dx - \int_D c(x) p_2(x) \tilde{z}_2(x; w) dx =$$

$$= -\int_D (C_1^t J_{H_1} w_1, \mathbf{p}_1(x))_{\mathbb{R}^n} dx - \int_D (C_2^t J_{H_2} w_2) p_2\, dx =$$

$$= -(w_1, C_1 \mathbf{p}_1)_{H_1} - (w_2, C_2 p_2)_{H_2}.$$

The last relation and (59) imply

$$(w_1, C_1\mathbf{p}_1)_{H_1} + (w_2, C_2p_2)_{H_2} = (\tilde{Q}_1^{-1}\hat{u}_1, w_1)_{H_1} + (\tilde{Q}_2^{-1}\hat{u}_2, w_2)_{H_2}.$$

Hence,

$$\hat{u}_1 = \tilde{Q}_1C_1\mathbf{p}_1, \quad \hat{u}_2 = \tilde{Q}_2C_2p_2\,. \tag{64}$$

Substituting these expressions into (29)–(30) and denoting $\mathbf{z}_1(x;\hat{u}) =: \hat{\mathbf{z}}_1(x)$, $z_2(x;\hat{u}) =: \hat{z}_2(x)$, we establish that the functions $\hat{\mathbf{z}}_1$, $\hat{z}_2$, $\mathbf{p}_1$, and p_2 satisfy (49)–(52); the unique solvability of the problem (49)–(52) follows from the existence of the unique minimum point $\hat{u}$ of the functional $I(u)$.

Now let us establish the validity of formula (53). From (32) at $u = \hat{u}$ and (64), it follows

$$\sigma^2 = I(\hat{u}) = (Q^{-1}z_2(\cdot;\hat{u}), z_2(\cdot;\hat{u}))_{L^2(D)} + (\tilde{Q}_1^{-1}\hat{u}_1, \hat{u}_1)_{H_1} + (\tilde{Q}_2^{-1}\hat{u}_2, \hat{u}_2)_{H_2} =$$

$$= (Q^{-1}\hat{z}_2, \hat{z}_2)_{L^2(D)} + (C_1\mathbf{p}_1, \tilde{Q}_1C_1\mathbf{p}_1)_{H_1} + (C_2p_2, \tilde{Q}_2C_2p_2)_{H_2}. \tag{65}$$

Transform the first term in (65). Setting in (60) and (61) $\mathbf{q}_2 = \hat{\mathbf{z}}_1$ and $v_2 = \hat{z}_2$, we find

$$\int_D ((\mathbf{A}(x))^{-1}\mathbf{p}_1(x), \hat{\mathbf{z}}_1(x))_{\mathbb{R}^n} dx - \int_D p_2(x)\mathrm{div}\,\hat{\mathbf{z}}_1(x)\,dx = 0, \text{ and}$$

$$-\int_D \hat{z}_2(x)\mathrm{div}\,\mathbf{p}_1(x)\,dx - \int_D c(x)p_2(x)\hat{z}_2(x)\,dx = \int_D \hat{z}_2(x)Q^{-1}\hat{z}_2(x)\,dx\,.$$

From the latter relations and from equations (49) and (50) with $\mathbf{q}_1 = \mathbf{p}_1$ and $v_1 = p_2$, we have

$$(Q^{-1}\hat{z}_2, \hat{z}_2)_{L^2(D)} =$$

$$= -\int_D \hat{z}_2(x)\mathrm{div}\,\mathbf{p}_1(x)\,dx - \int_D c(x)p_2(x)\hat{z}_2(x)\,dx +$$

$$+ \int_D ((\mathbf{A}(x))^{-1}\mathbf{p}_1(x), \hat{\mathbf{z}}_1(x))_{\mathbb{R}^n} dx - \int_D p_2(x)\,\mathrm{div}\,\hat{\mathbf{z}}_1(x)\,dx =$$

$$= \int_D (((\mathbf{A}(x))^{-1})^T\,\hat{\mathbf{z}}_1(x), \mathbf{p}_1(x))_{\mathbb{R}^n} dx - \int_D \hat{z}_2(x)\,\mathrm{div}\,\mathbf{p}_1(x)\,dx -$$

$$- \int_D p_2(x)\,\mathrm{div}\,\hat{\mathbf{z}}_1(x)\,dx - \int_D c(x)p_2(x)\hat{z}_2(x)\,dx =$$

$$= \int_D (\mathbf{l}_1(x) - C_1^tJ_{H_1}\tilde{Q}_1C_1\mathbf{p}_1(x), \mathbf{p}_1(x))_{\mathbb{R}^n}\,dx + \int_D (l_2(x) - C_2^tJ_{H_2}\tilde{Q}_2C_2p_2(x))p_2(x)\,dx$$

$$= \int_D (\mathbf{l}_1(x), \mathbf{p}_1(x))_{\mathbb{R}^n}\,dx + \int_D l_2(x)p_2(x)\,dx -$$

$$- (C_1\mathbf{p}_1, \tilde{Q}_1C_1\mathbf{p}_1)_{H_1} - (C_2p_2, \tilde{Q}_2C_2p_2)_{H_2}. \tag{66}$$

From (65) and (66), we obtain (53), and the Theorem is proved. □

Note that the pair of functions $(\hat{\mathbf{z}}(x), \hat{z}_2(x)) = (\mathbf{z}_1(x;\hat{u}), z_2(x;\hat{u}))$ and the element $u = \hat{u} \in H$ is a solution of optimal control problem (29), (30), (32).

In the following theorem we obtain an alternative representation for the guaranteed estimate of the quantity $l(\mathbf{j}, \varphi)$ which is expressed via a solution of certain system of mixed variational equations not depending on $\mathbf{l}_1$ and l_2.

Theorem 4.2. *The guaranteed estimate of* $l(\mathbf{j}, \varphi)$ *has the form*

$$\widehat{\widehat{l(\mathbf{j}, \varphi)}} = l(\hat{\mathbf{j}}, \hat{\varphi}), \tag{67}$$

where the pair $(\hat{\mathbf{j}}, \hat{\varphi}) \in H(\mathrm{div}, D) \times L^2(D)$ *is a solution to the following problem:*

$$\int_D (((\mathbf{A}(x))^{-1})^T \hat{\mathbf{p}}_1(x), \mathbf{q}_1(x))_{\mathbb{R}^n} dx - \int_D \hat{p}_2(x) \operatorname{div} \mathbf{q}_1(x)\, dx =$$
$$= \int_D (C_1^t J_{H_1} \tilde{Q}_1(y_1 - C_1\hat{\mathbf{j}})(x), \mathbf{q}_1(x))_{\mathbb{R}^n} dx \quad \forall \mathbf{q}_1 \in H(\mathrm{div}, D), \tag{68}$$

$$-\int_D v_1(x)\ \operatorname{div} \hat{\mathbf{p}}_1(x)\, dx - \int_D c(x)\hat{p}_2(x) v_1(x)\, dx =$$
$$= \int_D C_2^t J_{H_2} \tilde{Q}_2(y_2 - C_2\hat{\varphi})(x) v_1(x)\, dx \quad \forall v_1 \in L^2(D), \tag{69}$$

$$\int_D ((\mathbf{A}(x))^{-1} \hat{\mathbf{j}}(x), \mathbf{q}_2(x))_{\mathbb{R}^n}\, dx - \int_D \hat{\varphi}(x) \operatorname{div} \mathbf{q}_2(x)\, dx = 0 \quad \forall \mathbf{q}_2 \in H(\mathrm{div}, D), \tag{70}$$

$$-\int_D v_2(x) \operatorname{div} \hat{\mathbf{j}}(x)\, dx - \int_D c(x)\hat{\varphi}(x) v_2(x)\, dx =$$
$$= \int_D v_2(x)(Q^{-1}\hat{p}_2(x) - f_0(x))\, dx \quad \forall v_2 \in L^2(D), \tag{71}$$

where the equalities (68)–(71) *are fulfilled with probability* 1. *Problem* (68)–(71) *is uniquely solvable. The random fields* $\hat{\mathbf{j}}$, $\hat{\mathbf{p}}_1$ *and* $\hat{\varphi}$, $\hat{p}_2$, *whose realizations satisfy problem* (68)–(71), *belong to the spaces* $L^2(\Omega, H(\mathrm{div}, D))$ *and* $L^2(\Omega, L^2(D))$, *respectively.*

Proof. Note that the unique solvability of problem (68)–(71) at the realizations y_1 and y_2 that belong with probability 1 to the spaces H_1 and H_2, respectively, can be proved similarly as to the problem (49)–(52).

Namely, consider the optimal control problem of the system described by [2]

$$\hat{\mathbf{p}}_1 \in L^2(\Omega, H(\mathrm{div}, D)), \quad \hat{p}_2 \in L^2(\Omega, L^2(D)), \tag{72}$$

---

[2] The unique solvability of problem (72)–(74) for every fixed $u = (u_1, u_2)$ follows from correctness of stochastic statement of mixed variational problem (2.2) on page 1427 in [3].

$$\mathbb{E}\Big[\int_D (((\mathbf{A}(x))^{-1})^T \hat{\mathbf{p}}_1(x;u), \mathbf{q}_1(x))_{\mathbb{R}^n} dx\Big] - \mathbb{E}\Big[\int_D \hat{p}_2(x;u) \operatorname{div} \mathbf{q}_1(x)\, dx\Big] =$$

$$= \mathbb{E}\Big[\int_D (\mathbf{d}_1(x) - (C_1^t J_{H_1} u_1)(x)), \mathbf{q}_1(x))_{\mathbb{R}^n} dx\Big] \quad \forall \mathbf{q}_1 \in L^2(\Omega, H(\operatorname{div}, D)), \quad (73)$$

$$- \mathbb{E}\Big[\int_D v_1(x) \operatorname{div} \hat{\mathbf{p}}_1(x;u)\, dx\Big] - \mathbb{E}\Big[\int_D c(x)\hat{p}_2(x;u) v_1(x)\, dx\Big] =$$

$$= \mathbb{E}\Big[\int_D (d_2(x) - (C_2^t J_{H_2} u_2)(x)) v_1(x)\, dx\Big] \quad \forall v_1 \in L^2(\Omega, L^2(D)), \quad (74)$$

with the cost function

$$I(u) = \mathbb{E}\Big[\int_D Q^{-1}(\hat{p}_2(\cdot;u) - Qf_0)(x)(\hat{p}_2(\cdot;u) - Qf_0)(x)\, dx\Big] +$$

$$+ (\tilde{Q}_1^{-1} u_1, u_1)_{L^2(\Omega, H_1)} + (\tilde{Q}_2^{-1} u_2, u_2)_{L^2(\Omega, H_2)} \to \inf_{u=(u_1,u_2)\in L^2(\Omega,H)=L^2(\Omega,H_1\times H_2)},$$

where $\mathbf{d}_1(x) = C_1^t J_{H_1} \tilde{Q}_1 y_1(x)$, and $d_2(x) = C_2^t J_{H_2} \tilde{Q}_2 y_2(x)$.

The functional $I(u)$ is quadratic and coercive on the space $L^2(\Omega, H)$. Therefore, there exists a unique element $\hat{u} \in L^2(\Omega, H)$ such that

$$I(\hat{u}) = \inf_{u \in L^2(\Omega, H)} I(u).$$

Next, denoting by $(\hat{\mathbf{j}}, \hat{\varphi}) \in L^2(\Omega, H(\operatorname{div}, D)) \times L^2(\Omega, L^2(D))$ a unique solution of the problem

$$\mathbb{E}\Big[\int_D ((\mathbf{A}(x))^{-1} \hat{\mathbf{j}}(x), \mathbf{q}_2(x))_{\mathbb{R}^n}\, dx\Big] - \mathbb{E}\Big[\int_D \hat{\varphi}(x) \operatorname{div} \mathbf{q}_2(x)\, dx\Big] = 0$$

for all $\mathbf{q}_2 \in L^2(\Omega, H(\operatorname{div}, D))$, and for all $v_2 \in L^2(\Omega, L^2(D))$

$$- \mathbb{E}\Big[\int_D v_2(x) \operatorname{div} \hat{\mathbf{j}}(x)\, dx\Big] - \mathbb{E}\Big[\int_D c(x)\hat{\varphi}(x) v_2(x)\, dx\Big] =$$

$$= \mathbb{E}\Big[\int_D v_2(x)(Q^{-1}\hat{p}_2(x;\hat{u}) - f_0(x))\, dx\Big],$$

and making use of virtually the same reasoning that led to the proof of Theorem 4.1, we arrive at the equalities $\hat{u}_1 = \tilde{Q}_1 C_1 \hat{\mathbf{j}}$ and $\hat{u}_2 = \tilde{Q}_2 C_2 \hat{\varphi}$. Denoting $\hat{\mathbf{p}}_1(x) = \hat{\mathbf{p}}_1(x;\hat{u})$, $\hat{p}_2(x) = \hat{p}_2(x;\hat{u})$, we deduce from the latter statement the unique solvability of problem

$$\mathbb{E}\Big[\int_D (((\mathbf{A}(x))^{-1})^T \hat{\mathbf{p}}_1(x), \mathbf{q}_1(x))_{\mathbb{R}^n} dx\Big] - \mathbb{E}\Big[\int_D \hat{p}_2(x) \operatorname{div} \mathbf{q}_1(x)\, dx\Big]$$

$$= \mathbb{E}\Big[\int_D (C_1^t J_{H_1} \tilde{Q}_1 (y_1 - C_1 \hat{\mathbf{j}})(x), \mathbf{q}_1(x))_{\mathbb{R}^n} dx\Big] \quad \forall \mathbf{q}_1 \in L^2(\Omega, H(\operatorname{div}, D)),$$

$$-\mathbb{E}\Big[\int_D v_1(x)\mathrm{div}\,\hat{\mathbf{p}}_1(x)\,dx\Big] - \mathbb{E}\Big[\int_D c(x)\hat{p}_2(x)v_1(x)\,dx\Big] =$$
$$= \mathbb{E}\Big[\int_D C_2^t J_{H_2}\tilde{Q}_2(y_2 - C_2\hat{\varphi})(x)v_1(x)\,dx\Big] \quad \forall v_1 \in L^2(\Omega, L^2(D)),$$

$$\mathbb{E}\Big[\int_D ((\mathbf{A}(x))^{-1}\hat{\mathbf{j}}(x), \mathbf{q}_2(x))_{\mathbb{R}^n}\,dx\Big] -$$
$$-\mathbb{E}\Big[\int_D \hat{\varphi}(x)\,\mathrm{div}\,\mathbf{q}_2(x)\,dx\Big] = 0 \quad \forall \mathbf{q}_2 \in L^2(\Omega, H(\mathrm{div}, D)),$$

$$-\mathbb{E}\Big[\int_D v_2(x)\mathrm{div}\,\hat{\mathbf{j}}(x)\,dx\Big] - \mathbb{E}\Big[\int_D c(x)\hat{\varphi}(x)v_2(x)\,dx\Big] =$$
$$= \mathbb{E}\Big[\int_D v_2(x)(Q^{-1}\hat{p}_2(x) - f_0(x))\,dx\Big] \quad \forall v_2 \in L^2(\Omega, L^2(D)).$$

From here, following the argumentation in paper [3], we conclude that problem (68)–(71) is uniquely solvable.

Now let us prove the representation (67). By virtue of (23) and (48),

$$\widehat{\widehat{l(\mathbf{j},\varphi)}} = (y_1, \hat{u}_1)_{H_1} + (y_2, \hat{u}_2)_{H_2} + \hat{c} =$$
$$= (y_1, \tilde{Q}_1 C_1 \mathbf{p}_1)_{H_1} + (y_2, \tilde{Q}_2 C_2 p_2)_{H_2} - (\hat{z}_2, f_0)_{L^2(D)}. \tag{75}$$

Putting in (68) and (69) $\mathbf{q}_1 = \mathbf{p}_1$ and $v_1 = p_2$, we obtain

$$\int_D (((\mathbf{A}(x))^{-1})^T\hat{\mathbf{p}}_1(x), \mathbf{p}_1(x))_{\mathbb{R}^n}dx - \int_D \hat{p}_2(x)\,\mathrm{div}\,\mathbf{p}_1(x)\,dx =$$
$$= \int_D (C_1^t J_{H_1}\tilde{Q}_1(y_1 - C_1\hat{\mathbf{j}})(x), \mathbf{p}_1(x))_{\mathbb{R}^n}dx, \tag{76}$$

$$-\int_D p_2(x)\,\mathrm{div}\,\hat{\mathbf{p}}_1(x)\,dx - \int_D c(x)\hat{p}_2(x)p_2(x)\,dx =$$
$$= \int_D (C_2^t J_{H_2}\tilde{Q}_2(y_2 - C_2\hat{\varphi})(x)p_2(x)\,dx. \tag{77}$$

Putting in (51) and (52) $\mathbf{q}_2 = \hat{\mathbf{p}}_1$ and $v_2 = \hat{p}_2$, we find

$$\int_D ((\mathbf{A}(x))^{-1}\mathbf{p}_1(x), \hat{\mathbf{p}}_1(x))_{\mathbb{R}^n}dx - \int_D p_2(x)\mathrm{div}\,\hat{\mathbf{p}}_1(x)\,dx = 0, \tag{78}$$

$$-\int_D \hat{p}_2(x)\mathrm{div}\,\mathbf{p}_1(x)\,dx - \int_D c(x)p_2(x)\hat{p}_2(x)\,dx = \int_D \hat{p}_2(x)Q^{-1}\hat{z}_2(x)\,dx. \tag{79}$$

Since the sum of the left-hand sides of equalities (76) and (77) is equal to the sum of the left-hand sides of (78) and (79), we find from (75)

$$\widehat{\widehat{l(\mathbf{j},\varphi)}} = (C_1\hat{\mathbf{j}}, \tilde{Q}_1 C_1 \mathbf{p}_1)_{H_1} + (C_2\hat{\varphi}, \tilde{Q}_2 C_2 p_2)_{H_2} + (Q^{-1}\hat{p}_2 - f_0, \hat{z}_2)_{L^2(D)}. \tag{80}$$

Next, putting in (70)–(71) $\mathbf{q}_2 = \hat{\mathbf{z}}_1$, $v_2 = \hat{z}_2$ and in (49)–(50) $\mathbf{q}_1 = \hat{\mathbf{j}}$, $v_1 = \hat{\varphi}$, we obtain

$$\int_D ((\mathbf{A}(x))^{-1}\hat{\mathbf{j}}(x), \hat{\mathbf{z}}_1(x))_{\mathbb{R}^n}\,dx - \int_D \hat{\varphi}(x)\,\operatorname{div}\hat{\mathbf{z}}_1(x)\,dx = 0, \tag{81}$$

$$-\int_D \hat{z}_2(x)\,\operatorname{div}\hat{\mathbf{j}}(x)\,dx - \int_D c(x)\hat{\varphi}(x)\hat{z}_2(x)\,dx = \\ = \int_D \hat{z}_2(x)(Q^{-1}\hat{p}_2(x) - f_0(x))\,dx, \tag{82}$$

and

$$\int_D (((\mathbf{A}(x))^{-1})^T\hat{\mathbf{z}}_1(x), \hat{\mathbf{j}}(x))_{\mathbb{R}^n}dx - \int_D \hat{z}_2(x)\,\operatorname{div}\hat{\mathbf{j}}(x)\,dx = \\ = \int_D (\mathbf{l}_1(x) - C_1^t J_{H_1}\tilde{Q}_1 C_1\mathbf{p}_1(x), \hat{\mathbf{j}}(x))_{\mathbb{R}^n}\,dx, \tag{83}$$

$$-\int_D \hat{\varphi}(x)\operatorname{div}\hat{\mathbf{z}}_1(x)\,dx - \int_D c(x)z_2(x)\hat{\varphi}(x)\,dx = \\ = \int_D (l_2(x) - C_2^t J_{H_2}\tilde{Q}_2 C_2 p_2(x))\hat{\varphi}(x)\,dx. \tag{84}$$

The relations (81)–(84) imply

$$\int_D \hat{z}_2(x)(Q^{-1}\hat{p}_2(x) - f_0(x))\,dx = (\mathbf{l}_1 - \tilde{Q}_1 C_1\hat{\mathbf{p}}_1, C_1\hat{\mathbf{j}}(x))_{H_1} + (l_2 - \tilde{Q}_2 C_2\hat{p}_2, C_2\hat{\varphi}(x))_{H_2}.$$

By virtue of (80), we receive from here the representation (67). ☐

Remark 4.3. Notice that in the representation $l(\hat{\mathbf{j}}, \hat{\varphi})$ for the minimax estimate $\widehat{\widehat{l(\mathbf{j}, \varphi)}}$ the functions $\hat{\mathbf{j}}, \hat{\varphi}$ which are defined from the equations (68)–(71) do not depend on the specific form of the functional l and hence can be taken as a good estimate for the unknown solution $\mathbf{j}, \varphi$ of the Dirichlet problem (10)–(11).

5. Approximate guaranteed estimates: theorems on convergence

In this section we introduce the notion of approximate guaranteed estimates of $l(\mathbf{j}, \varphi)$ and prove their convergence to $\widehat{\widehat{l(\mathbf{j}, \varphi)}}$. To do this, we use the mixed finite element method for solving the aforementioned problems (49)–(52) and (68)–(71) and obtain approximate estimates via solutions of linear algebraic equations. We show their convergence to the optimal estimates.

In this section D is supposed to be a bounded and connected domain of $\mathbb{R}^n$ with polyhedral boundary Γ. First, we note that according to the mixed finite element method, an approximation $(\mathbf{j}^h, \varphi^h)$ to the solution $(\mathbf{j}, \varphi)$ of the problem (18)–(19) is sought in the finite element space $V_1^h \times V_2^h$ given by

$$V_1^h = \{\mathbf{q}^h \in H(\mathrm{div};\, D) : \mathbf{q}^h|_K \in (P^k(K))^n + \mathbf{x}P^k(K) \quad \forall K \in \mathcal{T}_h\},$$
$$V_2^h = \{v^h \in L^2(D) : v^h|_K \in P^k(K) \quad \forall K \in \mathcal{T}_h\},$$

where $\mathcal{T}_h$ is a simplicial triangulation of D, $P^k(K)$ denotes the space of polynomials on K of degree at most k, $k \geq 0$, $\mathbf{x} := (x_1, \dots, x_n)$, and is defined by requiring that

$$a(\mathbf{j}^h, \mathbf{q}^h) + b(\mathbf{q}^h, \varphi^h) = 0 \quad \forall \mathbf{q}^h \in V_1^h, \tag{85}$$

$$b(\mathbf{j}^h, v^h) - c(\varphi^h, v^h) = (f, v^h)_{L^2(D)} \quad \forall v^h \in V_2^h \tag{86}$$

Here the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $c(\cdot,\cdot)$ are defined by (14)–(16). Hence system (85)–(86) can be rewritten in the form

$$\int_D ((\mathbf{A}(x))^{-1}\mathbf{j}^h(x), \mathbf{q}^h(x))_{\mathbb{R}^n} dx - \int_D \varphi^h(x) \operatorname{div} \mathbf{q}^h(x) dx = 0 \quad \forall \mathbf{q}^h \in V_1^h, \tag{87}$$

$$\int_D v(x) \operatorname{div} \mathbf{j}^h(x) dx + \int_D c(x)\varphi^h(x) v^h(x) dx = \int_D f(x) v^h(x)\, dx \quad \forall v^h \in V_2^h. \tag{88}$$

It can be easily verified that the bilinear form $a|_{V_1^h \times V_1^h}$ is uniformly coercive on $\operatorname{Ker} B|_{V_1^h}$ and that the bilinear form $b|_{V_1^h \times V_2^h}$ satisfies the inf-sup condition (Babuska-Brezzi condition). Moreover, we have $\operatorname{Ker} B^t|_{V_2^h} = \emptyset$ and therefore, the mixed discretization (85)–(86) (or what is the same (87)–(88)) is uniquely solvable and the following estimates are valid

$$\begin{aligned} &\|\mathbf{j} - \mathbf{j}^h\|_{H(\mathrm{div},D)} + \|\varphi - \varphi^h\|_{L^2(D)} \leq \\ &\qquad \leq \tilde{c} \left(\inf_{\mathbf{q}^h \in V_1^h} \|\mathbf{j} - \mathbf{q}^h\|_{H(\mathrm{div},D)} + \inf_{v^h \in V_2^h} \|\varphi - v^h\|_{L^2(D)} \right), \end{aligned} \tag{89}$$

$$\|\mathbf{j}^h\|_{H(\mathrm{div},D)} + \|\varphi^h\|_{L^2(D)} \leq \tilde{\tilde{c}} \|f\|_{L^2(D)}, \tag{90}$$

where $\tilde{c}$ and $\tilde{\tilde{c}}$ are constant not depending on h (cf. e.g. [2]; §II, Prop. 2.11, and [4], p. 102).

Note that since $\operatorname{div} \mathbf{q}_h|_K \in P^k(K)$, $K \in \mathcal{T}_h$, then a natural choice for the approximation of the variable φ is to use piecewise polynomials of degree at most k leading to the space V_2^h defined above. Due to Proposition 3.9 of [2], p. 132, it follows that the sequences of the subspaces $\{V_1^h\}$ and $\{V_2^h\}$ are complete in $H(\mathrm{div}; \mathrm{D})$ and $L^2(D)$, respectively, in the following sense.

Definition 5.1. Let V be a Hilbert space. We consider a sequence of finite-dimensional subspaces V^h in V, defined by an infinite set of parameters $h_1, h_2, \dots$ with $\lim_{k\to\infty} h_k = 0$. We say that the sequence $\{V^h\}$ is *complete in* V, if for any $v \in V$ and $\epsilon > 0$ there exists an $\hat{h} = \hat{h}(v, \epsilon) > 0$ such that $\inf_{w \in V^h} \|v - w\|_H < \varepsilon$ for any $h < \hat{h}$. In other words, the completeness of the sequence $\{V^h\}$ means that any element $v \in V$ may be approximated with any degree of accuracy by elements of $\{V^h\}$.

Completeness of $\{V_1^h\}$ and $\{V_2^h\}$ in $H(\mathrm{div}; \mathrm{D})$ and $L^2(D)$ together with the estimate (89) imply that

$$\lim_{h\to 0} \big(\|\mathbf{j} - \mathbf{j}^h\|_{H(\mathrm{div},D)} + \|\varphi - \varphi^h\|_{L^2(D)} \big) = 0. \tag{91}$$

Now we are in a position to consider the following situation. Take an approximate guaranteed estimate of $l(\mathbf{j},\varphi)$ as

$$\widehat{l^h(\mathbf{j},\varphi)} = (u_1^h, y_1)_{H_1} + (u_2^h, y_2)_{H_1} + c^h, \tag{92}$$

where $u_1^h = \tilde{Q}_1 C_1 \mathbf{p}_1^h$, $u_2^h = \tilde{Q}_2 C_2 p_2^h$, $c^h = \int_D \hat{z}_2^h(x) f_0(x)\,dx$, and functions $\hat{\mathbf{z}}_1^h, \mathbf{p}_1^h \in V_1^h$ and $\hat{z}_2^h, p_2^h \in V_2^h$ which are determined from the following uniquely solvable system of variational equalities

$$\int_D (\mathbf{A}^{-1}(x)\,\hat{\mathbf{z}}_1^h(x), \mathbf{q}_1^h(x))_{\mathbb{R}^n} dx + \int_D \hat{z}_2^h(x) \operatorname{div} \mathbf{q}_1^h(x)\,dx =$$
$$= \int_D (\mathbf{l}_1(x) - C_1^t J_{H_1} \tilde{Q}_1 C_1 \mathbf{p}_1^h(x), \mathbf{q}_1^h(x))_{\mathbb{R}^n}\,dx \quad \forall \mathbf{q}_1^h \in V_1^h, \tag{93}$$

$$\int_D v_1^h(x) \operatorname{div} \hat{\mathbf{z}}_1^h(x)\,dx = \int_D (l_2(x) - C_2^t J_{H_2} \tilde{Q}_2 C_2 p_2^h(x)) v_1^h(x)\,dx \quad \forall v_1^h \in V_2^h, \tag{94}$$

$$\int_D (\mathbf{A}^{-1}(x)\,\mathbf{p}_1^h(x), \mathbf{q}_2^h(x))_{\mathbb{R}^n} dx + \int_D p_2^h(x) \operatorname{div} \mathbf{q}_2^h(x)\,dx = 0 \quad \forall \mathbf{q}_2^h \in V_1^h, \tag{95}$$

$$\int_D v_2^h(x) \operatorname{div} \mathbf{p}_1^h(x)\,dx = \int_D v_2^h(x) Q^{-1} \hat{z}_2^h(x)\,dx \quad \forall v_2^h \in V_2^h. \tag{96}$$

The unique solvability of system (93)–(96) follows from the same reasoning of the the previous sections which led to the proof of Theorem 4.1 with $H(\operatorname{div}, D)$ and $L^2(D)$ being replaced by V_1^h and V_2^h, respectively.

Theorem 5.2. *Let* $\hat{\mathbf{z}}_1, \mathbf{p}_1 \in H(\operatorname{div}, D)$, $\hat{z}_2, p_2 \in L^2(D)$ *and* $\hat{\mathbf{z}}_1^h, \mathbf{p}_1^h \in V_1^h$, $\hat{z}_2^h, p_2^h \in V_2^h$ *be solutions of problems* (49)–(52) *and* (93)–(96), *respectively.*

Then the following hold:

i)
$$\|\hat{\mathbf{z}}_1 - \hat{\mathbf{z}}_1^h\|_{H(\operatorname{div},D)} + \|\hat{z}_2 - \hat{z}_2^h\|_{L^2(D)} \to 0 \quad \text{as} \quad h \to 0, \ \textit{and} \tag{97}$$

$$\|\mathbf{p}_1 - \mathbf{p}_1^h\|_{H(\operatorname{div},D)} + \|p_2 - p_2^h\|_{L^2(D)} \to 0 \quad \text{as} \quad h \to 0. \tag{98}$$

ii) The approximate guaranteed estimate $\widehat{l^h(\mathbf{j},\varphi)}$ *of* $l(\mathbf{j},\varphi)$ *tends to a guaranteed estimate* $\widehat{\widehat{l(\mathbf{j},\varphi)}}$ *of this expression as* $h \to 0$ *in the sense that*

$$\lim_{h\to 0} \mathbb{E}|\widehat{l^h(\mathbf{j},\varphi)} - \widehat{\widehat{l(\mathbf{j},\varphi)}}|^2 = 0.$$

Moreover,

$$\lim_{h\to 0} \sup_{\tilde{f}\in G_0, (\tilde{\eta}_1,\tilde{\eta}_2)\in G_1} \mathbb{E}|\widehat{l^h(\tilde{\mathbf{j}},\tilde{\varphi})} - \widehat{\widehat{l(\tilde{\mathbf{j}},\tilde{\varphi})}}|^2 = 0, \quad \textit{and} \tag{99}$$

$$\lim_{h\to 0} \sup_{\tilde{f}\in G_0, (\tilde{\eta}_1,\tilde{\eta}_2)\in G_1} \mathbb{E}|\widehat{l^h(\tilde{\mathbf{j}},\tilde{\varphi})} - l(\tilde{\mathbf{j}},\tilde{\varphi})|^2 = \sup_{\tilde{f}\in G_0, (\tilde{\eta}_1,\tilde{\eta}_2)\in G_1} \mathbb{E}|\widehat{\widehat{l(\tilde{\mathbf{j}},\tilde{\varphi})}} - l(\tilde{\mathbf{j}},\tilde{\varphi}))|^2, \tag{100}$$

where $\tilde{f}$, $\tilde{\mathbf{j}}$, *and* $\tilde{\varphi}$ *have the same meaning as in Definition 2.1, and*

$$\widehat{l^h(\tilde{\mathbf{j}},\tilde{\varphi})} = (u_1^h,\tilde{y}_1)_{H_1} + (u_2^h,\tilde{y}_2)_{H_1} + c^h, \quad \tilde{y}_1 = C_1\tilde{\mathbf{j}} + \tilde{\eta}_1, \quad \tilde{y}_2 = C_2\tilde{\varphi} + \tilde{\eta}_2\,.$$

Proof. Denote by $\{h_n\}$ any sequence of positive numbers such that $h_n \to 0$ as $n \to \infty$. Let $\mathbf{z}_1^{h_n}(\cdot;u) \in V_1^{h_n}$ and $z_2^{h_n}(\cdot;u) \in V_2^{h_n}$ be a solution of the problem

$$\int_D (((\mathbf{A}(x))^{-1})^T \mathbf{z}_1^{h_n}(x;u), \mathbf{q}^{h_n}(x))_{\mathbb{R}^n} dx - \int_D z_2^{h_n}(x;u) \operatorname{div} \mathbf{q}^{h_n}(x)\, dx =$$

$$= \int_D (\mathbf{l}_1(x) - (C_1^t J_{H_1} u_1)(x), \mathbf{q}^{h_n}(x))_{\mathbb{R}^n} dx \quad \forall \mathbf{q}^{h_n} \in V_1^{h_n}, \tag{101}$$

$$-\int_D v^{h_n}(x) \operatorname{div} \mathbf{z}_1^{h_n}(x;u)\, dx - \int_D c(x) z_2^{h_n}(x;u) v^{h_n}(x)\, dx =$$

$$= \int_D (l_2(x) - (C_2^t J_{H_2} u_2)(x)) v^{h_n}(x)\, dx \quad \forall v^{h_n} \in V_2^{h_n}. \tag{102}$$

Then
$$\hat{\mathbf{z}}_1^{h_n}(x) = \mathbf{z}_1^{h_n}(x;u^{h_n}), \quad \hat{z}_2^{h_n}(x) = z_2^{h_n}(x;u^{h_n}). \tag{103}$$

Problem (101)–(102) can be rewritten as

$$a^*(\mathbf{z}_1^{h_n}(\cdot\,;u), \mathbf{q}^{h_n}) + b(\mathbf{q}^{h_n}, z_2^{h_n}(\cdot\,;u)) = \int_D (\mathbf{l}_1(x) - (C_1^t J_{H_1} u_1)(x), \mathbf{q}^{h_n}(x))_{\mathbb{R}^n}\, dx$$

for all $\mathbf{q}^{h_n} \in V_1^{h_n}$, and for all $v^{h_n} \in V_2^{h_n}$

$$b(\mathbf{z}_1^{h_n}(\cdot\,;u), v^{h_n}) - c(z_2^{h_n}(\cdot\,;u), v^{h_n}) = \int_D (l_2(x) - (C_2^t J_{H_2} u_2)(x)) v^{h_n}(x)\, dx,$$

where

$$a^*(\mathbf{z}_1^{h_n}(\cdot\,;u), \mathbf{q}^{h_n}) = a(\mathbf{q}^{h_n}, \mathbf{z}_1^{h_n}(\cdot\,;u)) = \int_D (((\mathbf{A}(x))^{-1})^T \mathbf{z}_1^{h_n}(x;u), \mathbf{q}^{h_n}(x))_{\mathbb{R}^n}\, dx$$

and the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $c(\cdot,\cdot)$ are defined by (14)–(16).

Since the bilinear form $a^*(\mathbf{z}_1^{h_n}(\cdot\,;u), \mathbf{q}^{h_n}) = a(\mathbf{q}^{h_n}, \mathbf{z}_1^{h_n}(\cdot\,;u))$ is uniformly coercive on $\operatorname{Ker} B|_{V_1^{h_n}}$ the system (101)–(102) is uniquely solvable. Theorem 1.2, Prop. 2.11 in §2 from [2] (see also [4], page 102), and uniform coerciveness of the form $a^*(\mathbf{z}_1^{h_n}(\cdot\,;u), \mathbf{q}^{h_n})$ on $\operatorname{Ker} B|_{V_1^{h_n}}$ imply that the following estimates are valid:

$$\|\mathbf{z}_1(\cdot;u) - \mathbf{z}_1^{h_n}(\cdot;u)\|_{H(\mathrm{div},D)} + \|z_2(\cdot;u) - z_2^{h_n}(\cdot;u)\|_{L^2(D)}$$

$$\leq \tilde{c}\left(\inf_{\mathbf{q}^{h_n} \in V_1^{h_n}} \|\mathbf{z}_1(\cdot;u) - \mathbf{q}^{h_n}\|_{H(\mathrm{div},D)} + \inf_{v^{h_n} \in V_2^{h_n}} \|z_2(\cdot;u) - v^{h_n}\|_{L^2(D)} \right), \tag{104}$$

$$\|\mathbf{z}_1^{h_n}(\cdot;u)\|_{H(\mathrm{div},D)} + \|z_2^{h_n}(\cdot;u)\|_{L^2(D)}$$

$$\leq \tilde{\tilde{c}}\left(\|\mathbf{l}_1 - C_1^t J_{H_1} u_1\|_{L^2(D)^n} + \|l_2 - C_2^t J_{H_2} u_2\|_{L^2(D)} \right), \tag{105}$$

where $\tilde{c}, \tilde{\tilde{c}}$ are constants not depending on h_n and $(\mathbf{z}_1(\cdot;u), z_2(\cdot;u))$ is a solution of the system of variational equations (29)–(30).

From the estimate (104) and the completeness of $\{V_1^h\}$ and $\{V_2^h\}$ in $H(\mathrm{div};\mathrm{D})$ and $L^2(D)$, it follows that

$$\|\mathbf{z}_1(\cdot;u)-\mathbf{z}_1^{h_n}(\cdot;u)\|_{H(\mathrm{div},D)}+\|z_2(\cdot;u)-z_2^{h_n}(\cdot;u)\|_{L^2(D)}\to 0 \tag{106}$$

as $n\to\infty$. We now prove that

$$\lim_{n\to\infty}\|u^{h_n}-\hat{u}\|_H=\lim_{n\to\infty}\left(\|u_1^{h_n}-\hat{u}_1\|_{H_1}^2+\|u_2^{h_n}-\hat{u}_2\|_{H_2}^2\right)^{1/2}=0,$$

where $u^{h_n}=(u_1^{h_n},u_2^{h_n})$, $\hat{u}=(\hat{u}_1,\hat{u}_2)$, $H=H_1\times H_2$. Set

$$I_n(u)=(Q^{-1}z_2^{h_n}(\cdot;u),z_2^{h_n}(\cdot;u))_{L^2(D)}+(\tilde{Q}_1^{-1}u_1,u_1)_{H_1}+(\tilde{Q}_2^{-1}u_2,u_2)_{H_2}.$$

It is clear that $\inf_{u\in H}I_n(u)=I_n(u^{h_n})$ and $I_n(u^{h_n})\le I_n(\hat{u})$. From the strong convergence of the sequence $\{(\mathbf{z}_1^{h_n}(\cdot;\hat{u}),z_2^{h_n}(\cdot;\hat{u}))\}$ to $(\mathbf{z}_1(\hat{u}),z_2(\hat{u}))$ in the space $H(\mathrm{div},D)\times L^2(D)$, which follows from (106), we have

$$\lim_{n\to\infty}I_n(\hat{u})=I(\hat{u}),$$

and, hence $\overline{\lim}_{n\to\infty}I_n(u^{h_n})\le I(\hat{u})$. Since

$$I_n(u^{h_n})\ge(\tilde{Q}_1^{-1}u_1^{h_n},u_1^{h_n})_{H_1}+(\tilde{Q}_2^{-1}u_2^{h_n},u_2^{h_n})_{H_2}\ge\alpha\|u^{h_n}\|_H^2,$$

where $\alpha>0$ is the constant from (27), then $\|u^{h_n}\|_H\le C$ ($C=\mathrm{const}$) and we can extract from the sequence $\{u^{h_n}\}$ a subsequence $\{u^{h_{n_k}}\}$ such that $u^{h_{n_k}}\to\tilde{u}$ weakly in H (see [19], Theorem 1, p. 180).

We now prove that the sequence $\{(\mathbf{z}_1^{h_{n_k}}(\cdot;u^{h_{n_k}}),z_2^{h_{n_k}}(\cdot;u^{h_{n_k}}))\}$ weakly converges to $(\mathbf{z}_1(\tilde{u}),z_2(\tilde{u}))$ in $H(\mathrm{div},D)\times L^2(D)$.

In fact, take a subsequence $\{(\mathbf{z}_1^{h_{n_{k_i}}}(\cdot;u^{h_{n_{k_i}}}),z_2^{h_{n_{k_i}}}(\cdot;u^{h_{n_{k_i}}}))\}$ of the sequence $\{(\mathbf{z}_1^{h_{n_k}}(\cdot;u^{h_{n_k}}),z_2^{h_{n_k}}(\cdot;u^{h_{n_k}}))\}$ which weakly converges to some $(\tilde{\mathbf{z}}_1,\tilde{z}_2)$ in the space $H(\mathrm{div},D)\times L^2(D)$ and for an arbitrary $(\mathbf{q},v)$ from $H(\mathrm{div},D)\times L^2(D)$ take a sequence $\{(\mathbf{q}^{h_{n_{k_i}}},v^{h_{n_{k_i}}})\}$, $(\mathbf{q}^{h_{n_{k_i}}},v^{h_{n_{k_i}}})\in V_1^{h_{n_{k_i}}}\times V_2^{h_{n_{k_i}}}$ which strongly converges to $(\mathbf{q},v)$ in $H(\mathrm{div},D)\times L^2(D)$ [3] and pass to the limit in both sides of the equations

$$\int_D(((\mathbf{A}(x))^{-1})^T\mathbf{z}_1^{h_{n_{k_i}}}(x;u^{h_{n_{k_i}}}),\mathbf{q}^{h_{n_{k_i}}}(x))_{\mathbb{R}^n}dx-\int_D z_2^{h_{n_{k_i}}}(x;u^{h_{n_{k_i}}})\,\mathrm{div}\,\mathbf{q}^{h_{n_{k_i}}}(x)\,dx$$
$$=\int_D(\mathbf{l}_1(x)-(C_1^tJ_{H_1}u_1^{h_{n_{k_i}}})(x),\mathbf{q}^{h_{n_{k_i}}}(x))_{\mathbb{R}^n}dx, \tag{107}$$

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[3] Such sequences exist due to the boundedness of the sequence $\{(\mathbf{z}_1^{h_{n_k}}(\cdot;u^{h_{n_k}}),z_2^{h_{n_k}}(\cdot;u^{h_{n_k}}))\}$ in the space $H(\mathrm{div};\mathrm{D})\times\mathrm{L}^2(\mathrm{D})$, which follows from inequality (105) and the boundedness of the sequence $\{u^{h_{n_k}}\}$ in the space H, and from the completeness of the sequence of the subspaces $\{V_1^h\times V_2^h\}$ in $H(\mathrm{div};\mathrm{D})\times\mathrm{L}^2(\mathrm{D})$.

$$-\int_D v^{h_{n_{k_i}}}(x)\operatorname{div}\mathbf{z}_1^{h_{n_{k_i}}}(x;u^{h_{n_{k_i}}})\,dx-\int_D c(x)z_2^{h_{n_{k_i}}}(x;u^{h_{n_{k_i}}})v^{h_{n_{k_i}}}(x)\,dx=$$
$$=\int_D (l_2(x)-(C_2^tJ_{H_2}u_2^{h_{n_{k_i}}})(x))v^{h_{n_{k_i}}}(x)\,dx \quad (108)$$

(which follows from (101)–(102)) when $i\to\infty$. Taking into account that [4]

$$\lim_{i\to\infty}\Big(\int_D(((\mathbf{A}(x))^{-1})^T\mathbf{z}_1^{h_{n_{k_i}}}(x;u^{h_{n_{k_i}}}),\mathbf{q}^{h_{n_{k_i}}}(x))_{\mathbb{R}^n}dx-$$
$$-\int_D z_2^{h_{n_{k_i}}}(x;u^{h_{n_{k_i}}})\ \operatorname{div}\mathbf{q}^{h_{n_{k_i}}}(x)\,dx\Big)=$$
$$=\lim_{i\to\infty}a(\mathbf{q}^{h_{n_{k_i}}},\mathbf{z}_1^{h_{n_{k_i}}}(\cdot\,;u^{h_{n_{k_i}}}))+\lim_{i\to\infty}b(\mathbf{q}^{h_{n_{k_i}}},z_2^{h_{n_{k_i}}}(\cdot\,;u^{h_{n_{k_i}}}))=$$
$$=\lim_{i\to\infty}<A\mathbf{q}_1^{h_{n_{k_i}}},\mathbf{z}_1^{h_{n_{k_i}}}(\cdot\,;u^{h_{n_{k_i}}})>_{H(\operatorname{div},D)'\times H(\operatorname{div},D)}+$$
$$+\lim\nolimits_{i\to\infty}<B\mathbf{q}^{h_{n_{k_i}}},z_2^{h_{n_{k_i}}}(\cdot\,;u^{h_{n_{k_i}}})>_{L^2(D)'\times L^2(D)}=$$
$$=<A\mathbf{q},\tilde{\mathbf{z}}_1>_{H(\operatorname{div},D)'\times H(\operatorname{div},D)}+<B\mathbf{q},\tilde{z}_2>_{L^2(D)'\times L^2(D)}=$$
$$=a(\mathbf{q},\tilde{\mathbf{z}}_1)+b(\mathbf{q},\tilde{z}_2)=$$
$$=\int_D(((\mathbf{A}(x))^{-1})^T\tilde{\mathbf{z}}_1(x),\mathbf{q}(x))_{\mathbb{R}^n}dx-\int_D\tilde{z}_2(x)\operatorname{div}\mathbf{q}(x)\,dx, \quad (109)$$

where by $A:H(\operatorname{div},D)\to H(\operatorname{div},D)'$ we denote the bounded operator associated with the bilinear form $a(\cdot,\cdot)$, defined by $a(u,v)=<Au,v>\ \ \forall u,v\in H(\operatorname{div},D)$,

$$-\lim_{i\to\infty}\int_D v^{h_{n_{k_i}}}(x)\ \operatorname{div}\mathbf{z}_1^{h_{n_{k_i}}}(x;u^{h_{n_{k_i}}})\,dx=\lim_{i\to\infty}b(\mathbf{z}_1^{h_{n_{k_i}}}(\cdot\,;u^{h_{n_{k_i}}}),v^{h_{n_{k_i}}})=$$
$$=\lim_{i\to\infty}<B^tv^{h_{n_{k_i}}},\mathbf{z}_1^{h_{n_{k_i}}}(\cdot;u^{h_{n_{k_i}}}))>_{H(\operatorname{div},D)'\times H(\operatorname{div},D)}=$$
$$=b(\tilde{\mathbf{z}}_1,v)=-\int_D v(x)\operatorname{div}\tilde{\mathbf{z}}_1(x)\,dx, \quad (110)$$

$$\lim_{i\to\infty}\int_D c(x)z_2^{h_{n_{k_i}}}(x;u^{h_{n_{k_i}}})v^{h_{n_{k_i}}}(x)\,dx=\lim_{i\to\infty}(z_2^{h_{n_{k_i}}}(\cdot;u^{h_{n_{k_i}}}),cv^{h_{n_{k_i}}})_{L^2(D)}=$$
$$=(\tilde{z}_2,cv)_{L^2(D)}=\int_D c(x)\tilde{z}_2(x)v(x)\,dx\,, \quad (111)$$

---

[4] Passage to the limit in (109)–(113) is justified by the following assertion (see, for example [1], p. 12): *Let a sequence* $\{v_n\}$ *weakly converge to* v_0 *in some linear normed space* X *and a sequence* $\{F_n\}$ *strongly converge to* F_0 *in the space* X'*, dual of* X*. Then* $\lim_{n\to\infty}<F_n,u_n>_{X'\times X}=<F_0,u_0>_{X'\times X}$.

$$\lim_{i\to\infty}\int_D(\mathbf{l}_1(x)-(C_1^tJ_{H_1}u_1^{h_{n_{k_i}}})(x),\mathbf{q}^{h_{n_{k_i}}}(x))_{\mathbb{R}^n}dx=$$

$$=\lim_{i\to\infty}\Big((\mathbf{l}_1,\mathbf{q}^{h_{n_{k_i}}})_{L^2(D)^n}-<J_{H_1}C_1\mathbf{q}^{h_{n_{k_i}}},u_1^{h_{n_{k_i}}}>_{H_1'\times H_1}\Big)=$$

$$=(\mathbf{l}_1,\mathbf{q})_{L^2(D)^n}-<J_{H_1}C_1\mathbf{q},\tilde{u}_1>H_1'\times H_1=$$

$$=\int_D(\mathbf{l}_1(x)-(C_1^tJ_{H_1}\tilde{u}_1)(x),\mathbf{q}(x))_{\mathbb{R}^n}dx\,, \tag{112}$$

$$\lim_{i\to\infty}\int_D(l_2(x)-(C_2^tJ_{H_2}u_2^{h_{n_{k_i}}})(x))v^{h_{n_{k_i}}}(x)\,dx=$$

$$=\int_D(l_2(x)-(C_2^tJ_{H_2}\tilde{u}_2)(x))v(x)\,dx, \tag{113}$$

we see, from (107)–(113), that $(\tilde{\mathbf{z}}_1,\tilde{z}_2)\in H(\mathrm{div},D)\times L^2(D)$ satisfies the equations (29) and (30) at $u=\tilde{u}$. But problem (29)–(30) has a unique solution $(\mathbf{z}_1(\tilde{u}),z_2(\tilde{u}))$ at $u=\tilde{u}$. Hence $(\tilde{\mathbf{z}}_1,\tilde{z}_2)=(\mathbf{z}_1(\tilde{u}),z_2(\tilde{u}))$ and

$$(\mathbf{z}_1^{h_{n_k}}(\cdot;u^{h_{n_k}}),z_2^{h_{n_k}}(\cdot;u^{h_{n_k}}))\to(\mathbf{z}_1(\tilde{u}),z_2(\tilde{u}))\quad\text{weakly in}\quad H(\mathrm{div},D)\times L^2(D).$$

Then, since the functionals $F_1(z_2):=(Q^{-1}z_2,z_2)_{L^2(D)}$ and $F_2(u):=(\tilde{Q}^{-1}u,u)_H:=(\tilde{Q}_1^{-1}u_1,u_1)_{H_1}+(\tilde{Q}_2^{-1}u_2,u_2)_{H_2}$ are weakly lower semicontinuous in the spaces $L^2(D)$ and H, respectively, [5] we obtain

$$I(\tilde{u})=(Q^{-1}z_2(\cdot;\tilde{u}),z_2(\cdot;\tilde{u}))_{L^2(D)}+(\tilde{Q}^{-1}\tilde{u},\tilde{u})_H\le$$

$$\le\underline{\lim}_{k\to\infty}(Q^{-1}z_2^{h_{n_k}}(\cdot\,;u^{h_{n_k}}),z_2^{h_{n_k}}(\cdot\,;u^{h_{n_k}}))_{L^2(D)}+\underline{\lim}_{k\to\infty}(\tilde{Q}^{-1}u^{h_{n_k}},u^{h_{n_k}})_H\le$$

$$\le\underline{\lim}_{k\to\infty}\Big[(Q^{-1}z_2^{h_{n_k}}(\cdot\,;u^{h_{n_k}}),z_2^{h_{n_k}}(\cdot\,;u^{h_{n_k}}))_{L^2(D)}+(\tilde{Q}^{-1}u^{h_{n_k}},u^{h_{n_k}})_H\Big]=$$

$$=\underline{\lim}_{k\to\infty}I_{n_k}(u^{h_{n_k}})\le\overline{\lim}_{k\to\infty}I_{n_k}(u^{h_{n_k}})\le I(\hat{u}). \tag{114}$$

Here $\tilde{Q}^{-1}:H\to H$ is the bounded selfadjoint positive definite operator defined by

$$\tilde{Q}^{-1}u=\tilde{Q}_1^{-1}u_1+\tilde{Q}_2^{-1}u_2,\quad u=(u_1,u_2)\in H=H_1\times H_2,$$

satisfying the inequality

$$(\tilde{Q}^{-1}u,u)_H\ge\alpha\|u\|_H^2\quad\forall u\in H, \tag{115}$$

where α is a constant from (27). Taking into account the uniqueness of an element on which the minimum of functional $I(u)$ is attained, we find from (114) that $\tilde{u}=\hat{u}$. This implies that

$$\lim_{n\to\infty}I_n(u^{h_n})=I(\hat{u}) \tag{116}$$

and $u^{h_n}\xrightarrow{\text{weakly}}\hat{u}$ in H, $\hat{z}_2^{h_n}=z_2^{h_n}(\cdot;u^{h_n})\xrightarrow{\text{weakly}}z_2(\cdot;\hat{u})=\hat{z}_2$ in $L^2(D)$ as $n\to\infty$.

---

[5] These assertions are the corollary of a more general statement (that can be found, for example, in [1], p. 41): *Let X be a reflexive Banach space and $B:X\to X^*$ a linear bounded nonnegative selfadjoint operator. Then the functional $F(u):=<Bu,u>_{X^*\times X}$ is weakly lower semicontinuous on X.*

Hence,

$$(Q^{-1}z_2(\cdot\,;\hat{u}), z_2(\cdot\,;\hat{u}))_{L^2(D)} \le \underline{\lim}_{n\to\infty}(Q^{-1}z_2^{h_n}(\cdot\,;u^{h_n}), z_2^{h_n}(\cdot\,;u^{h_n}))_{L^2(D)}, \qquad (117)$$

$$(\tilde{Q}^{-1}\hat{u}, \hat{u})_H \le \underline{\lim}_{n\to\infty}(\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H\,, \qquad (118)$$

and from (117)–(118), we have

$$\underline{\lim}_{n\to\infty}(Q^{-1}z_2^{h_n}(\cdot\,;u^{h_n}), z_2^{h_n}(\cdot\,;u^{h_n}))_{L^2(D)} + \underline{\lim}_{n\to\infty}(\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H \ge$$

$$\ge (Q^{-1}z_2(\cdot\,;\hat{u}), z_2(\cdot\,;\hat{u}))_{L^2(D)} + (\tilde{Q}^{-1}\hat{u}, \hat{u})_H = I(\hat{u}) =$$

$$= \lim_{n\to\infty}\Big[Q^{-1}z_2^{h_n}(\cdot\,;u^{h_n}), z_2^{h_n}(\cdot\,;u^{h_n}))_{L^2(D)} + (\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H\Big] =$$

$$= \overline{\lim}_{n\to\infty}\Big[Q^{-1}z_2^{h_n}(\cdot\,;u^{h_n}), z_2^{h_n}(\cdot\,;u^{h_n}))_{L^2(D)} + (\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H\Big] \ge$$

$$\ge \underline{\lim}_{n\to\infty}Q^{-1}z_2^{h_n}(\cdot\,;u^{h_n}), z_2^{h_n}(\cdot\,;u^{h_n}))_{L^2(D)} + \overline{\lim}_{n\to\infty}(\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H.$$

Whence $\quad \underline{\lim}_{n\to\infty}(\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H \ge \overline{\lim}_{n\to\infty}(\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H\,.$

The last inequality shows that the sequence $\{(\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H\}$ is convergent. This fact and (116) also imply the convergence of the sequence $\{(Q^{-1}z_2^{h_n}(\cdot\,;u^{h_n}), z_2^{h_n}(\cdot;u^{h_n}))_{L^2(D)}\}$ and the validity of the equality

$$I(\hat{u}) = \lim_{n\to\infty}(Q^{-1}z_2^{h_n}(\cdot\,;u^{h_n}), z_2^{h_n}(\cdot\,;u^{h_n}))_{L^2(D)} + \lim_{n\to\infty}(\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H. \qquad (119)$$

It is easy to see that

$$\lim_{n\to\infty}(\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H = (\tilde{Q}^{-1}\hat{u}, \hat{u})_H. \qquad (120)$$

In fact, if we suppose that (120) does not hold, i.e.

$$\lim_{n\to\infty}(\tilde{Q}^{-1}u^{h_n}, u^{h_n})_H = (\tilde{Q}^{-1}\hat{u}, \hat{u})_H + a,$$

where a is a certain positive number, then (due to (119)) we conclude

$$\underline{\lim}_{n\to\infty}(Q^{-1}z_2^{h_n}(\cdot\,;u^{h_n}), z_2^{h_n}(\cdot\,;u^{h_n}))_{L^2(D)} =$$

$$= \lim_{n\to\infty}(Q^{-1}z_2^{h_n}(\cdot\,;u^{h_n}), z_2^{h_n}(\cdot\,;u^{h_n}))_{L^2(D)} = (Q^{-1}z_2(\cdot\,;\hat{u}, z_2(\cdot\,;\hat{u}))_{L^2(D)} - a\,. \qquad (121)$$

But this is impossible since (121) leads to the contradictory inequality

$$\underline{\lim}_{n\to\infty}(Q^{-1}z_2^{h_n}(\cdot\,;u^{h_n}), z_2^{h_n}(\cdot\,;u^{h_n}))_{L^2(D)} < (Q^{-1}z_2(\cdot\,;\hat{u}, z_2(\cdot\,;\hat{u}))_{L^2(D)}.$$

Hence, (120) is proved.

Now let us show that $u^{h_n} \to \hat{u}$ strongly in H. To this end we introduce the Hilbert space $\tilde{H}$ consisting of elements of H endowed with the norm

$$\|v\|_{\tilde{H}} := (\tilde{Q}^{-1}v, v)_H^{1/2}.$$

Then from the weak convergence of the sequence $\{u^{h_n}\}$ to $\hat{u}$ as $n \to \infty$, it follows, obviously, that

$$u^{h_n} \to \hat{u} \text{ weakly in } \tilde{H} \text{ as } n \to \infty\,. \tag{122}$$

Since (120) means that

$$\|u^{h_n}\|_{\tilde{H}} \to \|\hat{u}\|_{\tilde{H}} \text{ as } n \to \infty\,, \tag{123}$$

we obtain from (122) and (123) that $u^{h_n} \to \hat{u}$ strongly in $\tilde{H}$, i.e.,[6]

$$\lim_{n\to\infty} \|u^{h_n} - \hat{u}\|_{\tilde{H}} = \lim_{n\to\infty} (\tilde{Q}^{-1}(u^{h_n} - \hat{u}), u^{h_n} - \hat{u})_H^{1/2} = 0\,.$$

From here, due to the inequality

$$\|u^{h_n} - \hat{u}\|_H \le \frac{1}{\alpha}(\tilde{Q}^{-1}(u^{h_n} - \hat{u}), u^{h_n} - \hat{u})_H^{1/2},$$

following from (115), we find that $\lim\limits_{n\to\infty} \|u^{h_n} - \hat{u}\|_H = 0$, i.e. the sequence $\{u^{h_n}\}$ strongly converges to $\hat{u}$ in H. In order to get estimate (97), we note that

$$(\mathbf{z}_1^{h_n}(\cdot\,;\hat{u}) - \mathbf{z}_1^{h_n}(\cdot\,;u^{h_n}), z_2^{h_n}(\cdot\,;\hat{u}) - z_2^{h_n}(\cdot\,;u^{h_n}))$$

is a solution of the following problem

$$\int_D (((\mathbf{A}(x))^{-1})^T(\mathbf{z}_1^{h_n}(x;\hat{u}) - \mathbf{z}_1^{h_n}(x;u^{h_n})), \mathbf{q}^{h_n}(x))_{\mathbb{R}^n}dx -$$
$$- \int_D (z_2^{h_n}(x;\hat{u}) - z_2^{h_n}(x;u^{h_n}))\ \mathrm{div}\,\mathbf{q}^{h_n}(x)\,dx =$$
$$= \int_D ((C_1^t J_{H_1}(u_1^{h_n} - \hat{u}_1))(x), \mathbf{q}^{h_n}(x))_{\mathbb{R}^n}dx \quad \forall \mathbf{q}^{h_n} \in V_1^{h_n}, \tag{124}$$

$$- \int_D v^{h_n}(x)\ \mathrm{div}\,(\mathbf{z}_1^{h_n}(x;\hat{u}) - \mathbf{z}_1^{h_n}(x;u^{h_n}))\,dx -$$
$$- \int_D c(x)(z_2^{h_n}(x;\hat{u}) - z_2^{h_n}(x;u^{h_n}))v^{h_n}(x)\,dx =$$
$$= \int_D (C_2^t J_{H_2}(u_2^{h_n} - \hat{u}_2))(x)v^{h_n}(x)\,dx \quad \forall v^{h_n} \in V_2^{h_n}. \tag{125}$$

Applying estimate (105) to the solution of problem (124)–(125), we obtain

$$\|\mathbf{z}_1^{h_n}(\cdot\,;\hat{u}) - \mathbf{z}_1^{h_n}(\cdot;u^{h_n})\|_{H(\mathrm{div},D)} + \|z_2^{h_n}(\cdot\,;\hat{u}) - z_2^{h_n}(\cdot\,;u^{h_n})\|_{L^2(D)} \le$$
$$\le C\|u^{h_n} - \hat{u}\|_H. \tag{126}$$

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[6] Here we use the following statement (see, for example [19], p. 124): *Let* $\{f_n\}$ *be a sequence in the Hilbert space* X*. If* $f_n \to f$ *weakly in* X *and* $\|f_n\|_X \to \|f\|_X$ *as* $n \to \infty$*, then* $f_n \to f$ *strongly in* X*.*

As a consequence of the triangle inequality, (103), (126), and the fact that the sequence $\{(\mathbf{z}_1^{h_n}(\cdot;\hat{u}), z_2^{h_n}(\cdot;\hat{u}))\}$ strongly converges to $(\mathbf{z}_1(\hat{u}), z_2(\hat{u}))$ in the space $H(\mathrm{div}, D) \times L^2(D)$, we get

$$\|\hat{\mathbf{z}}_1 - \hat{\mathbf{z}}_1^{h_n}\|_{H(\mathrm{div},D)} + \|\hat{z}_2 - \hat{z}_2^{h_n}\|_{L^2(D)} \leq \|\mathbf{z}_1(\cdot;\hat{u}) - \mathbf{z}_1^{h_n}(\cdot;\hat{u})\|_{H(\mathrm{div},D)} +$$

$$+ \|z_2(\cdot;\hat{u}) - z_2^{h_n}(\cdot;\hat{u})\|_{L^2(D)} + \|\mathbf{z}_1^{h_n}(\cdot;\hat{u}) - \mathbf{z}_1^{h_n}(\cdot;u^{h_n})\|_{H(\mathrm{div},D)} +$$

$$+ \|z_2^{h_n}(\cdot;\hat{u}) - z_2^{h_n}(\cdot;u^{h_n})\|_{L^2(D)} \to 0 \quad \text{as} \quad n \to \infty. \tag{127}$$

Analogously, in order to obtain estimate (98), we note that

$$\|\mathbf{p}_1 - \mathbf{p}_1^{h_n}\|_{H(\mathrm{div},D)} + \|p_2 - p_2^{h_n}\|_{L^2(D)} \leq \|\mathbf{p}_1 - \mathbf{p}_1^{h_n}(\cdot;\hat{u})\|_{H(\mathrm{div},D)} +$$

$$+ \|p_2 - p_2^{h_n}(\cdot;\hat{u})\|_{L^2(D)} + \|\mathbf{p}_1^{h_n}(\cdot;\hat{u}) - \mathbf{p}_1^{h_n}\|_{H(\mathrm{div},D)} + \|p_2^{h_n}(\cdot;\hat{u}) - p_2^{h_n}\|_{L^2(D)}, \tag{128}$$

where $(\mathbf{p}_1^{h_n}(\cdot;\hat{u}), p_2^{h_n}(\cdot;\hat{u}))$ is a solution of the problem

$$\int_D ((\mathbf{A}(x))^{-1}\mathbf{p}_1^{h_n}(x;\hat{u}), \mathbf{q}_2^{h_n}(x))_{\mathbb{R}^n} dx -$$

$$- \int_D p_2^{h_n}(x;\hat{u}) \mathrm{div}\, \mathbf{q}_2^{h}(x)\, dx = 0 \quad \forall \mathbf{q}_2^{h_n} \in V_1^{h_n}, \tag{129}$$

$$- \int_D v_2^{h_n}(x) \mathrm{div}\, \mathbf{p}_1^{h_n}(x;\hat{u})\, dx - \int_D c(x) p_2^{h_n}(x;\hat{u}) v_2^{h_n}(x)\, dx =$$

$$= \int_D v_2^{h_n}(x) Q^{-1} z_2(x;\hat{u})\, dx \quad \forall v_2^{h_n} \in V_2^{h_n}. \tag{130}$$

Taking into account that, due to (95)–(96) and (129)–(130),

$$(\mathbf{p}_1^{h_n}(\cdot\,;\hat{u}) - \mathbf{p}_1^{h_n}, p_2^{h_n}(\cdot\,;\hat{u}) - p_2^{h_n})$$

is a solution of the following problem

$$\int_D ((\mathbf{A}(x))^{-1}(\mathbf{p}_1^{h_n}(x;\hat{u}) - \mathbf{p}_1^{h_n}(x)), \mathbf{q}_2^{h_n}(x))_{\mathbb{R}^n} dx -$$

$$- \int_D (p_2^{h_n}(x;\hat{u}) - p_2^{h_n}(x))\ \mathrm{div}\, \mathbf{q}_2^{h_n}(x)\, dx = 0 \quad \forall \mathbf{q}_2^{h_n} \in V_1^{h_n}, \tag{131}$$

$$- \int_D v_2^{h_n}(x)\, \mathrm{div}\, (\mathbf{p}_1^{h_n}(x;\hat{u}) - \mathbf{p}_1^{h_n}(x))\, dx - \int_D c(x)(p_2^{h_n}(x;\hat{u}) - p_2^{h_n}(x)) v_2^{h_n}(x)\, dx =$$

$$= \int_D v_2^{h_n}(x) Q^{-1}(z_2(\cdot\,;\hat{u}) - z_2^{h_n}(\cdot\,;u^{h_n}))(x)\, dx \quad \forall v_2^{h_n} \in V_2^{h_n}, \tag{132}$$

and applying relationship (91) to the solution of problem (129)–(130) and the estimate (90) to the solution of problem (131)–(132), respectively, we obtain, in view of (127) that

$$\|\mathbf{p}_1 - \mathbf{p}_1^{h_n}(\cdot;\hat{u})\|_{H(\mathrm{div},D)} + \|p_2 - p_2^{h_n}(\cdot;\hat{u})\|_{L^2(D)} \to 0 \quad \text{as} \quad n \to \infty, \text{ and} \tag{133}$$

$$\|\mathbf{p}_1^{h_n}(\cdot;\hat{u}) - \mathbf{p}_1^{h_n})\|_{H(\mathrm{div},D)} + \|p_2^{h_n}(\cdot;\hat{u}) - p_2^{h_n})\|_{L^2(D)} \\ \leq C\|z_2(\cdot;\hat{u}) - \hat{z}_2^{h_n}\|_{L^2(D)} \to 0 \quad \text{as} \quad n \to \infty\,. \qquad (134)$$

From (134), (133), and (128), we find

$$\|\mathbf{p}_1 - \mathbf{p}_1^{h_n}\|_{H(\mathrm{div},D)} + \|p_2 - p_2^{h_n}\|_{L^2(D)} \to 0 \quad \text{as} \quad n \to \infty\,. \qquad (135)$$

Relationships (135) and (127) mean that (97) and (98) are proved.

We now show the validity of (99) and (100). Let $(\tilde{\mathbf{j}}, \tilde{\varphi})$ be a solution of problem (12)–(13) at $f(x) = \tilde{f}(x)$. Then from (92) and (47), we obtain

$$\mathbb{E}|\widehat{l^{h_n}(\tilde{\mathbf{j}}, \tilde{\varphi})} - \widehat{\widehat{l(\tilde{\mathbf{j}}, \tilde{\varphi})}}|^2 = \mathbb{E}[(u_1^{h_n}, \tilde{y}_1)_{H_1} + (u_2^{h_n}, \tilde{y}_2)_{H_1} + c^{h_n} - (\hat{u}_1, \tilde{y}_1)_{H_1} - (\hat{u}_2, \tilde{y}_2)_{H_2} - \hat{c}]^2$$

$$= \mathbb{E}[(u_1^{h_n} - \hat{u}_1, \tilde{y}_1)_{H_1} + (u_2^{h_n} - \hat{u}_2, \tilde{y}_2)_{H_2} + c^{h_n} - \hat{c}]^2 =$$

$$= [(u_1^{h_n} - \hat{u}_1, C_1\tilde{\mathbf{j}})_{H_1} + (u_2^{h_n} - \hat{u}_2, C_2\tilde{\varphi})_{H_2} + c^{h_n} - \hat{c}]^2 + \\ + \mathbb{E}[(u_1^{h_n} - \hat{u}_1, \tilde{\eta}_1)_{H_1} + (u_2^{h_n} - \hat{u}_2, \tilde{\eta}_2)_{H_2}]. \qquad (136)$$

Weak convergence of the sequence $\{\hat{z}_2^{h_n}\}$ to $\hat{z}_2$ in the space $L^2(D)$ implies that $c^{h_n} \to \hat{c}$ as $n \to \infty$. Then from the fact that $\tilde{f} \in G_0$ and the inequality

$$[(u_1^{h_n} - \hat{u}_1, C_1\tilde{\mathbf{j}})_{H_1} + (u_2^{h_n} - \hat{u}_2, C_2\tilde{\varphi})_{H_2} + c^{h_n} - \hat{c}]^2 \leq$$

$$\leq C\left(\|u_1^{h_n} - \hat{u}_1\|_{H_1}^2 + \|u_2^{h_n} - \hat{u}_2\|_{H_2}^2 + (c^{h_n} - \hat{c})^2\right)\left(\|\tilde{\mathbf{j}}\|_{H(\mathrm{div},D)}^2 + \|\tilde{\varphi}\|_{L^2(D)}^2 + 1\right) \leq$$

$$\leq C\left(\|u^{h_n} - \hat{u}\|_H^2 + (c^{h_n} - \hat{c})^2\right)\left(\tilde{C}\,\|\tilde{f}\|_{L^2(D)}^2 + 1\right) \leq$$

$$\leq \tilde{\tilde{C}}\left(\|u^{h_n} - \hat{u}\|_H^2 + (c^{h_n} - \hat{c})^2\right) \quad (C, \tilde{C}, \tilde{\tilde{C}} = \text{const.}),$$

we see that the first term in the r.h.s. of (136) tends to 0 as $n \to \infty$. Analogously, we show that for the last term in the r.h.s. of (136) the following estimate is valid

$$\mathbb{E}[(u_1^{h_n} - \hat{u}_1, \tilde{\eta}_1)_{H_1} + (u_2^{h_n} - \hat{u}_2, \tilde{\eta}_2)_{H_2}] \leq C\|u^{h_n} - \hat{u}\|_H^2 \quad (C = \text{const})$$

and therefore this term also tends to 0 as $n \to \infty$. From here and the inequality

$$\mathbb{E}|\widehat{l^{h_n}(\tilde{\mathbf{j}}, \tilde{\varphi})} - l(\tilde{\mathbf{j}}, \tilde{\varphi})|^{1/2} = \mathbb{E}[\widehat{l^{h_n}(\tilde{\mathbf{j}}, \tilde{\varphi})} - \widehat{\widehat{l(\tilde{\mathbf{j}}, \tilde{\varphi})}} + \widehat{\widehat{l(\mathbf{j}, \varphi)}} - l(\tilde{\mathbf{j}}, \tilde{\varphi})|^{1/2} \leq$$

$$\leq \left\{\mathbb{E}|\widehat{l^{h_n}(\tilde{\mathbf{j}}, \tilde{\varphi})} - \widehat{\widehat{l(\tilde{\mathbf{j}}, \tilde{\varphi})}}|^2\right\}^{1/2} + \left\{\mathbb{E}[\widehat{\widehat{l(\tilde{\mathbf{j}}, \tilde{\varphi})}} - l(\tilde{\mathbf{j}}, \tilde{\varphi})]^2|\right\}^{1/2},$$

follows the validity of the conclusion of the theorem. □

A similar result for the case when an estimate $(\hat{\mathbf{j}}, \hat{\varphi})$ of the state $(\mathbf{j}, \varphi)$ is directly determined from the solution to problem (68)–(71) can be formulated.

Theorem 5.3. *Let* $(\hat{\mathbf{j}}^h, \hat{\varphi}^h) \in V_1^h \times V_2^h$ *be an approximate estimate of* $(\hat{\mathbf{j}}, \hat{\varphi})$ *determined from the solution to the variational problem*

$$\int_D (((\mathbf{A}(x))^{-1})^T \hat{\mathbf{p}}_1^h(x), \mathbf{q}_1^h(x))_{\mathbb{R}^n} dx - \int_D \hat{p}_2^h(x) \operatorname{div} \mathbf{q}_1^h(x)\, dx = \\ = \int_D (C_1^t J_{H_1} \tilde{Q}_1(y_1(x) - C_1 \hat{\mathbf{j}}^h), \mathbf{q}_1^h(x))_{\mathbb{R}^n} dx \quad \forall \mathbf{q}_1^h \in V_1^h, \tag{137}$$

$$-\int_D v_1^h(x) \operatorname{div} \hat{\mathbf{p}}_1^h(x)\, dx - \int_D c(x) \hat{p}_2^h(x) v_1^h(x)\, dx = \\ = \int_D (C_2^t J_{H_2} \tilde{Q}_2(y_2(x) - C_2 \hat{\varphi}^h(x)) v_1^h(x)\, dx \quad \forall v_1^h \in V_2^h, \tag{138}$$

$$\int_D ((\mathbf{A}(x))^{-1} \hat{\mathbf{j}}^h(x), \mathbf{q}_2^h(x))_{\mathbb{R}^n}\, dx - \int_D \hat{\varphi}^h(x) \operatorname{div} \mathbf{q}_2^h(x)\, dx = 0 \quad \forall \mathbf{q}_2^h \in V_1^h, \tag{139}$$

$$-\int_D v_2^h(x) \operatorname{div} \hat{\mathbf{j}}^h(x)\, dx - \int_D c(x) \hat{\varphi}^h(x) v_2^h(x)\, dx = \\ = \int_D v_2(x)(Q^{-1} \hat{p}_2^h(x) + f_0(x))\, dx \quad \forall v_2^h \in V_2^h. \tag{140}$$

Then $\quad \|\hat{\mathbf{j}} - \hat{\mathbf{j}}^h\|_{H(\mathrm{div},D)} + \|\hat{\varphi} - \hat{\varphi}^h\|_{L^2(D)} \to 0 \quad \text{as} \quad h \to 0,$

and $\quad \|\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_1^h\|_{H(\mathrm{div},D)} + \|\hat{p}_2 - \hat{p}_2^h\|_{L^2(D)} \to 0 \quad \text{as} \quad h \to 0.$

The proof of this theorem is similar to the proof of Theorem 5.2.

Introducing bases in the spaces V_1^h and V_2^h, problem (93)–(96) can be rewritten as a system of liner algebraic equations. To do this, we denote the elements of the bases of V_1^h and V_2^h by $\boldsymbol{\xi}_i$ $(i = 1, \ldots, n_1)$ and η_i $(i = 1, \ldots, n_2)$, respectively, where $n_1 = \dim V_1^h$, $n_2 = \dim V_2^h$. The fact that $\hat{\mathbf{z}}_1^h$, $\mathbf{p}_1^h$ and $\hat{z}_2^h$, p_2^h belong to the spaces V_1^h and V_2^h leads to the existence of constants $\hat{z}_i^{(1)}$, $p_i^{(1)}$ and $\hat{z}_i^{(2)}$, $p_i^{(2)}$ such that

$$\hat{\mathbf{z}}_1^h = \sum_{j=1}^{n_1} \hat{z}_j^{(1)} \boldsymbol{\xi}_j, \quad \mathbf{p}_1^h = \sum_{j=1}^{n_1} p_j^{(1)} \boldsymbol{\xi}_j \tag{141}$$

and

$$\hat{z}_2^h = \sum_{j=1}^{n_2} \hat{z}_j^{(2)} \eta_j, \quad p_2^h = \sum_{j=1}^{n_2} p_j^{(2)} \eta_j. \tag{142}$$

Setting in (93) and (95) $\mathbf{q}_1^h = \mathbf{q}_2^h = \boldsymbol{\xi}_i$ $(i = 1, \ldots, n_1)$ and in (94), (96) $v_1^h = v_2^h = \eta_i$ $(i = 1, \ldots, n_2)$ respectively, we obtain that finding $\hat{\mathbf{z}}_1^h$, $\mathbf{p}_1^h$, $\hat{z}_2^h$ and p_2^h from (93)–(96) is equivalent to solving the following system of linear algebraic equations with respect to the coefficients $\hat{z}_j^{(1)}$, $p_j^{(1)}$, $\hat{z}_j^{(2)}$, and $p_j^{(2)}$ of the expansions (141)–(142):

$$\sum_{j=1}^{n_1} \bar{a}_{ij}^{(1)} \hat{z}_j^{(1)} + \sum_{j=1}^{n_2} a_{ji}^{(2)} \hat{z}_j^{(2)} + \sum_{j=1}^{n_1} a_{ij}^{(3)} p_j^{(1)} = b_i^{(1)}, \quad i = 1, \ldots, n_1, \tag{143}$$

$$\sum_{j=1}^{n_1} a_{ij}^{(2)} \hat{z}_j^{(1)} + \sum_{j=1}^{n_2} a_{ij}^{(6)} \hat{z}_j^{(2)} + \sum_{j=1}^{n_2} a_{ij}^{(4)} p_j^{(2)} = b_i^{(1)}, \quad i = 1, \ldots, n_2, \tag{144}$$

$$\sum_{j=1}^{n_1} a_{ij}^{(1)} p_j^{(1)} + \sum_{j=1}^{n_2} a_{ji}^{(2)} p_j^{(2)} = 0, \quad i = 1, \ldots, n_1, \tag{145}$$

$$\sum_{j=1}^{n_1} a_{ij}^{(2)} p_j^{(1)} + \sum_{j=1}^{n_2} a_{ij}^{(6)} p_j^{(2)} + \sum_{j=1}^{n_2} a_{ij}^{(5)} \hat{z}_j^{(2)} = 0, \quad i = 1, \ldots, n_2, \tag{146}$$

where

$$\bar{a}_{ij}^{(1)} = \int_D (((\mathbf{A}(x))^{-1})^T \boldsymbol{\xi}_i(x), \boldsymbol{\xi}_j(x))_{\mathbb{R}^n} dx, \quad i, j = 1, \ldots, n_1,$$

$$a_{ij}^{(1)} = \int_D ((\mathbf{A}(x))^{-1} \boldsymbol{\xi}_i(x), \boldsymbol{\xi}_j(x))_{\mathbb{R}^n} dx, \quad i, j = 1, \ldots, n_1,$$

$$a_{ij}^{(2)} = -\int_D \eta_i(x) \operatorname{div} \boldsymbol{\xi}_j(x)\, dx, \quad i = 1, \ldots, n_2, \quad j = 1, \ldots, n_1,$$

$$a_{ij}^{(3)} = \int_D (C_1^t J_{H_1} \tilde{Q}_1 C_1 \boldsymbol{\xi}_i(x), \boldsymbol{\xi}_j(x))_{\mathbb{R}^n}\, dx, \quad i, j = 1, \ldots, n_1,$$

$$a_{ij}^{(4)} = \int_D C_2^t J_{H_2} \tilde{Q}_2 C_2 \eta_i(x) \eta_j(x)\, dx, \quad i, j = 1, \ldots, n_2,$$

$$a_{ij}^{(5)} = -\int_D \eta_j(x) Q^{-1} \eta_i(x)\, dx, \quad i, j = 1, \ldots, n_2,$$

$$a_{ij}^{(6)} = -\int_D c(x) \eta_i(x) \eta_j(x)\, dx, \quad i, j = 1, \ldots, n_2,$$

$$b_i^{(1)} = \int_D (\mathbf{l}_1(x), \boldsymbol{\xi}_i(x))_{\mathbb{R}^n}\, dx, \quad i = 1, \ldots, n_1,$$

$$b_i^{(2)} = \int_D l_2(x) \eta_i(x)\, dx, \quad i = 1, \ldots, n_2.$$

An analogous system of linear algebraic equations can also be obtained for problem (137)–(140).

6. The case of integral observation operators

As an example we consider the case when $H_1 = L^2\big(D_1^{(1)}\big)^n \times \cdots \times L^2\big(D_{i_1}^{(1)}\big)^n \times \cdots \times L^2\big(D_{n_1}^{(1)}\big)^n$, $H_2 = L^2\big(D_1^{(2)}\big) \times \cdots \times L^2\big(D_{i_2}^{(2)}\big) \times \cdots \times L^2\big(D_{n_2}^{(2)}\big)$. Then $J_{H_1} = I_{H_1}$, $J_{H_2} = I_{H_2}$, where I_{H_1} and I_{H_2} are the identity operators in H_1 and H_2, respectively,

$$y_1(x) = \Big(\mathbf{y}_1^{(1)}(x), \dots, \mathbf{y}_{i_1}^{(1)}(x), \dots, \mathbf{y}_{n_1}^{(1)}(x)\Big),$$

$$\eta_1(x) = \Big(\boldsymbol{\eta}_1^{(1)}(x), \dots, \boldsymbol{\eta}_{i_1}^{(1)}(x), \dots, \boldsymbol{\eta}_{n_1}^{(1)}(x)\Big),$$

where $\mathbf{y}_{i_1}^{(1)}(x) = (y_{i_1,1}^{(1)}(x), \dots, y_{i_1,n}^{(1)}(x))^T \in L^2\big(D_{i_1}^{(1)}\big)^n$ and $\boldsymbol{\eta}_{i_1}^{(1)}(x) = (\eta_{i_1,1}^{(1)}(x), \dots, \eta_{i_1,n}^{(1)}(x))^T$ is a stochastic vector process with components $\eta_{i_1,j}^{(1)}(x)$ ($j = 1, \dots, n$; $i_1 = 1, \dots, n_1$) that are stochastic processes with zero expectations and finite second moments,

$$y_2(x) = \Big(y_1^1(x), \dots, y_{i_2}^{(2)}(x), \dots, y_{n_2}^{(2)}(x)\Big),$$

$$\eta_2(x) = \Big(\eta_1^{(2)}(x), \dots, \eta_{i_2}^{(2)}(x), \dots, \eta_{n_2}^{(2)}(x)\Big), \tag{147}$$

where $y_{i_2}^{(2)} \in L^2(D)$, $\eta_{i_2}^{(2)}(x)$ ($i_2 = 1, \dots, n_2$) is a stochastic process with zero expectation and finite second moment. Let in the observations (21) the operators $C_1 : L^2(D)^n \to H_1$ and $C_2 : L^2(D) \to H_2$ be defined by

$$C_1\mathbf{j}(x) = \Big(C_1^{(1)}\mathbf{j}(x), \dots, C_{i_1}^{(1)}\mathbf{j}(x) \dots, C_{n_1}^{(1)}\mathbf{j}(x)\Big),$$

$$C_2\varphi(x) = \Big(C_1^{(2)}\varphi(x), \dots, C_{i_2}^{(2)}\varphi(x) \dots, C_{n_2}^{(2)}\varphi(x)\Big),$$

where $C_{i_1}^{(1)} : L^2(D)^n \to L^2(D_{i_1}^{(1)})^n$ and $C_{i_2}^{(2)} : L^2(D) \to L^2(D_{i_2}^{(2)})$ are integral operators defined by

$$C_{i_1}^{(1)}\mathbf{j}(x) := \int_{D_{i_1}^{(1)}} \mathbf{K}_{i_1}^{(1)}(x,\xi)\mathbf{j}(\xi)\,d\xi,$$

and

$$C_{i_2}^{(2)}\varphi(x) := \int_{D_{i_2}^{(2)}} K_{i_2}^{(2)}(x,\xi)\varphi(\xi)\,d\xi,$$

correspondingly, $\mathbf{K}_{i_1}^{(1)}(x,\xi) = \{k_{is}^{(i_1)}(x,\xi)\}_{i,j=1}^n$ is a matrix with entries $k_{is}^{(i_1)} \in L^2(D_{i_1}^{(1)}) \times L^2(D_{i_1}^{(1)})$, $i_1 = 1, \dots, n_1$, $K_{i_2}^{(2)}(x,\xi) \in L^2(D_{i_2}^{(2)}) \times L^2(D_{i_2}^{(2)})$ is a given function, $i_2 = 1, \dots, n_2$.

As a result, the observations y_1 and y_2 in (21) take the form

$$y_1 = (\mathbf{y}_1^{(1)}(x), \dots, \mathbf{y}_{i_1}^{(1)}(x), \dots, \mathbf{y}_{n_1}^{(1)}(x)),$$

$$y_2 = (y_1^{(2)}(x), \dots, y_{i_2}^{(2)}(x), \dots, \mathbf{y}_{n_2}^{(2)}(x)),$$

where

$$\mathbf{y}_{i_1}^{(1)}(x) = \int_{D_{i_1}^{(1)}} \mathbf{K}_{i_1}^{(1)}(x,\xi)\mathbf{j}(\xi)\,d\xi + \boldsymbol{\eta}_{i_1}^{(1)}(x), \quad i_1 = \overline{1,n_1}, \tag{148}$$

$$y_{i_2}^{(2)}(x) = \int_{D_{i_2}^{(2)}} K_{i_2}^{(2)}(x,\xi)\varphi(\xi)\,d\xi + \eta_{i_2}^{(2)}(x), \quad i_2 = \overline{1,n_2}, \tag{149}$$

and the operators

$$\tilde{Q}_1 \in \mathcal{L}\Big(L^2\big(D_1^{(1)}\big)^n \times \dots \times L^2\big(D_{i_1}^{(1)}\big)^n \times \dots \times L^2\big(D_{n_1}^{(1)}\big)^n, \\ L^2\big(D_1^{(1)}\big)^n \times \dots \times L^2\big(D_{i_1}^{(1)}\big)^n \times \dots \times L^2\big(D_{n_1}^{(1)}\big)^n\Big)$$

and

$$\tilde{Q}_2 \in \mathcal{L}\Big(L^2\big(D_1^{(2)}\big) \times \dots \times L^2\big(D_{i_2}^{(2)}\big) \times \dots \times L^2\big(D_{n_2}^{(2)}\big), \\ L^2\big(D_1^{(2)}\big) \times \dots \times L^2\big(D_{i_2}^{(2)}\big) \times \dots \times L^2\big(D_{n_2}^{(2)}\big)\Big)$$

in (25), which is contained in the definition of the set G_1, are given by

$$\tilde{Q}_1\tilde{\eta}_1 = (\tilde{\mathbf{Q}}_1^{(1)}\tilde{\boldsymbol{\eta}}_1^{(1)}, \dots, \tilde{\mathbf{Q}}_{r_1}^{(1)}\tilde{\boldsymbol{\eta}}_{r_1}^{(1)}, \dots, \tilde{\mathbf{Q}}_{n_1}^{(1)}\tilde{\boldsymbol{\eta}}_{n_1}^{(1)})$$

and

$$\tilde{Q}_2\tilde{\eta}_2 = (\tilde{Q}_2^{(2)}\tilde{\eta}_2^{(2)}, \dots, \tilde{Q}_{r_2}^{(2)}\tilde{\eta}_{r_2}^{(2)}, \dots, \tilde{Q}_{n_2}^{(2)}\tilde{\eta}_{n_2}^{(2)}),$$

where $\tilde{\mathbf{Q}}_{r_1}^{(1)}(x)$ is a symmetric positive definite $n \times n$-matrix with entries $\tilde{q}_{ij}^{(r_1)} \in C(\bar{D}_{r_1}^{(1)})$, [7] $i,j = 1,\dots,n$, $\tilde{\boldsymbol{\eta}}_{r_1}^{(1)} \in L^2(\Omega, L^2(D_{r_1}^{(1)})^n)$, $r_1 = 1,\dots,n_1$, $\tilde{Q}_{r_2}^{(2)}(x)$ is a continuous positive function defined in the domain $\bar{D}_{r_2}^{(2)}$, $\tilde{\eta}_{r_2}^{(2)} \in L^2(\Omega, L^2(D_{r_2}^{(2)}))$, $r_2 = 1,\dots,n_2$.

In this case condition (25) takes the form [8]

$$\sum_{r_1=1}^{n_1}\int_{D_{r_1}^{(1)}} \mathrm{Sp}(\tilde{\mathbf{Q}}_{r_1}^{(1)}(x)\tilde{\mathbf{R}}_{r_1}^{(1)}(x,x))\,dx \le 1, \quad \sum_{r_2=1}^{n_2}\int_{D_{r_2}^{(2)}} \tilde{Q}_{r_2}^{(2)}(x)\tilde{R}_{r_2}^{(2)}(x,x)\,dx \le 1,$$

where by $\tilde{\mathbf{R}}_{r_1}^{(1)}(x,y) = [\tilde{b}_{i,j}^{(r_1)}(x,y)]_{i,j=1}^n$ we denote the correlation matrix of the vector process $\tilde{\boldsymbol{\eta}}_{r_1}^{(1)}(x) = (\tilde{\eta}_{r_1,1}^{(1)}(x), \dots, \tilde{\eta}_{r_1,n}^{(1)}(x))$ with the components

$$\tilde{b}_{i,j}^{(r_1)}(x,y) = \mathbb{E}\Big(\tilde{\eta}_{r_1,i}^{(1)}(x)\tilde{\eta}_{r_1,j}^{(1)}(y)\Big), \quad (x,y) \in D_{i_1}^{(1)} \times D_{i_1}^{(1)},$$

and by $\tilde{R}_{r_2}^{(2)}(x,y) = \mathbb{E}\tilde{\eta}_{r_2}^{(2)}(x)\tilde{\eta}_{r_2}^{(2)}(y)$ we denote the correlation function of process $\tilde{\eta}_{r_2}^{(2)}(x)$, $(x,y) \in D_{r_2}^{(2)} \times D_{r_2}^{(2)}$.

[7] Here and below we denote by $C(\bar{D})$ a class of functions continuous in the domain $\bar{D}$.

[8] By $\mathrm{Sp}(\tilde{\mathbf{Q}}_{r_1}^{(1)}(x)\tilde{\mathbf{R}}_{r_1}^{(1)}(x,x))$ we denote the trace of the matrix $\tilde{\mathbf{Q}}_{r_1}^{(1)}(x)\tilde{\mathbf{R}}_{r_1}^{(1)}(x,x)$, i.e. the sum of the diagonal elements of this matrix.

In fact,

$$\mathbb{E}(\tilde{Q}_1\tilde{\eta}_1,\tilde{\eta}_1)_{H_1}=\sum_{r_1=1}^{n_1}\mathbb{E}(\tilde{\mathbf{Q}}_{r_1}^{(1)}(x)\boldsymbol{\eta}_{r_1}^{(1)}(x),\boldsymbol{\eta}_{r_1}^{(1)}(x))_{L^2\left(D_{r_1}^{(1)}\right)^n}$$

$$=\sum_{r_1=1}^{n_1}\mathbb{E}\Big(\int_{D_{r_1}^{(1)}}\tilde{\mathbf{Q}}_{r_1}^{(1)}(x)\boldsymbol{\eta}_{r_1}^{(1)}(x),\boldsymbol{\eta}_{r_1}^{(1)}(x))_{\mathbb{R}^n}dx\Big)$$

$$=\sum_{r_1=1}^{n_1}\sum_{i=1}^{n}\int_{D_{r_1}^{(1)}}\sum_{j=1}^{n}\mathbb{E}(\tilde{q}_{ij}^{(r_1)}(x)\eta_{j,r_1}^{(1)}(x)\eta_{i,r_1}^{(1)}(x))\,dx$$

$$=\sum_{r_1=1}^{n_1}\sum_{i=1}^{n}\int_{D_{r_1}^{(1)}}\sum_{j=1}^{n}\tilde{q}_{ij}^{(r_1)}(x)\mathbb{E}(\eta_{j,r_1}^{(1)}(x)\eta_{i,r_1}^{(1)}(x))\,dx$$

$$=\sum_{r_1=1}^{n_1}\int_{D_{r_1}^{(1)}}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{q}_{ij}^{(r_1)}(x)\tilde{b}_{j,i}^{(r_1)}(x,x)\,dx$$

$$=\sum_{r_1=1}^{n_1}\int_{D_{r_1}^{(1)}}\mathrm{Sp}(\tilde{\mathbf{Q}}_{r_1}^{(1)}(x)\tilde{\mathbf{R}}_{r_1}^{(1)}(x,x))\,dx\,.$$

Analogously,

$$\mathbb{E}(\tilde{Q}_2\tilde{\eta}_2,\tilde{\eta}_2)_{H_2}=\sum_{r_2=1}^{n_2}\mathbb{E}(\tilde{Q}_{r_2}^{(2)}(x)\eta_{r_2}^{(2)}(x),\eta_{r_2}^{(2)}(x))_{L^2\left(D_{r_2}^{(2)}\right)}=$$

$$=\sum_{r_2=1}^{n_2}\int_{D_{r_2}^{(2)}}\tilde{Q}_{r_2}^{(2)}(x)\mathbb{E}(\eta_{r_2}^{(2)}(x)\eta_{r_2}^{(2)}(x))\,dx=\sum_{r_2=1}^{n_2}\int_{D_{r_2}^{(2)}}\tilde{Q}_{r_2}^{(2)}(x)\tilde{R}_{r_2}^{(2)}(x,x)\,dx.$$

Uncorrelatedness of random variables $\tilde{\eta}_1$ and $\tilde{\eta}_2$ reduces in this case to the condition of uncorrelatedness of the componets $\tilde{\eta}_{i_1,j}^{(1)}$ of random vector fields $\tilde{\boldsymbol{\eta}}_{i_1}^{(1)}$, $i_1=\overline{1,n_1}$, $j=\overline{1,n}$, with random fields $\tilde{\eta}_{i_2}^{(2)}$, $i_2=\overline{1,n_2}$, and hence the set G_1 is described by the formula

$$G_1=\Big\{\tilde{\eta}=(\tilde{\eta}_1,\tilde{\eta}_2):\tilde{\eta}_1=(\tilde{\boldsymbol{\eta}}_1^{(1)},\dots,\tilde{\boldsymbol{\eta}}_{i_1,1}^{(1)},\dots\tilde{\boldsymbol{\eta}}_{n_1}^{(1)}),\ \tilde{\boldsymbol{\eta}}_{i_1}^{(1)}=(\tilde{\eta}_{i_1,1}^{(1)},\dots,\tilde{\eta}_{i_1,n}^{(1)})^T$$

$$\in L^2(\Omega,L^2(D_{i_1}^{(1)})^n),\ \tilde{\eta}_2=(\tilde{\eta}_1^{(2)},\dots,\tilde{\eta}_{i_2}^{(2)},\dots,\tilde{\eta}_{n_2}^{(2)}),\ \tilde{\eta}_{i_2}^{(2)}\in L^2(\Omega,L^2(D_{i_2}^{(2)})),$$

$$\mathbb{E}\tilde{\boldsymbol{\eta}}_{i_1}^{(1)}=0,\ \mathbb{E}\tilde{\eta}_{i_2}^{(2)}=0,\ \tilde{\eta}_{i_1,j}^{(1)}\text{ and }\tilde{\eta}_{i_2}^{(2)}\text{ are uncorrelated},\ j=\overline{1,n},\ i_1=\overline{1,n_1},$$

$$i_2=\overline{1,n_2},\quad\sum_{i_1=1}^{n_1}\int_{D_{i_1}^{(1)}}\mathrm{Sp}\,[\tilde{\mathbf{Q}}_{i_1}^{(1)}(x)\tilde{\mathbf{R}}_{i_1}^{(1)}(x,x)]\,dx\le 1\,,$$

$$\sum_{i_1=1}^{n_1}\int_{D_{i_2}^{(2)}}\tilde{Q}_{i_2}^{(2)}(x)\tilde{R}_{i_2}^{(2)}(x,x)\,dx\le 1\Big\}\,. \qquad (150)$$

The transpose operators C_1^t and C_2^t of C_1 and C_2 have the form

$$C_1^t : L^2\big(D_1^{(1)}\big)^n \times \cdots \times L^2\big(D_{i_1}^{(1)}\big)^n \times \cdots \times L^2\big(D_{n_1}^{(1)}\big)^n \to L^2(D)^n,$$

$$C_1^t\psi_1(x) = \sum_{l_1=1}^{n_1} \chi_{D_{l_1}^{(1)}}(x) \int_{D_{l_1}^{(1)}} [\mathbf{K}_{l_1}^{(1)}(\xi, x)]^T \boldsymbol{\psi}_{l_1}^{(1)}(\xi)\, d\xi,$$

where $\psi_1 = (\boldsymbol{\psi}_1^{(1)}, \ldots, \boldsymbol{\psi}_{l_1}^{(1)}, \ldots, \boldsymbol{\psi}_{n_1}^{(1)})$, $\boldsymbol{\psi}_{l_1} \in L^2\big(D_{l_1}^{(1)}\big)^n$, $l_1 = 1, \ldots, n_1$, and

$$C_2^t : L^2\big(D_1^{(2)}\big) \times \cdots \times L^2\big(D_{i_2}^{(2)}\big) \times \cdots \times L^2\big(D_{n_2}^{(2)}\big) \to L^2(D),$$

$$C_2^t\psi_2(x) = \sum_{l_2=1}^{n_2} \chi_{D_{l_2}^{(2)}}(x) \int_{D_{l_2}^{(2)}} K_{l_2}^{(2)}(\xi, x)\psi_{l_2}^{(2)}(\xi)\, d\xi,$$

where $\psi_2 = (\psi_1^{(2)}, \ldots, \psi_{l_2}^{(2)}, \ldots, \psi_{n_2}^{(2)})$, $\psi_2 \in L^2(D_{l_2}^{(2)})$, $l_2 = 1, \ldots, n_2$, and $\chi(M)$ is a characteristic function of the set $M \subset \mathbb{R}^n$. Since

$$\hat{u}_1 = \tilde{Q}_1 C_1 \mathbf{p}_1 = (\hat{\mathbf{u}}_1^1, \ldots, \hat{\mathbf{u}}_{l_1}^1, \ldots, \hat{\mathbf{u}}_{n_1}^1), \quad \hat{\mathbf{u}}_{l_1}^1 \in L^2\big(D_{l_1}^{(1)}\big)^n, \quad l_1 = 1, \ldots, n_1,$$

$$\hat{u}_2 = \tilde{Q}_2 C_2 p_2 = (\hat{u}_1^2, \ldots, \hat{u}_{l_2}^2, \ldots, \hat{u}_{n_2}^2), \quad \hat{u}_{l_2}^2 \in L^2(D_{l_2}^{(2)}), \quad l_2 = 1, \ldots, n_2,$$

where

$$\hat{\mathbf{u}}_{i_1}^{(1)}(\cdot) = \tilde{\mathbf{Q}}_{i_1}^{(1)}(\cdot) \int_{D_{l_1}^{(1)}} \mathbf{K}_{i_1}^{(1)}(\cdot, \eta)\mathbf{p}_1(\eta)\, d\eta, \quad i_1 = 1, \ldots, n_1, \tag{151}$$

$$\hat{u}_{i_2}^{(2)}(\cdot) = \tilde{Q}_{i_2}^{(2)}(\cdot) \int_{D_{l_2}^{(2)}} K_{i_2}^{(2)}(\cdot, \eta)p_2(\eta)\, d\eta, \quad i_2 = 1, \ldots, n_2, \tag{152}$$

we find

$$C_1^t J_{H_1} \tilde{Q}_1 C_1 \mathbf{p}_1(\cdot) = C_1^t \hat{u}_1(\cdot) = \sum_{l_1=1}^{n_1} \chi_{D_{l_1}^{(1)}}(\cdot) \int_{D_{l_1}^{(1)}} [\mathbf{K}_{l_1}^{(1)}(\xi, \cdot)]^T \hat{\mathbf{u}}_{l_1}(\xi)\, d\xi$$

$$= \sum_{l_1=1}^{n_1} \chi_{D_{l_1}^{(1)}}(\cdot) \int_{D_{l_1}^{(1)}} [\mathbf{K}_{l_1}^{(1)}(\xi, \cdot)]^T \tilde{\mathbf{Q}}_{l_1}^{(1)}(\xi) \int_{D_{l_1}^{(1)}} \mathbf{K}_{l_1}^{(1)}(\xi, \xi_1)\mathbf{p}_1(\xi_1)\, d\xi_1\, d\xi =$$

$$= \sum_{l_1=1}^{n_1} \chi_{D_{l_1}^{(1)}}(\cdot) \int_{D_{l_1}^{(1)}} \tilde{\mathbf{K}}_{l_1}^{(1)}(\cdot, \xi_1)\mathbf{p}_1(\xi_1)\, d\xi_1, \tag{153}$$

$$C_2^t J_{H_2} \tilde{Q}_2 C_2 p_2(\cdot) = \sum_{l_2=1}^{n_2} \chi_{D_{l_2}^{(2)}}(\cdot) \int_{D_{l_2}^{(2)}} \tilde{K}_{l_2}^{(2)}(\cdot, \xi_1)p_2(\xi_1)\, d\xi_1, \tag{154}$$

where [9]

$$\tilde{\mathbf{K}}^{(1)}_{l_1}(\cdot,\xi_1)=\int_{D^{(1)}_{l_1}}(\mathbf{K}^{(1)}_{l_1}(\xi,\cdot))^T\tilde{\mathbf{Q}}^{(1)}_{l_1}(\xi)\mathbf{K}^{(1)}_{l_1}(\xi,\xi_1)d\xi,$$

$$\tilde{K}^{(2)}_{l_2}(\cdot,\xi_1)=\int_{D^{(1)}_{l_1}}K^{(2)}_{l_2}(\xi,\cdot))\tilde{Q}^{(1)}_{l_2}(\xi)K^{(2)}_{l_2}(\xi,\xi_1)d\xi.$$

A class of estimates $\widehat{l(\mathbf{j},\varphi)}$, linear with respect to the observations (148) and (149), takes the form

$$\widehat{l(\mathbf{j},\varphi)}=\sum_{i_1=1}^{n_1}\int_{D^{(1)}_{i_1}}(\mathbf{u}^{(1)}_{i_1}(x),\mathbf{y}^{(1)}_{i_1}(x))_{\mathbb{R}^n}\,dx+\sum_{i_2=1}^{n_2}\int_{D^{(2)}_{i_2}}u^{(2)}_{i_2}(x)y^{(2)}_{i_2}(x)\,dx+c. \quad (155)$$

Thus, taking into account (151)–(155), we obtain that for observations of the form (148)–(149), under assumptions (20), (150), and (24), the following result is valid as a corollary from Theorems 4.1 and 4.2.

Theorem 6.1. *The guaranteed estimate* $\widehat{\widehat{l(\mathbf{j},\varphi)}}$ *of* $l(\mathbf{j},\varphi)$ *is determined by the formula*

$$\widehat{\widehat{l(\mathbf{j},\varphi)}}=\sum_{i_1=1}^{n_1}\int_{D^{(1)}_{i_1}}(\hat{\mathbf{u}}^{(1)}_{i_1}(x),\mathbf{y}^{(1)}_{i_1}(x))_{\mathbb{R}^n}\,dx+\sum_{i_2=1}^{n_2}\int_{D^{(2)}_{i_2}}\hat{u}^{(2)}_{i_2}(x)y^{(2)}_{i_2}(x)\,dx+\hat{c}=l(\hat{\mathbf{j}},\hat{\varphi}),$$

where

$$\hat{c}=-\int_D\hat{z}_2(x)f_0(x)\,dx,$$

$$\hat{\mathbf{u}}^{(1)}_{i_1}(x)=\tilde{\mathbf{Q}}^{(1)}_{i_1}(x)\int_{D^{(1)}_{i_1}}\mathbf{K}^{(1)}_{i_1}(x,\eta)\mathbf{p}_1(\eta)\,d\eta,\quad i_1=\overline{1,n_1},$$

$$\hat{u}^{(2)}_{i_2}(x)=\tilde{Q}^{(2)}_{i_2}(x)\int_{D^{(2)}_{i_2}}K^{(2)}_{i_2}(x,\eta)p_2(\eta)\,d\eta,\quad i_2=\overline{1,n_2},$$

and functions $\mathbf{p}_1\in H(\mathrm{div},D)$, $\hat{z}_2,p_2\in L^2(D)$ *and* $\hat{\mathbf{j}}\in H(\mathrm{div},D)$, $\hat{\varphi}\in L^2(D)$ *are found from the solution to the systems of variational equations*

$$\int_D(((\mathbf{A}(x))^{-1})^T\hat{\mathbf{z}}_1(x),\mathbf{q}_1(x))_{\mathbb{R}^n}dx-\int_D\hat{z}_2(x)\,\mathrm{div}\,\mathbf{q}_1(x)\,dx=$$

$$=\int_D\Big(\mathbf{l}_1(x)-\sum_{i_1=1}^{n_1}\chi_{D^{(1)}_{i_1}}(x)\int_{D^{(1)}_{i_1}}\tilde{\mathbf{K}}^{(1)}_{i_1}(x,\xi_1)\mathbf{p}_1(\xi_1)\,d\xi_1,\mathbf{q}_1(x)\Big)_{\mathbb{R}^n}dx\quad\forall\mathbf{q}_1\in H(\mathrm{div},D),$$

---

[9] We use the following notation: if $\mathbf{A}(\xi)=[a_{ij}(\xi)]_{i,j=1}^N$ is a matrix dependig on variable ξ that varies on measurable set Ω, then we define $\int_\Omega\mathbf{A}(\xi)\,d\xi$ by the equality

$$\int_\Omega\mathbf{A}(\xi)\,d\xi=\left[\int_\Omega a_{ij}(\xi)\,d\xi\right]_{i,j=1}^N.$$

$$-\int_D v_1(x)\operatorname{div}\hat{\mathbf{z}}_1(x)\,dx-\int_D c(x)\hat{z}_2(x)v_1(x)\,dx=$$

$$=\int_D\Big(l_2(x)-\sum_{i_2=1}^{n_2}\chi_{D^{(2)}_{i_2}}(x)\int_{D^{(2)}_{i_2}}\tilde{K}^{(2)}_{i_2}(x,\xi_1)p_2(\xi_1)\,d\xi_1\Big)v_1(x)\,dx\quad\forall v_1\in L^2(D),$$

$$\int_D((\mathbf{A}(x))^{-1}\mathbf{p}_1(x),\mathbf{q}_2(x))_{\mathbb{R}^n}dx-\int_D p_2(x)\operatorname{div}\mathbf{q}_2(x)\,dx=0\quad\forall\mathbf{q}_2\in H(\operatorname{div},D),$$

$$-\int_D v_2(x)\ \operatorname{div}\mathbf{p}_1(x)\,dx-\int_D c(x)\,p_2(x)\,v_2(x)\,dx=$$

$$=\int_D v_2(x)\,Q^{-1}\hat{z}_2(x)\,dx\quad\forall v_2\in L^2(D).$$

and

$$\int_D(((\mathbf{A}(x))^{-1})^T\hat{\mathbf{p}}_1(x),\mathbf{q}_1(x))_{\mathbb{R}^n}dx-\int_D\hat{p}_2(x)\operatorname{div}\mathbf{q}_1(x)\,dx=$$

$$=\int_D\Big(\mathbf{d}_1(x)-\sum_{i_1=1}^{n_1}\chi_{D^{(1)}_{i_1}}(x)\int_{D^{(1)}_{i_1}}\tilde{\mathbf{K}}^{(1)}_{i_1}(x,\xi_1)\hat{\mathbf{j}}_1(\xi_1)\,d\xi_1,\mathbf{q}_1(x)\Big)_{\mathbb{R}^n}dx\quad\forall\mathbf{q}_1\in H(\operatorname{div},D),$$

$$-\int_D v_1(x)\operatorname{div}\hat{\mathbf{p}}_1(x)\,dx-\int_D c(x)\hat{p}_2(x)v_1(x)\,dx=$$

$$=\int_D\Big(d_2(x)-\sum_{i_2=1}^{n_2}\chi_{D^{(2)}_{i_2}}(x)\int_{D^{(2)}_{i_2}}\tilde{K}^{(2)}_{i_2}(x,\xi_1)\hat{\varphi}(\xi_1)\,d\xi_1\Big)v_1(x)\,dx\quad\forall v_1\in L^2(D),$$

$$\int_D((\mathbf{A}(x))^{-1}\hat{\mathbf{j}}(x),\mathbf{q}_2(x))_{\mathbb{R}^n}\,dx-\int_D\hat{\varphi}(x)\operatorname{div}\mathbf{q}_2(x)\,dx=0\quad\forall\mathbf{q}_2\in H(\operatorname{div},D),$$

$$-\int_D v_2(x)\operatorname{div}\hat{\mathbf{j}}(x)\,dx-\int_D c(x)\hat{\varphi}(x)v_2(x)\,dx=$$

$$=\int_D v_2(x)(Q^{-1}\hat{p}_2(x)+f_0(x))\,dx\quad\forall v_2\in L^2(D),$$

respectively. Here $\hat{\mathbf{z}}_1,\hat{\mathbf{p}}_1\in H(\operatorname{div},D),\ \hat{p}_2\in L^2(D)$ *and*

$$\mathbf{d}_1(x)=\sum_{i_1=1}^{n_1}\chi_{D^{(1)}_{i_1}}(x)\int_{D^{(1)}_{i_1}}(\mathbf{K}^{(1)}_{i_1}(\xi,x))^T\tilde{\mathbf{Q}}^{(1)}_{i_1}(\xi)\mathbf{y}^{(1)}_{i_1}(\xi)\,d\xi\,,$$

$$d_2(x)=\sum_{i_2=1}^{n_2}\chi_{D^{(2)}_{i_2}}(x)\int_{D^{(2)}_{i_2}}K^{(2)}_{i_2}(\xi,x)\tilde{Q}^{(2)}_{i_2}(\xi)y^{(2)}_{i_2}(\xi)\,d\xi\,.$$

The estimation error σ *is given by the expression*

$$\sigma=l(\mathbf{p}_1,p_2)^{1/2}.$$

7. Minimax estimation of linear functionals from right-hand sides of elliptic equations: Representations for guaranteed estimates and estimation errors

The problem is to determine a minimax estimate of the value of the functional

$$l(f) := \int_D l_0(x)f(x)\,dx \tag{156}$$

from observations (21) in the class of estimates, linear with respect to observations,

$$\widehat{l(f)} := (y_1, u_1)_{H_1} + (y_2, u_2)_{H_2} + c, \tag{157}$$

where u_1 and u_2 are elements from Hilbert spaces H_1 and H_2, respectively, $c \in \mathbb{R}$, $l_0 \in L^2(D)$ is a given function, under the assumption that $f \in G_0$ and $\eta \in G_1$, where sets G_0 and G_1 are defined on page 183.

Definition 7.1. The estimate of the form

$$\widehat{\widehat{l(f)}} = (y_1, \hat{u}_1)_{H_1} + (y_2, \hat{u}_2)_{H_2} + \hat{c} \tag{158}$$

will be called the *guaranteed estimate* of $l(f)$ if the elements $\hat{u}_1 \in H_1$, $\hat{u}_2 \in H_2$ and a number $\hat{c}$ are determined from the condition

$$\inf_{u \in H,\, c \in \mathbb{R}} \sigma(u, c) = \sigma(\hat{u}, \hat{c}),$$

where $u = (u_1, u_2) \in H = H_1 \times H_2$, $\hat{u} = (\hat{u}_1, \hat{u}_2) \in H$,

$$\sigma(u, c) := \sup_{\tilde{f} \in G_0, (\tilde{\eta}_1, \tilde{\eta}_2) \in G_1} \mathbb{E}|l(\tilde{f}) - \widehat{l(\tilde{f})}|^2,$$

$$\widehat{l(\tilde{f})} := (\tilde{y}_1, u_1)_{H_1} + (\tilde{y}_2, u_2)_{H_2} + c, \tag{159}$$

$\tilde{y}_1 = C_1\tilde{\mathbf{j}} + \tilde{\eta}_1$, $\tilde{y}_2 = C_2\tilde{\varphi} + \tilde{\eta}_2$, and $(\tilde{\mathbf{j}}, \tilde{\varphi})$ is a solution to problem (10)–(11) when $f(x) = \tilde{f}(x)$. The quantity

$$\sigma := [\sigma(\hat{u}, \hat{c})]^{1/2}$$

is called the *error of the guaranteed estimation* of $l(f)$.

For any fixed $u \in H$ introduce a pair of functions $(\mathbf{z}_1(\cdot; u), z_2(\cdot; u)) \in H(\mathrm{div}; D) \times L^2(D)$ as a unique solution of the following problem:

$$\int_D (((\mathbf{A}(x))^{-1})^T \mathbf{z}_1(x; u), \mathbf{q}(x))_{\mathbb{R}^n} dx - \int_D z_2(x; u)\mathrm{div}\,\mathbf{q}(x)\,dx =$$
$$= -\int_D ((C_1^t J_{H_1} u_1)(x), \mathbf{q}(x))_{\mathbb{R}^n} dx \quad \forall \mathbf{q} \in H(\mathrm{div}, D), \tag{160}$$

$$\int_D v(x)\mathrm{div}\,\mathbf{z}_1(x; u)\,dx + \int_D c(x) z_2(x; u) v(x)\,dx =$$
$$= \int_D (C_2^t J_{H_2} u_2)(x) v(x)\,dx \quad \forall v \in L^2(D). \tag{161}$$

Lemma 7.2. *Finding the guaranteed estimate of $l(f)$ is equivalent to the problem of optimal control of a system described by the problem (160), (161) with cost function*

$$I(u) = \big(Q^{-1}(l_0 - z_2(\cdot;u)), l_0 - z_2(\cdot;u)\big)_{L^2(D)} \\ + (\tilde{Q}_1^{-1}u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2} \to \inf_{u\in H}. \quad (162)$$

Proof. Taking into account (156) with $f = \tilde{f}$ and (159), we have

$$\begin{gathered} l(\tilde{f}) - \widehat{l(\tilde{f})} = (l_0, \tilde{f})_{L^2(D)} - (\tilde{y}_1, u_1)_{H_1} - (\tilde{y}_2, u_2)_{H_2} - c = \\ = (l_0, \tilde{f})_{L^2(D)} - (u_1, C_1\tilde{\mathbf{j}} + \tilde{\eta}_1)_{H_1} - (u_2, C_2\tilde{\varphi} + \tilde{\eta}_2)_{H_2} - c = \\ = (l_0, \tilde{f})_{L^2(D)} - < J_{H_1}u_1, C_1\tilde{\mathbf{j}} >_{H_1'\times H_1} - < J_{H_2}u_2, C_2\tilde{\varphi} >_{H_2'\times H_2} - \\ -(u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c = \\ = -(C_1^t J_{H_1}u_1, \tilde{\mathbf{j}})_{L^2(D)^n} - (C_2^t J_{H_2}u_2, \tilde{\varphi})_{L^2(D)} + \\ +(l_0, \tilde{f})_{L^2(D)} - (u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c\,. \quad (163) \end{gathered}$$

Using a similar argument as in the proof of Lemma 1 in which the solution of problem (29), (30) is substituted by the solution of problem (160), (161), we obtain from (163) the following representation

$$\begin{gathered} l(\tilde{f}) - \widehat{l(\tilde{f})} = (\tilde{f}, l_0 - z_2(\cdot;u)_{L^2(D)} - (u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c = \\ = (\tilde{f} - f_0, l_0 - z_2(\cdot;u))_{L^2(D)} + (f_0, l_0 - z_2(\cdot;u))_{L^2(D)} - \\ -(u_1, \tilde{\eta}_1)_{H_1} - (u_2, \tilde{\eta}_2)_{H_2} - c. \end{gathered}$$

By virtue of (42), we find from here

$$\mathbb{E}\left|l(\tilde{f}) - \widehat{(l(\tilde{f})}\right|^2 = \left|(\tilde{f}_2 - f_0, l_0 - z_2(\cdot;u))_{L^2(D)} + (f_0, l_0 - z_2(\cdot;u))_{L^2(D)} - c\right|^2 + \\ +\mathbb{E}[(u_1, \tilde{\eta}_1)_{H_1} + (u_2, \tilde{\eta}_2)_{H_2}]^2.$$

From the latter equality, we obtain

$$\begin{gathered} \inf_{c\in\mathbb{R}} \sup_{\tilde{f}\in G_0, (\tilde{\eta}_1,\tilde{\eta})\in G_1} \mathbb{E}|l(\tilde{f}) - \widehat{l(\tilde{f})}|^2 = \\ = \inf_{c\in\mathbb{R}} \sup_{\tilde{f}\in G_0} \Big[(\tilde{f} - f_0, l_0 - z_2(\cdot;u))_{L^2(D)} + (f_0, l_0 - z_2(\cdot;u))_{L^2(D)} - c\Big]^2 + \\ + \sup_{(\tilde{\eta}_1,\tilde{\eta}_2)\in G_1} \mathbb{E}[(\tilde{\eta}_1, u_1)_{H_1} + (\tilde{\eta}_2, u_2)_{H_2}]^2 = \\ = \sup_{\tilde{f}\in G_0} \Big[(\tilde{f} - f^{(0)}, l_0 - z_2(\cdot;u))_{L^2(D)}\Big]^2 + \sup_{(\tilde{\eta}_1,\tilde{\eta}_2)\in G_1} \mathbb{E}[(\tilde{\eta}_1, u_1)_{H_1} + (\tilde{\eta}_2, u_2)_{H_2}]^2, \quad (164) \end{gathered}$$

where the infimum over c is attained at $c = (f_0, l_0 - z_2(\cdot;u))_{L^2(D)}$. The Cauchy-Bunyakovsky inequality and (20) imply

$$|(\tilde{f} - f_0, l_0 - z_2(\cdot;u))_{L^2(D)}|^2 \leq$$
$$\leq (Q^{-1}(l_0 - z_2(\cdot;u)), l_0 - z_2(\cdot;u))_{L^2(D)}(Q(\tilde{f} - f_0), \tilde{f} - f_0)_{L^2(D)}$$
$$\leq (Q^{-1}(l_0 - z_2(\cdot;u)), l_0 - z_2(\cdot;u))_{L^2(D)},$$

where the inequality becomes an equality at

$$\tilde{f} = f_0 + \frac{Q^{-1}(l_0 - z_2(\cdot;u))}{(Q^{-1}(l_0 - z_2(\cdot;u)), l_0 - z(\cdot;u))^{1/2}_{L^2(D)}} \ .$$

Hence

$$\sup_{\tilde{f}\in G_0} \left[(\tilde{f}_2 - f_2^{(0)}, l_0 - z_2(\cdot;u))_{L^2(D)}\right]^2 = (Q^{-1}(l_0 - z_2(\cdot;u)), l_0 - z_2(\cdot;u))_{L^2(D)} \ .$$

Analoguosly, due to (25), (6), and (26), we have

$$\sup_{(\tilde{\eta}_1,\tilde{\eta}_2)\in G_1} \mathbb{E}[(\tilde{\eta}_1, u_1)_{H_1} + (\tilde{\eta}_2, u_2)_{H_2}]^2 = (\tilde{Q}_1^{-1}u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2} \ .$$

From two latter relations and (164), we get

$$\inf_{c\in\mathbb{R}} \sup_{\tilde{f}\in G_0, (\tilde{\eta}_1,\tilde{\eta})\in G_1} \mathbb{E}|l(\tilde{f}) - \widehat{l(\tilde{f})}|^2 = I(u),$$

at $c = (l_0 - z_2(\cdot;u), f_0)_{L^2(D)}$, where $I(u)$ is determined by (162). □

As a result of solving of optimal control problem (160)–(162), we come to the following assertion.

Theorem 7.3. *There exists a unique estimate of $l(f)$ which has the form*

$$\widehat{\widehat{l(f)}} = (y_1, \hat{u}_1)_{H_1} + (y_2, \hat{u}_2)_{H_2} + \hat{c}, \tag{165}$$

where

$$\hat{c} = \int_D (l_0(x) - \hat{z}_2(x)) f_0(x)\,dx, \quad \hat{u}_1 = \tilde{Q}_1 C_1 \mathbf{p}_1, \quad \hat{u}_2 = \tilde{Q}_2 C_2 p_2, \tag{166}$$

and the functions $\mathbf{p}_1 \in H(\mathrm{div}, D)$ and $\hat{z}_2, p_2 \in L^2(D)$ are found from the solution of the following variational problem:

$$\int_D (((\mathbf{A}(x))^{-1})^T \hat{\mathbf{z}}_1(x), \mathbf{q}_1(x))_{\mathbb{R}^n} dx - \int_D \hat{z}_2(x)\ \mathrm{div}\,\mathbf{q}_1(x)\,dx =$$
$$= -\int_D (C_1^t J_{H_1} \tilde{Q}_1 C_1 \mathbf{p}_1(x), \mathbf{q}_1(x))_{\mathbb{R}^n}\,dx \quad \forall \mathbf{q}_1 \in H(\mathrm{div}, D), \tag{167}$$

$$\int_D v_1(x)\mathrm{div}\,\hat{\mathbf{z}}_1(x)\,dx + \int_D c(x)\hat{z}_2(x)v_1(x)\,dx =$$
$$= \int_D (C_2^t J_{H_2}\tilde{Q}_2 C_2 p_2)(x)v_1(x)\,dx \quad \forall v_1 \in L^2(D), \tag{168}$$

$$\int_D ((\mathbf{A}(x))^{-1}\mathbf{p}_1(x), \mathbf{q}_2(x))_{\mathbb{R}^n} dx - \int_D p_2(x)\ \mathrm{div}\,\mathbf{q}_2(x)\,dx = 0 \tag{169}$$

for all $\mathbf{q}_2 \in H(\mathrm{div}, D)$*, and for all* $v_2 \in L^2(D)$

$$\int_D v_2(x)\,\mathrm{div}\,\mathbf{p}_1(x)\,dx + \int_D c(x)p_2(x)v_2(x)\,dx = \int_D v_2(x)Q^{-1}(l_0 - \hat{z}_2(\cdot))(x)\,dx, \tag{170}$$

where $\hat{\mathbf{z}}_1 \in H(\mathrm{div}, D)$*. Problem* (167)–(170) *is uniquely solvable. The error of estimation* σ *is given by the expression*

$$\sigma = \left(l(Q^{-1}(l_0 - \hat{z}_2))\right)^{1/2}. \tag{171}$$

Proof. Show that the solution to the optimal control problem (160)–(162) can be reduced to the solution of the system (167)–(170).

First, we note that the fuctional $I(u)$, defined by (162), can be represented in the form

$$I(u) = \tilde{I}(u) - L(u) + \int_D Q^{-1} l_0(x) l_0(x)\,dx, \tag{172}$$

where

$$\tilde{I}(u) = \int_D Q^{-1} z_2(x;u) z_2(x;u)\,dx + (\tilde{Q}_1^{-1}u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2}$$

is a quadratic form corresponding to a symmetric continuous bilinear form

$$\pi(u,v) := \int_D Q^{-1} z_2(x;u) z_2(x;v)\,dx + (\tilde{Q}_1^{-1}u_1, v_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, v_2)_{H_2},$$

defined on $H \times H$ and

$$L(u) = 2\int_D Q^{-1} z_2(x;u) l_0(x)\,dx$$

is a linear continuous functional defined on H.

The representation of $I(u)$ in the form (172) follows from the reasoning similar to that in the proof of Theorem 4.1 (replacing $\tilde{z}_2(x;u)$ by $z_2(x;u)$ and $z_2^{(0)}(x)$ by $l_0(x)$, correspondingly). Since

$$\tilde{I}(u) = \pi(u,u) \geq (Q_1^{-1}u_1, u_1)_{H_1} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2} \geq \alpha \|u\|_H^2 \quad \forall u \in H,$$

where α is a constant from (27), then the bilinear form $\pi(u,v)$ and the linear functional $L(u)$ satisfy the condition of Theorem 1.1 from [9]. Therefore, by this theorem, there exists a unique element $\hat{u} := (\hat{u}_1, \hat{u}_2) \in H$ on which the minimum of the functional $I(u)$ is attained, i.e. $I(\hat{u}) = \inf_{u\in H} I(u)$. This implies that for any fixed $w = (w_1, w_2) \in H$ and $\tau \in \mathbb{R}$ the function $s(\tau) := I(\hat{u} + \tau w)$ reaches its minimum at the point $\tau = 0$, so that

$$\frac{d}{d\tau} I(\hat{u} + \tau w) \,|_{\tau=0} = 0. \tag{173}$$

Taking into account that $z_2(x; \hat{u} + \tau w) = z_2(x; \hat{u}) + \tau z_2(x; w)$, we obtain from (173)

$$\frac{1}{2}\frac{d}{dt} I(\hat{u} + \tau w)\Big|_{\tau=0} = 0 =$$

$$= -(Q^{-1}(l_0 - z_2(\cdot; \hat{u})), z_2(\cdot; w))_{L^2(D)} + (\tilde{Q}_1^{-1}\hat{u}_1, w_1)_{H_1} + (\tilde{Q}_2^{-1}\hat{u}_2, w_2)_{H_2}. \tag{174}$$

Further, introducing a pair of functions $(\mathbf{p}_1, p_2) \in H(\mathrm{div}, D) \times L^2(D)$ as a unique solution of the problem

$$\int_D ((\mathbf{A}(x))^{-1}\mathbf{p}_1(x), \mathbf{q}_2(x))_{\mathbb{R}^n} dx -$$

$$- \int_D p_2(x)\, \mathrm{div}\, \mathbf{q}_2(x)\, dx = 0 \quad \forall \mathbf{q}_2 \in H(\mathrm{div}, D), \tag{175}$$

$$\int_D v_2(x)\, \mathrm{div}\, \mathbf{p}_1(x)\, dx + \int_D c(x) p_2(x) v_2(x)\, dx =$$

$$= \int_D v_2(x) Q^{-1}(l_0 - z_2(\cdot; \hat{u}))(x)\, dx \quad \forall v_2 \in L^2(D) \tag{176}$$

and reasoning analogously as in the proof of Theorem 4.1, we arrive at the following relation

$$-(Q^{-1}(l_0 - z_2(\cdot; \hat{u})), z_2(\cdot; w))_{L^2(D)} = -(w_1, C_1\mathbf{p}_1)_{H_1} - (w_2, C_2 p_2)_{H_2}\,.$$

From the latter we find

$$(w_1, C_1\mathbf{p}_1)_{H_1} + (w_2, C_2 p_2)_{H_2} = (\tilde{Q}_1^{-1}\hat{u}_1, w_1)_{H_1} + (\tilde{Q}_2^{-1}\hat{u}_2, w_2)_{H_2}\,,$$

whence it follows that $\hat{u}_1 = \tilde{Q}_1 C_1 \mathbf{p}_1$ and $\hat{u}_2 = \tilde{Q}_2 C_2 p_2$. Substituting these expressions into (160) and (161) and setting $\mathbf{z}_1(x; \hat{u}) =: \hat{\mathbf{z}}_1(x)$, $z_2(x; \hat{u}) =: \hat{z}_2(x)$, we establish that the functions $\hat{\mathbf{z}}_1$, $\hat{z}_2$ and $\mathbf{p}_1$, p_2 satisfy the system (167)–(170) of variational equations and the validity of the equalities (165), (166) is guaranteed. The unique solvability of this system follows from the existence of a unique minimum point $\hat{u}$ of the functional $I(u)$.

Next we determine the error of estimation. From (162) at $u = \hat{u}$ and (166), it follows

$$\sigma^2 = I(\hat{u}) =$$

$$= (Q^{-1}(l_0 - z_2(\cdot; \hat{u})), l_0 - z_2(\cdot; \hat{u}))_{L^2(D)} + (\tilde{Q}_1^{-1}\hat{u}_1, \hat{u}_1)_{H_1} + (\tilde{Q}_2^{-1}\hat{u}_2, \hat{u}_2)_{H_2} =$$

$$= (Q^{-1}(l_0 - \hat{z}_2), l_0 - \hat{z}_2)_{L^2(D)} + (C_1\mathbf{p}_1, \tilde{Q}_1 C_1 \mathbf{p}_1)_{H_1} + (C_2 p_2, \tilde{Q}_2 C_2 p_2)_{H_2}. \tag{177}$$

Setting in (169) and (170) $\mathbf{q}_2 = \hat{\mathbf{z}}_1$ and $v_2 = \hat{z}_2$, we find

$$\int_D ((\mathbf{A}(x))^{-1}\mathbf{p}_1(x), \hat{\mathbf{z}}_1(x))_{\mathbb{R}^n} dx - \int_D p_2(x)\mathrm{div}\,\hat{\mathbf{z}}_1(x)\,dx = 0,$$

$$\int_D \hat{z}_2(x)\mathrm{div}\,\mathbf{p}_1(x)\,dx + \int_D c(x)p_2(x)\hat{z}_2(x)\,dx = \int_D \hat{z}_2(x)Q^{-1}(l_0 - \hat{z}_2)(x)\,dx\,.$$

Setting in (167)–(168) $\mathbf{q}_1 = \mathbf{p}_1$ and $v_1 = p_2$, we derive from two latter relations

$$(Q^{-1}(l_0 - \hat{z}_2), l_0 - \hat{z}_2)_{L^2(D)} = (Q^{-1}l_0, l_0 - \hat{z}_2)_{L^2(D)} - \int_D \hat{z}_2(x)\ \mathrm{div}\,\hat{\mathbf{p}}_1(x)\,dx -$$

$$-\int_D c(x)p_2(x)\hat{z}_2(x)\,dx + \int_D ((\mathbf{A}(x))^{-1}\mathbf{p}_1(x), \hat{\mathbf{z}}_1(x))_{\mathbb{R}^n} dx - \int_D p_2(x)\ \mathrm{div}\,\hat{\mathbf{z}}_1(x)\,dx =$$

$$= (l_0, Q^{-1}(l_0 - \hat{z}_2))_{L^2(D)} + \int_D (((\mathbf{A}(x))^{-1})^T\hat{\mathbf{z}}_1(x), \mathbf{p}_1(x))_{\mathbb{R}^n} dx - \int_D \hat{z}_2(x)\mathrm{div}\,\mathbf{p}_1(x)\,dx$$

$$-\int_D p_2(x)\mathrm{div}\,\hat{\mathbf{z}}_1(x)\,dx - \int_D c(x)p_2(x)\hat{z}_2(x)\,dx = (l_0, Q^{-1}(l_0 - \hat{z}_2))_{L^2(D)} -$$

$$-\int_D (C_1^t J_{H_1}\tilde{Q}_1 C_1\mathbf{p}_1(x), \mathbf{p}_1(x))_{\mathbb{R}^n}\,dx - \int_D C_2^t J_{H_2}\tilde{Q}_2 C_2 p_2(x)p_2(x)\,dx =$$

$$= l(Q^{-1}(l_0 - \hat{z}_2)) - (C_1\mathbf{p}_1, \tilde{Q}_1 C_1\mathbf{p}_1)_{H_1} - (C_2p_2, \tilde{Q}_2C_2p_2)_{H_2}.$$

From here and (177) we get the representation (171) for the estimation error. □

In the following theorem we obtain another representation for the guaranteed estimate $\widehat{\widehat{l(f)}}$ of quantity $l(f)$ similar to (67).

Theorem 7.4. *The guaranteed estimate of* $l(f)$ *has the form*

$$\widehat{\widehat{l(f)}} = l(\hat{f}), \tag{178}$$

where $\hat{f}(x) = f_0(x) - Q^{-1}\hat{p}_2(x)$ *and* $\hat{p}_2 \in L^2(D)$ *is determined from solution of problem* (68)–(71).

Proof. From (165) and (166), we have

$$\widehat{\widehat{l(f)}} = (y_1, \hat{u}_1)_{H_1} + (y_2, \hat{u}_2)_{H_2} + \hat{c} =$$

$$= (y_1, \tilde{Q}_1C_1\mathbf{p}_1)_{H_1} + (y_2, \tilde{Q}_2C_2p_2)_{H_2} + (l_0 - \hat{z}_2, f_0)_{L^2(D)}. \tag{179}$$

Putting in (68)–(69) $\mathbf{q}_1 = \mathbf{p}_1$ and $v_1 = p_2$, respectively, we obtain

$$\int_D (((\mathbf{A}(x))^{-1})^T\hat{\mathbf{p}}_1(x), \mathbf{p}_1(x))_{\mathbb{R}^n} dx - \int_D \hat{p}_2(x)\ \mathrm{div}\,\mathbf{p}_1(x)\,dx =$$

$$= \int_D (C_1^t J_{H_1}\tilde{Q}_1(y_1 - C_1\hat{\mathbf{j}})(x), \mathbf{p}_1(x))_{\mathbb{R}^n} dx, \tag{180}$$

$$-\int_D p_2(x)\operatorname{div}\hat{\mathbf{p}}_1(x)\,dx-\int_D c(x)\hat{p}_2(x)p_2(x)\,dx=$$
$$=\int_D (C_2^t J_{H_2}\tilde{Q}_2(y_2-C_2\hat{\varphi})(x)p_2(x)\,dx\,. \tag{181}$$

Putting $\mathbf{q}_2=\hat{\mathbf{p}}_1$ and $v_2=\hat{p}_2$ in (169) and (170), we have

$$\int_D ((\mathbf{A}(x))^{-1}\,\mathbf{p}_1(x),\hat{\mathbf{p}}_1(x))_{\mathbb{R}^n}dx-\int_D p_2(x)\operatorname{div}\hat{\mathbf{p}}_1(x)\,dx=0, \tag{182}$$

$$-\int_D \hat{p}_2(x)\operatorname{div}\mathbf{p}_1(x)\,dx-\int_D c(x)p_2(x)\hat{p}_2(x)\,dx=$$
$$=-\int_D \hat{p}_2(x)Q^{-1}(l_0-\hat{z}_2)(x)\,dx\,. \tag{183}$$

The relations (180)–(183) and (179) imply

$$\widehat{\widehat{l(f)}}=(C_1\hat{\mathbf{j}},\tilde{Q}_1C_1\mathbf{p}_1)_{H_1}+(C_2\hat{\varphi},\tilde{Q}_2C_2p_2)_{H_2}-(Q^{-1}\hat{p}_2-f_0,l_0-\hat{z}_2)_{L^2(D)}\,. \tag{184}$$

Setting $\mathbf{q}_2=\hat{\mathbf{z}}_1$, $v_2=\hat{z}_2$ and $\mathbf{q}_1=\hat{\mathbf{j}}$, $v_1=\hat{\varphi}$ in the equations (70), (71) and (167), (168), respectively, we obtain

$$\int_D ((\mathbf{A}(x))^{-1}\hat{\mathbf{j}}(x),\hat{\mathbf{z}}_1(x))_{\mathbb{R}^n}\,dx-\int_D \hat{\varphi}(x)\operatorname{div}\hat{\mathbf{z}}_1(x)\,dx=0\,, \tag{185}$$

$$-\int_D \hat{z}_2(x)\operatorname{div}\hat{\mathbf{j}}(x)\,dx-\int_D c(x)\hat{\varphi}(x)\hat{z}_2(x)\,dx=\int_D \hat{z}_2(x)(Q^{-1}\hat{p}_2(x)-f_0(x))\,dx, \tag{186}$$

and

$$\int_D (((\mathbf{A}(x))^{-1})^T\hat{\mathbf{z}}_1(x),\hat{\mathbf{j}}(x))_{\mathbb{R}^n}dx-\int_D \hat{z}_2(x)\ \operatorname{div}\hat{\mathbf{j}}(x)\,dx=$$
$$=-\int_D (C_1^t J_{H_1}\tilde{Q}_1C_1\mathbf{p}_1(x),\hat{\mathbf{j}}(x))_{\mathbb{R}^n}\,dx, \tag{187}$$

$$-\int_D \hat{\varphi}(x)\operatorname{div}\hat{\mathbf{z}}_1(x)\,dx-\int_D c(x)\hat{z}_2(x)\hat{\varphi}(x)\,dx=-\int_D (C_2^t J_{H_2}\tilde{Q}_2C_2p_2(x)\hat{\varphi}(x)\,dx\,. \tag{188}$$

From (185) and (188), we deduce

$$\int_D \hat{z}_2(x)(Q^{-1}\hat{p}_2(x)-f_0(x))\,dx=-(\tilde{Q}_1C_1\hat{\mathbf{p}}_1,C_1\hat{\mathbf{j}}(x))_{H_1}-(\tilde{Q}_2C_2\hat{p}_2,C_2\hat{\varphi}(x))_{H_2},$$

whence, by virtue of (184), we receive the representation (178). □

Remark 7.5. In the representation $l(\hat{f})$ for the minimax estimate $\widehat{\widehat{l(f)}}$ the function $\hat{f}(x)=f_0(x)-Q^{-1}\hat{p}_2(x)$, where $\hat{p}_2$ is defined from the equations (68)–(71), can be taken as a good estimate for the unknown function f entering the right-hand side of equation (11) (for explanations, see Remark 4.3).

8. Approximate guaranteed estimates of linear functionals from the right-sides of elliptic equations

In this section we introduce the notions of approximate guaranteed estimates of $l(\mathbf{j},\varphi)$ and prove their convergence to $\widehat{l(\mathbf{j},\varphi)}$.

As in section 6, D is supposed to be a bounded and connected domain of $\mathbb{R}^n$ with polyhedral boundary Γ. Take an approximate minimax estimate of $l(f)$ as

$$\widehat{l^h(f)} = (u_1^h, y_1)_{H_1} + (u_2^h, y_2)_{H_1} + c^h, \tag{189}$$

where $u_1^h = \tilde{Q}_1 C_1 \mathbf{p}_1^h$, $u_2^h = \tilde{Q}_2 C_2 p_2^h$, $c^h = \int_D (l_0(x) - \hat{z}_2^h(x)) f_0(x)\,dx$, and the functions $\hat{\mathbf{z}}_1^h, \mathbf{p}_1^h \in V_1^h$ and $\hat{z}_2^h, p_2^h \in V_2^h$ are determined[10] from the following uniquely solvable system of variational equalities:

$$\int_D (((\mathbf{A}(x))^{-1})^T\, \hat{\mathbf{z}}_1^h(x), \mathbf{q}_1^h(x))_{\mathbb{R}^n} dx - \int_D \hat{z}_2^h(x)\ \operatorname{div} \mathbf{q}_1^h(x)\,dx =$$
$$= - \int_D (C_1^t J_{H_1} \tilde{Q}_1 C_1 \mathbf{p}_1^h(x), \mathbf{q}_1^h(x))_{\mathbb{R}^n}\,dx \quad \forall \mathbf{q}_1^h \in V_1^h, \tag{190}$$

$$\int_D v_1^h(x) \operatorname{div} \hat{\mathbf{z}}_1^h(x)\,dx + \int_D c(x) \hat{z}_2^h(x) v_1^h(x)\,dx =$$
$$= \int_D (C_2^t J_{H_2} \tilde{Q}_2 C_2 p_2^h(x) v_1^h(x)\,dx \quad \forall v_1^h \in V_2^h, \tag{191}$$

$$\int_D ((\mathbf{A}(x))^{-1} \mathbf{p}_1^h(x), \mathbf{q}_2^h(x))_{\mathbb{R}^n} dx - \int_D p_2^h(x) \operatorname{div} \mathbf{q}_2^h(x)\,dx = 0 \quad \forall \mathbf{q}_2^h \in V_1^h, \tag{192}$$

$$\int_D v_2^h(x) \operatorname{div} \mathbf{p}_1^h(x)\,dx + \int_D c(x) p_2^h(x) v_2^h(x)\,dx =$$
$$= \int_D v_2^h(x) Q^{-1}(l_0 - \hat{z}_2^h)(x)\,dx \quad \forall v_2^h \in V_2^h. \tag{193}$$

The quantity $\sigma^h = (I(u^h))^{1/2}$, where

$$I(u^h) = (Q^{-1}(l_0 - \hat{z}_2^h), l_0 - \hat{z}_2^h)_{L^2(D)} + (\tilde{Q}_1^{-1} u_1^h, u_1^h)_{H_1} + (\tilde{Q}_2^{-1} u_2^h, u_2^h)_{H_2},$$

is called the *approximate error of the guaranteed estimation of* $l(f)$.

Theorem 8.1. *Approximate guaranteed estimate* $\widehat{l^h(f)}$ *of* $l(f)$ *which is defined by* (189) *can be represented in the form* $\widehat{l^h(f)} = l(\hat{f}^h)$, *where* $\hat{f}^h = f_0(x) - Q^{-1}\hat{p}_2^h(x)$, *and the function* $\hat{p}_2^h \in Q^h$ *is determined from the solution of problem* (137)–(140). *The approximate error of estimation has the form*

$$\sigma^h = \left(l(Q^{-1}(l_0 - \hat{z}_2^h)) \right)^{1/2}.$$

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[10] The spaces V_1^h and V_2^h are described on page 197.

In addition,

$$\lim_{h\to 0}\mathbb{E}|\widehat{l^h(f)}-\widehat{\widehat{l(f)}}|^2=0,\quad \lim_{h\to\infty}\sigma^h=\sigma,$$

and

$$\|\hat{\mathbf{z}}_1-\hat{\mathbf{z}}_1^h\|_{H(\mathrm{div},D)}+\|\hat{z}_2-\hat{z}_2^h\|_{L^2(D)}\to 0\quad\text{as}\quad h\to 0,$$

$$\|\mathbf{p}_1-\mathbf{p}_1^h\|_{H(\mathrm{div},D)}+\|p_2-p_2^h\|_{L^2(D)}\to 0\quad\text{as}\quad h\to 0,$$

$$\|\hat{\mathbf{j}}-\hat{\mathbf{j}}^h\|_{H(\mathrm{div},D)}+\|\hat{\varphi}-\hat{\varphi}^h\|_{L^2(D)}\to 0\quad\text{as}\quad h\to 0,$$

$$\|\hat{\mathbf{p}}_1-\hat{\mathbf{p}}_1^h\|_{H(\mathrm{div},D)}+\|\hat{p}_2-\hat{p}_2^h\|_{L^2(D)}\to 0\quad\text{as}\quad h\to 0.$$

Proof. The proof of this theorem is similar to those of Theorems 5.3 and 6.1. □

The system of linear algebraic equations with respect to the coefficients of the expansions (141)–(142) of the functions $\hat{\mathbf{z}}_1^h$, $\hat{z}_2^h$, $\mathbf{p}_1^h$, and p_2^h, analogous to (143)–(146), can be also obtained for problem (190)–(193).

9. Corollary from the obtained results

Note in conclusion that the above results generalize, for the class of estimation problems for systems described by boundary value problems considered in this work, the results by A. G. Nakonechnyi [11], [12]. To see this, suppose, as in these papers, that from the observations of a random variable of the form

$$y_2=C_2\varphi+\eta_2, \tag{194}$$

it is necessary to estimate the expression

$$l(\varphi):=\int_D l_2(x)\varphi(x)\,dx \tag{195}$$

in the class of estimates of the form

$$\widehat{l(\varphi)}:=(y_2,u_2)_{H_2}+c, \tag{196}$$

where φ is a solution to the problem (8)–(9), l_2 is a given function from $L^2(D)$, $u_2\in H_2$, $c\in\mathbb{R}$, $C_2\in\mathcal{L}(L^2(D),H_2)$ is a linear operator.

The case considered here corresponds to setting $C_1=0$, $\eta_1=0$, $\mathbf{l}_1=0$, $u_1=0$, respectively in (21), (22), (23), (25), and Lemma 3.1 can be stated as follows.

Lemma 9.1. *Finding the minimax estimate of $l(\varphi)$ is equivalent to the problem of optimal control of the system described by the equations*

$$\int_D(((\mathbf{A}(x))^{-1})^T\mathbf{z}_1(x;u_2),\mathbf{q}(x))_{\mathbb{R}^n}dx-\int_D z_2(x;u_2)\mathrm{div}\,\mathbf{q}(x)\,dx=0 \tag{197}$$

for all $\mathbf{q} \in H(\mathrm{div}, D)$*, and for all* $v \in L^2(D)$

$$-\int_D v(x)\mathrm{div}\,\mathbf{z}_1(x;u_2)\,dx - \int_D c(x)z_2(\cdot;u_2)v(x)\,dx = \\ = \int_D (l_2(x) - (C_2^t J_{H_2} u_2)(x))v(x)\,dx \quad (198)$$

with the cost function

$$I(u_2) = (Q^{-1}z_2(\cdot;u_2), z_2(\cdot;u_2))_{L^2(D)} + (\tilde{Q}_2^{-1}u_2, u_2)_{H_2} \to \inf_{u_2 \in H_2}. \quad (199)$$

It is easy to see that the second component $z_2(\cdot;u)$ of the solution $(\mathbf{z}_1(\cdot;u), z_2(\cdot;u))$ to this problem belongs to the space $H_0^1(D)$ and satisfies the integral identity

$$-(\mathbf{A}^T\,\mathbf{grad}\,z_2, \mathbf{grad}\,v)_{L^2(D)^n} - (cz_2, v)_{L^2(D)} = ((l_2 - (C_2^t J_{H_2} u_2), v)_{L^2(D)} \quad (200)$$

for all $v \in H_0^1(D)$. Therefore, Lemma 9.1 takes the form:

Lemma 9.2. *Finding the minimax estimate of* $l(\varphi)$ *is equivalent to solving the optimal control problem described by equation* (200) *with the cost function* (199).

Theorems 4.1 and 4.2 are transformed into the following assertions.

Theorem 9.3. *There exists a unique minimax estimate of* $l(\mathbf{j}, \varphi)$ *which has the form*

$$\widehat{\widehat{l(\mathbf{j}, \varphi)}} = (y_2, \hat{u}_2)_{H_2} + \hat{c}, \quad (201)$$

where

$$\hat{c} = -\int_D \hat{z}_2(x) f_0(x)\,dx, \quad \hat{u}_2 = \tilde{Q}_2 C_2 p_2, \quad (202)$$

and the functions $\hat{z}_2$ *and* $p_2 \in H_0^1(D)$ *are determined from the solution of the following problem:*

$$-\int_D (\mathbf{A}^T(x)\,\mathbf{grad}\,\hat{z}_2(x), \mathbf{grad}\,v_1(x))_{\mathbb{R}^n}\,dx - \int_D c(x)\hat{z}_2(x)v_1(x)\,dx = \\ = \int_D (l_2(x) - C_2^t J_{H_2}\tilde{Q}_2 C_2 p_2(x))v_1(x)\,dx \quad \forall v_1 \in H_0^1(D), \quad (203)$$

$$-\int_D (\mathbf{A}(x)\,\mathbf{grad}\,p_2(x), \mathbf{grad}\,v_2(x))_{\mathbb{R}^n}\,dx - \int_D c(x)p_2(x)v_2(x)\,dx = \\ = \int_D v_2(x)Q^{-1}\hat{z}_2(x)\,dx \quad \forall v_2 \in H_0^1(D). \quad (204)$$

Problem (203)–(204) *is uniquely solvable. The error of estimation* σ *is given by the expression*

$$\sigma = l(p_2)^{1/2}. \quad (205)$$

Theorem 9.4. *The minimax estimate of* $l(\mathbf{j},\varphi)$ *has the form*

$$\widehat{\widehat{l(\mathbf{j},\varphi)}} = l(\hat{\mathbf{j}},\hat{\varphi})),$$

where the function $\hat{\varphi} \in H_0^1(D)$ *is determined from the solution of the following problem:*

$$-\int_D (\mathbf{A}^T(x)\,\mathbf{grad}\,\hat{p}_2(x),\mathbf{grad}\,v_1(x))_{\mathbb{R}^n}\,dx - \int_D c(x)\hat{p}_2(x)v_1(x)\,dx =$$
$$= \int_D C_2^t J_{H_2}\tilde{Q}_2(y_2 - C_2\hat{\varphi})(x)v_1(x)\,dx \quad \forall v_1 \in H_0^1(D), \quad (206)$$

$$-(\mathbf{A}^T(x)\,\mathbf{grad}\,\hat{\varphi}(x),\mathbf{grad}\,v_2(x))_{\mathbb{R}^n}\,dx - \int_D c(x)\hat{\varphi}(x)v_2(x)\,dx =$$
$$= \int_D v_2(x)(Q^{-1}\hat{p}_2(x) - f_0(x))\,dx \quad \forall v_2 \in H_0^1(D), \quad (207)$$

where the equalities (206)–(207) *are fulfilled with probability* 1. *Problem* (206)–(207) *is uniquely solvable. The random fields* $\hat{\varphi}$ *and* $\hat{p}_2$, *whose realizations satisfy the equations* (206)–(207), *belong to the space* $L^2(\Omega, H_0^1(D))$.

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