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An Upper Bound for the Least Energy of a Nodal Solution to the Yamabe Equation on the Sphere

Mónica Clapp*
Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad de México
monica.clapp@im.unam.mx

Angela Pistoia
Dipartimento SBAI, La Sapienza Università di Roma, Italy
angela.pistoia@uniroma1.it

Tobias Weth
Institut für Mathematik, Goethe-Universität, Fankfurt am Main, Germany
weth@math.uni-frankfurt.de

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For each $n \geq 3$ we establish the existence of a nodal solution u to the Yamabe problem on the round sphere $(\mathbb{S}^n, g)$ which satisfies

$$\int_{\mathbb{S}^n} |u|^{2^*} dV_g < 2m_n \mathrm{vol}(\mathbb{S}^n),$$

where $m_3 = 9$, $m_4 = 7$, $m_5 = m_6 = 6$, and $m_n = 5$ if $n \geq 7$.

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1. Introduction

We consider the Yamabe problem on the round n-sphere

$$\frac{4(n-1)}{n-2}\Delta_g u + n(n-1)u = n(n-1)|u|^{2^*-2}u \qquad \text{on } \mathbb{S}^n, \tag{1}$$

where $n \geq 3$, g is the standard metric, $\Delta_g u = -\mathrm{div}_g \nabla_g$ is the Laplace-Beltrami operator and $2^* := \frac{2n}{n-2}$ is the critical Sobolev exponent.

The existence of positive and sign-changing solutions to this problem is well known. Different types of nodal solutions have been exhibited in [1, 3, 7, 6, 9].

The constant function $u_0 \equiv 1$ solves (1) and it is a least energy solution. It is easily observed that the L^{2^*}-norm of any nodal solution u of (1) is larger than twice the L^{2^*}-norm of u_0, i.e.,

$$\int_{\mathbb{S}^n} |u|^{2^*} dV_g > 2\mathrm{vol}(\mathbb{S}^n), \tag{2}$$

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see e.g. [10, Chapter III.3]. This estimate has been slightly improved in [11], where it is shown that

$$\inf\Big\{ \int_{\mathbb{S}^n} |u|^{2^*} dV_g \ :\ u \text{ nodal solution of } (1) \Big\} > 2\mathrm{vol}(\mathbb{S}^n). \tag{3}$$

We note that (3) is not a direct consequence of (2), since it is unknown if the infimum in (3) is attained.

Estimates for the least energy of nodal solutions to problem (1) are of interest, since they are related to compactness properties of semilinear elliptic boundary value problems with critically growing nonlinearities via Struwe's compactness lemma. See [10, Chapter III.3] for a discussion of this aspect.

The aim of this note is to give an upper bound for the least energy of a nodal solution to problem (1). Set

$$m_n := \begin{cases} 9 & \text{if } n = 3, \\ 7 & \text{if } n = 4, \\ 6 & \text{if } n = 5, 6, \\ 5 & \text{if } n \geq 7. \end{cases} \tag{4}$$

We prove the following result.

Theorem 1.1. *The Yamabe equation* (1) *has a nodal solution u which satisfies*

$$u(z_1, z_2, x) = u(\mathrm{e}^{2\pi \mathrm{i}/m_n} z_1, \mathrm{e}^{2\pi \mathrm{i}/m_n} z_2, x)$$
$$u(z_1, z_2, x) = -u(z_2, z_1, x),$$

for all $(z_1, z_2, x) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-3} \cong \mathbb{R}^{n+1}$, and

$$\int_{\mathbb{S}^n} |u|^{2^*} dV_g < 2m_n \mathrm{vol}(\mathbb{S}^n).$$

The solution given by Theorem 1.1 might be the same as the one obtained in [3, Theorem 1.1] for $n \geq 4$ or one of those obtained in [1, Remark 3.3] for the Laplace operator, but it is different from those obtained by Ding in [7], as shown in Proposition 4.2 below. Estimates for the energy of some of Ding's solutions are listed in [9], but no information is given allowing to verify them.

Our approach is as follows. First, we give a condition for the existence of a least energy solution to the Yamabe problem (1) having certain symmetries (see Theorem 2.2). The symmetries are chosen in such a way that they yield sign-changing solutions by construction. Then, we estimate the energy of a specific ansatz and derive an explicit condition on the symmetries which guarantees the validity of the requirement (10) in Theorem 2.2 (see Proposition 3.1). Finally, we prove that the condition on the symmetries holds true for the particular example considered in Theorem 1.1.

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2. Symmetric nodal solutions

The group $O(n+1)$ of linear isometries of $\mathbb{R}^{n+1}$ acts isometrically on $\mathbb{S}^n$. We fix a closed subgroup Γ of $O(n+1)$ and, as usual, we denote by

$$\Gamma p := \{\gamma p : \gamma \in \Gamma\} \qquad \text{and} \qquad \Gamma_p := \{\gamma \in \Gamma : \gamma p = p\}$$

the Γ-orbit and the Γ-isotropy subgroup of a point p in $\mathbb{S}^n$. Recall that Γp is Γ-diffeomorphic to the homogeneous space Γ/Γ_p. So, they have the same cardinality, i.e., $\#\Gamma p = |\Gamma/\Gamma_p|$, the index of Γ_p in Γ.

Let $\phi : \Gamma \to \mathbb{Z}_2 := \{1, -1\}$ be a continuous homomorphism of groups. We shall look for solutions $u : \mathbb{S}^n \to \mathbb{R}$ to the Yamabe equation (1) which satisfy

$$u(\gamma p) = \phi(\gamma) u(p) \qquad \forall \gamma \in \Gamma, \quad p \in \mathbb{S}^n. \tag{5}$$

A function u with this property will be called *ϕ-equivariant*. It might occur that the only function u satisfying (5) is the trivial function. This happens, e.g., if $\Gamma = O(n+1)$ and $\phi(\gamma)$ is the determinant of γ. To avoid this behavior, we will assume, from now on, that

(A_0) Γ does not act transitively on $\mathbb{S}^n$, and there exists $p_0 \in \mathbb{S}^n$ such that we have $\Gamma_{p_0} \subset \ker \phi =: G$.

This assumption guarantees that the space

$$H_g^1(\mathbb{S}^n)^\phi := \{u \in H_g^1(\mathbb{S}^n) : u \text{ is } \phi\text{-equivariant}\}$$

is infinite dimensional; see [2].

If $\phi \equiv 1$, then (5) simply says that u is a Γ-invariant function. On the other hand, if ϕ is surjective and u is nontrivial, then (5) implies that u is sign-changing and G-invariant, where $G = \ker \phi$.

Set $a_n := n(n-2)/4$. We take

$$\|u\| := \left(\int_{\mathbb{S}^n} \left[|\nabla_g u|_g^2 + a_n u^2 \right] dV_g \right)^{\frac{1}{2}}, \quad |u|_{2^*} := \left(\int_{\mathbb{S}^n} a_n |u|^{2^*} dV_g \right)^{\frac{1}{2^*}} \tag{6}$$

as the norms in $H_g^1(\mathbb{S}^n)$ and $L_g^{2^*}(\mathbb{S}^n)$, respectively.

The ϕ-equivariant solutions to the Yamabe equation (1) are the critical points of the functional $J_n : H_g^1(\mathbb{S}^n)^\phi \to \mathbb{R}$ given by

$$J_n(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}|u|_{2^*}^{2^*}.$$

The nontrivial ones lie on the Nehari manifold

$$\mathcal{N}^\phi(\mathbb{S}^n) := \{u \in H_g^1(\mathbb{S}^n)^\phi : u \neq 0, \ \|u\|^2 = |u|_{2^*}^{2^*}\}.$$

Define

$$c_n^\phi := \inf_{u \in \mathcal{N}^\phi(\mathbb{S}^n)} J_n(u).$$

Assumption (A_0) implies that $\mathcal{N}^\phi(\mathbb{S}^n) \neq \emptyset$. Therefore, $c_n^\phi \in \mathbb{R}$.

If $\phi \equiv 1$, then $u_0 \equiv 1$ belongs to $\mathcal{N}^\phi(\mathbb{S}^n)$ and minimizes J_n on $\mathcal{N}^\phi(\mathbb{S}^n)$. Hence,

$$c_n^\phi = \frac{n-2}{4}\mathrm{vol}(\mathbb{S}^n) =: c_n \qquad \text{if } \phi \equiv 1. \tag{7}$$

If K is a closed subgroup of Γ we write $\phi|K$ for the restriction of the homomorphism ϕ to K. Then we have that

$$c_n^\phi \geq c_n^{\phi|K} \geq c_n^{\phi|\{1\}} = c_n.$$

The following result gives conditions for the existence of a minimizer.

Theorem 2.1. *Assume* (A_0) *holds true and there exists* $q \in \mathbb{S}^n$ *satisfying* $\Gamma_q = \Gamma$ *(i.e., q is a Γ-fixed point). If*

$$c_n^\phi < \min\{(\#\Gamma p)\, c_n^{\phi|\Gamma_p} : p \in \mathbb{S}^n \text{ and } \Gamma_p \neq \Gamma\}, \tag{8}$$

then there exists $u \in \mathcal{N}^\phi(\mathbb{S}^n)$ *such that* $J_n(u) = c_n^\phi$*, i.e., the Yamabe problem* (1) *has a nontrivial least energy ϕ-equivariant solution. This solution changes sign if ϕ is surjective.*

Proof. After a change of coordinates, we may assume that $q = (0, \dots, 0, 1)$. Then, Γ acts trivially on the second factor of $\mathbb{R}^n \times \mathbb{R} \equiv \mathbb{R}^{n+1}$ and the stereographic projection from the point q induces an orthogonal action of Γ on $\mathbb{R}^n$. It is well known that there is a one-to-one correspondence between solutions to the Yamabe problem (1) on the round sphere and solutions to the problem

$$-\Delta v = a_n |v|^{2^*-2} v, \qquad v \in D^{1,2}(\mathbb{R}^n). \tag{9}$$

This correspondence is given explicitly and one can verify directly that ϕ-equivariant solutions to (1) correspond to ϕ-equivariant solutions to (9); see for example [5, Section 3]. So the statement follows from [3, Theorem 3.3]. □

We denote by 1 the identity in $O(n+1)$. The symmetries we shall consider in this paper satisfy the following additional assumptions.

(A_1) For any $p \in \mathbb{S}^n$, either $\Gamma_p = \Gamma$, or $\Gamma_p = \{1\}$, or $\phi : \Gamma_p \to \mathbb{Z}_2$ is an isomorphism.
(A_2) $\phi : \Gamma \to \mathbb{Z}_2$ is surjective.

Theorem 2.2. *Assume* (A_0), (A_1) *and* (A_2)*. If*

$$c_n^\phi < |\Gamma|\, c_n = \frac{n-2}{4}\, |\Gamma|\, \mathrm{vol}(\mathbb{S}^n), \tag{10}$$

then the Yamabe problem (1) *has a nontrivial least energy ϕ-equivariant solution. This solution changes sign.*

Proof. We distinguish two cases.

Case 1. $\mathbb{S}^n$ has a Γ-fixed point.

Then, we may apply Theorem 2.1. Note that, by (A_0) and (A_2), there exists $p_0 \in \mathbb{S}^n$ such that $\Gamma_{p_0} \subset \ker\phi \neq \Gamma$. From (A_1) we conclude that $\Gamma_{p_0} = \{1\}$.

If $p \in \mathbb{S}^n$ and $\Gamma_p \neq \Gamma$ assumption (A_1) leaves two possibilities: either $\Gamma_p = \{1\}$, in which case $c_n^{\phi|\Gamma_p} = c_n$, or $\phi : \Gamma_p \to \mathbb{Z}_2$ is an isomorphism, in which case every function in $\mathcal{N}^{\phi|\Gamma_p}(\mathbb{S}^n)$ changes sign and, from (2), $c_n^{\phi|\Gamma_p} > 2c_n$. So (10) implies (8) and our claim follows from Theorem 2.1.

Case 2. $\mathbb{S}^n$ does not have a Γ-fixed point.

Let $p \in \mathbb{S}^n$. As $z \cdot p = \gamma z \cdot \gamma p = \gamma z \cdot p$ for every $\gamma \in \Gamma_p$, the tangent space $T_p(\mathbb{S}^n)$ to $\mathbb{S}^n$ at p is Γ_p-invariant. Consider the functional $I : H_g^1(\mathbb{S}^n) \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2}\int_{\mathbb{S}^n} |\nabla_g u|_g^2 \, dV_g - \frac{1}{2^*}\int_{\mathbb{S}^n} a_n |u|^{2^*} dV_g.$$

Claim. *Let (u_k) be a sequence in $H_g^1(\mathbb{S}^n)^\phi$ such that $I(u_k) \to c$, $I'(u_k) \to 0$ in $(H_g^1(\mathbb{S}^n))'$ and $u_k \rightharpoonup 0$ weakly but not strongly in $H_g^1(\mathbb{S}^n)$. Then there exist $p \in \mathbb{S}^n$ with $\#\Gamma p < \infty$, a subgroup K of Γ_p and a nontrivial solution v to (9) such that v is $(\phi|K)$-equivariant on $T_p(\mathbb{S}^n) \equiv \mathbb{R}^n$ and*

$$c \geq (\#\Gamma p)\frac{1}{m}\int_{\mathbb{R}^n} |\nabla v|^2 \geq |\Gamma| c_n. \tag{11}$$

To prove this claim we follow the proof of [4, Proposition 4.2], but this time we need to take extra care in the choice of the concentration points p_k. So after choosing $p_k \in \mathbb{S}^n$ and $r_k > 0$ as in [4, Equation (4.8)], we use [3, Lemma 2.4] to find a subgroup K of Γ and points $q_k \in \mathbb{S}^n$ such that, after passing to a subsequence, $q_k \to p$ and

(a) the sequence $\left(r_k^{-1} \mathrm{d}_g(\Gamma p_k, q_k)\right)$ is bounded, where d_g denotes the geodesic distance on $\mathbb{S}^n$,

(b) $\Gamma_{q_k} = K$ for all $k \in \mathbb{N}$,

(c) if $|\Gamma/K| < \infty$, then $r_k^{-1}\mathrm{d}_g(\alpha q_k, \beta q_k) \to \infty$ for any $\alpha, \beta \in \Gamma$ with $\alpha^{-1}\beta \notin K$,

(d) if $|\Gamma/K| = \infty$ then there exists a closed subgroup K' of Γ such that $K \subset K'$, $|\Gamma/K'| = \infty$ and $r_k^{-1}\mathrm{d}_g(\alpha q_k, \beta q_k) \to \infty$ for any $\alpha, \beta \in \Gamma$ with $\alpha^{-1}\beta \notin K'$.

Defining v_k as in [4] with p_k replaced by q_k we have that v_k is $(\phi|K)$-equivariant on $T_{q_k}(\mathbb{S}^n)$ for k sufficiently large and, so, the weak limit v of (v_k) is $(\phi|K)$-equivariant on $T_p(\mathbb{S}^n)$. From this point on the proof of the claim proceeds like that of [4, Proposition 4.2] with the obvious changes.

Let now (u_k) be a sequence in $\mathcal{N}^\phi(\mathbb{S}^n)$ such that $J_n(u_k) \to c_n^\phi$. Then $u_k \rightharpoonup u$ weakly $H_g^1(\mathbb{S}^n)$. If $u \neq 0$ a standard argument shows that u is a least energy ϕ-equivariant solution to (1). If $u = 0$, then $u_k \rightharpoonup 0$ weakly but not strongly in $H_g^1(\mathbb{S}^n)$ and (11) yields a contradiction. This completes the proof. □

The following statement is an immediate consequence of Theorem 2.2.

Corollary 2.3. *Assume (A_0), (A_1) and (A_2). If $|\Gamma| = \infty$, then the Yamabe problem (1) has a nontrivial least energy ϕ-equivariant solution. This solution changes sign.*

Next, we give some examples. We write $\mathbb{R}^{n+1} \equiv \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-3}$, and the points in $\mathbb{R}^{n+1}$ as (z_1, z_2, x) with $z_i \in \mathbb{C}$ and $x \in \mathbb{R}^{n-3}$.

Example 2.4. For $m \in \mathbf{N}$, let $G_m := \{e^{2\pi i j/m} : j = 0, \dots, m-1\}$, Γ_m be the group generated by $G_m \cup \{\tau\}$, acting on $\mathbb{R}^{n+1}$ as

$$e^{2\pi i j/m}(z_1, z_2, x) := (e^{2\pi i j/m} z_1, e^{2\pi i j/m} z_2, x), \qquad \tau(z_1, z_2, x) := (z_2, z_1, x),$$

and $\phi_m : \Gamma_m \to \mathbb{Z}_2$ be the homomorphism determined by $\phi_m(e^{2\pi i j/m}) := 1$ and by $\phi_m(\tau) := -1$. Then, $G_m = \ker \phi_m$, $|\Gamma_m| = 2m$ and, for any $p = (z_1, z_2, x)$, the Γ_m-orbit of p is

$$\Gamma_m p \cong \Gamma_m / (\Gamma_m)_p = \begin{cases} \{1\} & \text{if } z_1 = z_2 = 0, \\ G_m & \text{if } z_1 = z_2 \neq 0, \\ \Gamma_m & \text{if } z_1 \neq z_2. \end{cases}$$

So (A_0), (A_1) and (A_2) are satisfied. Note that $\mathbb{S}^n$ has Γ_m-fixed points iff $n \geq 4$. □

Example 2.5. Let Γ_∞ be the group generated by $\{e^{i\vartheta} : \vartheta \in [0, 2\pi)\} \cup \{\tau\}$, acting on $\mathbb{R}^{n+1}$ as

$$e^{i\vartheta}(z_1, z_2, x) := (e^{i\vartheta} z_1, e^{i\vartheta} z_2, x), \qquad \tau(z_1, z_2, x) := (z_2, z_1, x),$$

and let $\phi_\infty : \Gamma_\infty \to \mathbb{Z}_2$ be the homomorphism determinded by $\phi_\infty(e^{i\vartheta}) := 1$ and by $\phi_\infty(\tau) := -1$. Then, the assumptions of Corollary 2.3 are satisfied and the Yamabe problem (1) has a nontrivial least energy ϕ_∞-equivariant solution, which changes sign. □

For $\phi_m : \Gamma_m \to \mathbb{Z}_2$ as in Example 2.4, Theorem 2.1 yields a sign-changing solution to (1) whose energy is $c_n^{\phi_m}$, if the inequality

$$c_n^{\phi_m} < 2m\, c_n \tag{12}$$

is satisfied. As Γ_m is a subgroup of Γ_∞, we have that $c_n^{\phi_m} \leq c_n^{\phi_\infty}$ for all $m \in \mathbb{N}$. So (12) holds true for sufficiently large m. On the other hand, as shown in [11], the least energy of a sign-changing solution to (1) is strictly larger than $2c_n$, so (12) is not satisfied for $m = 1$.

In the next section we give a condition for (12) to hold true.

3. Estimates for the energy of nodal solutions

Let Γ be a finite subgroup of $O(n+1)$ and $\phi : \Gamma \to \mathbb{Z}_2$ be a homomorphism satisfying (A_0), (A_1) and (A_2). Fix $\hat\gamma \in \Gamma$ with $\phi(\hat\gamma) = -1$ and write

$$\Gamma = \{g_1, \dots, g_m, \hat\gamma g_1, \dots, \hat\gamma g_m\} \quad \text{with } g_1 = 1,\ \phi(g_i) = 1 \text{ for } i = 1, \dots, m.$$

The Γ-orbit of $p \in \mathbb{S}^n$ is

$$\Gamma p = \{p_1, \dots, p_m, q_1, \dots, q_m\} \qquad \text{with} \quad p_j := g_j p \quad \text{and} \quad q_j := \hat\gamma p_j.$$

We set

$$\mu_p := \sum_{1 \leq i \neq j \leq k} (1 - \cos d_g(p_i, p_j))^{\frac{2-n}{2}}, \qquad \widehat{\mu}_p := \sum_{1 \leq i, j \leq k} (1 - \cos d_g(p_i, q_j))^{\frac{2-n}{2}},$$

where $d_g(p, p') = \arccos\langle p, p'\rangle$ is the geodesic distance from p to p' on $\mathbb{S}^n$.

Proposition 3.1. *If* $\mu_p - \widehat{\mu}_p > 0$ *for some* $p \in \mathbb{S}^n$, *then* $c_n^\phi < 2m\, c_n$.

To prove this proposition, we fix $p \in \mathbb{S}^n$ and, for each $\beta > 1$, we define

$$u_\beta(q) := (\beta^2 - 1)^{\frac{n-2}{4}} (\beta - \cos \mathrm{d}_g(p,q))^{-\frac{n-2}{2}}, \qquad q \in \mathbb{S}^n.$$

The function u_β is a positive least energy solution of the Yamabe equation (1); see for example [8, Equation (8.2)]. Hence,

$$J_n(u_\beta) = \frac{1}{n}\|u_\beta\|^2 = \frac{1}{n}|u_\beta|^{2^*} = c_n = \frac{n-2}{4}\mathrm{vol}(\mathbb{S}^n).$$

We denote by $B_\delta(p)$ the geodesic ball of radius δ centered at p in $\mathbb{S}^n$, and set $\omega_n := \mathrm{vol}(\mathbb{S}^n)$.

Lemma 3.2. *For any* $f \in \mathcal{C}^0(\mathbb{S}^n)$ *and* $\delta \in (0,\pi)$, *we have*

$$\int_{B_\delta(p)} f u_\beta^{2^*-1} \mathrm{d}V_g = \frac{2^{\frac{3n+2}{4}}\omega_{n-1}}{n} f(p)(\beta-1)^{\frac{n-2}{4}} + o\left((\beta-1)^{\frac{n-2}{4}}\right) \quad \text{as } \beta \to 1,$$

$$\int_{\mathbb{S}^n \smallsetminus B_\delta(p)} f u_\beta^{2^*-1} \mathrm{d}V_g = O\left((\beta-1)^{\frac{n+2}{4}}\right) \quad \text{for } \beta \text{ close to } 1.$$

Proof. Let $\sigma : \mathbb{S}^n \smallsetminus \{p\} \to \mathbb{R}^n$ be the stereographic projection. Then,

$$|x| = \cot\frac{r}{2} \quad \text{and} \quad r = \arccos\left(\frac{|x|^2-1}{|x|^2+1}\right), \quad \text{if } x = \sigma(q),\ r := \mathrm{d}_g(p,q).$$

The pullback of the round metric in the local coordinates $\sigma^{-1} : \mathbb{R}^n \to \mathbb{S}^n \smallsetminus \{p\}$ is $(\sigma^{-1})^* g = \frac{4}{(1+|x|^2)^2}\bar{g}$, where $\bar{g}$ is the Euclidean metric. Writing $x = \sqrt{\frac{\beta+1}{\beta-1}}\frac{\xi}{\rho}$ with $\xi \in \mathbb{S}^{n-1}$, we obtain

$$\begin{aligned}
\int_{B_\delta(p)} f u_\beta^{2^*-1} \mathrm{d}V_g &= \int_{\sigma(B_\delta(p))} (f u_\beta^{2^*-1})(\sigma^{-1}(x)) \frac{2^n}{(1+|x|^2)^n} \mathrm{d}x \\
&= 2^n(\beta^2-1)^{\frac{n+2}{4}} \int_{\{|x|>\cot\frac{\delta}{2}\}} \frac{f(\sigma^{-1}(x))}{((\beta-1)|x|^2+\beta+1)^{\frac{n+2}{2}}(1+|x|^2)^{\frac{n-2}{2}}} \mathrm{d}x \\
&= 2^n\left(\frac{\beta-1}{\beta+1}\right)^{\frac{n-2}{4}} \int_{\mathbb{S}^{n-1}} \int_0^{\sqrt{\frac{\beta+1}{\beta-1}}\tan\frac{\delta}{2}} \frac{f\left(\sigma^{-1}(\sqrt{\frac{\beta+1}{\beta-1}}\frac{\xi}{\rho})\, \rho^{n-1}\right)}{(\rho^2+1)^{\frac{n+2}{2}}(\frac{\beta-1}{\beta+1}\rho^2+1)^{\frac{n-2}{2}}} \mathrm{d}\rho\, \mathrm{d}V_{g_{n-1}},
\end{aligned}$$

where g_{n-1} is the round metric on $\mathbb{S}^{n-1}$. As $n\int_0^\infty \rho^{n-1}(1+\rho^2)^{-\frac{n+2}{2}}\mathrm{d}\rho = 1$, we deduce

$$\lim_{\beta\to1} \frac{1}{(\beta-1)^{\frac{n-2}{4}}} \int_{B_\delta(p)} f u_\beta^{2^*-1} \mathrm{d}V_g = 2^{\frac{3n+2}{4}}\,\omega_{n-1}\,\frac{f(p)}{n}.$$

This is the first statement. The second one can be obtained easily, since

$$\lim_{\beta\to1} \frac{1}{(\beta-1)^{\frac{2+n}{4}}} \int_{\mathbb{S}^n\smallsetminus B_\delta(p)} f u_\beta^{2^*-1} \mathrm{d}V_g = 2^{\frac{n+2}{4}} \int_{\mathbb{S}^n\smallsetminus B_\delta(p)} f(1-\cos r)^{-\frac{n+2}{4}} \mathrm{d}V_g.$$

This completes the proof. □

Proof of Proposition 3.1. We fix $p \in \mathbb{S}^n$ with $\Gamma_p = \{1\}$. For each $\beta > 1$ we set

$$u_{j,\beta} := u_\beta \circ g_j^{-1}, \qquad u_{m+j,\beta} := u_\beta \circ (\hat{\gamma} g_j)^{-1}, \qquad j = 1, \ldots, m,$$

and we define
$$w_\beta := \sum_{j=1}^{m} (u_{j,\beta} - u_{m+j,\beta}).$$

Since (A_0), (A_1) and (A_2) hold true and $\Gamma_p = \{1\}$, we have that $w_\beta \neq 0$. Hence, there exists $t_\beta \in (0, \infty)$ such that $t_\beta w_\beta \in \mathcal{N}^\phi(\mathbb{S}^n)$, and

$$c_n^\phi \leq J_n(t_\beta w_\beta) = \frac{1}{n} [Y_n(w_\beta)]^{n/2}, \qquad \text{where } \; Y_n(u) := \frac{\|u\|^2}{|u|_{2^*}^2}$$

and $\|u\|$ and $|u|_{2^*}$ are the norms defined in (6).

Since $u_{j,\beta}$ solves (1), using Lemma 3.2 we estimate

$$\begin{aligned}
\|w_\beta\|^2 &= a_n \sum_{i,j=1}^{m} \int_{\mathbb{S}^n} [u_{i,\beta} u_{j,\beta}^{2^*-1} + u_{i+m,\beta} u_{j+m,\beta}^{2^*-1} - u_{i,\beta} u_{j+m,\beta}^{2^*-1} - u_{i+m,\beta} u_{j,\beta}^{2^*-1}] \mathrm{d}V_g \\
&= a_n \sum_{i,j=1}^{m} \int_{\mathbb{S}^n} (u_{i,\beta} \circ g_j + u_{i+m,\beta} \circ \hat{\gamma} g_j - u_{i,\beta} \circ \hat{\gamma} g_j - u_{i+m,\beta} \circ g_j) u_\beta^{2^*-1} \mathrm{d}V_g \\
&= 2m a_n \omega_n + \frac{2^{n+1} a_n \omega_{n-1}}{n} (\mu_p - \widehat{\mu}_p)(\beta - 1)^{\frac{n-2}{2}} + o(\beta - 1)^{\frac{n-2}{2}},
\end{aligned}$$

We choose $\delta > 0$ such that $B_\delta(q) \cap B_\delta(q') = \emptyset$ for all points $q, q' \in \Gamma p$ with $q \neq q'$. Then we obtain

$$\begin{aligned}
\int_{\mathbb{S}^n} |w_\beta|^{2^*} \mathrm{d}V_g &\geq \sum_{j=1}^{m} \int_{B_\delta(p_j)} |u_{j,\beta} + \sum_{i \neq j} u_{i,\beta} - \sum_i u_{i+m,\beta}|^{2^*} \mathrm{d}V_g \\
&\quad + \sum_{j=1}^{m} \int_{B_\delta(q_j)} |u_{j+m,\beta} + \sum_{i \neq j} u_{i+m,\beta} - \sum_i u_{i,\beta}|^{2^*} \mathrm{d}V_g \\
&\geq 2m \int_{B_\delta(p)} u_\beta^{2^*} \mathrm{d}V_g + 2^* \int_{B_\delta(p)} \Bigg(\sum_{1 \leq i \neq j \leq m} u_{i,\beta} \circ g_j + u_{i+m,\beta} \circ \hat{\gamma} g_j \\
&\quad - \sum_{1 \leq i,j \leq m} u_{i+k,\beta} \circ g_j + u_{i,\beta} \circ \tilde{\gamma} g_j \Bigg) u_\beta^{2^*-1} \mathrm{d}V_g \\
&\geq 2m\omega_n + \frac{2^{n+2}}{n-2} \omega_{n-1} (\mu_p - \widehat{\mu}_p)(\beta - 1)^{\frac{n-2}{2}} + o\left((\beta - 1)^{\frac{n-2}{4}}\right),
\end{aligned}$$

where we used the inequality $|a + b|^p \geq a^p + p a^{p-1} b$ for $a \geq 0$, $b \in \mathbb{R}$ and $p \geq 1$, and Lemma 3.2. Thus,

$$|w_\beta|_{2^*}^{-2} \leq (2m a_n \omega_n)^{\frac{2-n}{n}} - \frac{2^{n+2} a_n^{\frac{2-n}{n}} \omega_{n-1}}{n} (2m\omega_n)^{\frac{2-2n}{n}} (\mu_p - \widehat{\mu}_p)(\beta - 1)^{\frac{n-2}{2}} + o\left((\beta - 1)^{\frac{n-2}{4}}\right).$$

We conclude that

$$Y_n(w_\beta) \le (2ma_n\omega_n)^{\frac{2}{n}} - C_{n,k}(\mu_p - \widehat{\mu}_p)(\beta-1)^{\frac{n-2}{2}} + o\left((\beta-1)^{\frac{n-2}{4}}\right),$$

where $C_{n,k} := \frac{2^{n+1}a_n\omega_{n-1}}{n}(2ma_n\omega_n)^{\frac{2-n}{n}}$.

If $\mu_p - \widehat{\mu}_p > 0$, then $Y_n(w_\beta) < (2ma_n\omega_n)^{\frac{2}{n}} = (2mnc_n)^{\frac{2}{n}}$ for $\beta > 1$ sufficiently close to 1. Therefore,

$$c_n^\phi \le J_n(t_\beta w_\beta) = \frac{1}{n}[Y_n(w_\beta)]^{n/2} < 2mc_n \qquad \text{for } \beta \text{ sufficiently close to 1.}$$

This completes the proof. □

4. The proof of the main result

Next, we compute the sign of $\mu_p - \widehat{\mu}_p$ for $\phi_m : \Gamma_m \to \mathbb{Z}_2$ as in Example 2.4.

Lemma 4.1. *Let* $\phi_m : \Gamma_m \to \mathbb{Z}_2$ *be as in* Example 2.4 *and let*

$$p = (1,0,0) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{n-3} \equiv \mathbb{R}^{n+1}.$$

Then, $\mu_p - \widehat{\mu}_p > 0$ *if and only if* $m \ge m_n$*, with* m_n *as in* (4).

Proof. $\Gamma p = \{p_1, \dots, p_m, q_1, \dots, q_m\}$ with $p_j := (\mathrm{e}^{2\pi \mathrm{i} j/m}, 0, 0)$ and $q_j := \tau p_j$. Note that q_j is orthogonal to p_i for all $i, j = 1, \dots, m-1$. So the quantities defined at the beginning of Section 3 are

$$\mu_p = m \sum_{j=1}^{m-1} \left(1 - \cos\frac{2\pi j}{m}\right)^{\frac{2-n}{2}} \qquad \text{and} \qquad \widehat{\mu}_p = m^2.$$

As $1 - \cos\frac{2\pi j}{m} = 2\sin^2(\frac{\pi j}{m})$, we get that

$$\frac{1}{m}(\mu_p - \widehat{\mu}_p) = \sum_{j=1}^{m-1} \left(\frac{1}{\sqrt{2}\sin\frac{\pi j}{m}}\right)^{n-2} - m =: a_{n,m}.$$

Next, for each $n \ge 3$, we describe the set $A_{n,m} := \{m \in \mathbf{N} : a_{n,m} > 0\}$.

If $2 \le m \le 4$, then $\sqrt{2}\sin\frac{\pi}{m} \ge \sqrt{2}\sin\frac{\pi}{4} = 1$. Hence,

$$a_{n,m} \le \frac{m-1}{(\sqrt{2}\sin\frac{\pi}{m})^{n-2}} - m \le -1 \quad \forall n \ge 3.$$

If $m \ge 5$, then $\sqrt{2}\sin\frac{\pi}{m} \le \sqrt{2}\sin\frac{\pi}{5} < 1$. Hence,

$$a_{n,m} > \frac{2}{(\sqrt{2}\sin\frac{\pi}{m})^{n-2}} - m \ge \frac{2}{(\sqrt{2}\sin\frac{\pi}{m})^{n_0}} - m \quad \text{if } n \ge n_0 + 2.$$

We have that

$$\frac{2}{(\sqrt{2}\sin\frac{\pi}{m})^{n_0}} - m > 0$$

if, either $n_0 = 5$ and $m \geq 5$, or $n_0 = 4$ and $m \geq 6$, or $n_0 = 3$ and $m \geq 7$. Indeed, setting $x := \frac{1}{m}$ and $f_{n_0}(x) := (2x)^{1/n_0} - \sqrt{2}\sin(\pi x)$, looking at the graph of f_{n_0} (see Figure 4.1) and computing its value at $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}$ respectively, we get that

$$(2x)^{1/n_0} - \sqrt{2}\sin(\pi x) > 0 \qquad \begin{cases} \text{if } n_0 = 5 \text{ and } 0 \leq x \leq \frac{1}{5}, \\ \text{if } n_0 = 4 \text{ and } 0 \leq x \leq \frac{1}{6}, \\ \text{if } n_0 = 3 \text{ and } 0 \leq x \leq \frac{1}{7}. \end{cases}$$

On the other hand, we have that

$$a_{5,6} = \frac{1}{2^{3/2}}\left(\frac{2}{(\sin\frac{\pi}{6})^3} + \frac{2}{(\sin\frac{\pi}{3})^3} + 1\right) - 6 \approx 1.09907,$$

$$a_{5,5} = \frac{1}{2^{3/2}}\left(\frac{2}{(\sin\frac{\pi}{5})^3} + \frac{2}{(\sin\frac{2\pi}{5})^3}\right) - 5 \approx -0.69601,$$

$$a_{6,5} = \frac{1}{4}\left(\frac{2}{(\sin\frac{\pi}{5})^4} + \frac{2}{(\sin\frac{2\pi}{5})^4}\right) - 5 = -\frac{1}{5}.$$

Hence, $A_{n,m} := \{m \in \mathbf{N} : m \geq 6\}$ if $n = 5, 6$ and $A_{n,m} := \{m \in \mathbf{N} : m \geq 5\}$ if $n \geq 7$. This completes the proof for $n \geq 5$.

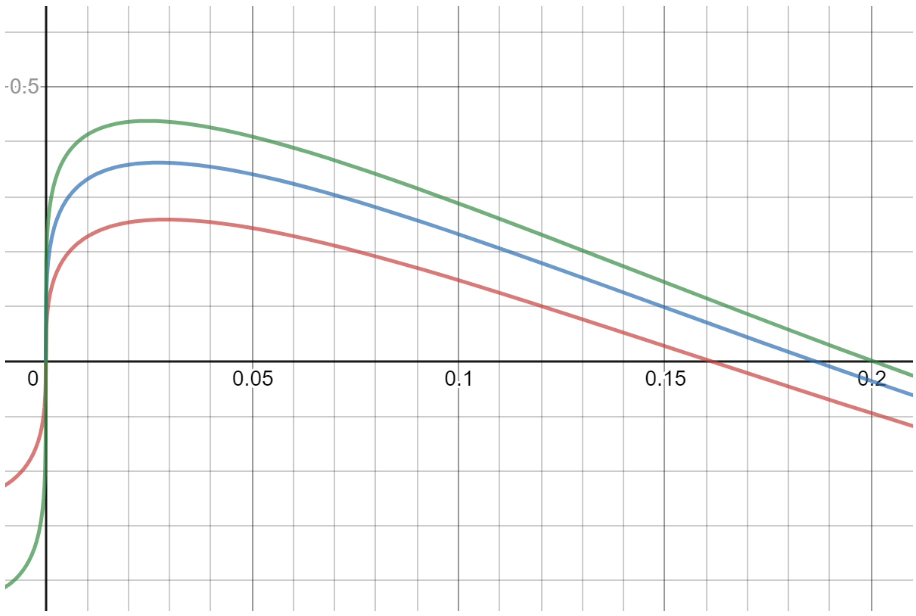

Figure 4.1: The graph of f_{n_0} for $n_0 = 3, 4, 5$.

If $n = 4$, then $a_{4,m} = \frac{1}{6}(m^2 - 1) - m$. Hence, $a_{4,m} > 0$ if and only if $m \geq 7$.

If $n = 3$, direct calculations show that $a_{3,m} < 0$ if $m = 5, 6, 7, 8$.

Note that $t \mapsto \sin t$ is increasing if $t \in [0, \frac{\pi}{2}]$.

So, for m even we have that

$$\begin{aligned} a_{3,m+1} &= \sum_{j=1}^{m/2} \frac{2}{\sqrt{2}\sin\frac{\pi j}{m+1}} - m - 1 \geq \sum_{j=2}^{m/2} \frac{2}{\sqrt{2}\sin\frac{\pi j}{m}} - m - 1 + \frac{2}{\sqrt{2}\sin\frac{\pi}{m+1}} \\ &= \sum_{j=1}^{\frac{m}{2}-1} \frac{2}{\sqrt{2}\sin\frac{\pi j}{m}} + \frac{2}{\sqrt{2}} - m - 1 + \frac{2}{\sqrt{2}\sin\frac{\pi}{m+1}} - \frac{2}{\sqrt{2}\sin\frac{\pi}{m}} \\ &= a_{3,m} + \frac{2}{\sqrt{2}}\left(\frac{1}{2} - \frac{\sqrt{2}}{2} + \frac{1}{\sin\frac{\pi}{m+1}} - \frac{1}{\sin\frac{\pi}{m}}\right). \end{aligned}$$

A similar computation shows that, also for m odd,

$$a_{3,m+1} \geq a_{3,m} + \frac{2}{\sqrt{2}}\left(\frac{1}{2} - \frac{\sqrt{2}}{2} + \frac{1}{\sin\frac{\pi}{m+1}} - \frac{1}{\sin\frac{\pi}{m}}\right).$$

We claim that

$$\frac{1}{\sin\frac{\pi}{m+1}} - \frac{1}{\sin\frac{\pi}{m}} > \frac{\sqrt{2}-1}{2} \qquad \forall m \geq 9. \tag{13}$$

If this is true, then $a_{3,m} > 0$ for all $m \geq 9$, and the proof of the lemma is complete. To prove (13) note that, since $\frac{t(6-t^2)}{6} = t - \frac{t^3}{6} \leq \sin t \leq t$,

$$\begin{aligned} \frac{1}{\sin\frac{\pi}{m+1}} - \frac{1}{\sin\frac{\pi}{m}} &\geq \frac{m+1}{\pi} - \frac{6}{\frac{\pi}{m}(6-(\frac{\pi}{m})^2} \\ &= \frac{1}{\pi} - \left(\frac{\frac{\pi}{m}}{6-(\frac{\pi}{m})^2}\right) \geq \frac{1}{\pi} - \left(\frac{\frac{\pi}{9}}{6-(\frac{\pi}{9})^2}\right) \qquad \forall m \geq 9. \end{aligned}$$

A direct calculation gives

$$\frac{\frac{\pi}{9}}{6-(\frac{\pi}{9})^2} \approx 0.059383 < 0.111203 \approx \frac{1}{\pi} - \frac{\sqrt{2}-1}{2},$$

which yields (13). □

Proof of Theorem 1.1. This result follows from Lemma 4.1, Proposition 3.1 and Corollary 2.2. □

To conclude, we show that our solutions are different from those of Ding [7]. We write $\mathbb{R}^{n+1} \equiv \mathbb{C}\times\mathbb{R}^{k-2}\times\mathbb{C}\times\mathbb{R}^{m-2}$ with $k, m \geq 2$ and $k+m = n+1$ and, accordingly, we write the points in $\mathbb{R}^{n+1}$ as (z_1, x_1, z_2, x_2).

Proposition 4.2. *Let $n > 3$. If $u : \mathbb{S}^n \to \mathbb{R}$ is $[O(k)\times O(m)]$-invariant and*

$$u(z_1, x_1, z_2, x_2) = -u(z_2, x_1, z_1, x_2) \qquad \forall (z_1, x_1, z_2, x_2) \in \mathbb{R}^{n+1},$$

then $u \equiv 0$.

Proof. Without loss of generality, we may assume that $k \leq m$. Since the mapping u is $[O(k) \times O(m)]$-invariant it can be written as

$$u(z_1, x_1, z_2, x_2) = w(|(z_1, x_1)|, |(z_2, x_2)|).$$

Then, for every $(z_1, x_1, z_2, x_2) \in \mathbb{S}^n$, we have that

$$w(|(z_1, x_1)|, |(z_2, x_2)|) = -w(|(z_2, x_1)|, |(z_1, x_2)|)$$

and, taking $z_1 = z_2 = 0$, we get that

$$w(|x_1|, |x_2|) = -w(|x_1|, |x_2|) \qquad \forall (x_1, x_2) \in \mathbb{R}^{k-2} \times \mathbb{R}^{m-2}.$$

If $k > 2$, this implies that $w \equiv 0$.

On the other hand, if $k = 2$ then $m > 2$ and, taking $z_1 = z_2 = 0$, we get that $w(0, 1) = -w(0, 1) = 0$. Setting $z_1 = 0$ we conclude that

$$0 = w(0, 1) = -w(|z_2|, |x_2|) \qquad \forall (z_2, x_2) \in \mathbb{C} \times \mathbb{R}^{m-2}.$$

Hence, $w \equiv 0$. □

Remark 4.3. The same argument shows that the solutions given by [3, Theorem 1.1] are different from those of Ding [7].

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