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Generalized Telegraph Equation with Fractional $p(x)$-Laplacian

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The purpose of this paper is devoted to discussing the existence of solutions for a generalized fractional telegraph equation involving a class of ψ-Hilfer fractional with $p(x)$-Laplacian differential equation.

Keywords: ψ-Hilfer fractional derivative, $p(x)$-Laplacian,generalized telegraph equation.

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1. Introduction and motivation

Over the last few decades problems involving wave equations have been paid considerable attention due to their utilization in modeling mathematical physics problems and view of their applications.

The study of quasilinear wave equations involving the p-Laplacian operator in the principal part and viscosity damping has emerged from the nonlinear Voigt model of longitudinal motion of a rod made from viscoelastic material [45]. Specifically, for the so-called Ludwick materials, it can be shown that they obey the following equations, under the effect of an external force f. For the Euler rod

$$\rho\frac{\partial^2 u}{\partial x^2} = K\frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-1}\frac{\partial u}{\partial x}\right) + f.$$

And for the Euler bean

$$\rho A\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(KI_n\left|\frac{\partial^2 u}{\partial x^2}\right|^{p-1}\frac{\partial^2 u}{\partial x^2}\right) + f,$$

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where $u = u(x,t)$ means the displacement at the time t and the space coordinate x, ρ represents the density of the material, K is the engineering constant, p is the strain-hardening exponent, A is the cross-sectional area and I_n designates the second moment of the cross-section for the material.

Henceforth, it is well known that an improvement in the convergence rate can be achieved by considering the corresponding second order damped problem, [4, 15, 32]. For instance, the second-order damped problem naturally appears in modeling mechanical systems. Moreover, the study of problems involving the p-Laplacian operator is a well-consolidated and rich topic of PDEs due to its well-founded and highly-impact results, see for instance [16, 22, 23, 26, 36, 47, 48] and the references therein. As a result, there are numerous works that address questions about the telegraph equation, viscoelastic hyperbolic equation, viscoelastic wave equation, and wave equation [1, 3, 25, 31], among others. These are elaborated via Dirichlet problems and variational techniques. Also, there is a wide literature about p-Laplacian problems involving fractional operators [6, 7, 8, 17, 24].

On the other hand, scientists have been considering partial differential equations involving variable exponents due to their applicability in several relevant models. One of the main applications of such equations has appeared in the study of electrorheological fluids. As mentioned in [34], the study of these fluids originated with the discovery of Bingham fluids, which spontaneously stop moving. Indeed, in the classical reference [46], W. Winslow presented one of the main properties of electrorheological fluids, which is the emergence of parallel and string-like formations in the fluid, under the influence of an electrical field. This phenomenon is known in the literature as the Winslow effect. Furthermore, the electrical field can increase the fluid's viscosity by up to five orders of magnitude, as stated in [34]. Additionally, NASA laboratories have conducted several studies on electrorheological fluids, as highlighted in the interesting paper [33].

There is a growing interest in problems involving variable exponents from a mathematical point of view. For example, in the reference [1], the regularity of solutions for a stationary system related to electrorheological fluids is proven. In the paper [14], a strong maximum principle for a variable exponent operator is obtained, which generalizes the classical case of the Laplacian operator. Moreover, several applications are explored. We also highlight [30], which provides an overview of elliptic variational problems with general growth conditions and variable exponents. For a complete presentation of the theory of Sobolev spaces in the context of variable exponents and their applications, we refer to the references [34, 35].

In 2011, Stavrakakis-Stylianou [44], considered the following problem

$$\begin{cases} u_{tt} - div(|\nabla u|^{p(x)-2}\nabla u) - \Delta u_t + g(u) = f(x,t), \quad (x,t) \in \Omega \times (0,T) := Q_T \\ u(x,t) = 0 \qquad (x,t) \in \partial\Omega \times (0,T) := \Gamma_T \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x) \;\; x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $\mathbb{N} \geq 1$ is a bounded domain, $\partial\Omega$ is Lipschitz continuous. The equation (1) is related to several physical applications in viscoelasticity and processes of filtration through source terms.

In 2017, Messaoudi-Talahmed [28], treated the following problem

$$\begin{cases} u_{tt} - div(|\nabla u|^{m(x)-2}\nabla u) + \mu u_t = |u|^{p(x)-2}u, & in\ \Omega \times (0,T) \\ u(x,t) = 0 \ \ on\, \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) \ \ in\, \Omega, \end{cases} \tag{2}$$

where $\mu \geq 0$. For more details about (2), see [28]. In the same year, Messaoudi and Talahmed [29] established the same problem by adding a constant $b > 0$ to the right-hand side of the equation.

In 2018, Messaoudi et al. [27], studied the problem

$$\begin{cases} u_{tt} - div(|\nabla u|^{r(\cdot)-2}\nabla u) + |u_t|^{m(\cdot)-2}u_t = 0, & in\ \Omega \times (0,T) \\ u(x,t) = 0 \quad on\ \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x) \ in\, \Omega, \end{cases} \tag{3}$$

where the exponents $m(\cdot)$ and $r(\cdot)$ are given measurable functions on Ω. See [27], for more details about the problem (3).

On the other hand, when dealing with fractional differential equations, we are considering a well-established theory with numerous applications and important consequences that are related real phenomena [11, 18, 19]. Moreover, it is interesting to consider a general notion of fractional derivative of a function with respect to another function, due to the several definitions of integrals and fractional derivatives. Such questions was recently considered in Sousa & Oliveira [38], where the authors introduced the $\psi-$Hilfer fractional derivative and provided several examples. We also point out that there has been a recent interest in equations with the ψ-Hilfer fractional operator, see for example [2, 12, 13, 40, 41, 42, 49].

So, motivated by the problem (1)–(3), we are concerned in this paper with the generalized fractional telegraph equation given by

$$\begin{cases} \varepsilon u_{tt} - {}^{\mathbf{H}}\mathbf{D}_{T}^{\alpha,\beta;\psi}\left(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right|^{p(x)-2}\, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right) + u_t = f(x,t) \\ \forall (x,t) \in \mathbf{Q}_T := \Omega \times (0,T) \end{cases} \tag{4}$$

where $f(x,t)$ is a extremal force, $\varepsilon > 0$, ${}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}(\cdot)$ and ${}^{\mathbf{H}}\mathbf{D}_{T}^{\alpha,\beta;\psi}(\cdot)$ are the ψ-Hilfer partial fractional derivative of order $\frac{1}{p(x)} < \alpha < 1$ and of type $0 \leq \beta \leq 1$, with the conditions

$$\begin{cases} u(x,t) = 0, & (x,t) \in \Gamma_T := \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), & \forall x \in \Omega \\ u_t(x,0) = u_1(x), & \forall x \in \Omega, \end{cases} \tag{5}$$

where $\Omega = (0,L)$.

Throughout this work, the functions $p(x), u_0, u_1$ and g satisfy the following condition:

$$\begin{cases} 2 \leq p^- \leq p(x) \leq p^+ < \infty \\ u_0 \in \mathcal{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega), \ u_1 \in L^2(\Omega), \ g(x) \in L^{\frac{p(x)}{p(x)-1}}(\Omega). \end{cases} \tag{6}$$

When working with problems with variable exponents, the principal difficulty is dealing with non-homogeneity, which imply that the application of the Komornik's inequality is not immediately clear. In order to overcome such difficulty, the problem considered will be studied by means of an alternative method that consists in considering a second-order dynamical system that establishes the strong convergence of the difference norm.

The main result of the present manuscript is given as follow:

Theorem 1.1. *Assume that the condition* (6) *holds. If* $u(x,t)$ *satisfies* (4) *and* (5), *then for* $t>0$ *there exists* $u^*(x)$ *satisfying*

$$\int_0^L \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u(x,t) - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^*(x) \right|^{p(x)} dx \leq \Theta t^{\frac{1}{1-p^+}},$$

where $\Theta = \left[c_2^{-p^+} (p^+ - 1) \right]^{\frac{1}{1-p^+}}$, *which is a constant independent of time the variable but depends only on the coefficient* ε. *Here* u^* *is the solution the steady state equation.*

Note that the result above also holds for the equations

$$\varepsilon u_{tt} - {}^{\mathbf{H}}\mathbf{D}_{T}^{\alpha,\beta;\psi} \left(\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right) + u_t = f(x,t).$$

and
$$\varepsilon u_{tt} - \frac{\partial}{\partial x} \left(\left| \frac{\partial u}{\partial x} \right|^{p(x)-2} \frac{\partial u}{\partial x} \right) + u_t = f(x,t); \tag{7}$$

with the conditions (5).

Consequently, all results investigated here are valid for the classic case.

As noted earlier, one of the advantages of the ψ-Hilfer fractional operator is that it is possible to consider a wide class of cases. Namely, choosing $\beta = 1$ and $\psi(t) = t$ yields the problem in the Caputo fractional operator version. Even though, choosing $\beta = 0$ and $\psi(t) = t$ yields the problem in the fractional Riemann-Liouville version. Furthermore, the results investigated in this paper are valid for both of these particular cases, as well as other specific cases.

The rest of the paper is structured as follows: In Section 2, we present some concepts and prove some important ones to discuss the main result of the paper. Finally, in Section 3, we investigate the main contribution of this present manuscript, i.e., the proof of Theorem 1.1.

2. Mathematical background: preliminaries

Let $\Omega \subset \mathbb{R}^N$ be an open set. We denote by $|\Omega|$ the N-dimensional Lebesgue measure of Ω. This section is devoted to recall some preliminary definitions and results that we need in the rest of the paper.

For this aim, let us introduce the space

$$C_+(\overline{\Omega}) = \{ p \in C(\overline{\Omega}; \mathbb{R}) : \inf_{x \in \Omega} p(x) > 1 \}.$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$\mathscr{L}^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable: } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}.$$

$\mathscr{L}^{p(\cdot)}$ is a Banach space when endowed with the *Luxemburg norm* defined by

$$|u|_{p(\cdot)} := \inf \left\{ \mu > 0 : \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

The variable exponent Lebesgue space $\mathscr{L}^{p(\cdot)}(\Omega)$ is a special case of an Orlicz-Musielak space.

For any Lipschitz continuous function $p : \overline{\Omega} \to (1, \infty)$, let [5]

$$p^{-} := \inf_{x \in \Omega} p(x) \text{ and } p^{+} := \sup_{x \in \Omega} p(x).$$

It is well known (see [7]) that for each p_1, $p_2 \in C_{+}(\overline{\Omega})$ such that $p_1 \leq p_2$ in Ω, the embedding $\mathscr{L}^{p_2(\cdot)}(\Omega) \hookrightarrow \mathscr{L}^{p_1(\cdot)}(\Omega)$ is continuous. Furthermore, For $p(x) \in C_{+}(\overline{\Omega})$, if we take p' such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, then the Hölder inequality (see [7, 6]) is given as follows

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^{-}} + \frac{1}{p'^{-}} \right) ||u||_{p(x)} ||v||_{p'(x)} \leq 2 ||u||_{p(x)} ||v||_{p'(x)},$$

for any $u \in \mathscr{L}^{p(x)}(\Omega)$ and $v \in \mathscr{L}^{p'(x)}(\Omega)$. In addition, the relation between the norm $\|u\|_{p(x)}$ and its modular function $\displaystyle\int_{\Omega} |u(x)|^{p(x)} dx$ is stated as follows

$$||u||_{p(x)} \geq 1 \ \Rightarrow \ ||u||_{p(x)}^{p^{-}} \leq \int_{\Omega} |u(x)|^{p(x)} dx \leq ||u||_{p(x)}^{p^{+}}.$$

$$||u||_{p(x)} < 1 \ \Rightarrow \ ||u||_{p(x)}^{p^{+}} \leq \int_{\Omega} |u(x)|^{p(x)} dx \leq ||u||_{p(x)}^{p^{-}}.$$

The ψ-fractional space is given by [42]

$$\mathcal{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega) = \left\{ u \in \mathscr{L}^{p(x)}(\Omega) \ : \ \left| {}^{\mathrm{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right| \in \mathscr{L}^{p(x)}(\Omega) \right\}$$

with the norm $||u|| = ||u||_{\mathcal{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)} = ||u||_{\mathscr{L}^{p(x)}(\Omega)} + \left\| {}^{\mathrm{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right\|_{\mathscr{L}^{p(x)}(\Omega)}$.

Let $\theta = (\theta_1, \theta_2, \theta_3)$, $T = (T_1, T_2, T_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ where $0 < \alpha_1, \alpha_2, \alpha_3 < 1$ with $\theta_j < T_j$, for all $j \in \{1, 2, 3\}$. Also put

$$\Lambda = I_1 \times I_2 \times I_3 = [\theta_1, T_1] \times [\theta_2, T_2] \times [\theta_3, T_3],$$

where T_1, T_2, T_3 and $\theta_1, \theta_2, \theta_3$ positive constants. Consider also $\psi(\cdot)$ be an increasing and positive monotone function on $(\theta_1, T_1), (\theta_2, T_2), (\theta_3, T_3)$, having a continuous derivative $\psi'(\cdot)$ on $(\theta_1, T_1], (\theta_2, T_2], (\theta_3, T_3]$.

On the other hand, let $u, \psi \in C^n(\Lambda)$ two functions such that ψ is increasing and $\psi'(x_j) \neq 0$ with $x_j \in [\theta_j, T_j]$, $j \in \{1,2,3\}$. The left and right-sided ψ-Hilfer fractional partial derivative of 3-variables of $u \in AC^n(\Lambda)$ of order $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ $(0 < \alpha_1, \alpha_2, \alpha_3 \leq 1)$ and type $\beta = (\beta_1, \beta_2, \beta_3)$ where $0 \leq \beta_1, \beta_2, \beta_3 \leq 1$, are defined by [43, 38, 39]

$$\begin{aligned}&{}^{\mathbf{H}}\mathbf{D}_{\theta}^{\alpha,\beta;\psi} u(x_1, x_2, x_3)\\ &= \mathbf{I}_{\theta}^{\beta(1-\alpha),\psi}\left(\frac{1}{\psi'(x_1)\psi'(x_2)\psi'(x_3)}\left(\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3}\right)\right)\mathbf{I}_{\theta}^{(1-\beta)(1-\alpha),\psi} u(x_1, x_2, x_3)\end{aligned}$$

and

$$\begin{aligned}&{}^{\mathbf{H}}\mathbf{D}_{T}^{\alpha,\beta;\psi} u(x_1, x_2, x_3)\\ &= \mathbf{I}_{T}^{\beta(1-\alpha),\psi}\left(-\frac{1}{\psi'(x_1)\psi'(x_2)\psi'(x_3)}\left(\frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3}\right)\right)\mathbf{I}_{T}^{(1-\beta)(1-\alpha),\psi} u(x_1, x_2, x_3),\end{aligned}$$

where $\mathbf{I}_{\theta}^{\alpha,\psi} u(x_1, x_2, x_3)$ and $\mathbf{I}_{T}^{\alpha,\psi} u(x_1, x_2, x_3)$ there are the ψ-Riemann-Liouville fractional integrals of $u \in \mathscr{L}^1(\Lambda)$ of order α $(0 < \alpha < 1)$ given by [43, 38, 39]

$$\begin{aligned}&\mathbf{I}_{\theta}^{\alpha,\psi} u(x_1, x_2, x_3)\\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_{\theta_1}^{x_1}\int_{\theta_2}^{x_2}\int_{\theta_3}^{x_3} \psi'(s_1)\psi'(s_2)\psi'(s_3)(\psi(x_1) - \psi(s_1))^{\alpha_1 - 1}\\ &\quad \times (\psi(x_2) - \psi(s_2))^{\alpha_2 - 1}(\psi(x_3) - \psi(s_3))^{\alpha_3 - 1} u(s_1, s_2, s_3) ds_3 ds_2 ds_1,\end{aligned}$$

to $\theta_1 < s_1 < x_1$, $\theta_2 < s_2 < x_2$, $\theta_3 < s_3 < x_3$ and

$$\begin{aligned}&\mathbf{I}_{T}^{\alpha,\psi} u(x_1, x_2, x_3)\\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_{x_1}^{T_1}\int_{x_2}^{T_2}\int_{x_3}^{T_3} \psi'(s_1)\psi'(s_2)\psi'(s_3)(\psi(s_1) - \psi(x_1))^{\alpha_1 - 1}\\ &\times (\psi(s_2) - \psi(x_2))^{\alpha_2 - 1}(\psi(s_3) - \psi(x_3))^{\alpha_3 - 1} u(s_1, s_2, s_3) ds_3 ds_2 ds_1,\end{aligned}$$

with $x_1 < s_1 < T_1$, $x_2 < s_2 < T_2$, $x_3 < s_3 < T_3$, $x_1 \in [\theta_1, T_1]$, $x_2 \in [\theta_2, T_2]$ and $x_3 \in [\theta_3, T_3]$. For a study of N-variables, see [43].

Let $\theta = (\theta_1, \theta_2, \theta_3)$, $T = (T_1, T_2, T_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. The relation

$$\begin{aligned}&\int_a^b\int_c^d\int_e^f \left(\mathbf{I}_{\theta}^{\alpha;\psi}\varphi(\xi_1, \xi_2, \xi_3)\right)\phi(\xi_1, \xi_2, \xi_3)\, d\xi_3 d\xi_2 d\xi_1 \qquad (8)\\ &= \int_a^b\int_c^d\int_e^f \varphi(\xi_1, \xi_2, \xi_3)\psi'(\xi_1)\psi'(\xi_2)\psi'(\xi_3)\mathbf{I}_{T}^{\alpha;\psi}\left(\frac{\phi(\xi_1, \xi_2, \xi_3)}{\psi'(\xi_1)\psi'(\xi_2)\psi'(\xi_3)}\right) d\xi_3 d\xi_2 d\xi_1\end{aligned}$$

is valid. Whereas, let $\psi(\cdot)$ be an increasing and positive monotone function on $[a,b] \times [c,d] \times [e,f]$, having a continuous derivative $\psi'(\cdot) \neq 0$ on $(a,b) \times (c,d) \times (e,f)$.

If $0 < \alpha = (\alpha_1, \alpha_2, \alpha_3) < 1$ and $0 \leq \beta = (\beta_1, \beta_2, \beta_3) \leq 1$, then

$$\int_a^b \int_c^d \int_e^f \left({}^{\mathbf{H}}\mathbf{D}_{\theta}^{\alpha,\beta;\psi} \varphi\left(\xi_1, \xi_2, \xi_3\right) \right) \phi\left(\xi_1, \xi_2, \xi_3\right) d\xi_2 d\xi_1 \tag{9}$$

$$= \int_a^b \int_c^d \int_e^f \varphi\left(\xi_1, \xi_2, \xi_3\right) \psi'\left(\xi_1\right) \psi'\left(\xi_2\right) \psi'(\xi_3)\; {}^{\mathbf{H}}\mathbf{D}_{T}^{\alpha,\beta;\psi} \left(\frac{\phi\left(\xi_1, \xi_2, \xi_3\right)}{\psi'\left(\xi_1\right) \psi'\left(\xi_2\right) \psi'(\xi_3)} \right) d\xi_3 d\xi_2 d\xi_1$$

for any $\varphi \in AC^1$ and $\phi \in C^1$ satisfying the boundary conditions $\varphi\left(a, c, e\right) = 0 = \varphi\left(b, d, f\right)$.

For the proof of the next result, we will use the notation ${}^{\mathrm{H}}\mathbf{D}_{0+}^{\alpha,\beta;,\psi}(\cdot) = \partial^{\alpha,\beta;\psi} x_j(\cdot)$, where it is the ψ-Hilfer fractional derivative with respect to the variable x_j, with $j = 1, 2, 3, ..., n$.

Note that for $u \in \mathcal{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)$, $\partial^{\alpha,\beta;\psi} x_j u$ denotes the j-th weak fractional derivative of u, which is given, for all $\phi \in C_0^{\infty}(\Omega)$, by

$$\int_{\Omega} u\, \partial_{x_j}^{\alpha,\beta;\psi} \phi dx = -\int_{\Omega} \partial_{x_j}^{\alpha,\beta;\psi} u\, \phi dx.$$

Proposition 2.1. [42, 43] *The space $\mathscr{L}^{p(x)}(\Omega)$ is a separable and reflexive Banach space.*

Proposition 2.2. *The space $\mathcal{H}_{p(x)}^{\alpha,\beta;\,\psi}(\Omega)$ is a separable and reflexive Banach space if $p^- > 1$.*

Proof. Let $\{u_m\} \subset \mathcal{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)$ a Cauchy sequence. Then $\{u_m\}$ and $\left\{\partial_{x_j}^{\alpha,\beta;\psi} u_m\right\}$, $j = 1, 2, ..., N$ are Cauchy sequences in $\mathscr{L}^{p(x)}(\Omega)$. Since $\mathscr{L}^{p(x)}(\Omega)$ is a Banach space, there are $u, w_j \in \mathscr{L}^{p(x)}(\Omega)$ such that

$$u_n \to u \textit{ and } \partial_{x_j}^{\alpha,\beta;\psi} u_n \to w_j \textit{ in } \mathscr{L}^{p(x)}(\Omega), \tag{10}$$

when $n \to \infty$ with $j = 1, 2, ..., N$. Using Holder's inequality, we have

$$\int_{\Omega} \left(u_n - u\right) \partial_{x_j}^{\alpha,\beta;\psi} \phi dx \leq C ||u_n - u||_{p(x)} \left\| \partial_{x_j}^{\alpha,\beta;\psi} \phi \right\| \tag{11}$$

to $\phi \in C_0^{\infty}(\Omega)$, where $C = \dfrac{1}{p^-} + \dfrac{1}{q^-}$. It follows from Eq.(10) and Eq.(11) that

$$\int_{\Omega} u_n \partial_{x_j}^{\alpha,\beta;\psi} \phi dx \to \int_{\Omega} u\, \partial_{x_j}^{\alpha,\beta;\psi} \phi dx \tag{12}$$

to $\phi \in C_0^{\infty}(\Omega)$, when $n \to \infty$. Similarly,

$$\int_{\Omega} \partial_{x_j}^{\alpha,\beta;\psi} u_n\, \phi dx \to \int_{\Omega} w_j\, \phi dx \tag{13}$$

to $\phi \in C_0^{\infty}(\Omega)$, when $n \to \infty$ since we have

$$\int_{\Omega} u_n \partial_{x_j}^{\alpha,\beta;\psi} \phi dx = -\int_{\Omega} \partial_{x_j}^{\alpha,\beta;\psi} u_n\, \phi dx \tag{14}$$

for all $\phi \in C_0^{\infty}(\Omega)$.

Passing to the limit in (14) of $n \to \infty$ and using Eq.(12) and Eq.(13), yields

$$\int_{\Omega} u\partial_{x_j}^{\alpha,\beta;\psi}\phi dx = -\int_{\Omega} w_j\,\phi dx \tag{15}$$

to $\phi \in C_0^{\infty}(\Omega)$. Using the Du Bois-Reymond lemma in (15), we conclude that $u \in \mathcal{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)$ and $w_j = \partial_{x_j}^{\alpha,\beta;\psi}u$, $j = 1,2,...,N$. So, using Eq.(10), we have

$$\|u_n - u\|_{*} = \|u_n - u\|_{p(x)} + \sum_{j=1}^{N}\left\|\partial_{x_j}^{\alpha,\beta;\psi}u_n - \partial_{x_j}^{\alpha,\beta;\psi}u\right\|_{p(x)} \to 0$$

when $n \to 0$. Therefore, $\mathcal{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)$ is a Banach space.

Now, let us prove that it is separable and reflexive, but first note that the space $E = L^{p(x)}(\Omega)\times\cdots\times L^{p(x)}(\Omega)$, $(n+1)$-times, equipped with the norm $||\cdot||$ is reflective and separable. Define the linear operator $T:\mathcal{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)\to E$ to $T(u) = \big(u, \partial_{x}^{\alpha,\beta;\psi}u\big)$. Note that $||Tu|| = ||u||$. From this last equality we conclude that $T\big(\mathcal{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)\big)$ is a closed subspace of E. From the Proposition III.17 and III. 22 of [10] we have $T\big(\mathcal{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)\big)$ it is reflexive and separable. Therefore, $\mathcal{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega)$ is reflexive and separable. ☐

To understand our technique easier, we only consider $f(x,t) = g(x) = Ap_0(x)$ in which $p_0(x)$ represents the static pressure.

Consider $\varepsilon u_{tt} = \mathcal{M} + g(x,t)$ where $\mathcal{M} := {}^{\mathbf{H}}\mathbf{D}_{T}^{\alpha,\beta;\psi}\Big(\big|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\big|^{p(x)-2}\,{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\Big) - u_t$.

The Galerkin approximations of solutions to problem (4) with the condition (5) are sought in the form

$$u^{(n)} = \sum_{k=1}^{n} u_k(t)\psi_k(t),\;\; u_k = (u(x,t),\psi_k(t))_{\mathcal{H}_2^{\alpha,\beta;\psi}(\Omega)}\,. \tag{16}$$

We assume also

$$u_1^{(n)} \to u_1,\;\text{strongly in } L^2(\Omega),\;\; u_0^{(n)} \to u_0,\;\text{strongly in } \mathcal{H}_2^{\alpha,\beta;\psi}(\Omega). \tag{17}$$

The coefficient $u_k(t)$ are defined from the relation

$$\int_0^L \Big(\varepsilon u_{tt}^{(n)} - \mathcal{M}u^{(n)} - g\Big)\psi_k = 0,\;\; k = 1,2,...,n. \tag{18}$$

The last equalities and the initial conditions lead us to the Cauchy problem for the system of n ordinary differential equations of the second order for the coefficients $u_k(t)$

$$u_k'' = \mathcal{G}_k(t, u(t),...,u_n(t)) \tag{19}$$

and

$$u_k(0) = \int_0^L u_0\psi_k,\;\; u_k'(0) = \int_0^L u_1\psi_k,\;\; k = 1,2,...,n \tag{20}$$

where

$$\mathcal{G}_k = \int_0^L \left(\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^{(n)}(x) \right|^{p(x)-2} {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^{(n)} \right) {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} \psi_k dx + \int_0^L (g\psi_k - u_t^{(n)}\psi_k) dx.$$

Using the Peano's Theorem, for every finite n the problem (19), (20) has a solution $u_k(t)$, $k = 1, ..., n$ on an interval $(0, T_n)$ for each n. The estimates below allow one to take $T_n = T$ for all n.

Multiplying each of equations (19) by $c_k'(t)$ and summing over $k = 1, ..., n$, we arrive at the relation

$$\begin{aligned} &\frac{d}{dt} \int_0^L \left(\frac{\varepsilon}{2} \left| u_t^{(n)} \right|^2 + \frac{\left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^{(n)} \right|^{p(x)}}{p(x)} - u^{(n)} g(x) \right) dx + \int_0^L \int_0^L \left| u^{(n)} \right|^2 dx \\ &= \int_0^L g(x) u_t^{(n)} dx. \end{aligned} \tag{21}$$

Omitting the index n for simplicity, we define the energy functional associated to the problem (4) as follows

$$\mathbb{E}_{p(x)}^{\alpha,\beta;\psi}(t) = \frac{\varepsilon}{2} \int_0^L |u_t|^2 dx + \int_0^L \frac{1}{p(x)} \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx - \int_0^L u g(x) dx. \tag{22}$$

Note that, the Eq.(21) can write the following form

$$\left(\mathbb{E}_{p(x)}^{\alpha,\beta;\psi}(t) \right)' + \int_0^L |u_t|^2 dx = \int_0^L g(x) u_t dx. \tag{23}$$

Taking $g = 0$ and using Eq.(23), we have the following result:

Lemma 2.3. *Assume that condition* (6) *holds and that* $u \in L^\infty(0, T; \mathcal{H}_{p(x)}^{\alpha,\beta;\psi}(\Omega))$ *is a solution of problem* (4) *such that* $u_t \in L^\infty(0, T; L^2(\Omega))$ *and* $u_{tt} \in L^\infty\left(0, T; \mathcal{H}_{\frac{p(x)}{p(x)-1}}^{\alpha,\beta;\psi}(\Omega)\right)$. *Then the energy functional* $\mathbb{E}_{p(x)}^{\alpha,\beta;\psi}(t)$ *satisfies the following inequality*

$$\mathbb{E}_{p(x)}^{\alpha,\beta;\psi}(t) + \int_0^t \int_0^L |u_s|^2 dx ds \leq \mathbb{E}_{p(x)}^{\alpha,\beta;\psi}(0), \quad 0 \leq t \leq T. \tag{24}$$

Obviously, with the help of Eq. (24) and the Poincaré inequality [40, 41],

$$\|u\|_{p(\cdot)} \leq C \left\| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right\|_{p(\cdot)}, \text{ with } \quad u \in \mathcal{H}_{p(x)}^{\alpha,\beta;\psi} \tag{25}$$

we have the following result:

Proposition 2.4. *If all the conditions of Lemma* 2.3 *are satisfied, then there exists* $\Theta_0 > 0$ *(constant) such that for* $t > 0$, *we have*

$$\varepsilon \int_0^L |u_t|^2 dx + \int_0^L \frac{1}{p(x)} \left| {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \right|^{p(x)} dx + 2 \int_0^t \int_0^L |u_\tau|^2 dx d\tau \leq \Theta_0. \tag{26}$$

Proof. Using the Young inequality, there exists $C>0$ (constant), such that

$$\begin{aligned}
\mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(t) &\geq \frac{\varepsilon}{2}\int_0^L |u_t|^2 dx + \int_0^L \frac{1}{p(x)}\left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}u\right|^{p(x)} dx - C\|g\|_{p'(\cdot)}\left\|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}u\right\|_{p(\cdot)}\\
&\geq \frac{\varepsilon}{2}\int_0^L |u_t|^2 dx + \int_0^L \frac{1}{p(x)}\left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}u\right|^{p(x)} dx - C\|g\|_{p'(\cdot)}\\
&\quad\times max\left\{\left(\int_0^L\left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}u\right|^{p(x)}dx\right)^{1/p^+},\left(\int_0^L\left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}u\right|^{p(x)}dx\right)^{1/p^-}\right\}\\
&\geq \frac{\varepsilon}{2}\int_0^L |u_t|^2 dx + \int_0^L \left(\frac{1}{p(x)}-\frac{1}{2p(x)}\right)\left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}u\right|^{p(x)} dx\\
&\quad - Cmax\left\{||g||^{p'^+}_{p'},||g||^{p'^-}_{p'}\right\} \qquad (27)\\
&\geq \frac{\varepsilon}{2}\int_0^L |u_t|^2 dx + \int_0^L \frac{1}{2p(x)}\left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}u\right|^{p(x)} dx - Cmax\left\{||g||^{p'^+}_{p'(\cdot)},||g||^{p'^-}_{p'(\cdot)}\right\}
\end{aligned}$$

where $p'(x)=\frac{p(x)}{p(x)-1}$, $p'^+=\underset{x\in\Omega}{max}\,p'(x)$, $p'^-=\underset{x\in\Omega}{min}\,p'(x)$. On the other hand, from the inequalities (24) and (27), one has

$$\begin{aligned}
&\varepsilon\int_0^L |u_t|^2 dx + \int_0^L \frac{1}{p(x)}\left|{}^{\mathrm{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0+}u\right|^{p(x)} dx + 2\int_0^t\int_0^L |u_\tau|^2 dx d\tau\\
&\leq 2\left|\mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(0)\right| + 2C\,max\left\{||g||^{p'_+}_{p'(\cdot)},||g||^{p'_-}_{p'(\cdot)}\right\} := \Theta_0,
\end{aligned}$$

where $p'(\cdot)$ is the conjugate of $p(\cdot)$. □

Next, we will obtain the existence of solutions to the stationary problem.

Lemma 2.5. *If* $g(x)\in L^{\frac{p(x)}{p(x)-1}}(\Omega)$, *then* $u^*(x)$ *is the minimizer of the following functional*

$$\mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(v)=\int_0^L \frac{1}{p(x)}\left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}v\right|^{p(x)} dx - \int_0^L g(x)v dx. \qquad (28)$$

Moreover, it satisfies the following fractional Euler-Lagrange equation in the sense of distribution

$$\begin{cases} -{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{T}\left(\left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}u\right|^{p(x)-2}\ {}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}u^*\right)=g(x),\ x\in\Omega\\ u^*(x)=0,\ x\in\partial\Omega. \end{cases} \qquad (29)$$

3. Main results

Before attacking the main result of the paper, it is necessary to investigate other results that are of paramount importance for the proof of Theorem 1.1.

Lemma 3.1. *Let* u *and* u^* *be solutions of* Eq. (4) *and* Eq. (28) *respectively, then we have*

$$\lim_{t\to\infty+}\mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u(\cdot,t))=\mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u^*(\cdot)).$$

Proof. For any $t > 0$ we have $u(\cdot, t) \in \mathcal{H}^{\alpha,\beta;\psi}_{p(x)}(\Omega)$. Now, due to Lemma 2.5 we have

$$\mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u^*(\cdot)) \leq \mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u(\cdot, t)).$$

This implies that
$$\mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u^*(\cdot)) \leq \liminf_{t\to\infty+} \mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u(\cdot, t)). \tag{30}$$

Conversely, we claim that $\limsup_{t\to\infty} \mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u(\cdot, t)) \leq \mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u^*(\cdot))$.
To prove this inequality, set

$$\phi(t) := \frac{1}{2}\int_0^L \big(u(x,t) - u^*(x)\big)^2 dx.$$

This yields

$$\begin{aligned} \varepsilon\phi''(t) + \phi'(t) = & -\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+} u\right|^{p(x)-2} {}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+} u \; {}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}(u - u^*) dx \\ & + \int_0^L g(x)(u - u^*) dx + \varepsilon \int_0^L |u_t|^2 dx. \end{aligned} \tag{31}$$

Using the Young inequality and the monotonicity of the energy functional $\mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(t)$, Eq. (31) can be rewritten as

$$\begin{aligned} & \varepsilon\phi''(t) + \phi'(t) \\ & \leq -\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+} u\right|^{p(x)} dx + \int_0^L \frac{p(x)-1}{p(x)} \left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+} u\right|^{p(x)} dx \\ & \quad + \int_0^L \frac{1}{p(x)} \left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+} u^*\right|^{p(x)} dx + \int_0^L g(x)(u - u^*) dx + \varepsilon \int_0^L |u_t|^2 dx \\ & \leq \mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u^*(\cdot)) - \mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u(\cdot, t)) + \varepsilon \int_0^L |u_t|^2 dx \\ & = \mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u^*(\cdot)) + \frac{3\varepsilon}{2}\int_0^L |u_t|^2 dx - \mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(t) \\ & \leq \mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u^*(\cdot)) + \frac{3\varepsilon}{2}\int_0^L |u_t|^2 dx - \mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(T), \quad 0 < t < T. \end{aligned} \tag{32}$$

Furthermore, on the left side of the inequality (32), we can write

$$\begin{aligned} & \phi(t) + \mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(T)\big(T + \varepsilon e^{-\frac{T}{\varepsilon}} - \varepsilon\big) \leq \varepsilon\phi'(0)(1 - e^{-\frac{T}{\varepsilon}}) + \phi(0) \\ & + \mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u^*(\cdot))\big(T + \varepsilon e^{-\frac{T}{\varepsilon}} - \varepsilon\big) + \frac{3}{2}\int_0^T\int_0^t\int_0^L |u_t|^2 e^{\frac{\tau - t}{\varepsilon}} dx d\tau dt. \end{aligned} \tag{33}$$

Claim: $$\frac{3}{2}\int_0^T\int_0^t\int_0^L |u_t|^2 e^{\frac{\tau - t}{\varepsilon}} dx d\tau dt \leq \frac{3\Theta_0\varepsilon}{4}.$$

Note that, $\{(\tau,t):0\leq t\leq T,\ 0\leq\tau\leq t\}=\{(\tau,t):0\leq\tau\leq T,\ \tau\leq t\leq T\}$.

Thus, for any $T>0$, one has

$$\begin{aligned}\frac{3}{2}\int_0^T\int_0^t\int_0^L|u_t|^2e^{\frac{\tau-t}{\varepsilon}}dxd\tau dt&=\frac{3}{2}\int_0^T\int_\tau^T e^{\frac{\tau-t}{\varepsilon}}dt\int_0^L|u_\tau|^2dxd\tau\\&=\frac{3\varepsilon}{2}\int_0^T\int_0^L|u_\tau|^2dx\big(1-e^{\frac{\tau-T}{\varepsilon}}\big)d\tau\leq\frac{3\varepsilon}{2}\int_0^T\int_0^L|u_\tau|^2dxd\tau\leq\frac{3\Theta_0\varepsilon}{4}.\end{aligned}\tag{34}$$

Now, we multiply Eq. (31) by $\big(T+\varepsilon e^{-\frac{T}{\varepsilon}}-\varepsilon\big)^{-1}$ and let us not consider nonnegative terms. In this sense, we obtain

$$\mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(T)\leq\frac{\varepsilon\phi'(0)(1-e^{\frac{-T}{\varepsilon}})+\phi(0)}{T+\varepsilon e^{\frac{-T}{\varepsilon}}-\varepsilon}+\mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u^*(\cdot))+\frac{\frac{3\Theta_0\varepsilon}{4}}{T+\varepsilon e^{\frac{-T}{\varepsilon}}-\varepsilon},\ \ T>\varepsilon.\tag{35}$$

In addition, we use the fact that

$$\mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u(\cdot,T))=\mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(T)-\frac{\varepsilon}{2}\left\|u_t\right\|_2^2\leq\mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(T),$$

which implies

$$\limsup_{T\to\infty+}\mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u(\cdot,T))\leq\limsup_{T\to\infty+}\mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(T)\leq\mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u^*(\cdot)).\tag{36}$$

Our desired results are the consequence of Eq. (30) and Eq. (36) □

As a consequence of Lemma 2.5 it is possible to establish better estimates for $\|u_t\|_2^2$ and $\left\|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}u\right\|_{p(\cdot)}$ with $0\leq\beta\leq1$ and $\frac{1}{p(x)}<\alpha<1$.

Lemma 3.2. *Let u and u^* be solutions of problems* (4) *and* (29) *respectively. Then*

(i) $\|u_t(T)\|_{L^2(\Omega)}\overset{T\to\infty+}{\longrightarrow}0$,

(ii) $\displaystyle\int_0^L\left|{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}u(\cdot,T)-{}^{\mathbf{H}}\mathbf{D}^{\alpha,\beta;\psi}_{0^+}u^*(\cdot)\right|^{p(x)}dx\overset{T\to\infty+}{\longrightarrow}0.$

Proof. (i) Note that Eq. (27) shows that $\mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(t)$ is bounded from below. Furthermore, we have that $\lim_{t\to\infty+}\mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(t)$ is well defined, (just use the fact of $\mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(t)$ to be monotone and linear bounded), because every bounded monotone sequence converges. In this sense, using Lemma 2.5 and

$$\varepsilon\|u_t(\cdot,T)\|_2^2=2\ \mathbb{E}^{\alpha,\beta;\psi}_{p(x)}(T)-2\ \mathcal{I}^{\alpha,\beta;\psi}_{p(x)}(u(\cdot,T)),$$

we deduce that $\lim_{T\to\infty+}\|u_t(\cdot,T)\|_2^2$ exists.

Let us now define $a=\lim_{T\to\infty+}\|u_t(\cdot,T)\|_2^2\geq0$. Therefore $a=0$. In fact, if not, namely, $a\neq0$, then $\exists T>0$ such that for $T\geq T_0$, yields

$$\|u_t(\cdot,T)\|_2^2\geq\frac{a}{2}>0,\ \ \text{which implies}\ \int_{T_0}^T\|u_t(\cdot,T)\|_2^2\geq\frac{a}{2}(T-T_0).$$

But this is a contradiction since

$$\int_0^T \int_0^L |u_t|^2 dxdt \le \Theta_0.$$

(ii) Consider the following inequality

$$\langle |A|^{q-2}A - |B|^{q-2}B, A-B\rangle \ge 2^{2-q}|A-B|^q, \;\; A, B \in \mathbb{R}^N, \;\; q \ge 2.$$

For $p(x) \ge 2$, we have

$$\begin{aligned}
&\int_0^L \frac{1}{p(x)} \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u\right|^{p(x)} dx - \int_0^L \frac{1}{p(x)} \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^*\right|^{p(x)} dx \\
&= \int_0^L \frac{1}{p(x)} \int_0^1 \left|\theta \;\; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u + (1-\theta) \;\; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^*\right|^{p(x)} dxd\theta \\
&= \int_0^L \frac{1}{p(x)} \int_0^1 \left|\theta \;\; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u + (1-\theta) \;\; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^*\right|^{p(x)-2} \\
&\quad \left(\theta \;\; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u + (1-\theta) \;\; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^*\right)\left({}^{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u \; - \; {}^{H}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^*\right) dxd\theta \\
&\ge \int_0^L \frac{1}{p(x)} \Bigg(\int_0^1 \frac{2^{2-p(x)}}{\theta} \Big|\theta \;\; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u + (1-\theta) \;\; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^* \\
&\quad - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^*\Big|^{p(x)} + \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^*\right|^{p(x)-2} \;\; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^* \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}(u-u^*) d\theta \Bigg) dx \\
&\ge \int_0^L \int_0^1 \theta^{p(x)-1} 2^{2-p(x)} \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^*\right|^{p(x)} dxd\theta \\
&\quad + \int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^*\right|^{p(x)-2} \; {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^* \, {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}(u-u^*) dx \\
&= \frac{2^{2-p^+}}{p^+} \int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^*\right|^{p(x)} dx + \int_0^L g(x)(u-u^*)dx, \qquad (37)
\end{aligned}$$

where u^* is the solution of the problem (29). Note that, on the right side of the inequality (37)

$$\frac{2^{2-p^+}}{p^+} \int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u - {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi} u^*\right|^{p(x)} dx \le \mathcal{I}_{p(x)}^{\alpha,\beta;\psi}(u) - \mathcal{I}_{p(x)}^{\alpha,\beta;\psi}(u^*). \qquad (38)$$

So the result follows from Lemma 2.5. □

Let us now investigate the main contribution of the present paper, i.e., the proof of Theorem 1.1.

Proof of Theorem 1.1. Consider $w = u - u^*$. The proof of the theorem will be divided into three steps.

Step1: The purpose of this step is to discuss the sign of the error functional.

Let $$\mathcal{G}(t) = \int_0^L \left(\frac{1}{2\varepsilon}w^2 + ww_t + \varepsilon w_t^2\right) dx + 2\mathcal{I}_{p(x)}^{\alpha,\beta;\psi}(u) - 2\mathcal{I}_{p(x)}^{\alpha,\beta;\psi}(u^*).$$

Using problem (4), yields

$$\begin{aligned}
\mathcal{G}'(t) &= \frac{1}{\varepsilon}\int_0^L \left(g(x)w - \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right|^{p(x)-2} \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}w\right)dx - \int_0^L |w_t|^2 dx \\
&= \frac{1}{\varepsilon}\int_0^L \Bigg(\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u^*\right|^{p(x)-2} \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u^* \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}w \\
&\qquad - \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right|^{p(x)-2} \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u \ {}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}w\Bigg)dx - \int_0^L |w_t|^2 dx \\
&\leq -\frac{2^{2-p^+}}{\varepsilon}\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}w\right|^{p(x)} dx - \int_0^L |w_t|^2 dx \leq 0, \quad t > 0.
\end{aligned} \tag{39}$$

Step 2: The objective of this steps is to establish a relationship between

$$\mathcal{G}(t), \quad \int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}w\right|^{p(x)} dx \quad \text{and} \quad \int_0^L |w_t|^2 dx.$$

First, we need to prove that $\left\|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u^*\right\|_{p(\cdot)}$ is bounded from above.

Indeed, by multiplying the first identity of the problem (29) by u^*, integrate over Ω, using the Poincaré inequality and the Hölder inequality, one has

$$\begin{aligned}
&\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u^*\right|^{p(x)} dx = \int_0^L g(x)u^* dx \\
&\qquad \Rightarrow \int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u^*\right|^{p(x)} dx \leq C \max\left\{\|g\|_{p'(\cdot)}^{p'_+}, \|g\|_{p'(\cdot)}^{p'_-}\right\} \leq \Theta_0.
\end{aligned}$$

Using Eq. (26), we get

$$\begin{aligned}
\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}w\right|^{p(x)} dx &\leq 2^{p^+-1}\left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right|^{p(x)} dx + \int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u^*\right|^{p(x)} dx\right) \\
&\leq 2^{p^+}\Theta_0 = \Theta_1.
\end{aligned} \tag{40}$$

In the forthcoming proof, we first estimate $\mathcal{G}(t)$. By means of the Hölder inequality, we have

$$\begin{aligned}
\mathcal{G}(t) &\leq \frac{\varepsilon+1}{2\varepsilon}\int_0^L |w|^2 dx + \frac{2\varepsilon+1}{2\varepsilon}\int_0^L |w_t|^2 dx + 2\mathcal{I}_{p(x)}^{\alpha,\beta;\psi}(u) - 2\mathcal{I}_{p(x)}^{\alpha,\beta;\psi}(u^*) \\
&= \frac{\varepsilon+1}{2\varepsilon}\int_0^L |w|^2 dx + \frac{2\varepsilon+1}{2\varepsilon}\int_0^L |w_t|^2 dx + 2\int_0^L g(x)w\,dx \\
&\quad + \int_0^L \frac{1}{p(x)}\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u\right|^{p(x)} dx - \int_0^L \frac{1}{p(x)}\left|{}^{\mathbf{H}}\mathbf{D}_{0+}^{\alpha,\beta;\psi}u^*\right|^{p(x)} dx \\
&:= \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4.
\end{aligned} \tag{41}$$

Using inequality (25) and the Hölder inequality, it follows that

$$\begin{aligned}\mathcal{A}_1 &\leq \frac{(\varepsilon+1)}{2\varepsilon}C_3 \max\left\{\left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(x)}dx\right)^{\frac{2}{p^+}}, \left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(x)}dx\right)^{\frac{2}{p^-}}\right\}\\ &\leq \frac{(\varepsilon+1)}{2\varepsilon}C_3 \max\left\{\Theta_1^{\frac{2}{p^-}-\frac{1}{p^+}}, \Theta_1^{\frac{1}{p^+}}\right\}\left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(x)}dx\right)^{\frac{1}{p^+}}.\end{aligned}$$

Furthermore, one has

$$\begin{aligned}\mathcal{A}_3 &\leq 2C_3\|g\|_{p'(x)} \max\left\{\left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(x)}dx\right)^{\frac{1}{p^+}}, \left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(x)}dx\right)^{\frac{1}{p^-}}\right\}\\ &\leq 2C_3\|g\|_{p'(x)} \max\left\{\Theta_1^{\frac{1}{p^-}-\frac{1}{p^+}}, 1\right\}\left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(x)}dx\right)^{\frac{1}{p^+}}. \qquad (42)\end{aligned}$$

In the inequality above, we use

$$\left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(x)}dx\right)^{\eta} \leq \max\left\{\Theta_1^{\eta-\delta}, 1\right\}\left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(x)}dx\right)^{\delta}, \quad 0<\delta\leq\eta.$$

On the other hand, by using the following inequality

$$\begin{aligned}&\left|\frac{\left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}u\right|^{p(x)}}{p(x)} - \frac{\left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}u^*\right|^{p(x)}}{p(x)}\right|\\ &\leq \left(\left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}u\right| + \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}u^*\right|\right)^{p(x)-1}\left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}u - {}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}u^*\right|,\end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{A}_4 &\leq \int_0^L \left(\left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}u\right| + \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}u^*\right|\right)^{p(x)-1}\left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|dx\\ &\leq C_5(\Theta_0)\max\left\{\Theta_1^{\frac{1}{p^-}-\frac{1}{p^+}}, 1\right\}\left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(x)}dx\right)^{\frac{1}{p^+}}. \qquad (43)\end{aligned}$$

Using Eq. (41) and Eq. (43), yields

$$\begin{aligned}\mathcal{G}(t) &\leq \frac{(\varepsilon+1)C_3}{2\varepsilon}\max\left\{\Theta_1^{\frac{2}{p^-}-\frac{2}{p^+}}, 1\right\}\left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(x)}dx\right)^{\frac{2}{p^+}}\\ &\quad + \frac{2\varepsilon+1}{2}\int_0^L |u_t|^2dx\\ &\quad + C_3\|g\|_{p'(x)}\max\left\{\Theta_1^{\frac{1}{p^-}-\frac{1}{p^+}}, 1\right\}\left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(x)}dx\right)^{\frac{1}{p^+}}\\ &\quad + C_5(\Theta_0)\max\left\{\Theta_1^{\frac{1}{p^-}-\frac{1}{p^+}}, 1\right\}\left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}u\right|^{p(x)}dx\right)^{\frac{1}{p^+}}\\ &\leq \tilde{C}\left(\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi}w\right|^{p(x)}dx\right)^{\frac{1}{p^+}} + \frac{2\varepsilon+1}{2}\|u_t\|_2^2, \qquad (44)\end{aligned}$$

where

$$\tilde{C} = \frac{(\varepsilon+1)C_3}{2\varepsilon} \max\left\{\Theta_1^{\frac{2}{p^-}-\frac{1}{p^+}}, \Theta_1^{\frac{1}{p^+}}\right\} + 2C_3\|g\|_{p'(x)} \max\left\{\Theta_1^{\frac{1}{p^-}-\frac{1}{p^+}}, 1\right\}$$
$$+ C_3(\Theta_1) \max\left\{\Theta_1^{\frac{1}{p^-}-\frac{1}{p^+}}, 1\right\}.$$

So, from the Eq. (39) and Eq. (44), we obtain

$$\mathcal{G}(t) \leq \tilde{C}\left(\frac{-\varepsilon\mathcal{G}'(t)}{2^{2-p^+}}\right)^{\frac{1}{p^+}} - \mathcal{G}'(t). \tag{45}$$

Step 3: In this step, we establish the first-order differential inequality.
From Eq. (39) we have

$$|\mathcal{G}'(t)| \leq \frac{C_3}{\varepsilon}\|g\|_{p'(x)} \left\|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi} w\right\|_{p(x)} + \frac{1}{\varepsilon}\left\|\left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi} u\right|^{p(x)-1}\right\|_{p'(x)} \left\|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi} w\right\|_{p(x)}$$
$$+ \int_0^L |w_t|^2 dx \leq \Theta_2, \tag{46}$$

where $$\Theta_2 = \frac{C_3}{\varepsilon}\|g\|_{p'(x)} \max\left\{\Theta_1^{\frac{1}{p^-}}, \Theta_1^{\frac{1}{p^+}}\right\}$$
$$+ \frac{1}{\varepsilon}\max\left\{\Theta_1^{\frac{p^--1}{p^-}}, \Theta_1^{\frac{p^+-1}{p^+}}\right\} \max\left\{\Theta_1^{\frac{1}{p^-}}, \Theta_1^{\frac{1}{p^+}}\right\} + \frac{\Theta_0}{\varepsilon}.$$

In this sense, using inequality (46), Eq. (45) can be rewritten in the form

$$\mathcal{G}(t) \leq \left(\left(\frac{\varepsilon(\tilde{C})^{p^+}}{2^{2-p^+}}\right)^{\frac{1}{p^+}} + \Theta_2^{1-\frac{1}{p^+}}\right)\left(-\mathcal{G}'(t)\right)^{\frac{1}{p^+}} := C_4\left(-\mathcal{G}'(t)\right)^{\frac{1}{p^+}}. \tag{47}$$

To solve Eq. (47), we have

$$\mathcal{G}(t) \leq \left(\mathcal{G}^{1-p^+}(0)C_4^{-p^+}(p^+-1)t\right)^{\frac{1}{1-p^+}} \leq \left(C_4^{-p^+}(p^+-1)\right)^{\frac{1}{1-p^+}} t^{\frac{1}{1-p^+}}. \tag{48}$$

The inequality (48) together with the non negativity of the form

$$\frac{1}{2\varepsilon}w^2 + ww_t + \varepsilon w_t^2 = \left(\frac{1}{2\sqrt{\varepsilon}}w^2 + \sqrt{\varepsilon}w_t\right)^2 + \frac{1}{4\varepsilon}w^2,$$

we obtain

$$\mathcal{I}_{p(x)}^{\alpha,\beta;\psi}(u) - \mathcal{I}_{p(x)}^{\alpha,\beta;\psi}(u^*) \leq \frac{1}{2}\left(C_4^{-p^+}(p^+-1)\right)^{\frac{1}{1-p^+}} t^{\frac{1}{1-p^+}}.$$

Finally, from Eq. (38), one has

$$\int_0^L \left|{}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi} u^* - {}^{\mathbf{H}}\mathbf{D}_{0^+}^{\alpha,\beta;\psi} u(x,t)\right|^{p(x)} dx \leq \frac{p^+}{2^{3-p^+}}\left(C^{-p^+}(p^+-1)\right)^{\frac{1}{1-p^+}} t^{\frac{1}{1-p^+}}$$
$$= \Theta t^{\frac{1}{1-p^+}},$$

where
$$\Theta = \left(C_4^{-p^+}(p^+ - 1)\right)^{\frac{1}{1-p^+}}, \quad C_4 := \left(\left(\frac{\varepsilon(\tilde{C})^{p^+}}{2^{2-p^+}}\right)^{\frac{1}{p^+}} + \Theta_2^{1-\frac{1}{p^+}}\right). \tag{49}$$

The proof is complete. □

Declarations

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