Minimax Theory and its Applications
Volume 8 (2023), No. 2, 393–408

Existence and Uniqueness of Common Solutions of Strict Stampacchia and Minty Variational Inequalities with Non-Monotone Operators in Banach Spaces

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Received: June 9, 2023
Accepted: August 4, 2023

We study the existence of common solutions of the Stampacchia and Minty variational inequalities associated to non-monotone operators in Banach spaces, as a consequence of a general saddle-point theorem. We prove, in particular, that if $(X, \|\cdot\|)$ is a Banach space, whose norm has suitable convexity and differentiability properties, $B_\rho := \{x \in X : \|x\| \le \rho\}$, and $\Phi : B_\rho \to X^*$ is a C^1 function with Lipschitzian derivative, with $\Phi(0) \neq 0$, then for each $r > 0$ small enough, there exists a unique $x^* \in B_r$, with $\|x\| = r$, such that $\max\{\langle \Phi(x^*), x^* - x\rangle, \langle \Phi(x), x^* - x\rangle\} < 0$ for all $x \in B_r \setminus \{x^*\}$. Our results extend to the setting of Banach spaces some results previously obtained by B. Ricceri in the setting of Hilbert spaces.

Keywords: Saddle point, minimax theorem, Banach space, modulus of convexity, C^1 function, Stampacchia and Minty variational inequalities, ball, non-monotone operators.

2020 Mathematics Subject Classification: 47J20, 49J35, 49J40.

1. Introduction

Let $(X, \|\cdot\|_X)$ be a real Banach space, with topological dual $(X^*, \|\cdot\|_{X^*})$. For each $r > 0$, we put

$$B_r = \{x \in X : \|x\|_X \le r\}, \qquad S_r = \{x \in X : \|x\|_X = r\}.$$

Let $A \subseteq X$ be a nonempty convex set, and let $\Phi : A \to X^*$ be a given operator. As known, the *Stampacchia variational inequality* associated with A and Φ (in short, $\mathrm{SVI}(A, \Phi)$) is to find $x^* \in A$ such that

$$\langle \Phi(x^*), x^* - y\rangle \le 0 \quad \text{for all} \quad y \in A.$$

On the other side, the *Minty variational inequality* associated with A and Φ (in short, $\mathrm{MVI}(A, \Phi)$) is to find $x^* \in A$ such that

$$\langle \Phi(y), x^* - y\rangle \le 0 \quad \text{for all} \quad y \in A.$$

Actually, the literature concerning variational inequalities is really impressive, due to their wide range of applications, including partial differential equations, mechanics, engineering, optimization and complementarity problems, traffic equilibrium problems, economics, and many others. Just for instance, we refer the reader to [1, 12, 13, 14, 17, 23, 26, 27, 30] and to the references therein.

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

It is known that if X is finite-dimensional, A is compact (for instance, a closed ball), and $\Phi : A \to X^*$ is continuous, then the problem SVI(A, Φ) has a solution (see Lemma 3.1 of [27], or also Theorem 3.1 of [30]). When X is infinite-dimensional, the situation is more delicate. For instance, if X is infinite-dimensional and A is a closed ball, then the continuity of Φ is not enough to guarantee the existence of a solution. In this direction, Frasca and Villani [22] proved that if X is an infinite-dimensional Hilbert space and $A \subseteq X$ is a closed ball, then there exists a (strongly) continuous affine operator $\Phi : X \to X$ such that, for every $x \in A$, one has

$$\sup_{y \in A} \langle \Phi(x), x - y \rangle > 0.$$

When X is infinite-dimensional, the existence results for the SVI(A, Φ) are substantially divided into two branches: on one side, together with some continuity assumptions, some suitable monotonicity assumptions are made on the operator Φ (see, just for instance, [3, 4, 5, 6, 24, 25, 28, 32, 33, 34, 35, 44, 46] and the reference therein). On the other side, another branch of the research is concerned merely with the kind of continuity of the operator Φ (together with suitable assumptions on A), and no requirement of monotonicity is made. In this direction, we refer, for instance, to the papers [7, 8, 15, 16, 29, 37, 38, 42, 43].

It is well-known (see, for instance, Lemma 1 of [6] or also Lemma 1 of [36]) that when Φ is monotone and continuous from line segments to the weak-star topology of X^*, then the variational inequalities SVI(A, Φ) and MVI(A, Φ) are equivalent. That is, a point $x^* \in A$ solves SVI(A, Φ) if and only if it solves MVI(A, Φ). As far as we know, up to our days, monotonicity is always required in the literature in order to guarantee that a solution x^* to SVI(A, Φ) is also a solution to MVI(A, Φ) (see, for instance, [31, 45] and the references therein).

On the basis of the above facts, the following problem was proposed by B. Ricceri in the paper [40]: find classes of non-monotone operators Φ such that there is a solution of SVI(A, Φ) which also satisfies MVI(A, Φ). He also gave a first contribution in this direction (Theorem 2.3 of [40]), in the setting of Hilbert spaces, which was furtherly refined in Theorem 5 of [41]. In this latter result, B. Ricceri established a characterization of the existence and unicity of the solution (in closed balls of sufficiently small radius) of the Minty variational inequality in Hilbert spaces, under the unique assumption that the operator Φ is of class C^1, with Lipschitzian derivative. In particular, Theorem 5 of [41] guarantees that, if $\Phi : B_\rho \to X$ is a C^1 function with Lipschitzian derivative (for some $\rho > 0$) and $\Phi(0_X) \neq 0_X$, then for each $r > 0$ small enough, there exists a unique $x^* \in S_r$ such that, for all $y \in B_r \setminus \{x^*\}$,

$$\max \big\{ \langle \Phi(x^*), x^* - y \rangle, \ \langle \Phi(y), x^* - y \rangle \big\} < 0. \tag{1}$$

This is a remarkable and deep result, for the absolute minimality of the assumptions (excluding monotonicity), and for the strict inequality in (1), which, in particular, guarantees the unicity of the solution point x^* in the ball B_r. Ricceri's result is particularly meaningful if we consider that even for an extremely regular (non-monotone) operator Φ, the problem MVI(A, Φ) may admit no solutions under standard assumptions on the convex set A (closedness, boundedness, or even compactness). In particular, it may admit no solutions if $A = B_r$, with arbitrary $r > 0$, even in finite-dimensional setting. The following simple example illustrates this fact.

Example 1.1. Let $X = \mathbf{R}^2$, endowed with the Euclidean norm $\|\cdot\|$ induced by the usual inner product $(\cdot,\cdot)_{\mathbf{R}^2}$. Let $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $\Phi(x,y) = (y+1, x+1)$. Of course, by the classical result of Hartman and Stampacchia [27], for every $r > 0$ the problem SVI(B_r, Φ) has a solution. Now, fix $r > 0$ and let $(u,v) \in B_r$ be a solution of the problem MVI(B_r, Φ). Thus, we have

$$(y+1)(u-x) + (x+1)(v-y) \leq 0 \quad \text{for all} \quad (x,y) \in B_r. \tag{2}$$

If we choose $(x,y) = (v,u)$, we get $(u-v)^2 \leq 0$, hence $u = v$. By (2), we then get

$$u(x+y+2) - 2xy - x - y \leq 0 \quad \text{for all} \quad (x,y) \in B_r. \tag{3}$$

By choosing $(x,y) = (0,0)$, the inequality (3) immediately gives $u \leq 0$. We claim that $u < 0$. Indeed, assume that $u = 0$. In this case, by (3) we get

$$-2xy - x - y \leq 0 \quad \text{for all} \quad (x,y) \in B_r.$$

Choosing $(x,y) = (-r, 0)$, we then get $r \leq 0$, an absurd. Hence, $u < 0$, as claimed. Finally, if in (3) we choose $x \leq 0$ and $y = 0$, we get $u(x+2) \leq x \leq 0$, hence $x \geq -2$. This implies that we necessarily have $r \leq 2$. That is, the problem MVI(B_r, Φ) admits no solutions if $r > 2$. □

It is worth noticing that Ricceri's result (Theorem 5 of [41]) was proved as an application of a very general saddle point theorem in the setting of Hilbert spaces (Theorem 1 of [41]), which, in turn, makes use of a classical minimax theorem by Ky Fan (Theorem 2 of [21]). The proofs of the former results (Theorems 1 and 5 of [41]) make a deep use of the Hilbert space setting. Consequently, it is not possible to extend them naturally to the setting of Banach spaces.

The aim of the present paper is exactly to provide such an extension, giving a further contribution to the problem raised by Ricceri in [40]. The following theorem can be considered as the main result of this paper (for the definitions not recalled before, we refer to the next Section 2).

Theorem 1.2. *Let $(X, \|\cdot\|_X)$ be a real Banach space. Assume that the norm $\|\cdot\|_X$ is Gâteaux-differentiable on S_1 and has a modulus of convexity of power type 2. Let $\rho > 0$, and let $\Phi : B_\rho \to X^*$ be a C^1 function, with Φ' Lipschitzian.*
Then, the following assertions are equivalent:

(1) *for each $r > 0$ small enough, there exists a unique $x^* \in S_r$ such that*

$$\max\{\langle \Phi(x^*), x^* - x\rangle, \langle \Phi(x), x^* - x\rangle\} < 0 \quad \textit{for all } x \in B_r \setminus \{x^*\}. \tag{4}$$

(2) $\Phi(0_X) \neq 0_{X^*}$.

As for Theorem 5 of [41], the main peculiarity of Theorem 1.2 resides in the minimality of the assumptions (which do not include any kind of monotonicity), and for the strict inequality in (4). This last feature, in particular, guarantees the unicity of the solution x^* in the ball B_r.

It is worth noticing that Theorem 1.2 is not still true for any Banach space $(X, \|\cdot\|_X)$ (that is, if we do not assume that the norm $\|\cdot\|_X$ is Gâteaux-differentiable on S_1 and has a modulus of convexity of power type 2; see Example 3.9 below). In the case where $(X, \|\cdot\|_X)$ is a real Hilbert space, Theorem 1.2 gives back exactly Ricceri's result (Theorem 5 of [41]).

In order to prove Theorem 1.2, we shall need, in particular, to extend Ricceri's saddle point result (Theorem 1 of [41]) to the setting of Banach spaces, and to prove some intermediate results. We shall do this in Section 3, while in Section 2 we shall fix some notations and recall some definitions and some results that will be fundamental in the sequel.

2. Preliminaries

In what follows, $(X, \|\cdot\|_X)$ is a Banach space. For the notions of differentiability of the norm $\|\cdot\|_X$ and for related results, we refer to [19, 20]. As regards Gâteaux and Fréchet differentiability of operators and related results, we refer, for instance, to [18]. If V is a topological space, a function $g : V \to \mathbf{R}$ is said to be sup-compact if, for each $r \in \mathbf{R}$, the set $\{v \in V : g(v) \geq r\}$ is compact.

We recall (see [19]) that a function $\varphi : X \setminus \{0_X\} \to X^* \setminus \{0_{X^*}\}$ is said to be a *support function* if it satisfies the following conditions:

(i) for each $x \in S_1$, one has $\|\varphi(x)\|_{X^*} = 1 = \langle \varphi(x), x \rangle$;

(ii) for each $\lambda > 0$ and each $x \in X \setminus \{0_X\}$, one has $\varphi(\lambda x) = \lambda \varphi(x)$.

We denote by $\mathcal{L}(X, X^*)$ the space of all linear continuous functions from X to X^*, endowed with its standard norm $\|\cdot\|_{\mathcal{L}(X,X^*)}$. We recall that the *modulus of convexity* of the norm $\|\cdot\|_X$ is the function $\delta_{\|\cdot\|_X} :]0, 2] \to [0, 1]$ defined by

$$\delta_{\|\cdot\|_X}(\varepsilon) := \inf \left\{ 1 - \frac{\|x + z\|_X}{2} : \ \|x\|_X = \|z\|_X = 1, \ \|x - z\|_X \geq \varepsilon \right\}.$$

The norm $\|\cdot\|_X$ is said to have modulus of convexity *of power type 2* if there exists $a > 0$ such that $\delta_{\|\cdot\|_X}(\varepsilon) \geq a\,\varepsilon^2$ for each $\varepsilon \in]0, 2]$ (see [2, 11]). It is known (see Proposition 3.3 and Remark 3.4(ii) of [11]) that the norm $\|\cdot\|_X$ has modulus of convexity of power type 2 if and only if it satisfies the *lower weak parallelogram law*

$$2(\|x\|_X^2 + \|z\|_X^2) - \|x + z\|_X^2 \geq b\,\|x - z\|_X^2 \quad \text{for all} \quad x, z \in X, \tag{5}$$

for some $b \in]0, 1]$. It was proved in [9, 10] that if (Ω, Σ, μ) is any measure space, then for each $p \in]1, 2]$ the space $L^p(\Omega, \Sigma, \mu)$, endowed with the usual norm, satisfies the inequality (5) with $b = p - 1$. More precisely, $p - 1$ is the best constant b in (5). Hence, in particular, for each $p \in]1, 2]$ the usual norm of $L^p(\Omega, \Sigma, \mu)$ has modulus of convexity of power type 2.

Of course, if $(X, \|\cdot\|_X)$ is a real Hilbert space, then the norm $\|\cdot\|_X$ satisfies inequality (5) with $b = 1$, hence it has modulus of convexity of power type 2. More precisely, a Banach space $(X, \|\cdot\|_X)$ satisfies (5) with $b = 1$ if and only if $\|\cdot\|_X$ is an inner product norm (see Remark 3.4(ii) of [11]).

The following proposition follows by the proof of Proposition 3.3 of [11].

Proposition 2.1. *Let $(X, \|\cdot\|_X)$ be a real Banach space, and let $b > 0$. Then, the following facts are equivalent:*

(1) *the lower weak parallelogram law* (5) *is satisfied;*

(2) *for each convex $U \subseteq X$, and for each $f : U \to \mathbf{R}$ of class C^1, such that the derivative f' is Lipschitzian with constant L, one has that the functional $f + \frac{L}{2b}\|\cdot\|_X^2$ is convex.*

Finally, we remark explicitly (see Remark 3.4(ii) of [11]) that the constant $b > 0$ appearing in (5) cannot be greater than 1.

3. Results

The following is our general saddle-point result.

Theorem 3.1. *Let X be a real reflexive Banach space, whose norm $\|\cdot\|_X$ is Gâteaux-differentiable on S_1 and strictly convex. Let Y be a non-empty closed convex set in a Hausdorff real topological vector space. Let $\rho > 0$, and let $J : B_\rho \times Y \to \mathbf{R}$ be a function satisfying the following conditions:*

(i) *for each $y \in Y$, the function $J(\,\cdot, y)$ is of class C^1 and $J'_x(\,\cdot, y)$ is Lipschitzian with constant $L > 0$ (independent of y);*

(ii) *$J(x, \cdot)$ is upper semicontinuous and concave for all $x \in B_\rho$, and there exists $x_0 \in B_\rho$ such that $J(x_0, \cdot)$ is sup-compact;*

(iii) *one has $\delta := \inf_{y \in Y} \|J'_x(0_X, y)\|_{X^*} > 0$;*

(iv) *there exists $b > 0$ such that, for every $y \in Y$, the functional*

$$x \in B_\rho \;\to\; \frac{L}{2b}\|x\|_X^2 + J(x, y)$$

is convex.

Then, for each $r \in \left]0, \min\{\rho, \frac{\delta}{L(1 + 1/b)}\}\right]$, and for each nonempty closed convex $T \subseteq Y$, there exist $x^* \in S_r$ and $y^* \in T$ such that, for all $x \in B_r \setminus \{x^*\}$ and all $y \in T$,

$$J(x^*, y) \le J(x^*, y^*) < J(x, y^*).$$

Remark 3.2. If $(X, \|\cdot\|_X)$ is a real Hilbert space, assumption (iv) is satisfied with $b = 1$ (see, for instance, Remark 3.4(iii) of [11], or even the proof of Corollary 2.7 of [39]).

Proof of Theorem 3.1. Since the norm $\|\cdot\|_X$ is Gâteaux differentiable on S_1, there exists a support function $\varphi : X \setminus \{0_X\} \to X^* \setminus \{0_{X^*}\}$ which is norm to weak-star continuous (see [19], Chapter 2). As a matter of fact, by the inequality (1) at p. 21 of [19], it is easy to see that the functional $g(\cdot) = \|\cdot\|_X$ is Gâteaux-differentiable on the whole $X \setminus \{0_X\}$ with derivative

$$g'(x) = \frac{\varphi(x)}{\|x\|_X}.$$

This easily implies that the functional $p(x) = \|x\|_X^2/2$ is Gâteaux-differentiable on the whole X, with derivative

$$p'(x) = \begin{cases} \varphi(x) & \text{if } x \neq 0_X, \\ 0_{X^*} & \text{if } x = 0_X. \end{cases} \tag{6}$$

Now, fix $r \in \left]0, \min\{\rho, \frac{\delta}{L(1+1/b)}\}\right]$, and a nonempty closed convex set $T \subseteq Y$. Let $\psi : B_r \times T \to \mathbf{R}$ be the functional defined by putting, for each $(x, y) \in B_r \times T$,

$$\psi(x, y) = \frac{L}{2b} \|x\|_X^2 + J(x, y).$$

Of course, for every $y \in T$ the function $\psi(\cdot, y)$ is norm continuous. Moreover, it is convex by assumption (iv). Thus, if we consider B_r with the weak topology, it is routine matter to check that all the assumptions of Theorem 2 of [21] are satisfied. Consequently, one has

$$\min_{x \in B_r} \sup_{y \in T} \psi(x, y) = \sup_{y \in T} \min_{x \in B_r} \psi(x, y).$$

Now, observe that the function $y \in T \to \min_{x \in B_r} \psi(x, y)$ is sup-compact. Consequently, the exist $x^* \in B_r$ and $y^* \in T$ such that

$$\sup_{y \in T} \psi(x^*, y) = \min_{x \in B_r} \sup_{y \in T} \psi(x, y) = \sup_{y \in T} \min_{x \in B_r} \psi(x, y) = \min_{x \in B_r} \psi(x, y^*).$$

In particular, we get, for all $x \in B_r$ and all $y \in T$,

$$\psi(x^*, y) \le \psi(x^*, y^*) \le \psi(x, y^*). \tag{7}$$

By what precedes, we infer that the functional $\psi(\cdot, y^*)$ is Gâteaux-differentiable on B_r. We claim that it has no critical points belonging to the interior of B_r. Arguing by contradiction, assume that there exists $\hat{x} \in X$, with $\|\hat{x}\|_X < r$, such that

$$\frac{L}{b} p'(\hat{x}) + J'_x(\hat{x}, y^*) = 0_{X^*}. \tag{8}$$

If $\hat{x} = 0_X$, by (6) we get $J'_x(0_X, y^*) = 0_{X^*}$, against assumption (iii). If $\hat{x} \neq 0_X$, by (6) and (8) we get

$$\frac{L}{b}\varphi(\hat{x}) + J'_x(\hat{x}, y^*) = 0_{X^*}.$$

Hence, by assumption (i) we have

$$\begin{aligned} &\|\frac{L}{b}\varphi(\hat{x}) + J'_x(0_X, y^*)\|_{X^*} \\ &\le \|\frac{L}{b}\varphi(\hat{x}) + J'_x(\hat{x}, y^*)\|_{X^*} + \|J'_x(0_X, y^*) - J'_x(\hat{x}, y^*)\|_{X^*} \le L\|\hat{x}\|_X. \end{aligned}$$

Consequently, we get

$$\begin{aligned} \|J'_x(0_X, y^*)\|_{X^*} &= \|J'_x(0_X, y^*) - \frac{L}{b}\varphi(\hat{x}) + \frac{L}{b}\varphi(\hat{x})\|_{X^*} \\ &\le L\|\hat{x}\|_X + \frac{L}{b} \left\| \varphi\Big(\frac{\hat{x}}{\|\hat{x}\|_X}\Big) \right\|_{X^*} \|\hat{x}\|_X = \Big(L + \frac{L}{b}\Big) \|\hat{x}\|_X. \end{aligned}$$

Thus, by assumption (iii) we obtain the contradiction

$$\|\hat{x}\|_X \ge \frac{\|J'_x(0_X, y^*)\|_{X^*}}{L(1 + \frac{1}{b})} \ge \frac{\delta}{L(1 + \frac{1}{b})} \ge r.$$

Now, observe that the set

$$\Gamma = \{x \in B_r : \psi(x, y^*) = \min_{v \in B_r} \psi(v, y^*)\}$$

is nonempty and convex. By what precedes, we have $\Gamma \subseteq S_r$. Hence, since the norm $\|\cdot\|_X$ is strictly convex, the set Γ contains only one point, hence $\Gamma = \{x^*\}$. Consequently, by (7) we get

$$\frac{L}{2b}\|x^*\|_X^2 + J(x^*, y) \le \frac{L}{2b}\|x^*\|_X^2 + J(x^*, y^*) <$$
$$< \frac{L}{2b}\|x\|_X^2 + J(x, y^*) \le \frac{L}{2b}\|x^*\|_X^2 + J(x, y^*)$$

for all $x \in B_r \setminus \{x^*\}$ and all $y \in T$. The conclusion follows at once. □

The following result is an immediate consequence of Theorem 3.1.

Corollary 3.3. *Let $(X, \|\cdot\|_X)$ be a real Banach space. Assume that the norm $\|\cdot\|_X$ is Gâteaux-differentiable on S_1 and has a modulus of convexity of power type 2. Let Y be a non-empty closed convex set in a Hausdorff real topological vector space. Let $\rho > 0$, and let $J : B_\rho \times Y \to \mathbf{R}$ be a function satisfying the following conditions:*

- (i) *for each $y \in Y$, the function $J(\cdot, y)$ is of class C^1 and $J'_x(\cdot, y)$ is Lipschitzian with constant $L > 0$ (independent of y);*
- (ii) *$J(x, \cdot)$ is upper semicontinuous and concave for all $x \in B_\rho$, and there exists $x_0 \in B_\rho$ such that $J(x_0, \cdot)$ is sup-compact;*
- (iii) *one has $\delta := \inf_{y \in Y} \|J'_x(0_X, y)\|_{X^*} > 0$.*

Then, if $b \in]0, 1]$ makes inequality (5) true, for each $r \in \left]0, \min\{\rho, \frac{\delta}{L(1+1/b)}\}\right]$ and for each nonempty closed convex $T \subseteq Y$, there exist $x^* \in S_r$ and $y^* \in T$ such that

$$J(x^*, y) \le J(x^*, y^*) < J(x, y^*)$$

for all $x \in B_r \setminus \{x^*\}$ and all $y \in T$. In particular, if $(X, \|\cdot\|_X)$ is a real Hilbert space, the conclusion is valid with $b = 1$.

Proof. Since the norm $\|\cdot\|_X$ has modulus of convexity of power type 2, it is uniformly convex (see [2]), hence strictly convex. In particular, the Banach space $(X, \|\cdot\|_X)$ is reflexive. Moreover, by Proposition 3.3 of [11], there exists $b \in]0, 1]$ such that the inequality (5) is satisfied. By Proposition 2.1, it follows that, for each $y \in Y$, the functional

$$x \in B_\rho \ \to \ \frac{L}{2b}\|x\|_X^2 + J(x, y)$$

is convex. At this point, the conclusion follows at once by Theorem 3.1. In particular, if $(X, \|\cdot\|_X)$ is a real Hilbert space, we can obviously take $b = 1$. □

Remark 3.4. By what precedes, it is clear that, when $(X, \|\cdot\|_X)$ is a real Hilbert space, both Theorem 3.1 and Corollary 3.3 give back Theorem 1 of [41]. □

We now establish the following application of Theorem 3.1 to the existence and unicity of common solutions for strict Stampacchia and Minty variational inequalities.

Theorem 3.5. *Let X be a real reflexive Banach space, whose norm $\|\cdot\|_X$ is Gâteaux-differentiable on S_1 and strictly convex. Let $\rho > 0$, and let $\Phi : B_\rho \to X^*$ be a given function. Assume that:*

(i) Φ *is of class* C^1, *and* Φ' *is Lipschitzian with constant* $\gamma > 0$;

(ii) *one has* $\sigma := \inf_{y \in B_\rho} \sup_{\|u\|_X = 1} |\langle \Phi(0_X), u\rangle - \langle \Phi'(0_X)(u), y\rangle| > 0.$

Let $\theta := \sup_{x \in B_\rho} \|\Phi'(x)\|_{\mathcal{L}(X,X^*)}$ *and* $M := 2(\theta + \rho\gamma)$.

Moreover, assume that:

(iii) *there exists* $b \in]0,1]$ *such that, for every* $y \in B_\rho$, *the functional*

$$x \in B_\rho \ \to \ \frac{M}{2b}\|x\|_X^2 + \langle \Phi(x), x - y\rangle$$

is convex.

Then, for each $r \in \left]0, \min\{\rho, \frac{\sigma}{M(1+1/b)}\}\right]$, *there exists a unique* $x^* \in S_r$ *such that*

$$\max\{\langle \Phi(x^*), x^* - x\rangle, \langle \Phi(x), x^* - x\rangle\} < 0 \quad \text{for all } x \in B_r \setminus \{x^*\}.$$

Proof. We want to apply Theorem 3.1 to the functional $J : B_\rho \times B_\rho \to \mathbf{R}$ defined by putting, for each $(x,y) \in B_\rho \times B_\rho$,

$$J(x,y) = \langle \Phi(x), x - y\rangle.$$

To this aim, observe what follows.

(a) For each $y \in B_\rho$, the function $J(\cdot, y)$ is of class C^1 and $J'_x(\cdot, y)$ is Lipschitzian with constant M. To see this, fix $y \in B_\rho$. By Theorem 5.1.13 of [18], the function $J(\cdot, y)$ is Fréchet differentiable in B_ρ, and for all $x \in B_\rho$ and $u \in X$ one has

$$\langle J'_x(x,y), u\rangle = \langle \Phi'(x)(u), x - y\rangle + \langle \Phi(x), u\rangle. \tag{9}$$

Now, fix $x, z \in B_\rho$ and $u \in X$, with $\|u\|_X = 1$. By (9) we get

$$\begin{aligned}
&|\langle J'_x(x,y), u\rangle - \langle J'_x(z,y), u\rangle| \\
&= |\langle \Phi(x) - \Phi(z), u\rangle + \langle \Phi'(x)(u), x - y\rangle - \langle \Phi'(z)(u), z - y\rangle| \\
&\le \|\Phi(x) - \Phi(z)\|_{X^*} + |\langle \Phi'(x)(u) - \Phi'(z)(u), z - y\rangle + \langle \Phi'(x)(u), x - z\rangle|.
\end{aligned}$$

By Theorem 5.1.12 of [18], we then get

$$\begin{aligned}
&|\langle J'_x(x,y), u\rangle - \langle J'_x(z,y), u\rangle| \\
&\le \theta\|x - z\|_X + \|\Phi'(x) - \Phi'(z)\|_{\mathcal{L}(X,X^*)} \cdot \|z - y\|_X + \|\Phi'(x)\|_{\mathcal{L}(X,X^*)} \cdot \|x - z\|_X \\
&\le 2\,(\theta + \rho\gamma)\,\|x - z\|_X.
\end{aligned}$$

Consequently, we get

$$\|J'_x(x,y) - J'_x(z,y)\|_{X^*} \le 2\,(\theta + \rho\gamma)\,\|x - z\|_X,$$

as claimed.

(b) For every $x \in B_\rho$, the function $J(x,\cdot)$ is concave and weakly upper semicontinuous. Moreover, for each $x \in B_\rho$, the function $J(x,\cdot)$ is weakly sup-compact.

(c) One has $\inf_{y\in B_\rho} \|J'_x(0_X,y)\|_{X^*} = \sigma$. Indeed, by (9), we get

$$\begin{aligned}\inf_{y\in B_\rho} \|J'_x(0_X,y)\|_{X^*} &= \inf_{y\in B_\rho} \sup_{\|u\|_X=1} |\langle J'_x(0_X,y),u\rangle| \\ &= \inf_{y\in B_\rho} \sup_{\|u\|_X=1} |\langle \Phi(0_X),u\rangle - \langle \Phi'(0_X)(u),y\rangle| = \sigma,\end{aligned}$$

as claimed.

Consequently, all the assumptions of Theorem 3.1 (taking $Y = B_\rho$ with the weak topology) are satisfied. Hence, for each $r \in \,]\,0, \min\{\rho, \frac{\sigma}{M(1+1/b)}\}]$, there exist two points $x^* \in S_r$ and $y^* \in B_r$ such that

$$\langle \Phi(x^*), x^*-y\rangle \le \langle \Phi(x^*), x^*-y^*\rangle < \langle \Phi(x), x-y^*\rangle \tag{10}$$

for all $x \in B_r \setminus \{x^*\}$ and all $y \in B_r$.

Now, let $r \in \,]\,0, \min\{\rho, \frac{\sigma}{M(1+1/b)}\}]$ be fixed, and let $x^* \in S_r$ and $y^* \in B_r$ satisfy (10). We claim that $\Phi(x^*) \neq 0_{X^*}$. Arguing by contradiction, assume that $\Phi(x^*) = 0_{X^*}$. Hence, by Theorem 5.1.12 of [18], we get

$$\|\Phi(0_X)\|_{X^*} = \|\Phi(0_X) - \Phi(x^*)\|_{X^*} \le \theta\,\|x^*\|_X = \theta r.$$

Taking into account that $0 < \sigma \le \|\Phi(0_X)\|_{X^*}$, we have

$$r \le \frac{\sigma}{M(1+1/b)} \le \frac{\|\Phi(0_X)\|_{X^*}}{2M} < \frac{\|\Phi(0_X)\|_{X^*}}{4\theta} \le \frac{r}{4},$$

which is absurd.

Now we claim that $x^* = y^*$. Indeed, assume that $y^* \neq x^*$. Taking $y = x^*$, by the first part of the inequality (10) we get

$$\langle \Phi(x^*), x^*-y^*\rangle \ge 0.$$

Hence, taking $x = y^*$, by the last inequality in (10) we get

$$0 = \langle \Phi(y^*), y^*-y^*\rangle > \langle \Phi(x^*), x^*-y^*\rangle \ge 0,$$

which is absurd. Hence, $x^* = y^*$, as claimed. At this point, (10) implies that

$$\langle \Phi(x^*), x^*-y\rangle \le 0 < \langle \Phi(x), x-x^*\rangle \tag{11}$$

for all $x \in B_r \setminus \{x^*\}$ and all $y \in B_r$. In particular, we get

$$\langle \Phi(x^*), x^*\rangle \le \langle \Phi(x^*), y\rangle \quad \text{for all} \quad y \in B_r. \tag{12}$$

Since $\Phi(x^*) \neq 0_{X^*}$ and the norm is strictly convex, by (12) the point x^* is the unique global minimum in B_r of the functional $\Phi(x^*)$. In particular, we get

$$\langle \Phi(x^*), x^*\rangle < \langle \Phi(x^*), y\rangle \quad \text{for all} \quad y \in B_r \setminus \{x^*\}. \tag{13}$$

By (11) and (13) we immediately get that, for all $x \in B_r \setminus \{x^*\}$,

$$\max\{\langle \Phi(x^*), x^* - x\rangle, \langle \Phi(x), x^* - x\rangle\} < 0.$$

As regards the unicity, it suffices to argue as in the conclusion of the proof of Theorem 3 of [41]. □

Remark 3.6. If $(X, \|\cdot\|_X)$ is a real Hilbert space, then assumption (iii) of Theorem 3.5 is satisfied by choosing $b = 1$. Indeed, let

$$J : B_\rho \times B_\rho \to \mathbf{R}, \quad J(x,y) := \langle \Phi(x), x - y\rangle.$$

As we have seen in the proof of Theorem 3.5, the assumptions of Theorem 3.5 imply that for each $y \in B_\rho$, the function $J(\cdot, y)$ is of class C^1 and $J'_x(\cdot, y)$ is Lipschitzian with constant M. Hence (see Remark 3.4(iii) of [11]), for each $y \in Y$ the functional

$$x \in B_\rho \;\to\; \frac{M}{2}\|x\|_X^2 + J(x,y) = \frac{M}{2}\|x\|_X^2 + \langle \Phi(x), x - y\rangle$$

is convex. □

The following is an immediate consequence of Theorem 3.5.

Theorem 3.7. *Let $(X, \|\cdot\|_X)$ be a real Banach space. Assume that the norm $\|\cdot\|_X$ is Gâteaux-differentiable on S_1 and has a modulus of convexity of power type 2. Let $\rho > 0$, and let $\Phi : B_\rho \to X^*$ be a given function. Assume that:*

(i) *Φ is of class C^1, and Φ' is Lipschitzian with constant $\gamma > 0$;*

(ii) *one has* $\sigma := \inf_{y \in B_\rho} \sup_{\|u\|_X = 1} |\langle \Phi(0_X), u\rangle - \langle \Phi'(0_X)(u), y\rangle| > 0.$

Let $\theta := \sup_{x \in B_\rho} \|\Phi'(x)\|_{\mathcal{L}(X,X^*)}$ *and* $M := 2(\theta + \rho\gamma)$.

Let $b \in \,]0,1]$ be such that the inequality (5) is true.

Then, for each $r \in \,]0, \min\{\rho, \frac{\sigma}{M(1+1/b)}\}]$, there exists a unique $x^* \in S_r$ such that

$$\max\{\langle \Phi(x^*), x^* - x\rangle, \langle \Phi(x), x^* - x\rangle\} < 0$$

for all $x \in B_r \setminus \{x^*\}$.

Remark 3.8. Of course, if $(X, \|\cdot\|_X)$ is a real Hilbert space, then in Theorem 3.7 one can choose $b = 1$. Consequently, when $(X, \|\cdot\|_X)$ is a real Hilbert space, both Theorems 3.5 and 3.7 give back Theorem 3 of [41]. □

Proof of Theorem 3.7. Since the norm $\|\cdot\|_X$ has a modulus of convexity of power type 2, it is uniformly convex and the Banach space $(X, \|\cdot\|_X)$ is reflexive.

Let $J : B_\rho \times B_\rho \to \mathbf{R}$ be the functional defined by putting, for each $(x,y) \in B_\rho \times B_\rho$,

$$J(x,y) = \langle \Phi(x), x - y\rangle.$$

As we have already seen in the proof of Theorem 3.5, our assumptions imply that for each $y \in B_\rho$, the function $J(\cdot, y)$ is of class C^1 and $J'_x(\cdot, y)$ is Lipschitzian with constant M.

By Proposition 3.3 of [11], there exists $b \in]0,1]$ such that the inequality (5) is satisfied. By Proposition 2.1, it follows that for each $y \in Y$ the functional

$$x \in B_\rho \ \to \ \frac{M}{2b}\|x\|_X^2 + J(x,y) = \frac{M}{2b}\|x\|_X^2 + \langle \Phi(x), x-y\rangle$$

is convex. At this point, the conclusion follows at once by Theorem 3.5. □

We can now give the proof of Theorem 1.2, which extends Theorem 5 of [41] to the setting of Banach spaces.

Proof of Theorem 1.2. (1) $\Longrightarrow$ (2) Obvious.

(2) $\Longrightarrow$ (1) Let $f : X \to \mathbf{R}$ be the function defined by putting

$$f(y) = \sup_{\|u\|_X=1} |\langle \Phi(0_X), u\rangle - \langle \Phi'(0_X)(u), y\rangle| \,.$$

Observe that f is lower semicontinuous in X, and

$$f(0_X) = \|\Phi(0_X)\|_{X^*} > 0.$$

Hence, the set
$$A := \left\{ y \in X : f(y) > \frac{\|\Phi(0_X)\|_{X^*}}{2} \right\}$$

is an open neighborhood of 0_X. Consequently, there exists $\varepsilon \in]0,\rho]$ such that $B_\varepsilon \subseteq A$. Hence we get

$$\inf_{y\in B_\varepsilon} \sup_{\|u\|_X=1} |\langle \Phi(0_X), u\rangle - \langle \Phi'(0_X)(u), y\rangle| \geq \frac{\|\Phi(0_X)\|_{X^*}}{2} > 0.$$

Since the norm $\|\cdot\|_X$ has a modulus of convexity of power type 2, there exists $b \in]0,1]$ which makes inequality (5) true. At this point, it suffices to apply Theorem 3.7 to the function $\Phi|_{B_\varepsilon}$. □

It is worth noticing that Theorem 1.2 is not still true for a generic Banach space $(X, \|\cdot\|_X)$. That is, the conclusion may not hold if we do not assume that the norm $\|\cdot\|_X$ is Gâteaux-differentiable on S_1 and has a modulus of convexity of power type 2. The following example illustates this circumstance.

Example 3.9. Let $X = \mathbf{R}^2$, endowed with the norm $\|(x,y)\|_* = |x| + |y|$, and let $\Phi : \mathbf{R}^2 \to (\mathbf{R}^2)^*$ be defined by

$$\langle \Phi(x,y), (a,b)\rangle = a(y+1) + b(x+1)$$

for all $(x,y),(a,b) \in \mathbf{R}^2$. We claim that for every $r \in]0,2[$ the problem MVI(B_r,Φ) has no solution. To see this, fix an arbitrary $r \in]0,2[$. Arguing by contradiction, let $(u,v) \in B_r$ be a solution of the problem MVI(B_r,Φ). Arguing exactly as in Example 1.1, we get that $v = u$ and $u \leq 0$. Thus, we have

$$(y+1)(u-x) + (x+1)(u-y) \leq 0 \quad \text{for all} \quad (x,y) \in B_r. \tag{14}$$

By Lemma 1 of [6], the point (u,u) solves the problem SVI(B_r,Φ). By Corollary 3.2 of [44] (taking into account that $r < 2$), we get that $\|(u,u)\|_* = r$, hence $u = -r/2$. Consequently, by (14) we have

$$x + 2xy + y + r\left(1 + \frac{x}{2} + \frac{y}{2}\right) \geq 0 \quad \text{for all} \quad (x,y) \in B_r. \tag{15}$$

If in (15) we choose $(x, y) = (-r, 0)$, we get $-r^2/2 \geq 0$, which is absurd. □

We now prove the following further application of Theorem 3.5.

Theorem 3.10. *Let X be a real reflexive Banach space, whose norm $\|\cdot\|_X$ is Gâteaux-differentiable on S_1 and strictly convex. Let $\rho > 0$, and let $F : B_\rho \to X^*$ be a given function. Assume that:*

(i) *F is of class C^1, F' is Lipschitzian with constant $\gamma > 0$, and $F'(0_X) = 0_{\mathcal{L}(X,X^*)}$.*

Let $\theta := \sup_{x \in B_\rho} \|F'(x)\|_{\mathcal{L}(X,X^*)}$ and $M := 2(\theta + \rho\gamma)$. Moreover, assume that:

(ii) *there exists $b \in]0, 1]$ such that, for every $y \in B_\rho$, the functional*

$$x \in B_\rho \;\to\; \frac{M}{2b}\|x\|_X^2 + \langle F(x), x - y\rangle$$

is convex.

Finally, let $f \in X^*$ be such that

$$\|f - F(0_X)\|_{X^*} \geq M\rho\Big(1 + \frac{1}{b}\Big). \tag{16}$$

Then, for each $r \in]0, \rho]$, there exists a unique $x^* \in S_r$ such that

$$\max\{\langle F(x^*) - f, x^* - y\rangle, \langle F(y) - f, x^* - y\rangle\} < 0 \quad \textit{for all } y \in B_r \setminus \{x^*\}.$$

Remark 3.11. If $(X, \|\cdot\|_X)$ is a real Hilbert space, then assumption (ii) of Theorem 3.10 is satified by choosing $b = 1$. Indeed, reasoning as in the proof of Theorem 3.5, one can see that the assumptions of Theorem 3.10 imply that for each $y \in B_\rho$, the function

$$x \in B_\rho \to \langle F(x), x - y\rangle$$

is of class C^1, and its derivative is Lipschitzian with constant $M > 0$. Hence (see Remark 3.4(iii) of [11]), for each $y \in Y$ the functional

$$x \in B_\rho \;\to\; \frac{M}{2}\|x\|_X^2 + \langle F(x), x - y\rangle$$

is convex.

Proof of Theorem 3.10. We apply Theorem 3.5 to the operator $\Phi : B_\rho \to X^*$ defined by putting, for each $x \in X$, $\Phi(x) = F(x) - f$. To this aim, observe what follows.

(a) Φ is of class C^1, $\Phi' = F'$, and Φ' is Lipschitzian with constant γ.

(b) Since $\Phi'(0_X) = F'(0_X) = 0_{\mathcal{L}(X,X^*)}$, by (16) we have

$$\begin{aligned}\sigma &:= \inf_{y \in B_\rho} \sup_{\|u\|_X = 1} |\langle \Phi(0_X), u\rangle - \langle \Phi'(0_X)(u), y\rangle| \\ &= \|\Phi(0_X)\|_{X^*} = \|F(0_X) - f\|_{X^*} \geq M\rho\Big(1 + \frac{1}{b}\Big) > 0.\end{aligned} \tag{17}$$

(c) One has $\sup_{x\in B_\rho} \|\Phi'(x)\|_{\mathcal{L}(X,X^*)} = \sup_{x\in B_\rho} \|F'(x)\|_{\mathcal{L}(X,X^*)} = \theta.$

(d) By assumption (ii) for every $y \in B_\rho$, the functional

$$\begin{aligned} x \in B_\rho \ \to & \ \frac{M}{2b}\|x\|_X^2 + \langle \Phi(x), x-y\rangle \\ = & \ \frac{M}{2b}\|x\|_X^2 + \langle F(x), x-y\rangle + \langle -f, x-y\rangle \end{aligned}$$

is convex.

Then, all the assumptions of Theorem 3.5 are satisfied. Since by (17) we have

$$\frac{\sigma}{M(1+\frac{1}{b})} \geq \rho,$$

the conclusion follows at once. □

Theorem 3.10 immediately gives the following corollary.

Corollary 3.12. *Let $(X, \|\cdot\|_X)$ be a real Banach space. Assume that the norm $\|\cdot\|_X$ is Gâteaux-differentiable on S_1 and has a modulus of convexity of power type 2. Let $\rho > 0$, and let $F : B_\rho \to X^*$ be a given function, such that F is of class C^1, F' is Lipschitzian with constant $\gamma > 0$, and $F'(0_X) = 0_{\mathcal{L}(X,X^*)}$. Let*

$$\theta := \sup_{x\in B_\rho} \|F'(x)\|_{\mathcal{L}(X,X^*)}, \qquad M := 2(\theta + \rho\gamma).$$

Let $b \in]0, 1]$ be such that inequality (5) is true, and let $f \in X^*$ be such that

$$\|f - F(0_X)\|_{X^*} \geq M\rho\Big(1 + \frac{1}{b}\Big). \tag{18}$$

Then, for each $r \in]\,0, \rho]$, there exists a unique $x^* \in S_r$ such that

$$\max\{\langle F(x^*) - f, x^* - y\rangle, \langle F(y) - f, x^* - y\rangle\} < 0$$

for all $y \in B_r \setminus \{x^*\}$.

Proof. Since the norm $\|\cdot\|_X$ has a modulus of convexity of power type 2, it is uniformly convex and the Banach space $(X, \|\cdot\|_X)$ is reflexive. Reasoning as in the proof of Theorem 3.5, one can check that for each $y \in B_\rho$, the function

$$x \in B_\rho \to \langle F(x), x - y\rangle$$

is of class C^1, and its derivative is Lipschitzian with constant M. By Proposition 2.1, it follows that for each $y \in Y$ the functional

$$x \in B_\rho \ \to \ \frac{M}{2b}\|x\|_X^2 + \langle F(x), x-y\rangle$$

is convex. At this point, our conclusion follows by Theorem 3.10. □

Remark 3.13. When $(X, \|\cdot\|_X)$ is a real Hilbert space, then in the statement of Corollary 3.12 one can choose $b = 1$. □

Remark 3.14. Before concluding, we observe that it is quite natural to ask if Theorems 1.2 and 3.7, as well as Corollaries 3.3 and 3.12, can be extended to a larger class of spaces. In particular, taking into account Example 3.9, it would be interesting to know if Theorem 1.2 is still valid for uniformly convex Banach spaces with a Gâteaux differentiable norm.

We leave it as an open problem. □

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