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Existence and Local Uniqueness of Normalized Multi-Peak Solutions to a Class of Kirchhoff Type Equations

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We study the existence and local uniqueness of multi-peak solutions to the following Kirchhoff type equations

$$-\Big(a+b_\lambda\int_{\mathbb{R}^3}|\nabla u_\lambda|^2\Big)\Delta u_\lambda+\big(\lambda+V(x)\big)u_\lambda=\beta_\lambda u_\lambda^p,\ u_\lambda\in H^1(\mathbb{R}^3),\ u_\lambda>0\ \text{ in } \mathbb{R}^3,$$

with normalized L^2-constraint, that is,

$$\int_{\mathbb{R}^3}u_\lambda^2=1,$$

where $a>0,\ p\in(1,5)$ are constants, $\lambda,\ b_\lambda,\ \beta_\lambda>0$ are parameters, $V(x)\colon\mathbb{R}^3\to\mathbb{R}^1$ is a bounded continuous function. Physicists are very interested in normalized solutions. Compared to finding multi-pick solutions to the equation without normalized L^2-constraint one is facing here some new difficulties in getting normalized solutions to the equation. We first prove that for the case of $3<p<5$, there exist sequences $\{b_\lambda\}_\lambda$ and $\{\beta_\lambda\}_\lambda$ such that for any sufficiently large $\lambda>0$, one can construct multi-peak solutions u_λ of some given form to the above equation by using the Lyapunov-Schmidt reduction method under some mild assumptions on the function $V(x)$. In the proof of the above existence result, we consider the three cases of $p=11/3, 3<p<11/3$ and $11/3<p<5$ separately, which correspond to the cases of mass critical, subcritical and supercritical in physics respectively. Then, applying the blow-up technique and the local Pohozaev identities we obtain a uniqueness result of multi-peak solutions for the case of $3<p<5$. The difficulties caused by the nonlocal term and normalized L^2-constraint are overcome.

Keywords: Kirchhoff type equations, multi-peak normalized solutions, Lyapunov-Schmidt reduction, local Pohozaev identity, existence and local uniqueness.

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1. Introduction and main results

Let $a,b>0$ and $1<p<5$. We consider the following Kirchhoff type equations

$$-\Big(a+b\int_{\mathbb{R}^3}|\nabla u|^2\Big)\Delta u+\big(\lambda+V(x)\big)u=u^p,\ u\in H^1(\mathbb{R}^3),\ u>0\ \text{ in } \mathbb{R}^3,\tag{1.1}$$

where $\lambda>0$ is a parameter and $V(x):\mathbb{R}^3\to\mathbb{R}^1$ is a bounded continuous function.

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For any positive solution $u(x)$ to Eq. (1.1), denoting

$$u_\lambda(x) := \frac{u(x)}{\left(\int_{\mathbb{R}^3} u^2(x)\mathrm{d}x\right)^{\frac{1}{2}}},$$

then one can check easily that $u_\lambda(x)$ satisfies the following problem

$$-\Big(a+b_\lambda \int_{\mathbb{R}^3} |\nabla u_\lambda|^2\Big)\Delta u_\lambda + \big(\lambda+V(x)\big)u_\lambda = \beta_\lambda u_\lambda^p,\ u_\lambda \in H^1(\mathbb{R}^3),\ u_\lambda > 0 \ \text{ in } \mathbb{R}^3, \quad (1.2)$$

with normalized L^2-constraint

$$\int_{\mathbb{R}^3} u_\lambda^2 = 1, \quad (1.3)$$

where $b_\lambda = b\|u\|^2_{L^2(\mathbb{R}^3)}$, $\beta_\lambda = \|u\|^{p-1}_{L^2(\mathbb{R}^3)}$. In this sense, although the case of $b = 1$ is more concise, we do not assume that $b = 1$ for the sake of generality. We mainly aim to explore under what conditions on p, problem (1.2)–(1.3) admits multi-peak solutions (λ, u_λ) with normalized L^2-norm (that is, the L^2-norm of u_λ equals one) as $\lambda \to +\infty$.

Eq. (1.1) is related to the stationary solutions of

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0. \quad (1.4)$$

Problem (1.4) was proposed for the first time by G. Kirchhoff [16] for extending the classical D'Alembert's wave equations for free vibration of elastic strings. Readers can refer to Bernstein [6] and Pohozaev [24] for early research results about problem (1.4). After the pioneering work of Lions [20] much attention was paid to this field. The stationary analogue of Kirchhoff's wave equation leads to the Dirichlet problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^3$ is a bounded domain, and to equations of type

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u = f(x,u) \ \text{ in } \mathbb{R}^3,$$

respectively, where f always denotes some nonlinear functions. Notice that the nonlocal term $(\int_{\mathbb{R}^3} |\nabla u|^2)\Delta u$ depends not only on the pointwise value of Δu, but also on the integral $\int_{\mathbb{R}^3} |\nabla u|^2$, which produces much more mathematical difficulties. In recent years, this elliptic type problem has been studied by many researchers in the literature.

Eq. (1.1) is also closely related to the following singularly perturbed Kirchhoff problem

$$-\left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + V(x)u = u^p,\ u \in H^1(\mathbb{R}^3),\ u > 0 \ \text{ in } \mathbb{R}^3, \quad (1.5)$$

where $a, b > 0$, $1 < p < 5$ are constants, $\varepsilon > 0$ is a parameter and $V(x) : \mathbb{R}^3 \to \mathbb{R}^1$ is a positive bounded continuous function.

In fact, in order to be consistent with the form of Eq. (1.5), we perform a scaling on Eq. (1.1) as follows:

$$\varepsilon := \lambda^{-\frac{1}{2}} \to 0 \;\text{ as } \lambda \to +\infty \;\text{ and }\; w(x) := \varepsilon^{\frac{2}{p-1}} u(x).$$

Then Eq. (1.1) is equivalent to

$$-\Big(a\varepsilon^2 + b\varepsilon^{2-\frac{4}{p-1}} \int_{\mathbb{R}^3} |\nabla w|^2\Big)\Delta w + \Big(1 + \varepsilon^2 V(x)\Big) w = w^p, \tag{1.6}$$

where $w \in H^1(\mathbb{R}^3)$, $w > 0$ in $\mathbb{R}^3$.

We first review some known results on Schrödinger equations. Let $a = 1$, $b = 0$. Then Eq. (1.5) reduces to the problem

$$-\varepsilon^2 \Delta u + V(x)u = u^p, \; u \in H^1(\mathbb{R}^3), \; u > 0 \; \text{ in } \mathbb{R}^3,$$

which is a special case of the following perturbed Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = u^q, \; u \in H^1(\mathbb{R}^N), \; u > 0 \; \text{ in } \mathbb{R}^N, \tag{1.7}$$

where $1 < q$ is subcritical and $N \geq 1$. As we know, the following unperturbed Schrödinger equation

$$-\Delta w + w = w^p, \; w \in H^1(\mathbb{R}^N), \; w > 0 \; \text{ in } \mathbb{R}^N \tag{1.8}$$

admits a unique positive radial solution (up to translations) which is also nondegenerate (see e.g. [4, 5, 10, 17]). On the basis of this uniqueness and nondegeneracy property, Floer and Weinstein [11], Oh [23] and many others proved the existence of solutions to Eq. (1.7) for $\varepsilon > 0$ sufficiently small. It is worth mentioning that Oh [23] constructed multi-peak solutions to Eq. (1.7) by using the Lyapunov-Schmidt reduction method.

Then we turn to some results on singularly perturbed Kirchhoff equations. It seems that the paper [12] by He and Zou was the first to study singularly perturbed Kirchhoff problems. In [12], He and Zou considered the problem

$$-\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2\right) \Delta u + V(x)u = f(u), \; u \in H^1(\mathbb{R}^3), \; u > 0 \; \text{ in } \mathbb{R}^3, \tag{1.9}$$

where $f : \mathbb{R}^1 \to \mathbb{R}^1$ is a nonlinear C^1 function with subcritical growth of type u^q for some $3 < q < 5$. The potential function $V(x)$ is assumed to satisfy the global condition of Rabinowitz [25]

$$(V_R) \qquad \liminf_{|x| \to +\infty} V(x) > \inf_{x \in \mathbb{R}^3} V(x) > 0.$$

He and Zou [12] proved the existence of multiple positive solutions to Eq. (1.9) which concentrate at the minimum points of $V(x)$ for $\varepsilon > 0$ sufficiently small.

He, Li and Peng [14] established some existence results of positive solutions for the following Kirchhoff equations with critical growth

$$-\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2\right) \Delta u + V(x)u = f(u) + u^5, \; u \in H^1(\mathbb{R}^3), \; u > 0 \; \text{ in } \mathbb{R}^3, \tag{1.10}$$

where $a, b > 0$ are constants, $f : \mathbb{R}^1 \to \mathbb{R}^1$ is a nonlinear C^1 function satisfying some similar conditions given in [12].

At the same time, $V(x)$ only satisfies some local conditions:

(V_Ω) there exists a bounded open set $\Omega \subseteq \mathbb{R}^3$, such that $\inf_{x\in\Omega} V(x) < \inf_{x\in\partial\Omega} V(x)$,

(V_α) $V(x) \in C(\mathbb{R}^3,\ \mathbb{R}^1)$, $\inf_{x\in\mathbb{R}^3} V(x) = \alpha > 0$.

He, Li and Peng [14] proved that there exists a positive solution $u_\varepsilon \in H^1(\mathbb{R}^3)$ to the above problem for $\varepsilon > 0$ sufficiently small. Later, He and Li [13] proved the existence of positive solutions to Eq. (1.10) for $\varepsilon > 0$ sufficiently small with $f(u) = u^q$, where $1 < q < 3$ and $V(x)$ satisfies the local conditions (V_Ω) and (V_α).

Recently, Li et al. [18] established the uniqueness and nondegeneracy of positive energy solutions to the Kirchhoff equation

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + u = u^p,\ u \in H^1(\mathbb{R}^3),\ u > 0 \text{ in } \mathbb{R}^3, \tag{1.11}$$

where $a, b > 0, 1 < p < 5$ are constants. Then as an application, by applying the Lyapunov-Schmidt reduction method, they obtained the existence of single peak solutions ($k = 1$) to Eq. (1.5) for $\varepsilon > 0$ sufficiently small with $V(x)$ satisfying the assumptions (V_1)–(V_3) as follows:

(V_1) $V(x) \in C(\mathbb{R}^3,\ \mathbb{R}^1)$ is a bounded continuous function with $\inf_{x\in\mathbb{R}^3} V(x) > 0$.

(V_2) There exist k distinct points $\{a_1, a_2, \cdots, a_k\} \subseteq \mathbb{R}^3$ such that for every i with $1 \le i \le k$, $V(x) \in C^\theta\big(\overline{B}_{R_0}(a_i)\big)$ for some $\theta \in (0, 1)$, and

$$V(a_i) < V(x) \quad \text{for} \quad 0 < |x - a_i| < r$$

holds for some $0 < r < R_0 \equiv \frac{1}{2}\min\{|a_i - a_j| : 1 \le i, j \le k, i \ne j\}$.

(V_3) There exist $m > 1$, $\eta > 0$, $c_{i,j} \in \mathbb{R}^1$ with $c_{i,j} \ne 0$ for each $i = 1, 2, \cdots, k$ and $j = 1, 2, 3$ such that $V(x) \in C^1(B_\eta(a_i))$ and

$$\begin{cases} V(x) = V(a_i) + \displaystyle\sum_{j=1}^{3} c_{i,j}|x_j - a_{i,j}|^m + O(|x - a_i|^{m+1}), & x \in B_\eta(a_i), \\ \frac{\partial V}{\partial x_j} = mc_{i,j}|x_j - a_{i,j}|^{m-2}(x_j - a_{i,j}) + O(|x - a_i|^m), & x \in B_\eta(a_i), \end{cases}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $a_i = (a_{i,1}, a_{i,2}, a_{i,3})$.

In [18], Li et al. also established the uniqueness of single peak solutions to Eq. (1.5) by using a type of local Pohozaev identities. Furthermore, Luo et al. [21] and Li, Niu and Xiang [19] established the existence and local uniqueness results of multi-peak solutions to Eq.(1.5) under the assumptions (V_1)–(V_3) on $V(x)$, respectively. In [15], Hu and Shuai studied the problem with more general nonlinearity as follows:

$$-\left(\varepsilon^2 a + \varepsilon b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + V(x)u = f(u),\ u \in H^1(\mathbb{R}^3),\ u > 0 \text{ in } \mathbb{R}^3. \tag{1.12}$$

Here the nonlinearity $f(t) : \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and satisfy the following Berestycki-Lions condition

(f1) $f(t) = 0$ for $t < 0$ and $\lim_{t\to 0^+} f(t)/t = 0$;

(f2) there exists some $p \in (2, 6)$ such that $\lim_{t\to\infty} f(t)/t^{p-1} = 0$;

(f3) there exists $T > 0$ such that $\frac{1}{2}aT^2 < F(T)$, where $F(t) = \int_0^t f(s)ds$.

Under the above conditions on f and suitable assumptions on $V(x)$, Hu and Shuai [15] constructed the multi-peak solutions of Eq. (1.12).

The building block of the single peak solutions obtained by Li et al. in [18] is the unique positive radial solution of Eq. (1.11). However, to construct multi-peak solutions to Eq. (1.5), it was proved by Luo et al. in [21] that the limiting equation of Eq. (1.5) is a system of PDEs rather than a single equation. Thus, results of limiting system and existence of multi-peak solutions derived in [21] partially motivated us to consider the existence of multi-peak solutions to problem (1.6) which is equivalent to Eq. (1.1) under the assumptions (V_1)–(V_2).

Another motivation of this work comes from the work [22] of Luo et al., in which they studied the existence of solutions $(\mu, u) \in \mathbb{R}^1 \times H^1(\mathbb{R}^N)$ of the following Gross-Pitaevskii (GP) equation (in 2 or 3 dimensions):

$$-\Delta u + V(x)u = au^3 + \mu u, \quad u > 0 \ \text{ in } \mathbb{R}^N, \tag{1.13}$$

with the normalized L^2-constraint
$$\int_{\mathbb{R}^N} u^2 = 1, \tag{1.14}$$

where $a > 0$ is a constant, $\mu < 0$ is a parameter, and $V(x)$ is assumed to have non-degenerate critical points. In [22], Luo et al. first proved that if u_a is a solution to problem (1.13)–(1.14) concentrated at some points as $a \to a_0$, then it holds that

$$a_0 \geq 0 \text{ and } \mu_a \to -\infty, \text{ as } a \to a_0.$$

Moreover, if $N = 2$, then $a_0 = ka_* > 0$ for some integer $k > 0$, where $a^* = \int_{\mathbb{R}^N} Q^2$ and Q is the unique positive solution of $-\Delta u + u = u^2$, $u \in H^1(\mathbb{R}^N)$. This result greatly stimulates our putting forward reasonable conjecture: there holds $b_\lambda \to b_*, \beta_\lambda \to \beta_*$ as $\lambda \to +\infty$ for some constants b_*, β_* when constructing multi-peak solutions to problem (1.2)–(1.3).

Then, to study the existence and uniqueness of multi-peak solutions to problem (1.13)–(1.14), some additional conditions (non-degenerate critical points) were taken into consideration. Here, we will not repeat more details about that, since these results have little to do with the present paper. Interested readers can refer to Theorem 1.3 in [22] for more results.

Motivated by [21]–[22], we intend to investigate the existence and uniqueness of multi-peak solutions to problem (1.2) with normalized L^2-constraint (1.3).

Now let us maintain our focus on the problem (1.6). Notice that, if $1 < p < 3$, then $2 - \frac{4}{p-1} < 0$ and $\varepsilon^{2-\frac{4}{p-1}} \to +\infty$ as $\varepsilon \to 0$. The variational functional corresponding to (1.6) is given in the form below:

$$\begin{aligned} I_\varepsilon(w) = \frac{1}{2} \int_{\mathbb{R}^3} \left(a\varepsilon^2 |\nabla w|^2 + w^2 \right) + \frac{b\varepsilon^{2-\frac{4}{p-1}}}{4} \left(\int_{\mathbb{R}^3} |\nabla w|^2 \right)^2 \\ + \frac{\varepsilon^2}{2} \int_{\mathbb{R}^3} V(x) w^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} w_+^{p+1} \end{aligned} \tag{1.15}$$

for $w \in H^1(\mathbb{R}^3)$, where $w_+ = \max(w, 0)$.

Obviously, I_ε is not well-defined in $H^1(\mathbb{R}^3)$, since

$$\varepsilon^{2-\frac{4}{p-1}}\left(\int_{\mathbb{R}^3}|\nabla w|^2\right)^2 \to +\infty, \quad \text{as } \varepsilon \to 0,$$

for any fixed nonnegative $w \in H^1(\mathbb{R}^3), w \neq 0$. Hence, we cannot even construct multi-peak solutions to problem (1.6) by using Lyapunov-Schmidt reduction method. Therefore, to construct multi-peak solutions to problem (1.6), we always consider the case of $3 \leq p < 5$ in this paper. (As the paper unfolds, we will exclude the case of $p = 3$.)

For the case of $3 \leq p < 5$, the limiting equation of Eq. (1.6) must be of the form

$$-b\int_{\mathbb{R}^3}|\nabla v|^2\Delta v + v = v^p, \ v \in H^1(\mathbb{R}^3), \ v > 0 \ \text{ in } \mathbb{R}^3.$$

When constructing multi-peak solutions to Eq. (1.6), the limiting equation is a system of PDEs as below:

$$\begin{cases} -\left(a\varepsilon^{2-2q} + b\sum_{i=1}^{k}\int_{\mathbb{R}^3}|\nabla v^i|^2\right)\Delta v^i + (1+\varepsilon^2 V(a_i))v^i = (v^i)^p & \text{in } \mathbb{R}^3, \\ v^i(x) \in H^1(\mathbb{R}^3), \ v^i(x) > 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.16}$$

for some $q \in (0,1)$, where q depends only on p.

Next, we give the definition of multi-peak solutions to Eq. (1.6).

Definition 1.1. Let $k \in \mathbb{N}$, $b_i \in \mathbb{R}^3$ $(i = 1, \ldots, k)$. We say that $w_\varepsilon \in H^1(\mathbb{R}^3)$ is a *k-peak solution* of Eq. (1.6) concentrated at $\{b_1, b_2, \cdots, b_k\}$, if w_ε satisfies

(i) w_ε has k local maximum points $y_\varepsilon^i \in \mathbb{R}^3$ $(i = 1, \ldots, k)$, satisfying $y_\varepsilon^i \to b_i$ as $\varepsilon \to 0$ for each i;

(ii) For any given $\tau > 0$, there exists $R \gg 1$, such that
$$|w_\varepsilon(x)| \leq \tau \text{ for all } x \in \mathbb{R}^3 \setminus \cup_{i=1}^k B_{R\varepsilon^q}(y_\varepsilon^i);$$

(iii) There exists $C > 0$ such that $\displaystyle\int_{\mathbb{R}^3}\left(a\varepsilon^{2q}|\nabla w_\varepsilon|^2 + w_\varepsilon^2\right) \leq C\varepsilon^{3q}$.

To state our main results we introduce some notations that will be used throughout this paper. For any $\varepsilon > 0$ and $y \in \mathbb{R}^3$, denote

$$v_{\varepsilon,y}(x) = v((x-y)/\varepsilon^q), \quad x \in \mathbb{R}^3$$

for $v(x) \in H^1(\mathbb{R}^3)$. We assume throughout the paper that $V(x)$ satisfies the assumptions (V_1)–(V_3).

The assumption (V_1) allows us to introduce the inner products

$$\langle w, v\rangle_\varepsilon := \int_{\mathbb{R}^3}\left(a\varepsilon^{2q}\nabla w \cdot \nabla v + wv\right)$$

for any $w, v \in H^1(\mathbb{R}^3)$. We also write

$$H_\varepsilon := \{w \in H^1(\mathbb{R}^3) : \|w\|_\varepsilon \equiv \langle w, w\rangle_\varepsilon^{\frac{1}{2}} < +\infty\}.$$

The energy functional $I_\varepsilon : H_\varepsilon \to \mathbb{R}^1$ corresponding to Eq. (1.6) is given by (1.15). It is standard to verify that $I_\varepsilon \in C^2(H_\varepsilon)$.

Suppose that $(v^1, v^2, \cdots, v^k)$ is the unique positive radial solution to the system (1.16). We construct a k-peak solution to Eq. (1.6) of the form

$$w_\varepsilon(x) = \sum_{i=1}^{k} v^i \left(\frac{x - y_\varepsilon^i}{\varepsilon^q} \right) + \varphi_\varepsilon, \tag{1.17}$$

where $q \in (0,1)$, the local maximum points $y_\varepsilon^i \to b_i$ for some distinct $b_i \in \mathbb{R}^3$, $\varphi_\varepsilon \in H^1(\mathbb{R}^3)$ satisfying

$$\|\varphi_\varepsilon\|_\varepsilon = o(\varepsilon^{2+\frac{3q}{2}}). \tag{1.18}$$

We intend to verify $b_i \equiv a_i$ for each $i \in \{1, 2, \cdots, k\}$. In particular, q depends only on p. We identify the following result

$$q = 2 - \frac{4}{p-1}$$

until in Lemma 3.2. After that, we will always assume that $3 < p < 5$ since $q > 0$.

Now we state our main results. First, we have an existence result for multi-peak solutions of Eq. (1.6) as follows.

Theorem 1.2. *Assume that $a, b > 0$ are given constants, $V(x)$ satisfies (V_1)–(V_2) and $3 < p < 5$. Then for $\varepsilon > 0$ sufficiently small, Eq. (1.6) admits k-peak solutions w_ε concentrating at $\{a_i\}_{1 \le i \le k}$ defined as in Definition 1.1 of the form (1.17)–(1.18). More precisely, w_ε can be explicitly expressed by*

$$w_\varepsilon(x) = \sum_{i=1}^{k} v^i \left(\frac{x - y_\varepsilon^i}{\varepsilon^q} \right) + \varphi_\varepsilon,$$

where $q = 2 - \frac{4}{p-1}$, the local maximum points $y_\varepsilon^i \to a_i$, $\varphi_\varepsilon \in H^1(\mathbb{R}^3)$ satisfying

$$\|\varphi_\varepsilon\|_\varepsilon = o(\varepsilon^{2+\frac{3q}{2}}).$$

Next we give the uniqueness result corresponding to Theorem 1.2 as below.

Theorem 1.3. *Assume that $a, b > 0$ are given constants, $V(x)$ satisfies (V_1), (V_2) and (V_3), $3 < p < 5$ and $q = 2 - \frac{4}{p-1}$. If $w_\varepsilon^{(i)}$ $(i = 1, 2)$ are two solutions to Eq. (1.6) of the form (1.17)–(1.18), then $w_\varepsilon^{(1)} \equiv w_\varepsilon^{(2)}$ holds for $\varepsilon > 0$ sufficiently small. Moreover, let $w_\varepsilon(x) = \sum_{i=1}^{k} v^i_{\varepsilon, y_\varepsilon^i} + \varphi_\varepsilon$ be the unique solution, then there holds $|y_\varepsilon^i - a_i| = o(\varepsilon^q)$ for $i = 1, 2, \cdots, k$, and $\|\varphi_\varepsilon\|_\varepsilon = O\left(\varepsilon^{2+\frac{3q}{2}+qm}\right)$.*

By Theorem 1.2, we suppose that Eq. (1.6) admits a k-peak solution w_ε of the form (1.17)–(1.18). Let $u(x) > 0$ be defined as

$$u(x) := \lambda^{\frac{1}{p-1}} w_\varepsilon(x) \quad \text{where} \quad \lambda := \varepsilon^{-2}.$$

Then $u(x)$ satisfies Eq. (1.1).

Define $$u_\lambda(x) := \frac{u(x)}{\left(\int_{\mathbb{R}^3} u^2(x)\mathrm{d}x\right)^{\frac{1}{2}}} = \frac{w_\varepsilon(x)}{\left(\int_{\mathbb{R}^3} w_\varepsilon^2(x)\mathrm{d}x\right)^{\frac{1}{2}}},$$

then $u_\lambda(x)$ satisfies (1.2)–(1.3), where $b_\lambda = b\|u\|^2_{L^2(\mathbb{R}^3)}$, $\beta_\lambda = \|u\|^{p-1}_{L^2(\mathbb{R}^3)}$. Therefore, we can easily deduce the following conclusion which has been stated in the abstract.

Theorem 1.4. *Suppose that $a, b > 0$ are given constants, $V(x)$ is assumed to satisfy (V_1), (V_2) and (V_3), $3 < p < 5$ and $q = 2 - \frac{4}{p-1}$. Then there exist sequences $\{b_\lambda\}_\lambda$ and $\{\beta_\lambda\}_\lambda$ such that for any sufficiently large $\lambda > 0$, problem (1.2)–(1.3) admits multi-peak solutions u_λ of some form. In particular,*

(1) *if $3 < p < \frac{11}{3}$, then $b_\lambda = O(\lambda^{1-2q}) \to +\infty$, and $\beta_\lambda = O(\lambda^{\frac{11-3p}{2}}) \to +\infty$ as $\lambda \to +\infty$;*

(2) *if $\frac{11}{3} < p < 5$, then $b_\lambda = O(\lambda^{1-2q}) \to 0$, and $\beta_\lambda = O(\lambda^{\frac{11-3p}{2}}) \to 0$ as $\lambda \to +\infty$;*

(3) *if $p = \frac{11}{3}$, then $b_\lambda \to b_*$, $\beta_\lambda \to \beta_*$ as $\lambda \to +\infty$ for some constants $b_* > 0$, $\beta_* > 0$, where*
$$b_* = 6^3 b^4 k^4 \left(\int_{\mathbb{R}^3} Q^2\right)^4 \quad \text{and} \quad \beta_* = 6^4 b^4 k^{\frac{16}{3}} \left(\int_{\mathbb{R}^3} Q^2\right)^{\frac{16}{3}}.$$

Additionally, u_λ can be explicitly expressed by

$$\begin{aligned} u_\lambda(x) &= \frac{\lambda^{\frac{1}{p-1}}}{\lambda^{\frac{1-2q}{2}}(ka_*)^{\frac{1}{2}} + o\left(\lambda^{-q}\right)} \left[\sum_{i=1}^k v^i\left(\lambda^{\frac{q}{2}}(x - y^i_\lambda)\right) + \varphi_\lambda(x)\right] \\ &\approx (ka_*)^{-\frac{1}{2}} \lambda^{\frac{3(p-3)}{2(p-1)}} \left[\sum_{i=1}^k v^i\left(\lambda^{\frac{q}{2}}(x - y^i_\lambda)\right) + \varphi_\lambda(x)\right], \end{aligned}$$

where $a_* := \int_{\mathbb{R}^3}(U^i)^2 > 0$, $(U^1, U^2, \cdots, U^k)$ and $Q(x)$ are the unique positive radial solutions of the system (2.4) and Eq. (2.1), respectively, the local maximum points $y^i_\lambda \to a_i$ as $\lambda \to +\infty$ with $|y^i_\lambda - a_i| = o(\lambda^{-\frac{q}{2}})$, and $\varphi_\lambda(x) \in H^1(\mathbb{R}^3)$ with

$$\int_{\mathbb{R}^3}\left[\frac{1}{\lambda^q}|\nabla\varphi_\lambda|^2 + \varphi_\lambda^2\right] = \lambda^{-\left(2+\frac{3q}{2}+qm\right)}.$$

Finally, we give the uniqueness result corresponding to Theorem 1.4.

Theorem 1.5. *Suppose that $a, b > 0$ are given constants, $3 < p < 5$, and $V(x)$ is assumed to satisfy (V_1), (V_2) and (V_3). If $u^{(i)}_\lambda$ $(i = 1, 2)$ are two solutions to problem (1.2)–(1.3) derived as in Theorem 1.4, then $u^{(1)}_\lambda \equiv u^{(2)}_\lambda$ holds for $\lambda > 0$ sufficiently large.*

Remark 1.6. (Remark to Theorem 1.5) Suppose that $(\lambda, u^{(i)}_\lambda, b^{(i)}_\lambda, \beta^{(i)}_\lambda)$ $(i = 1, 2)$ are two solutions to problem (1.2)–(1.3) derived as in Theorem 1.4. Then by the construction of $u^{(i)}_\lambda$, there exist $w^{(i)}_\varepsilon$ $(i = 1, 2)$ such that

$$u_\lambda^{(i)}(x) = \frac{w_\varepsilon^{(i)}(x)}{\left(\int_{\mathbb{R}^3} \left(w_\varepsilon^{(i)}(x)\right)^2\right)^{\frac{1}{2}}},$$

and $w_\varepsilon^{(i)}$ $(i = 1, 2)$ satisfy Eq. (1.6), that is,

$$-\Big(a\varepsilon^2 + b\varepsilon^{2-\frac{4}{p-1}} \int_{\mathbb{R}^3} |\nabla w_\varepsilon^{(i)}|^2\Big)\Delta w_\varepsilon^{(i)} + \Big(1 + \varepsilon^2 V(x)\Big) w_\varepsilon^{(i)} = (w_\varepsilon^{(i)})^p \ \text{ in } \mathbb{R}^3,$$

where $\lambda = \varepsilon^{-2}$. By the uniqueness result derived as in Theorem 1.3,

$$w_\varepsilon^{(1)}(x) \equiv w_\varepsilon^{(2)}(x) \ \text{ in } \mathbb{R}^3 \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

Hence $u_\lambda^{(1)}(x) \equiv u_\lambda^{(2)}(x)$ in $\mathbb{R}^3$ for $\lambda > 0$ sufficiently large. □

Remark 1.7. Here we compare our results with some previous works (for example, those appeared in [15, 18, 19, 21]). [15] just considered the existence of multi-peak solutions of Eq. (1.12) for general nonlinearity without L^2 constraint; [18, 19, 21] proved the existence and local uniqueness of single peak and multi-peak solutions of Eq. (1.5) without L^2 constraint whereas we consider the multi-peak solutions of Eq. (1.1) with L^2 constraint and to consider L^2 constraint, we change our problem into Eq. (1.6) and then consider the concentration of solutions to Eq. (1.6). Besides, the limiting equation of (1.6) will be

$$-b\Big(\int_{\mathbb{R}^3} |\nabla v|^2\Big)\Delta v + v = v^p, \ v \in H^1(\mathbb{R}^3), \ v > 0 \ \text{ in } \mathbb{R}^3,$$

which is quite different from the previous works in [18, 19, 21]. □

At the end of this section, we give the main ideas. Firstly, we prove Theorem 1.2 by using Lyapunov-Schmidt reduction method. More precisely, first, it is easy to see that every solution to Eq. (1.6) is a critical point of the energy functional I_ε defined as in (1.15). So we are left to finding a critical point of I_ε. Then, we will follow the scheme of Cao and Peng [9], reducing the problem to finding a critical point of a finite dimensional function $J_\varepsilon(Y, \varphi)$ with respect to $Y \in \mathbb{R}^{3k}$ (see more details in Section 3 and Section 4). However, due to the presence of the nonlocal term $\int_{\mathbb{R}^3} |\nabla w|^2 \Delta w$, it requires more careful estimates on the orders of ε. Next, to prove Theorem 1.3, we will follow the idea of Li et al. [18] and Li, Niu and Xiang [19] to set

$$\xi_\varepsilon = \frac{w_\varepsilon^{(1)} - w_\varepsilon^{(2)}}{\|w_\varepsilon^{(1)} - w_\varepsilon^{(2)}\|_{L^\infty(\mathbb{R}^3)}},$$

and then obtain a contradiction by showing $\|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = o_\varepsilon(1)$. To deduce a contradiction, we will need delicate estimates on the asymptotic behaviors of solutions and the concentrating points y_ε^i (see Proposition 5.1). A main tool is a local Pohozaev type identity (see Proposition 2.4). Due to the presence of the nonlocal term $\int_{\mathbb{R}^3} |\nabla w|^2 \Delta w$, the local Pohozaev type identity is more complicated than the case of the Schödinger equation (1.7).

The paper is organized as follows. In Section 2, we explore the form of multi-peak solutions of Eq. (1.6) and locate the concentrating points. In section 3, we prepare some preliminaries for the proof of Theorem 1.2, and in Section 4 we prove Theorem 1.2. In Section 5, we further explore some properties of the solutions derived as in Theorem 1.2, and then prove Theorem 1.3. In Section 6, we prove Theorem 1.4 and so establish the existence result of normalized multi-peak solutions to problem (1.2)–(1.3). For brevity, some elementary but long calculations and analysis results are left for Appendix A and B.

Our notations are standard. Denote $u_+ = \max(u, 0)$ for $u \in H^1(\mathbb{R}^3)$. We use $B_R(x)(\overline{B}_R(x))$ to denote the open (closed) ball in $\mathbb{R}^3$ centered at x with radius R. For any $1 \le p \le \infty$, $L^p(\mathbb{R}^3)$ is the space of real-valued Lebesgue measurable functions with finite L^p norms. $H^1(\mathbb{R}^3)$ is the standard Sobolev space, and $H^{-1}(\mathbb{R}^3)$ is the dual space of $H^1(\mathbb{R}^3)$. By the usual abuse of notations, we write $u(x) = u(r)$ with $r = |x|$ whenever u is a radial function in $\mathbb{R}^3$. We will use C and $C_j (j \in \mathbb{N})$ to denote various positive constants, and $O(t)$, $o(t)$, $o_t(1)$ to mean $|O(t)| \le C|t|$, $o(t)/t \to 0$ and $o_t(1) \to 0$ as $t \to 0$, respectively.

2. The form and locations of multi-peak solutions

In this section, we give some priori estimates and locate the concentrating points of multi-peak solutions to Eq. (1.6) in the case of $3 \le p < 5$. In this case, $\frac{4}{p-1} \in (1, 2]$, $2 - \frac{4}{p-1} \in [0, 1)$ and $2 - \frac{4}{p-1} + q \in [q, q+1) \subseteq (0, 2)$.

Throughout this section, we denote by $Q(x) \in H^1(\mathbb{R}^3)$ the unique positive radial solution to the problem

$$\begin{cases} -\Delta Q + Q = Q^p, & Q(x) > 0 \ \text{ in } \mathbb{R}^3, \\ Q(0) = \max\limits_{x \in \mathbb{R}^3} Q(x), & Q(x) \in H^1(\mathbb{R}^3). \end{cases} \tag{2.1}$$

We refer to e.g. Berestycki and Lions [4] and Kwong [17] for the existence and uniqueness of $Q(x)$, respectively. It is well known that $Q(x)$ satisfies the following properties

$$\begin{cases} Q(x) = Q(|x|), \ Q(x) > 0, \ x \in \mathbb{R}^3; \\ \lim\limits_{|x| \to \infty} |x| e^{|x|} Q(x) = C, \ \lim\limits_{r \to \infty} \frac{Q'(r)}{Q(r)} = -1, \end{cases} \tag{2.2}$$

where $C > 0$ is a constant. Moreover, $Q(x)$ is non-degenerate, that is, the kernel of the operator $-\Delta w + w - pQ^{p-1} w$ in $H^1(\mathbb{R}^3)$ is spanned by $\{\partial_{x_j} Q : j = 1, 2, 3\}$. Additionally, $Q(x)$ satisfies the following Pohozaev identity.

Lemma 2.1. *Let $Q(x)$ be the unique positive radial function that satisfies* (2.1). *Then it holds*

$$(5 - p) \int_{\mathbb{R}^3} |\nabla Q|^2 = (3p - 3) \int_{\mathbb{R}^3} Q^2. \tag{2.3}$$

Proof. Invoking by the Pohozaev identity of elliptic equations, there holds

$$\int_{\mathbb{R}^3} |\nabla Q|^2 + \int_{\mathbb{R}^3} Q^2 = \int_{\mathbb{R}^3} Q^{p+1} \ \text{ and } \ (p+1) \int_{\mathbb{R}^3} |\nabla Q|^2 + 3(p+1) \int_{\mathbb{R}^3} Q^2 = 6 \int_{\mathbb{R}^3} Q^{p+1}.$$

The above system gives (2.3). The proof of Lemma 2.1 is complete. □

Next we give some properties including nondegeneracy and priori estimates of some form of multi-peak solutions as follows.

Proposition 2.2. (1) *For any $a, b > 0$, there exists a unique positive radial solution $(v^1, v^2, \ldots, v^k)$ to the limiting system* (1.16) *up to translations; In particular, suppose that $w_\varepsilon(x)$ is a k-peak solution to problem* (1.6) *of the form* (1.17)–(1.18), *then*

$$c_\varepsilon := \sum_{i=1}^{k} \int_{\mathbb{R}^3} |\nabla v^i|^2 \to c_*$$

as $\varepsilon \to 0$, for some $c_* > 0$, where

$$c_* := bk^2 \left(\frac{3p-3}{5-p}\right)^2 \left(\int_{\mathbb{R}^3} Q^2\right)^2;$$

(2) *There exists a unique positive radial solution $(U^1, U^2, \cdots, U^k)$ satisfying*

$$-b\sum_{i=1}^{k} \int_{\mathbb{R}^3} |\nabla U^i|^2 \Delta U^i + U^i = (U^i)^p, \; U^i \in H^1(\mathbb{R}^3), \; U^i > 0 \;\; in \; \mathbb{R}^3. \tag{2.4}$$

Moreover, $U^i(x)$ is non-degenerate in $H^1(\mathbb{R}^3)$ in the sense that

$$\mathrm{Ker}\mathcal{L}^i = \mathrm{span}\left\{\partial_{x_1}U^i, \partial_{x_2}U^i, \partial_{x_3}U^i\right\},$$

where $\mathcal{L}^i : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ is defined as

$$\mathcal{L}^i\varphi = -b\sum_{i=1}^{k} \int_{\mathbb{R}^3} |\nabla U^i|^2 \Delta\varphi - 2b\left(\int_{\mathbb{R}^3} \nabla U^i \cdot \nabla\varphi\right) \Delta U^i + \varphi - p(U^i)^{p-1}\varphi \tag{2.5}$$

for all $\varphi \in H^1(\mathbb{R}^3)$;

(3) *For each $i \in \{1, 2, \cdots, k\}$ we have*

$v^i(x) \to U^i(x)$ in $L^m(\mathbb{R}^3)$, $2 \le m \le 6$, as $\varepsilon \to 0$, and

$\nabla v^i(x) \to \nabla U^i(x)$ in $L^2(\mathbb{R}^3)$ as $\varepsilon \to 0$. In particular,

$v^i(x) \to U^i(x)$ in $H^1(\mathbb{R}^3)$ as $\varepsilon \to 0$.

Proof. (1) Suppose that $(v^1, v^2, \cdots, v^k)$ is a solution to the system (1.16). Define

$$c_\varepsilon = \sum_{i=1}^{k} \int_{\mathbb{R}^3} |\nabla v^i|^2.$$

Then v^i satisfies the following equation:

$$-\left(a\varepsilon^{2-2q} + bc_\varepsilon\right)\Delta v^i + (1 + \varepsilon^2 V(a_i))v^i = (v^i)^p \quad \text{in } \mathbb{R}^3.$$

By a scaling argument and the uniqueness result of positive solutions to Eq. (2.1), we have

$$v^i(x)=\left(\frac{1}{1+\varepsilon^2V(a_i)}\right)^{-\frac{1}{p-1}}Q\left(\frac{\sqrt{1+\varepsilon^2V(a_i)}(x-x_i)}{\sqrt{a\varepsilon^{2-2q}+bc_\varepsilon}}\right),$$

for some $x_i\in\mathbb{R}^3$ $(i=1,2,\cdots,k)$, and

$$c_\varepsilon=\sum_{i=1}^k\int_{\mathbb{R}^3}|\nabla v^i|^2=\sum_{i=1}^k\left(\frac{1}{1+\varepsilon^2V(a_i)}\right)^{-\frac{2}{p-1}}\frac{\sqrt{a\varepsilon^{2-2q}+bc_\varepsilon}}{\sqrt{1+\varepsilon^2V(a_i)}}\int_{\mathbb{R}^3}|\nabla Q|^2. \tag{2.6}$$

Meanwhile, we conclude that the set

$$\mathcal{N}=\left\{(v^1,v^2,\cdots,v^k):\begin{array}{l} v^i(x)=\left(\frac{1}{1+\varepsilon^2V(a_i)}\right)^{-\frac{1}{p-1}}Q\left(\frac{\sqrt{1+\varepsilon^2V(a_i)}(x-x_i)}{\sqrt{a\varepsilon^{2-2q}+bc_\varepsilon}}\right),\\ x_i\in\mathbb{R}^3,\ 1\le i\le k,\end{array}\right\}$$

consists of all solutions to the system (1.16).

Suppose that $w_\varepsilon(x)$ is a k-peak solution to problem (1.6) of the form (1.17)–(1.18). Then by the Definition 1.1 (iii), it yields

$$O(\varepsilon^{3q})=\int_{\mathbb{R}^3}w_\varepsilon^2=\varepsilon^{3q}\sum_{i=1}^k\left(\frac{1}{1+\varepsilon^2V(a_i)}\right)^{-\frac{2}{p-1}}\left(\frac{a\varepsilon^{2-2q}+bc_\varepsilon}{1+\varepsilon^2V(a_i)}\right)^{\frac{3}{2}}\int_{\mathbb{R}^3}Q^2\mathrm{d}x+o(\varepsilon^{3q}),$$

from which we assume that $c_\varepsilon\to c_*$ for some $c_*>0$ as $\varepsilon\to0$. Sending $\varepsilon\to0$, we conclude from (2.6) that

$$c_*=bk^2\left(\frac{3p-3}{5-p}\int_{\mathbb{R}^3}Q^2\right)^2.$$

Here we have used Eq. (2.3).

(2) Suppose that $(U^1,U^2,\cdots,U^k)$ is a solution to the system (2.4). Denote

$$c^*=\sum_{i=1}^k\int_{\mathbb{R}^3}|\nabla U^i|^2.$$

Then U^i satisfies $\quad -bc^*\Delta U^i+U^i=(U^i)^p\quad$ in $\mathbb{R}^3$.

By a scaling argument and the uniqueness result of positive solutions to Eq. (2.1), we have

$$U^i(x)=Q\left(\frac{x-y_i}{\sqrt{bc^*}}\right)$$

for some $y_i\in\mathbb{R}^3$ $(i=1,2,\cdots,k)$. Simultaneously, it also yields that the set

$$\mathcal{M}=\left\{(U^1,U^2,\cdots,U^k):U^i(x)=Q\left(\frac{x-y_i}{\sqrt{bc^*}}\right):y_i\in\mathbb{R}^3,\ 1\le i\le k\right\}$$

consists of all positive solutions of system (2.4). Notice that

$$c^*=\sum_{i=1}^k\int_{\mathbb{R}^3}|\nabla U^i|^2=k\sqrt{bc^*}\int_{\mathbb{R}^3}|\nabla Q|^2.$$

Combining the above equation and Eq. (2.3) yields

$$c^* = bk^2 \left(\frac{3p-3}{5-p} \int_{\mathbb{R}^3} Q^2 \right)^2 = c_*.$$

The nondegeneracy of U^i can be proved by the same argument as that of Li et al. in [18].

(3) Since $Q(x) \in H^1(\mathbb{R}^3)$, the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^m(\mathbb{R}^3)$ $(2 \le m \le 6)$ implies that $Q(x) \in L^m(\mathbb{R}^3)$. Then, it follows from Lemma B.1 that

$$v^i(x) \to U^i(x) \ \text{ in } L^m(\mathbb{R}^3),\ m \in [2,6].$$

We also have $\nabla Q \in L^2(\mathbb{R}^3)$, then there holds by Lemma B.1 again

$$\nabla v^i(x) \to \nabla U^i(x) \ \text{ in } L^2(\mathbb{R}^3).$$

Therefore, $v^i(x) \to U^i(x)$ in $H^1(\mathbb{R}^3)$ as $\varepsilon \to 0$. The proof of Proposition 2.2 is complete. □

Noting that since $Q(x)$ decays exponentially at infinity (see (2.2)), we infer that for $\varepsilon > 0$ sufficiently small,

$$\max_{x\in\mathbb{R}^3} \left(v^i(x) + |\nabla v^i(x)| \right) = O\left(|x|^{-\iota} e^{-\sigma|x|} \right) \tag{2.7}$$

holds for some constants $\iota, \sigma > 0$ independent of ε.

We will repeatedly use the following type of Sobolev inequality.

Lemma 2.3. *For any $2 \le m \le 6$, there exists a constant $C > 0$ depending only on a and m, but independent of ε, such that*

$$\|u\|_{L^m(\mathbb{R}^3)} \le C\varepsilon^{q\left(\frac{3}{m}-\frac{3}{2}\right)} \|u\|_\varepsilon \tag{2.8}$$

holds for all $u \in H_\varepsilon$. Specially, for the case of $m = 2$, $\|u\|_{L^2(\mathbb{R}^3)} \le C\|u\|_\varepsilon$.

Proof. The proof of Lemma 2.3 follows directly from a scaling argument and the Sobolev embedding theorems. We omit the details. □

Next we derive a local Pohozaev type identity (2.9) for solutions of Eq. (1.6).

Proposition 2.4. *Let w be a positive solution of Eq. (1.6). Let Ω be a bounded smooth domain in $\mathbb{R}^3$. Then we have*

$$\begin{aligned} \varepsilon^2 \int_\Omega \frac{\partial V}{\partial x_i} w^2 = &\left(a\varepsilon^2 + b\varepsilon^{2-\frac{4}{p-1}} \int_{\mathbb{R}^3} |\nabla w|^2 \right) \int_{\partial\Omega} \left(|\nabla w|^2 \nu_i - 2\frac{\partial w}{\partial \nu}\frac{\partial w}{\partial x_i} \right) \\ &+ \int_{\partial\Omega} \left(1 + \varepsilon^2 V(x) \right) w^2 \nu_i - \frac{2}{p+1} \int_{\partial\Omega} w^{p+1}\nu_i \end{aligned} \tag{2.9}$$

for each $i = 1,2,3$, where $\nu = (\nu_1, \nu_2, \nu_3)$ is the unit outward normal of $\partial\Omega$.

Proof. The proof is obtained by multiplying both sides of (1.6) by $\partial_{x_i} w$ for each $1 \le i \le 3$ and then integrating by parts. Interested readers can refer to Cao, Li and Luo [7] Proposition 2.3 for a similar proof. □

Now we locate the concentrating points of multi-peak solutions of the form (1.17)–(1.18).

Lemma 2.5. *Suppose $V(x)$ satisfies (V_1) and $V(x) \in C^1(\mathbb{R}^3)$. Let w_ε be a k-peak solution defined as in Definition 1.1 to Eq.* (1.6) *of the form* (1.17)–(1.18)*, i.e.,*

$$w_\varepsilon(x) = \sum_{i=1}^{k} v^i \left(\frac{x - y_\varepsilon^i}{\varepsilon^q} \right) + \varphi_\varepsilon,$$

where y_ε^i are local maximum points, $y_\varepsilon^i \to b_i$ as $\varepsilon \to 0$ for some distinct points $b_i \in \mathbb{R}^3$, with $\varphi_\varepsilon \in H^1(\mathbb{R}^3)$ satisfying

$$\|\varphi_\varepsilon\|_\varepsilon = o(\varepsilon^{2+\frac{3q}{2}}).$$

Then $\nabla V(y_\varepsilon^i) \to 0$ as $\varepsilon \to 0$, and hence $\nabla V(b_i) = 0$. As a consequence, if Eq. (1.6) *has a k-peak solution, then $V(x)$ must have at least one critical point.*

Proof. We only prove the result for $i = 1$. We use a contradiction argument. Assume, without loss of generality, that there exists a constant $C_0 > 0$ such that

$$|V_{x_1}(y_\varepsilon^1)| \geq C_0 > 0 \tag{2.10}$$

holds for any sufficiently small $\varepsilon > 0$. We will use the Pohozaev identity (2.9) to w_ε with $\Omega = B_r(y_\varepsilon^1)$ to deduce the contradiction. We choose the radius r as follows. Notice that $\{b_i\}$ are distinct points, let $r_1 = \min_{i \neq j}\{1, |b_i - b_j|/4\}$ and $r_2 = r_1/4$. For $\varepsilon > 0$ sufficiently small, $|y_\varepsilon^i - b_i| < r_2$, i.e., $B_{r_2}(y_\varepsilon^i) \subset B_{r_1}(b_i)$ and $|y_\varepsilon^i - y_\varepsilon^j| > 2r_1$. Here, we also point out that r_2 can be chosen as small as we wish. By (2.8) and the assumption $\|\varphi_\varepsilon\|_\varepsilon = o(\varepsilon^{2+\frac{3q}{2}})$, we have

$$\|\varphi_\varepsilon\|_{L^{p+1}(\mathbb{R}^3)} \leq C\varepsilon^{q\left(\frac{3}{p+1} - \frac{3}{2}\right)} \|\varphi_\varepsilon\|_\varepsilon = o\left(\varepsilon^{2+\frac{3q}{p+1}}\right).$$

Set $f = (\varepsilon^{2q}|\nabla\varphi_\varepsilon|^2 + |\varphi_\varepsilon|^2 + |\varphi_\varepsilon|^{p+1})$. Using the formula of polar coordinates, we get

$$\int_0^{r_2} \int_{\partial B_r(y_\varepsilon^1)} f = \int_{B_{r_2}(y_\varepsilon^1)} f,$$

so we can choose $r \in (0, r_2)$ such that

$$\int_{\partial B_r(y_\varepsilon^1)} \left(\varepsilon^{2q}|\nabla\varphi_\varepsilon|^2 + |\varphi_\varepsilon|^2 + |\varphi_\varepsilon|^{p+1}\right) = o(\varepsilon^{4+3q}),$$

since $p + 1 > 2$. It is obvious that r can be chosen as small as we wish, and $|y_\varepsilon^i - y_\varepsilon^j| > 2r$ for $i \neq j$.

Next we apply the local Pohozaev identity (2.9) to w_ε with $\Omega = B_r(y_\varepsilon^1)$ with r being chosen in the above. We obtain

$$\begin{aligned} \varepsilon^2 \int_{B_r(y_\varepsilon^1)} \frac{\partial V}{\partial x_1} w_\varepsilon^2 &= \varepsilon_1^1 \int_{\partial B_r(y_\varepsilon^1)} \left(|\nabla w_\varepsilon|^2 \nu_1 - 2\frac{\partial w_\varepsilon}{\partial \nu}\frac{\partial w_\varepsilon}{\partial x_1} \right) \\ &\quad + \int_{\partial B_r(y_\varepsilon^1)} \left(1 + \varepsilon^2 V(x)\right) w_\varepsilon^2 \nu_1 - \frac{2}{p+1} \int_{\partial B_r(y_\varepsilon^1)} w_\varepsilon^{p+1} \nu_1, \end{aligned} \tag{2.11}$$

where $\varepsilon_1^1 = \varepsilon^2 a + \varepsilon^{2-\frac{4}{p-1}} b \int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 = O(\varepsilon^{2-\frac{4}{p-1}+q})$, since $2 - \frac{4}{p-1} + q < 2$.

We estimate (2.11) term by term. To estimate $\int_{B_r(y_\varepsilon^1)} \frac{\partial V}{\partial x_1} w_\varepsilon^2$, we spilt it into

$$\int_{B_r(y_\varepsilon^1)} \frac{\partial V}{\partial x_1} w_\varepsilon^2 = \int_{B_r(y_\varepsilon^1)} \left(V_{x_1}(x) - V_{x_1}(y_\varepsilon^1)\right) w_\varepsilon^2 + V_{x_1}(y_\varepsilon^1) \int_{B_r(y_\varepsilon^1)} w_\varepsilon^2. \tag{2.12}$$

By continuity, we have

$$\left| \int_{B_r(y_\varepsilon^1)} \left(V_{x_1}(x) - V_{x_1}(y_\varepsilon^1)\right) w_\varepsilon^2 \right| \le \max_{x \in B_r(y_\varepsilon^1)} |V_{x_1}(x) - V_{x_1}(y_\varepsilon^1)| \int_{B_r(y_\varepsilon^1)} w_\varepsilon^2.$$

By Proposition 2.2 (3) and (2.7), we deduce

$$C_1 \varepsilon^{3q} \le \int_{B_r(y_\varepsilon^1)} w_\varepsilon^2 = \int_{B_r(y_\varepsilon^1)} \left(v_{\varepsilon, y_\varepsilon^1}^1\right)^2 + o(\varepsilon^{2+3q}) \le C_2 \varepsilon^{3q},$$

for $\varepsilon > 0$ sufficiently small, where $C_1, C_2 > 0$ are constants independent of ε. Hence, for $\varepsilon > 0$ sufficiently small, there holds

$$\left| \int_{B_r(y_\varepsilon^1)} \left(V_{x_1}(x) - V_{x_1}(y_\varepsilon^1)\right) w_\varepsilon^2 \right| \le C_2 \max_{x \in B_r(y_\varepsilon^1)} |V_{x_1}(x) - V_{x_1}(y_\varepsilon^1)| \varepsilon^{3q}, \tag{2.13}$$

and

$$|V_{x_1}(y_\varepsilon^1)| \int_{B_r(y_\varepsilon^1)} w_\varepsilon^2 \ge C_0 C_1 \varepsilon^{3q}. \tag{2.14}$$

Combining the above two estimates (2.13)–(2.14) and choosing r sufficiently small, we obtain

$$\begin{aligned} \left| \varepsilon^2 \int_{B_r(y_\varepsilon^1)} \frac{\partial V}{\partial x_1} w_\varepsilon^2 \right| &\ge \left(C_0 C_1 - C_2 \max_{x \in B_r(y_\varepsilon^1)} |V_{x_1}(x) - V_{x_1}(y_\varepsilon^1)| \right) \varepsilon^{2+3q} \\ &\ge \frac{C_0 C_1}{2} \varepsilon^{2+3q}. \end{aligned} \tag{2.15}$$

On the other hand, by (A.6) we have

$$\begin{aligned} \varepsilon_1^1 \left| \int_{\partial B_r(y_\varepsilon^1)} \left(|\nabla w_\varepsilon|^2 \nu_1 - 2 \frac{\partial w_\varepsilon}{\partial \nu} \frac{\partial w_\varepsilon}{\partial x_1} \right) \right| &\le C \varepsilon^{2 - \frac{4}{p-1} + q} \int_{\partial B_r(y_\varepsilon^1)} \left(\sum_{i=1}^k |\nabla v_{\varepsilon, y_\varepsilon^i}^i|^2 + |\nabla \varphi_\varepsilon|^2 \right) \\ &\le C \varepsilon^\gamma + o(\varepsilon^{4+3q}) = o(\varepsilon^{4+3q}), \end{aligned} \tag{2.16}$$

for sufficiently large $\gamma > 0$, and

$$\begin{aligned} &\left| \int_{\partial B_r(y_\varepsilon^1)} \left(1 + \varepsilon^2 V\right) w_\varepsilon^2 \nu_1 - \frac{2}{p+1} \int_{\partial B_r(y_\varepsilon^1)} w_\varepsilon^{p+1} \nu_1 \right| \\ &\le C \int_{\partial B_r(y_\varepsilon^1)} \left(\sum_{i=1}^k \left(v_{\varepsilon, y_\varepsilon^i}^i\right)^2 + |\varphi_\varepsilon|^2 + \sum_{i=1}^k \left(v_{\varepsilon, y_\varepsilon^i}^i\right)^{p+1} + |\varphi_\varepsilon|^{p+1} \right) \\ &\le O(\varepsilon^\gamma) + o(\varepsilon^{4+3q}) = o(\varepsilon^{4+3q}). \end{aligned} \tag{2.17}$$

Finally, combining (2.12) and (2.15)–(2.17), we obtain

$$\frac{C_0 C_1}{2} \varepsilon^{2+3q} \le o(\varepsilon^{4+3q}), \quad \text{as } \varepsilon \to 0,$$

contradicting (2.10). Hence $\nabla V(y_\varepsilon^i) \to 0$ as $\varepsilon \to 0$, and then $\nabla V(b_i) = 0$ since $V(x) \in C^1$ and $y_\varepsilon^i \to b_i$. The proof of Lemma 2.5 is complete. □

3. Some preliminaries

To obtain multi-peak solutions to Eq. (1.6) of the form (1.17)–(1.18), we first deduce some necessary estimates for the finite-dimensional reduction. To this end, let $(v^1, v^2, \cdots, v^k)$ be the unique positive radial solution to the system (1.16) and let

$$Y = (y^1, y^2, \dots, y^k) \in D_\delta^k := \overline{B}_\delta(a_1) \times \overline{B}_\delta(a_2) \times \cdots \times \overline{B}_\delta(a_k), \quad W_{\varepsilon,Y} := \sum_{i=1}^k v^i_{\varepsilon,y^i},$$

where $0 < \delta < \min\{|a_i - a_j|/4 : i \neq j\}$. Here we need to recall that we assume $a_i \neq a_j$ for $i \neq j$ in this paper.

Now if $Y = (y^1, y^2, \cdots, y^k) \in D_\delta^k$, then $|y^i - y^j| \geq |a_i - a_j|/2 \geq 2\delta$ with $i \neq j$, which implies by (2.7) that

$$\int_{\mathbb{R}^3} \nabla v^i_{\varepsilon,y^i} \cdot \nabla v^j_{\varepsilon,y^j} + \left(v^i_{\varepsilon,y^i}\right)^s \left(v^j_{\varepsilon,y^j}\right)^t = O(e^{-\frac{\gamma_0}{\varepsilon^q}}) \text{ with } i \neq j \tag{3.1}$$

for some constant $\gamma_0 > 0$ and for any given $s, t > 0$. Readers can refer to Lemma A.1 and Lemma A.2 for more estimates of v^i_{ε,y^i}.

To construct k-peak solutions to Eq. (1.6) of the form (1.17)–(1.18), we will follow the scheme of Cao and Peng [9], combining reduction method and variational method.

First, define
$$J_\varepsilon(Y, \varphi) := I_\varepsilon(W_{\varepsilon,Y} + \varphi),$$

for $(Y, \varphi) \in \mathbb{R}^{3k} \times H_\varepsilon$, where I_ε is defined as in (1.15). Then we can expand $J_\varepsilon(Y, \varphi)$ with respect to φ near $\varphi = 0$ for each fixed Y as follows

$$J_\varepsilon(Y, \varphi) = J_\varepsilon(Y, 0) + l_{\varepsilon,Y}(\varphi) + \frac{1}{2}\langle \mathcal{I}_\varepsilon \varphi, \varphi\rangle + R_{\varepsilon,Y}(\varphi), \tag{3.2}$$

where $J_\varepsilon(Y, 0) = I_\varepsilon(W_{\varepsilon,Y})$ and the operators $l_{\varepsilon,Y}$, $\mathcal{I}_\varepsilon$ and $R_{\varepsilon,Y}$ are given as follows:

$$\begin{aligned} l_{\varepsilon,Y}(\varphi) = &\int_{\mathbb{R}^3} \left(a\varepsilon^2 \nabla W_{\varepsilon,Y} \cdot \nabla\varphi + W_{\varepsilon,Y}\varphi\right) + \varepsilon^2 \int_{\mathbb{R}^3} V(x) W_{\varepsilon,Y}\varphi - \int_{\mathbb{R}^3} W^p_{\varepsilon,Y}\varphi \\ &+ b\varepsilon^{2-\frac{4}{p-1}} \int_{\mathbb{R}^3} |\nabla W_{\varepsilon,Y}|^2 \int_{\mathbb{R}^3} \nabla W_{\varepsilon,Y} \cdot \nabla\varphi, \end{aligned} \tag{3.3}$$

and
$$\begin{aligned} \langle \mathcal{I}_\varepsilon \varphi, \psi\rangle = \langle I_\varepsilon''(W_{\varepsilon,Y})[\varphi], \psi\rangle = &\int_{\mathbb{R}^3} \left(a\varepsilon^2 \nabla\varphi \cdot \nabla\psi + \varphi\psi\right) \\ &+ \varepsilon^2 \int_{\mathbb{R}^3} V(x)\varphi\psi + 2b\varepsilon^{2-\frac{4}{p-1}} \int_{\mathbb{R}^3} \nabla W_{\varepsilon,Y} \cdot \nabla\varphi \int_{\mathbb{R}^3} \nabla W_{\varepsilon,Y} \cdot \nabla\psi \\ &+ b\varepsilon^{2-\frac{4}{p-1}} \int_{\mathbb{R}^3} |\nabla W_{\varepsilon,Y}|^2 \int_{\mathbb{R}^3} \nabla\varphi \cdot \nabla\psi - p \int_{\mathbb{R}^3} W^{p-1}_{\varepsilon,Y}\varphi\psi, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} R_{\varepsilon,Y}(\varphi) = &\frac{b\varepsilon^{2-\frac{4}{p-1}}}{4} \left(\int_{\mathbb{R}^3} |\nabla\varphi|^2\right)^2 + b\varepsilon^{2-\frac{4}{p-1}} \int_{\mathbb{R}^3} \nabla W_{\varepsilon,Y} \cdot \nabla\varphi \int_{\mathbb{R}^3} |\nabla\varphi|^2 \\ &- \frac{1}{p+1} \int_{\mathbb{R}^3} \left((W_{\varepsilon,Y} + \varphi)_+^{p+1} - W^{p+1}_{\varepsilon,Y} - (p+1) W^p_{\varepsilon,Y}\varphi - \frac{1}{2}p(p+1) W^{p-1}_{\varepsilon,Y}\varphi^2 \right), \end{aligned} \tag{3.5}$$

for $\varphi, \psi \in H_\varepsilon$. Note that each term in (3.2) belongs to $C^2(H_\varepsilon)$. Following the scheme of Cao and Peng in [9], we divide the proof of Theorem 1.2 into two steps:

Step 1. For every $\varepsilon, \delta > 0$ sufficiently small and for any fixed $Y \in D^k_\delta$, we will prove that $J_\varepsilon(Y, \cdot) : E^k_{\varepsilon,Y} \to E^k_{\varepsilon,Y}$ has a unique critical point $\varphi_{\varepsilon,Y} \in E^k_{\varepsilon,Y}$, where

$$E^k_{\varepsilon,Y} = \cap^k_{i=1} E^i_{\varepsilon,y^i} := \cap^k_{i=1} \left\{ \varphi \in H_\varepsilon : \left\langle \varphi, \partial_{y^i_j} v^i_{\varepsilon,y^i} \right\rangle_\varepsilon = 0,\ j = 1,2,3 \right\}.$$

Step 2. Then, for each $\varepsilon, \delta > 0$ sufficiently small, we will find a critical point Y_ε for the function $j_\varepsilon : D^k_\delta \to \mathbb{R}^1$ induced by

$$Y \mapsto j_\varepsilon(Y) \equiv J_\varepsilon\left(Y, \varphi_{\varepsilon,Y}\right). \tag{3.6}$$

This gives a solution $w_\varepsilon \equiv W_{\varepsilon,Y_\varepsilon} + \varphi_{\varepsilon,Y_\varepsilon}$ to Eq. (1.6) in virtue of the following lemma.

Lemma 3.1. *There exist $\varepsilon_0 > 0, \delta_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, $Y_\varepsilon \in D^k_\delta$ is a critical point of the function $j_\varepsilon(Y) \equiv J_\varepsilon(Y, \varphi_{\varepsilon,Y})$ if and only if*

$$w_\varepsilon \equiv W_{\varepsilon,Y_\varepsilon} + \varphi_{\varepsilon,Y_\varepsilon}$$

is a critical point of I_ε.

Proof. This lemma can be proved in a standard way, see e.g. Cao and Peng [9], Cao, Noussair and Yan [8] and Bartsch and Peng [3]. □

In the lemma below, we deduce some necessary estimates for later use. First, we give estimates for $l_{\varepsilon,Y}$ (defined as in (3.3)).

Lemma 3.2. *Assume that $V(x)$ satisfies (V_1)–(V_2). Then there exists a constant $C > 0$, independent of ε, δ, such that for any $Y \in D^k_\delta$ and $\varphi \in H_\varepsilon$ we have*

$$|l_{\varepsilon,Y}(\varphi)| \le C\varepsilon^{2+\frac{3q}{2}} \left(\varepsilon^{q\theta} + \sum_{i=1}^k (V(y^i) - V(a_i)) \right) \|\varphi\|_\varepsilon \tag{3.7}$$

Furthermore, we conclude that $q = 2 - \frac{4}{p-1}$, and then $p \neq 3$ since $q > 0$.

Remark. After the proof of Lemma 3.2, we keep in mind that $q = 2 - \frac{4}{p-1}$ and $3 < p < 5$.

Proof. Since $(v^1, v^2, \cdots, v^k)$ is the unique positive radial solution to the system (1.16), v^i_{ε,y^i} satisfies

$$-\left(a\varepsilon^2 + bc_\varepsilon \varepsilon^{2q}\right) \Delta v^i_{\varepsilon,y^i} + (1 + \varepsilon^2 V(a_i)) v^i_{\varepsilon,y^i} = (v^i_{\varepsilon,y^i})^p \quad \text{in } \mathbb{R}^3.$$

Multiplying by $\varphi \in H_\varepsilon$ on both sides of the above equation and integrating by parts, we obtain

$$\left(a\varepsilon^2 + bc_\varepsilon \varepsilon^{2q}\right) \int_{\mathbb{R}^3} \nabla v^i_{\varepsilon,y^i} \cdot \nabla \varphi + (1 + \varepsilon^2 V(a_i)) \int_{\mathbb{R}^3} v^i_{\varepsilon,y^i} \varphi = \int_{\mathbb{R}^3} \left(v^i_{\varepsilon,y^i}\right)^p \varphi. \tag{3.8}$$

Summing the above equation over i, we deduce

$$\begin{aligned} &\int_{\mathbb{R}^3} \left(a\varepsilon^2 \nabla W_{\varepsilon,Y} \cdot \nabla \varphi + W_{\varepsilon,Y} \varphi\right) + \varepsilon^2 \int_{\mathbb{R}^3} V(x) W_{\varepsilon,Y} \varphi \qquad (3.9)\\ &= -bc_\varepsilon \varepsilon^{2q} \int_{\mathbb{R}^3} \nabla W_{\varepsilon,Y} \cdot \nabla \varphi + \varepsilon^2 \sum_{i=1}^k \int_{\mathbb{R}^3} (V(x) - V(a_i)) v^i_{\varepsilon,y^i} \varphi + \sum_{i=1}^k \int_{\mathbb{R}^3} \left(v^i_{\varepsilon,y^i}\right)^p \varphi. \end{aligned}$$

Then substituting (3.9) into (3.3), we obtain

$$\begin{aligned} l_{\varepsilon,Y}(\varphi) &= \left(b\varepsilon^{2-\frac{4}{p-1}}\int_{\mathbb{R}^3}|\nabla W_{\varepsilon,Y}|^2 - bc_\varepsilon\varepsilon^{2q}\right)\int_{\mathbb{R}^3}\nabla W_{\varepsilon,Y}\cdot\nabla\varphi \\ &\quad + \varepsilon^2\sum_{i=1}^{k}\int_{\mathbb{R}^3}(V(x)-V(a_i))v^i_{\varepsilon,y^i}\varphi - \int_{\mathbb{R}^3}\left(W^p_{\varepsilon,Y} - \sum_{i=1}^{k}\left(v^i_{\varepsilon,y^i}\right)^p\right)\varphi \\ &=: l_1 + l_2 - l_3. \end{aligned}$$

In our estimates particularly the estimate of l_2 plays a leading role in the finite-dimensional reduction method. To estimate l_2 just consider for each $i \in \{1, 2, \cdots, k\}$,

$$\begin{aligned} \int_{\mathbb{R}^3}(V(x)-V(a_i))v^i_{\varepsilon,y^i}\varphi &= \int_{\mathbb{R}^3}(V(x)-V(y^i))v^i_{\varepsilon,y^i}\varphi + (V(y^i)-V(a_i))\int_{\mathbb{R}^3}v^i_{\varepsilon,y^i}\varphi \\ &=: l_{21} + l_{22}. \end{aligned}$$

To estimate l_{21}, we spilt l_{21} into two parts:

$$l_{21} = \int\limits_{B_{R_0}(y^i)}(V(x)-V(y^i))v^i_{\varepsilon,y^i}\varphi + \int\limits_{B^c_{R_0}(y^i)}(V(x)-V(y^i))v^i_{\varepsilon,y^i}\varphi = l_{211} + l_{212}.$$

Combining the assumption (V_2), the exponential decay of v^i at infinity and (2.8), we easily derive by Hölder's inequality that

$$|l_{211}| \le C\varepsilon^{\frac{3q}{2}+q\theta}\|\varphi\|_\varepsilon.$$

By using the assumption (V_1), (2.7), applying Hölder's inequality and (2.8), it yields

$$|l_{212}| \le C\varepsilon^{\frac{3q}{2}+q\theta}\|\varphi\|_\varepsilon.$$

Therefore, $|l_{21}| \le |l_{211}| + |l_{212}| \le C\varepsilon^{\frac{3q}{2}+q\theta}\|\varphi\|_\varepsilon$.

To estimate l_{22}, we use a scaling argument and (2.8) to get

$$|l_{22}| \le C(V(y^i)-V(a_i))\varepsilon^{\frac{3q}{2}}\|\varphi\|_\varepsilon.$$

Hence

$$|l_2| \le C\varepsilon^{2+\frac{3q}{2}}\left(\varepsilon^{q\theta} + \sum_{i=1}^{k}(V(y^i)-V(a_i))\right)\|\varphi\|_\varepsilon.$$

To estimate l_1, at least we need $|l_1| = O(|l_2|)$. We obtain by Hölder's inequality and Lemma 2.3 that

$$\left|\int_{\mathbb{R}^3}\nabla v^i_{\varepsilon,y^i}\cdot\nabla\varphi\right| \le \|\nabla v^i_{\varepsilon,y^i}\|_{L^2(\mathbb{R}^3)}\|\nabla\varphi\|_{L^2(\mathbb{R}^3)} \le C\varepsilon^{-\frac{q}{2}}\|\varphi\|_\varepsilon,$$

and by (3.1) that

$$\begin{aligned} \left|b\varepsilon^{2-\frac{4}{p-1}}\int_{\mathbb{R}^3}|\nabla W_{\varepsilon,Y}|^2 - bc_\varepsilon\varepsilon^{2q}\right| &= \left|b\varepsilon^{2-\frac{4}{p-1}}\int_{\mathbb{R}^3}\sum_{i=1}^{k}|\nabla v^i_{\varepsilon,y^i}|^2 + O(e^{-\frac{\gamma_0}{\varepsilon^q}}) - bc_\varepsilon\varepsilon^{2q}\right| \\ &= \left|bc_\varepsilon\varepsilon^{2-\frac{4}{p-1}+q} + O(e^{-\frac{\gamma_0}{\varepsilon^q}}) - bc_\varepsilon\varepsilon^{2q}\right| \end{aligned}$$

for some constant $\gamma_0 > 0$.

Here we should notice that $c_\varepsilon \to c_* > 0$ as $\varepsilon \to 0$. We conclude that

$$\begin{cases} \text{if } 2-\frac{4}{p-1}+q > 2q, \text{ then } |l_1| = O(\varepsilon^{2q-\frac{q}{2}})\|\varphi\|_\varepsilon & \Longrightarrow |l_2| = o(|l_1|) \text{ false;} \\ \text{if } 2-\frac{4}{p-1}+q < 2q, \text{ then } |l_1| = O(\varepsilon^{2-\frac{4}{p-1}+q-\frac{q}{2}})\|\varphi\|_\varepsilon & \Longrightarrow |l_2| = o(|l_1|) \text{ false;} \\ \text{if } 2-\frac{4}{p-1}+q = 2q, \text{ then } |l_1| = O(\varepsilon^{-\frac{q}{2}}e^{-\frac{\gamma_0}{\varepsilon^q}})\|\varphi\|_\varepsilon & \Longrightarrow |l_1| = o(|l_2|) \text{ as required.} \end{cases}$$

In the case of $2-\frac{4}{p-1}+q = 2q$, one can check that $q = 2-\frac{4}{p-1}$ and

$$|l_1| \le C\varepsilon^{-\frac{q}{2}}e^{-\frac{\gamma_0}{\varepsilon^q}}\|\varphi\|_\varepsilon \le C\varepsilon^{2+\frac{3q}{2}+q\theta}\|\varphi\|_\varepsilon$$

for $\varepsilon > 0$ sufficiently small.

To estimate l_3, we deduce by Hölder's inequality and (3.1) that

$$\begin{aligned} |l_3| &= \left|\int_{\mathbb{R}^3}\left(W^p_{\varepsilon,Y} - \sum_{i=1}^k (v^i_{\varepsilon,y^i})^p\right)\varphi\right| \le \int_{\mathbb{R}^3}\left|\left(\sum_{i=1}^k v^i_{\varepsilon,y^i}\right)^p - \sum_{i=1}^k (v^i_{\varepsilon,y^i})^p\right||\varphi| \\ &\le C\int_{\mathbb{R}^3}\sum_{i\neq j}\left(\left(v^i_{\varepsilon,y^i}\right)^{p-1}\left(v^j_{\varepsilon,y^j}\right)^1 + \left(v^i_{\varepsilon,y^i}\right)^1\left(v^j_{\varepsilon,y^j}\right)^{p-1}\right)|\varphi| \\ &\le C\varepsilon^{\frac{3q}{2}}e^{-\frac{\gamma_0}{\varepsilon^q}}\|\varphi\|_\varepsilon \le C\varepsilon^{2+\frac{3q}{2}+q\theta}\|\varphi\|_\varepsilon \end{aligned}$$

for $\varepsilon > 0$ sufficiently small.

Finally, combining the above estimates of l_1, l_2 and l_3 gives (3.7). The proof of Lemma 3.2 is complete. □

Next, we give estimates of $R_{\varepsilon,Y}$ (defined as in (3.5)) and its derivatives $R^{(i)}_{\varepsilon,Y}$ for $i = 1, 2$.

Lemma 3.3. *There exist constants $C_1, C_2 > 0$, independent of ε, such that for $i \in \{0, 1, 2\}$ we have*

$$\|R^{(i)}_{\varepsilon,Y}(\varphi)\| \le C_1\varepsilon^{-3(p-3)}\|\varphi\|^{p+1-i}_\varepsilon + C_2\varepsilon^{-\frac{3q}{2}}\left(1+\varepsilon^{-\frac{3q}{2}}\|\varphi\|_\varepsilon\right)\|\varphi\|^{3-i}_\varepsilon \tag{3.10}$$

for all $\varphi \in H_\varepsilon$. Here, we denote $R^{(i)}_{\varepsilon,Y}$ $(i = 1, 2)$ the i-th derivative of $R_{\varepsilon,Y}$. $\|\cdot\|$ denotes $|\cdot|$ for $i = 0$ and operator-norms for $i = 1, 2$. Here $q = 2-\frac{4}{p-1}$.

Proof. This lemma can be proved by the same argument as that of Lemma 3.3 in [18]. We omit the details. □

4. Proof of Theorem 1.2

This section is devoted to proving Theorem 1.2. First, we consider the operator $\mathcal{I}_\varepsilon$ defined as in (3.4), that is,

$$\begin{aligned} \langle \mathcal{I}_\varepsilon\varphi, \psi\rangle &= \int_{\mathbb{R}^3}\left(a\varepsilon^2\nabla\varphi\cdot\nabla\psi + \varphi\psi\right) + \varepsilon^2\int_{\mathbb{R}^3}V(x)\varphi\psi + 2b\varepsilon^q\int_{\mathbb{R}^3}\nabla W_{\varepsilon,Y}\cdot\nabla\varphi\int_{\mathbb{R}^3}\nabla W_{\varepsilon,Y}\cdot\nabla\psi \\ &\quad + b\varepsilon^q\int_{\mathbb{R}^3}|\nabla W_{\varepsilon,Y}|^2\int_{\mathbb{R}^3}\nabla\varphi\cdot\nabla\psi - p\int_{\mathbb{R}^3}W^{p-1}_{\varepsilon,Y}\varphi\psi \end{aligned}$$

for $\varphi, \psi \in H_\varepsilon$. Here $q = 2-\frac{4}{p-1}$. When no confusion occurs, we always suppress $\varphi \in E^k_{\varepsilon,Y}, Y \in D^k_\delta$. Note that $E^k_{\varepsilon,Y}$ is a closed subspace of H_ε for every $\varepsilon > 0$ and $Y \in \mathbb{R}^{3k}$.

The following result shows that $\mathcal{I}_\varepsilon$ is invertible when restricted on $E^k_{\varepsilon,Y}$.

Lemma 4.1. *There exist* ε_0, δ_0 *and* $\rho > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \delta_0)$ *we have* $\|\mathcal{I}_\varepsilon \varphi\| \geq \rho \|\varphi\|_\varepsilon$, *for any* $\varphi \in E^k_{\varepsilon,Y}$ *uniformly with respect to* $Y \in D^k_\delta$.

Proof. The proof is standard, interested readers can refer to Luo et al. [21] Proposition 3.4 and Li et al. [18] Proposition 4.1 for a similar proof. □

Lemma 4.1 implies that the quadratic form $\mathcal{I}_\varepsilon : E^k_{\varepsilon,Y} \to E^k_{\varepsilon,Y}$ has a bounded inverse, with $\|\mathcal{I}_\varepsilon^{-1}\| \leq \rho^{-1}$ uniformly with respect to $Y \in D^k_\delta$.

Lemma 4.2. *There exist* $\varepsilon_0, \delta_0 > 0$ *sufficiently small such that for all* $\varepsilon \in (0, \varepsilon_0)$ *and* $\delta \in (0, \delta_0)$, *there exists a* C^1 *map* $\varphi_\varepsilon : D^k_\delta \to H_\varepsilon$, $Y \mapsto \varphi_{\varepsilon,Y}$ *satisfying*

$$\left\langle \frac{\partial J_\varepsilon(Y, \varphi_{\varepsilon,Y})}{\partial \varphi}, \psi \right\rangle_\varepsilon = 0, \quad \text{for all } \psi \in E^k_{\varepsilon,Y}, \tag{4.1}$$

with $\varphi_{\varepsilon,Y} \in N_\varepsilon$, *where*

$$N_\varepsilon = \left\{ \varphi \in E^k_{\varepsilon,Y} : \|\varphi\|_\varepsilon \leq \varepsilon^{2+\frac{3q}{2}} \left(\varepsilon^{q\theta-\tau} + \sum_{i=1}^k (V(y^i) - V(a_i))^{1-\tau} \right) \right\}$$

for some constant $\tau \in (0, q\theta/2)$. *Here* $q = 2 - \frac{4}{p-1}$. *Specially, we can choose* $\tau > 0$ *as small as we wish.*

Proof. Note that, for each $Y \in D^k_\delta$, we obtain from (3.2) that

$$J_\varepsilon(Y, \varphi) = J_\varepsilon(Y, 0) + \langle I'_\varepsilon(W_{\varepsilon,Y}), \varphi \rangle + \frac{1}{2} \langle I''_\varepsilon(W_{\varepsilon,Y})[\varphi], \varphi \rangle + R_{\varepsilon,Y}(\varphi).$$

Differentiating the above equation with respect to φ, we obtain

$$\left\langle \frac{\partial J_\varepsilon}{\partial \varphi}, \psi \right\rangle = \langle I'_\varepsilon(W_{\varepsilon,Y}), \psi \rangle + \langle I''_\varepsilon(W_{\varepsilon,Y})[\varphi], \psi \rangle + \langle R^{(1)}_{\varepsilon,Y}(\varphi), \psi \rangle$$

for $\psi \in E^k_{\varepsilon,Y}$, which implies that

$$\frac{\partial J_\varepsilon}{\partial \varphi} = I'_\varepsilon(W_{\varepsilon,Y}) + I''_\varepsilon(W_{\varepsilon,Y})[\varphi] + R^{(1)}_{\varepsilon,Y}(\varphi) = l_{\varepsilon,Y} + \mathcal{I}_\varepsilon \varphi + R^{(1)}_{\varepsilon,Y}(\varphi). \tag{4.2}$$

Since $E^k_{\varepsilon,Y}$ is a closed subspace of H_ε, Lemma 3.2 and Lemma 3.3 implies that $l_{\varepsilon,Y}$ and $R^{(1)}_{\varepsilon,Y}(\varphi)$ are bounded linear operators when restricted on $E^k_{\varepsilon,Y}$. Denote by $\mathfrak{M}$ the dual space of H_ε. By Riesz's representation theorem, for each $f \in \mathfrak{M}$, there exists a unique $\hat{f} \in H_\varepsilon$ such that

$$f(\psi) = \langle \hat{f}, \psi \rangle_\varepsilon, \quad \text{for all } \psi \in H_\varepsilon.$$

From this, we can define a map $\sigma : \mathfrak{M} \to H_\varepsilon$, $f \mapsto \hat{f}$. We claim that σ is a homeomorphism from $\mathfrak{M}$ to $\mathfrak{M}^* := \sigma(\mathfrak{M})$. In fact, suppose $\sigma(f_1) = \sigma(f_2)$, i.e., $\hat{f}_1 = \hat{f}_2$, then

$$f_1(\psi) = \langle \hat{f}_1, \psi \rangle_\varepsilon = \langle \hat{f}_2, \psi \rangle_\varepsilon = f_2(\psi), \text{ for all } \psi \in H_\varepsilon,$$

which implies that $f_1 = f_2$, and hence f is injective.

Since $\mathfrak{M}^* := \sigma(\mathfrak{M})$, σ is surjective. For any $f_1, f_2 \in \mathfrak{M}$, and any $\psi \in H_\varepsilon$,

$$\langle \widehat{f_1 + f_2}, \psi\rangle_\varepsilon = (f_1 + f_2)(\psi) = f_1(\psi) + f_2(\psi) = \langle \hat{f}_1, \psi\rangle_\varepsilon + \langle \hat{f}_2, \psi\rangle_\varepsilon = \langle \hat{f}_1 + \hat{f}_2, \psi\rangle_\varepsilon,$$

which implies that $\widehat{f_1 + f_2} = \hat{f}_1 + \hat{f}_2$, i.e., $\sigma(f_1 + f_2) = \sigma(f_1) + \sigma(f_2)$. And for any $k \in \mathbb{R}^1$, $f \in \mathfrak{M}$ and any $\psi \in H_\varepsilon$, we have

$$\langle \widehat{kf}, \psi\rangle_\varepsilon = (kf)(\psi) = kf(\psi) = k\langle \hat{f}, \psi\rangle_\varepsilon = \langle k\hat{f}, \psi\rangle_\varepsilon,$$

which implies that $\widehat{kf} = k\hat{f}$, i.e., $\sigma(kf) = k\sigma(f)$. The proof of the above claim is complete.

Based on the above claim, we can identify $l_{\varepsilon,Y}$ and $R^{(1)}_{\varepsilon,Y}$ with their representatives in $E^k_{\varepsilon,Y}$, and then Eq. (4.2) is equivalent to

$$\widehat{\frac{\partial J_\varepsilon}{\partial \varphi}} = \widehat{l}_{\varepsilon,Y} + \mathcal{I}_\varepsilon \varphi + \widehat{R}^{(1)}_{\varepsilon,Y}(\varphi). \tag{4.3}$$

Therefore, to prove Eq. (4.1), it is equivalent to find $\varphi \in E^k_{\varepsilon,Y}$ that satisfies

$$\widehat{l}_{\varepsilon,Y} + \mathcal{I}_\varepsilon \varphi + \widehat{R}^{(1)}_{\varepsilon,Y}(\varphi) = 0.$$

By Lemma 4.1, the operator $\mathcal{I}_\varepsilon$ is invertible, and $\|\mathcal{I}_\varepsilon^{-1}\| \le \rho^{-1}$. So Eq. (4.3) is equivalent to

$$\varphi = -\mathcal{I}_\varepsilon^{-1}(\widehat{l}_{\varepsilon,Y}) - \mathcal{I}_\varepsilon^{-1}(\widehat{R}^{(1)}_{\varepsilon,Y}(\varphi)) =: \mathcal{A}_{\varepsilon,Y}(\varphi). \tag{4.4}$$

Claim that $\mathcal{A}_{\varepsilon,Y}$ is a contraction map when restricted in N_ε. Indeed, for any $\varphi \in N_\varepsilon$, we have

$$\|\mathcal{A}_{\varepsilon,Y}(\varphi)\|_\varepsilon \le \rho^{-1}\|\widehat{l}_{\varepsilon,Y}\| + \rho^{-1}\|\widehat{R}^{(1)}_{\varepsilon,Y}(\varphi)\| = \rho^{-1}\|l_{\varepsilon,Y}\| + \rho^{-1}\|R^{(1)}_{\varepsilon,Y}(\varphi)\|.$$

By Lemma 3.2, $\|l_{\varepsilon,Y}\| \le C\varepsilon^{2+\frac{3q}{2}}\left(\varepsilon^{q\theta} + \sum_{i=1}^k \left(V(y^i) - V(a_i)\right)\right)$.

So we can choose $0 < \tau < q\theta/2$ sufficiently small in such a way that

$$\begin{cases} C\varepsilon^\tau < \frac{\rho}{2}, \\ C\left(V(y^i) - V(a_i)\right)^\tau < \frac{\rho}{2}, \quad \text{for each } i = 1, 2, \cdots, k. \end{cases}$$

We also notice that $q\theta/2 < q/2 = \frac{1}{2}\left(2 - \frac{4}{p-1}\right) < 1$. Then we obtain

$$\|l_{\varepsilon,Y}\| \le \frac{\rho}{2}\varepsilon^{2+\frac{3q}{2}}\left(\varepsilon^{q\theta-\tau} + \sum_{i=1}^k \left(V(y^i) - V(a_i)\right)^{1-\tau}\right). \tag{4.5}$$

For any $\varphi \in N_\varepsilon$, we have $\varepsilon^{-\frac{3q}{2}}\|\varphi\|_\varepsilon = o(\varepsilon^2)$, and then, by Lemma 3.3, for $\varepsilon > 0$ sufficiently small, there holds

$$\|R^{(1)}_{\varepsilon,Y}(\varphi)\| = o_\varepsilon(1)\|\varphi\|_\varepsilon \le \frac{\rho}{2}\|\varphi\|_\varepsilon. \tag{4.6}$$

Therefore, combining (4.5) and (4.6), we deduce

$$\|\mathcal{A}_{\varepsilon,Y}(\varphi)\|_\varepsilon \le \varepsilon^{2+\frac{3q}{2}}\Big(\varepsilon^{q\theta-\tau}+\sum_{i=1}^{k}(V(y^i)-V(a_i))^{1-\tau}\Big),$$

which implies that $\mathcal{A}_{\varepsilon,Y}(N_\varepsilon)\subset N_\varepsilon$. For any $\varphi,\psi\in N_\varepsilon$,

$$\begin{aligned}\|\mathcal{A}_{\varepsilon,Y}(\varphi)-\mathcal{A}_{\varepsilon,Y}(\psi)\|_\varepsilon &= \|\mathcal{I}_\varepsilon^{-1}\big(\widehat{R}^{(1)}_{\varepsilon,Y}(\varphi)\big)-\mathcal{I}_\varepsilon^{-1}\big(\widehat{R}^{(1)}_{\varepsilon,Y}(\psi)\big)\|_\varepsilon\\ &\le \rho^{-1}\|R^{(1)}_{\varepsilon,Y}(\varphi)-R^{(1)}_{\varepsilon,Y}(\psi)\|_\varepsilon\\ &= \rho^{-1}\|R^{(2)}_{\varepsilon,Y}(\xi\varphi+(1-\xi)\psi)\|\|\varphi-\psi\|_\varepsilon, \quad \text{for some } 0<\xi<1.\end{aligned}$$

Notice that

$$\begin{aligned}\|\xi\varphi+(1-\xi)\psi\|_\varepsilon &\le \xi\|\varphi\|_\varepsilon+(1-\xi)\|\psi\|_\varepsilon\\ &\le \varepsilon^{2+\frac{3q}{2}}\Big(\varepsilon^{q\theta-\tau}+\sum_{i=1}^{k}\big(V(y^i)-V(a_i)\big)^{1-\tau}\Big),\end{aligned}$$

i.e., $\xi\varphi+(1-\xi)\psi\in N_\varepsilon$, then from (3.10), we have

$$\begin{aligned}\|R^{(2)}_{\varepsilon,Y}(\xi\varphi+(1-\xi)\psi)\| &\le C_1\varepsilon^{-3(p-3)}\|\xi\varphi+(1-\xi)\psi\|_\varepsilon^{p-1}+\\ &+ C_2\varepsilon^{-\frac{3q}{2}}\Big(1+\varepsilon^{-\frac{3q}{2}}\|\xi\varphi+(1-\xi)\psi\|_\varepsilon\Big)\|\xi\varphi+(1-\xi)\psi\|_\varepsilon=o_\varepsilon(1).\end{aligned}$$

Hence for $\varepsilon>0$ sufficiently small,

$$\|\mathcal{A}_{\varepsilon,Y}(\varphi)-\mathcal{A}_{\varepsilon,Y}(\psi)\|_\varepsilon\le\frac{1}{2}\|\varphi-\psi\|_\varepsilon,$$

which implies that $\mathcal{A}_{\varepsilon,Y}:N_\varepsilon\to N_\varepsilon$ is a contraction map. We can apply fixed point theorem to find a unique solution to Eq. (4.4), as required. Thus, there exists a contraction map $Y\to\varphi_{\varepsilon,Y}$ such that (4.1) holds.

Finally, we claim that the map $Y\mapsto\varphi_{\varepsilon,Y}$ belongs to C^1. Indeed, by similar arguments as that of Cao, Noussair and Yan used in [8], we can deduce a unique C^1 map $\widetilde{\varphi}_\varepsilon: D^k_\delta\to E^k_{\varepsilon,Y}$ which satisfies (4.1). Therefore, by the uniqueness $\widetilde{\varphi}=\widetilde{\varphi}_\varepsilon$, and hence the claim follows. The proof of Lemma 4.2 is complete. □

Next we give the following observation.

Lemma 4.3. *For any $\varphi\in E^k_{\varepsilon,Y}$ we have*

$$\langle\mathcal{I}_\varepsilon\varphi,\varphi\rangle=O(\|\varphi\|_\varepsilon^2). \tag{4.7}$$

Proof. The proof is direct and we refer to the proof of Lemma 4.3 in [18]. □

Proposition 4.4. *Assume that $V(x)$ satisfies (V_1)–(V_2). Then for $\varepsilon>0$ sufficiently small, for any $Y\in D^k_\delta$, there holds*

$$I_\varepsilon(W_{\varepsilon,Y})=C_1\varepsilon^{3q}+\sum_{i=1}^{k}C_{2,i}\big(V(y^i)-V(a_i)\big)\varepsilon^{2+3q}+O(\varepsilon^{2+3q+q\theta}),$$

where

$$C_1=\Big(\frac{1}{2}-\frac{1}{p+1}\Big)\sum_{i=1}^{k}\int_{\mathbb{R}^3}(v^i)^{p+1}-\frac{bc_\varepsilon^2}{4}$$

and
$$C_{2,i} = \frac{1}{2}\int_{\mathbb{R}^3} (v^i)^2, \quad i = 1, 2, \cdots, k,$$

and θ is the Hölder continuity of $V(x)$ in the neighborhood of a_i $(1 \le i \le k)$.

Remark. Even though the constants $C_1, C_{2,i}$ depend on ε, we have $C_1 \to C_1^*$ and $C_{2,i} \to C_{2,i}^*$ as $\varepsilon \to 0$ by Proposition 2.2 (3), where

$$C_1^* = \left(\frac{1}{2} - \frac{1}{p+1}\right)\sum_{i=1}^{k}\int_{\mathbb{R}^3}(U^i)^{p+1} - \frac{bc_*^2}{4} \quad \text{and} \quad C_{2,i}^* = \frac{1}{2}\int_{\mathbb{R}^3}(U^i)^2.$$

Hence we can regard $C_1, C_{2,i}$ $(i = 1, 2, \cdots, k)$ as constants.

Proof. It follows from (1.15) and (3.1) that

$$\begin{aligned} I_\varepsilon(W_{\varepsilon,Y}) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(\varepsilon^2 a|\nabla W_{\varepsilon,Y}|^2 + W_{\varepsilon,Y}^2\right) + \frac{\varepsilon^2}{2}\int_{\mathbb{R}^3} V(x) W_{\varepsilon,Y}^2 \\ &\quad + \frac{b\varepsilon^q}{4}\left(\int_{\mathbb{R}^3}|\nabla W_{\varepsilon,Y}|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} W_{\varepsilon,Y}^{p+1} \\ &= \frac{1}{2}\sum_{i=1}^{k}\int_{\mathbb{R}^3}\left(\varepsilon^2 a|\nabla v_{\varepsilon,y^i}^i|^2 + \left(v_{\varepsilon,y^i}^i\right)^2\right) + \frac{\varepsilon^2}{2}\sum_{i=1}^{k}\int_{\mathbb{R}^3} V(x)\left(v_{\varepsilon,y^i}^i\right)^2 \\ &\quad + \frac{b\varepsilon^q}{4}\left(\sum_{i=1}^{k}\int_{\mathbb{R}^3}|\nabla v_{\varepsilon,y^i}^i|^2\right)^2 \frac{1}{p+1}\int_{\mathbb{R}^3}\sum_{i=1}^{k}\left(v_{\varepsilon,y^i}^i\right)^{p+1} + O(e^{-\frac{\gamma_0}{\varepsilon^q}}), \end{aligned}$$

where $\gamma_0 > 0$ is a constant. By (3.8), we obtain

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{k}\int_{\mathbb{R}^3}\left(\varepsilon^2 a|\nabla v_{\varepsilon,y^i}^i|^2 + \left(v_{\varepsilon,y^i}^i\right)^2\right) + \frac{\varepsilon^2}{2}\sum_{i=1}^{k}\int_{\mathbb{R}^3} V(x)\left(v_{\varepsilon,y^i}^i\right)^2 \\ &= -\frac{bc_\varepsilon \varepsilon^{2q}}{2}\sum_{i=1}^{k}\int_{\mathbb{R}^3}|\nabla v_{\varepsilon,y^i}^i|^2 + \frac{\varepsilon^2}{2}\sum_{i=1}^{k}\int_{\mathbb{R}^3}(V(x) - V(a_i))\left(v_{\varepsilon,y^i}^i\right)^2 + \frac{1}{2}\sum_{i=1}^{k}\int_{\mathbb{R}^3}\left(v_{\varepsilon,y^i}^i\right)^{p+1}. \end{aligned}$$

Notice that for each $i = 1, 2, \cdots, k$,

$$\int_{\mathbb{R}^3}(V(x) - V(a_i))(v_{\varepsilon,y^i}^i)^2 = \int_{\mathbb{R}^3}(V(x) - V(y^i))(v_{\varepsilon,y^i}^i)^2 + (V(y^i) - V(a_i))\int_{\mathbb{R}^3}(v_{\varepsilon,y^i}^i)^2.$$

Then similar arguments to the estimates of l_{21} and l_{22} in Lemma 3.2 imply that

$$\int_{\mathbb{R}^3}(V(x) - V(a_i))(v_{\varepsilon,y^i}^i)^2 = C\varepsilon^{3q+q\theta} + \left(V(y^i) - V(a_i)\right)\int_{\mathbb{R}^3}(v^i)^2\varepsilon^{3q} \tag{4.8}$$

for some constant $C > 0$.

Hence, by estimate (3.1), (4.8) and a scaling argument, we can deduce

$$I_\varepsilon(W_{\varepsilon,Y}) = -\frac{bc_\varepsilon\varepsilon^{2q}}{2}\sum_{i=1}^k\int_{\mathbb{R}^3}|\nabla v^i_{\varepsilon,y^i}|^2 + \frac{\varepsilon^2}{2}\sum_{i=1}^k\int_{\mathbb{R}^3}(V(x)-V(a_i))\left(v^i_{\varepsilon,y^i}\right)^2$$

$$+\frac{1}{2}\sum_{i=1}^k\int_{\mathbb{R}^3}\left(v^i_{\varepsilon,y^i}\right)^{p+1} + \frac{b\varepsilon^q}{4}\left(\sum_{i=1}^k\int_{\mathbb{R}^3}|\nabla v^i_{\varepsilon,y^i}|^2\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^3}\left(\sum_{i=1}^k v^i_{\varepsilon,y^i}\right)^{p+1} + O(e^{-\frac{\gamma_0}{\varepsilon^q}})$$

$$= \frac{b\varepsilon^q}{2}\sum_{i=1}^k\int_{\mathbb{R}^3}|\nabla v^i_{\varepsilon,y^i}|^2\left(\frac{1}{2}\sum_{i=1}^k\int_{\mathbb{R}^3}|\nabla v^i_{\varepsilon,y^i}|^2 - c_\varepsilon\varepsilon^q\right) + \frac{\varepsilon^2}{2}\sum_{i=1}^k\int_{\mathbb{R}^3}(V(x)-V(a_i))\left(v^i_{\varepsilon,y^i}\right)^2$$

$$+\int_{\mathbb{R}^3}\left(\frac{1}{2}\sum_{i=1}^k\left(v^i_{\varepsilon,y^i}\right)^{p+1} - \frac{1}{p+1}\sum_{i=1}^k\left(v^i_{\varepsilon,y^i}\right)^{p+1}\right) + O(e^{-\frac{\gamma_0}{\varepsilon^q}})$$

$$= C_1\varepsilon^{3q} + \sum_{i=1}^k C_{2,i}\left(V(y^i)-V(a_i)\right)\varepsilon^{2+3q} + O(\varepsilon^{2+3q+q\theta}).$$

The proof of Proposition 4.4 is complete. □

Now we prove Theorem 1.2.

Proof of Theorem 1.2. Let $Y \mapsto \varphi_{\varepsilon,Y}$ for $Y\in D^k_\delta$ be the map obtained in Lemma 4.2. We will find a critical point for the function j_ε defined as in (3.6) by Lemma 3.1. By the Taylor's expansion (3.2), we have

$$j_\varepsilon(Y) = J_\varepsilon(Y,\varphi_{\varepsilon,Y}) = I_\varepsilon(W_{\varepsilon,Y}) + l_{\varepsilon,Y}(\varphi_{\varepsilon,Y}) + \frac{1}{2}\langle\mathcal{I}_\varepsilon\varphi_{\varepsilon,Y},\varphi_{\varepsilon,Y}\rangle + R_{\varepsilon,Y}(\varphi_{\varepsilon,Y}).$$

Now by Proposition 4.4, we have

$$I_\varepsilon(W_{\varepsilon,Y}) = C_1\varepsilon^{3q} + \sum_{i=1}^k C_{2,i}\left(V(y^i)-V(a_i)\right)\varepsilon^{2+3q} + O(\varepsilon^{2+3q+q\theta}).$$

Lemma 3.2 and Lemma 4.2 give

$$|l_{\varepsilon,Y}(\varphi_{\varepsilon,Y})| \le C\varepsilon^{4+3q}\Big(\varepsilon^{q\theta} + \sum_{i=1}^k\left(V(y^i)-V(a_i)\right)\Big)\Big(\varepsilon^{q\theta-\tau} + \sum_{i=1}^k\left(V(y^i)-V(a_i)\right)^{1-\tau}\Big).$$

Lemma 4.3 gives $\langle\mathcal{I}_\varepsilon\varphi_{\varepsilon,Y},\varphi_{\varepsilon,Y}\rangle = O(\|\varphi_{\varepsilon,Y}\|^2_\varepsilon)$. Lemma 3.3 gives

$$|R_{\varepsilon,Y}(\varphi_{\varepsilon,Y})| = o_\varepsilon(1)\|\varphi_{\varepsilon,Y}\|^2_\varepsilon.$$

Combining the above estimates yields

$$j_\varepsilon(Y) = C_1\varepsilon^{3q} + \sum_{i=1}^k C_{2,i}\left(V(y^i)-V(a_i)\right)\varepsilon^{2+3q} + O(\varepsilon^{2+3q+q\theta})$$
$$+ O(\varepsilon^{4+3q})\Big(\varepsilon^{q\theta-\tau} + \sum_{i=1}^k\left(V(y^i)-V(a_i)\right)^{1-\tau}\Big)^2.$$

Next, we consider the minimizing problem

$$j_\varepsilon(Y_\varepsilon) \equiv \inf_{Y\in D^k_\delta} j_\varepsilon(Y).$$

We claim that Y_ε is an interior point of D_δ^k. To prove this claim, we apply a comparison argument. Let $e_j \in \mathbb{R}^3$ with $|e_j| = 1$, $e_i \neq e_j$ for $i \neq j$. We will choose $\eta > 1$ to be sufficiently large. Let $z_\varepsilon^j = a_j + \varepsilon^\eta e_j$ such that $Z_\varepsilon = (z_\varepsilon^1, z_\varepsilon^2, \cdots, z_\varepsilon^k) \in D_\delta^k$ for a sufficiently large $\eta > 1$. Then by the assumption (V_2), we have

$$\begin{aligned} j_\varepsilon(Z_\varepsilon) &= C_1\varepsilon^{3q} + \sum_{i=1}^k C_{2,i}\left(V(z_\varepsilon^i) - V(a_i)\right)\varepsilon^{2+3q} + O(\varepsilon^{2+3q+q\theta}) \\ &\qquad + O(\varepsilon^{4+3q})\Big(\varepsilon^{q\theta-\tau} + \sum_{i=1}^k \left(V(z_\varepsilon^i) - V(a_i)\right)^{1-\tau}\Big)^2 \\ &= C_1\varepsilon^{3q} + O(\varepsilon^{2+3q+\theta\eta}) + O(\varepsilon^{2+3q+q\theta}) + O(\varepsilon^{4+3q})(\varepsilon^{2q\theta-2\tau} + \varepsilon^{2\theta\eta(1-\tau)}) \\ &= C_1\varepsilon^{3q} + O(\varepsilon^{2+3q+q\theta}), \end{aligned}$$

where $\eta > 1$ is chosen to be sufficiently large accordingly. Here, we also have used the fact that $\tau \ll q\theta/2$. Thus, by using $j_\varepsilon(Y_\varepsilon) \leq j_\varepsilon(Z_\varepsilon)$ we deduce

$$\begin{aligned} \sum_{i=1}^k C_{2,i}\left(V(y_\varepsilon^i) - V(a_i)\right)\varepsilon^{2+3q} + O(\varepsilon^{4+3q})\Big(\varepsilon^{q\theta-\tau} + \sum_{i=1}^k \left(V(y_\varepsilon^i) - V(a_i)\right)^{1-\tau}\Big)^2 \\ \leq O(\varepsilon^{2+3q+q\theta}). \end{aligned}$$

That is,

$$\sum_{i=1}^k C_{2,i}\left(V(y_\varepsilon^i) - V(a_i)\right) + O(\varepsilon^2)\Big(\varepsilon^{q\theta-\tau} + \sum_{i=1}^k \left(V(y_\varepsilon^i) - V(a_i)\right)^{1-\tau}\Big)^2 \leq O(\varepsilon^{q\theta}).$$

If $Y_\varepsilon \in \partial D_\delta^k$, then by the condition (V_2), we have

$$V(y_\varepsilon^i) - V(a_i) \geq \widetilde{c}_i > 0$$

for some constants $\widetilde{c}_i > 0$ $(i = 1, 2, \cdots, k)$. Notice that $C_{2,i} \to C_{2,i}^* > 0$, then letting $\varepsilon \to 0$, we deduce that

$$0 < \sum_{i=1}^k C_{2,i}^* \widetilde{c}_i \leq 0,$$

a contradiction. This proves the claim. Therefore, $Y_\varepsilon = (y_\varepsilon^1, y_\varepsilon^2, \cdots, y_\varepsilon^k)$ is a critical point of j_ε in the interior of D_δ^k. Theorem 1.2 follows directly from this claim and Lemma 3.1. □

5. Proof of Theorem 1.3

In this section, we prove the local uniqueness result of multi-peak solutions to Eq. (1.6). We use a contradiction argument as that of Cao, Li and Luo [7]. Assume $w_\varepsilon^{(j)} = \sum_{i=1}^k v^i_{\varepsilon, y_\varepsilon^{i(j)}} + \varphi_\varepsilon^{(j)}$ $(j = 1, 2)$ are two distinct solutions concentrating at $\{a_1, \cdots, a_k\} \subseteq \mathbb{R}^3$ derived as in Theorem 1.2. Set

$$\xi_\varepsilon = \frac{w_\varepsilon^{(1)} - w_\varepsilon^{(2)}}{\|w_\varepsilon^{(1)} - w_\varepsilon^{(2)}\|_{L^\infty(\mathbb{R}^3)}}$$

and set $\widetilde{\xi}_\varepsilon(x) = \xi_\varepsilon(\varepsilon^q x + y_\varepsilon^{i_0(1)})$ for an arbitrary $i_0 \in \{1, \cdots, k\}$, where $q = 2 - \frac{4}{p-1}$.

In the following, i_0 will be fixed. It is clear that

$$\|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1 \quad \text{and} \quad \|\widetilde{\xi}_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1.$$

To deduce a contradiction, in the rest of this section, we will show

$$\|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = o_\varepsilon(1). \tag{5.1}$$

To prove (5.1), we will first prove that $|\xi_\varepsilon| \to 0$ holds in $\cup_{i=1}^k B_{R\varepsilon^q}(y_\varepsilon^{i(1)})$ with respect to sufficiently small $\varepsilon > 0$ (see Proposition 5.5), and then verify that it holds outside $\cup_{i=1}^k B_{R\varepsilon^q}(y_\varepsilon^{i(1)})$ (see Proposition 5.6). To this end, we will establish a series of results. First, we explore some properties of the solutions derived as in Theorem 1.2.

Proposition 5.1. *Assume that $V(x)$ satisfies (V_1), (V_2) and (V_3). Let*

$$w_\varepsilon(x) = W_{\varepsilon,Y_\varepsilon} + \varphi_\varepsilon = \sum_{i=1}^k v^i_{\varepsilon,y^i_\varepsilon} + \varphi_\varepsilon$$

be a solution derived as in Theorem 1.2. Then we have:

(i) *$\varphi_\varepsilon \in H_\varepsilon$ satisfies* $$\|\varphi_\varepsilon\|_\varepsilon = \varepsilon^{2+\frac{3q}{2}} O\Big(\varepsilon^{qm} + \sum_{i=1}^k |y^i_\varepsilon - a_i|^m\Big). \tag{5.2}$$

(ii) *Furthermore,* $$|y^i_\varepsilon - a_i| = o(\varepsilon^q), \tag{5.3}$$

for $i = 1, 2, \cdots, k$. Then a refinement of estimate (5.2) is given by

$$\|\varphi_\varepsilon\|_\varepsilon = O\left(\varepsilon^{2+\frac{3q}{2}+qm}\right); \tag{5.4}$$

(iii) *w_ε concentrates at $\{a_1, a_2, \cdots, a_k\}$ and decays exponentially at infinity.*

Proof. (i) By the same argument of (3.9), it yields

$$\begin{aligned} &\int_{\mathbb{R}^3} \big(\varepsilon^2 a \nabla W_{\varepsilon,Y_\varepsilon} \cdot \nabla\varphi_\varepsilon + W_{\varepsilon,Y_\varepsilon}\varphi_\varepsilon\big) + \varepsilon^2 \int_{\mathbb{R}^3} V(x) W_{\varepsilon,Y_\varepsilon}\varphi_\varepsilon \\ &= -bc_\varepsilon\varepsilon^{2q} \int_{\mathbb{R}^3} \nabla W_{\varepsilon,Y_\varepsilon} \cdot \nabla\varphi_\varepsilon + \varepsilon^2 \sum_{i=1}^k \int_{\mathbb{R}^3} (V(x) - V(a_i)) v^i_{\varepsilon,y^i_\varepsilon}\varphi_\varepsilon + \sum_{i=1}^k \int_{\mathbb{R}^3} \left(v^i_{\varepsilon,y^i_\varepsilon}\right)^p \varphi_\varepsilon. \end{aligned} \tag{5.5}$$

Since w_ε is a solution of Eq. (1.6), we have

$$\begin{aligned} &\int_{\mathbb{R}^3} \big(\varepsilon^2 a \nabla W_{\varepsilon,Y_\varepsilon} \cdot \nabla\varphi_\varepsilon + W_{\varepsilon,Y_\varepsilon}\varphi_\varepsilon\big) + \varepsilon^2 \int_{\mathbb{R}^3} V(x) W_{\varepsilon,Y_\varepsilon}\varphi_\varepsilon \\ &= -\int_{\mathbb{R}^3} \big(\varepsilon^2 a|\nabla\varphi_\varepsilon|^2 + \varphi_\varepsilon^2\big) - \varepsilon^2 \int_{\mathbb{R}^3} V(x)\varphi_\varepsilon^2 - b\varepsilon^q \int_{\mathbb{R}^3} |\nabla W_{\varepsilon,Y_\varepsilon}|^2 \int_{\mathbb{R}^3} \nabla W_{\varepsilon,Y_\varepsilon} \cdot \nabla\varphi_\varepsilon \\ &\quad - 2b\varepsilon^q \left(\int_{\mathbb{R}^3} \nabla W_{\varepsilon,Y_\varepsilon} \cdot \nabla\varphi_\varepsilon\right)^2 - 3b\varepsilon^q \int_{\mathbb{R}^3} \nabla W_{\varepsilon,Y_\varepsilon} \cdot \nabla\varphi_\varepsilon \int_{\mathbb{R}^3} |\nabla\varphi_\varepsilon|^2 \\ &\quad - b\varepsilon^q \int_{\mathbb{R}^3} |\nabla W_{\varepsilon,Y_\varepsilon}|^2 \int_{\mathbb{R}^3} |\nabla\varphi_\varepsilon|^2 - b\varepsilon^q \left(\int_{\mathbb{R}^3} |\nabla\varphi_\varepsilon|^2\right)^2 + \int_{\mathbb{R}^3} (W_{\varepsilon,Y_\varepsilon} + \varphi_\varepsilon)^p \varphi_\varepsilon. \end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6) implies that

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left(\varepsilon^2 a|\nabla\varphi_\varepsilon|^2+\varphi_\varepsilon^2\right)+\varepsilon^2\int_{\mathbb{R}^3}V(x)\varphi_\varepsilon^2+2b\varepsilon^q\left(\int_{\mathbb{R}^3}\nabla W_{\varepsilon,Y_\varepsilon}\cdot\nabla\varphi_\varepsilon\right)^2 \qquad (5.7)\\
&\quad+b\varepsilon^q\int_{\mathbb{R}^3}|\nabla W_{\varepsilon,Y_\varepsilon}|^2\int_{\mathbb{R}^3}|\nabla\varphi_\varepsilon|^2-p\int_{\mathbb{R}^3}W_{\varepsilon,Y_\varepsilon}^{p-1}\varphi_\varepsilon^2\\
&=-b\varepsilon^q\int_{\mathbb{R}^3}|\nabla W_{\varepsilon,Y_\varepsilon}|^2\int_{\mathbb{R}^3}\nabla W_{\varepsilon,Y_\varepsilon}\cdot\nabla\varphi_\varepsilon-3b\varepsilon^q\int_{\mathbb{R}^3}\nabla W_{\varepsilon,Y_\varepsilon}\cdot\nabla\varphi_\varepsilon\int_{\mathbb{R}^3}|\nabla\varphi_\varepsilon|^2\\
&\quad-b\varepsilon^q\left(\int_{\mathbb{R}^3}|\nabla\varphi_\varepsilon|^2\right)^2+\int_{\mathbb{R}^3}(W_{\varepsilon,Y_\varepsilon}+\varphi_\varepsilon)^p\varphi_\varepsilon-p\int_{\mathbb{R}^3}W_{\varepsilon,Y_\varepsilon}^{p-1}\varphi_\varepsilon^2\\
&\quad+bc_\varepsilon\varepsilon^{2q}\int_{\mathbb{R}^3}\nabla W_{\varepsilon,Y_\varepsilon}\cdot\nabla\varphi_\varepsilon-\varepsilon^2\sum_{i=1}^k\int_{\mathbb{R}^3}(V(x)-V(a_i))v^i_{\varepsilon,y^i_\varepsilon}\varphi_\varepsilon-\sum_{i=1}^k\int_{\mathbb{R}^3}\left(v^i_{\varepsilon,y^i_\varepsilon}\right)^p\varphi_\varepsilon.
\end{aligned}$$

From (3.4) and (5.7), we obtain

$$\begin{aligned}
\langle\mathcal{I}_\varepsilon\varphi_\varepsilon,\varphi_\varepsilon\rangle&=\left(bc_\varepsilon\varepsilon^{2q}-b\varepsilon^q\int_{\mathbb{R}^3}|\nabla W_{\varepsilon,Y_\varepsilon}|^2\right)\int_{\mathbb{R}^3}\nabla W_{\varepsilon,Y_\varepsilon}\cdot\nabla\varphi_\varepsilon \qquad (5.8)\\
&\quad-3b\varepsilon^q\int_{\mathbb{R}^3}\nabla W_{\varepsilon,Y_\varepsilon}\cdot\nabla\varphi_\varepsilon\int_{\mathbb{R}^3}|\nabla\varphi_\varepsilon|^2-b\varepsilon^q\left(\int_{\mathbb{R}^3}|\nabla\varphi_\varepsilon|^2\right)^2\\
&\quad+\int_{\mathbb{R}^3}(W_{\varepsilon,Y_\varepsilon}+\varphi_\varepsilon)^p\varphi_\varepsilon-p\int_{\mathbb{R}^3}W_{\varepsilon,Y_\varepsilon}^{p-1}\varphi_\varepsilon^2-\sum_{i=1}^k\int_{\mathbb{R}^3}\left(v^i_{\varepsilon,y^i_\varepsilon}\right)^p\varphi_\varepsilon\\
&\quad-\varepsilon^2\sum_{i=1}^k\int_{\mathbb{R}^3}(V(x)-V(a_i))v^i_{\varepsilon,y^i_\varepsilon}\varphi_\varepsilon=:\mathcal{P}_1-\mathcal{P}_2+\mathcal{P}_3-\mathcal{P}_4.
\end{aligned}$$

Similar to the argumention in the procedure of estimates of l_1 in Lemma 3.2, we get

$$|\mathcal{P}_1|=O\left(\varepsilon^{-\frac{q}{2}}e^{-\frac{\gamma_0}{\varepsilon^q}}\right)\|\varphi_\varepsilon\|_\varepsilon.$$

To estimate $\mathcal{P}_2$, just notice that

$$\left(\int_{\mathbb{R}^3}|\nabla\varphi_\varepsilon|^2\right)^2=O(\varepsilon^{-4q}\|\varphi_\varepsilon\|_\varepsilon^2),$$

and by Hölder's inequality that

$$\varepsilon^q\int_{\mathbb{R}^3}\nabla W_{\varepsilon,Y_\varepsilon}\cdot\nabla\varphi_\varepsilon\int_{\mathbb{R}^3}|\nabla\varphi_\varepsilon|^2=O\left(\varepsilon^{-\frac{3q}{2}}\|\varphi_\varepsilon\|_\varepsilon^3\right).$$

Hence, we deduce that $|\mathcal{P}_2|=O\left(\varepsilon^{-\frac{3q}{2}}\|\varphi_\varepsilon\|_\varepsilon^3+\varepsilon^{-3q}\|\varphi_\varepsilon\|_\varepsilon^2\right)$.

To estimate $\mathcal{P}_3$, applying Hölder's inequality and (2.8), we have

$$\int_{\mathbb{R}^3}W_{\varepsilon,Y_\varepsilon}^{p-1}\varphi_\varepsilon^2\le\left(\int_{\mathbb{R}^3}W_{\varepsilon,Y_\varepsilon}^{p+1}\right)^{\frac{p-1}{p+1}}\left(\int_{\mathbb{R}^3}|\varphi_\varepsilon|^{p+1}\right)^{\frac{2}{p+1}}\le C\|\varphi_\varepsilon\|_\varepsilon^2,$$

and $$\int_{\mathbb{R}^3}|\varphi_\varepsilon|^{p+1}\le C\varepsilon^{q\left(\frac{3}{p+1}-\frac{3}{2}\right)(p+1)}\|\varphi_\varepsilon\|_\varepsilon^{p+1}=C\varepsilon^{-3(p-3)}\|\varphi_\varepsilon\|_\varepsilon^{p+1}\le C\|\varphi_\varepsilon\|_\varepsilon^2.$$

Hence $\mathcal{P}_3=O(\|\varphi_\varepsilon\|_\varepsilon^2)$.

To estimate $\mathcal{P}_4$, we use the assumption (V_3). Just consider for each $1 \le i \le k$ and any fixed $d < 1/4 \min\{R_0, \eta\}$, that

$$\int_{\mathbb{R}^3} (V(x) - V(a_i)) v^i_{\varepsilon, y^i_\varepsilon} \varphi_\varepsilon = \int_{B_d(y^i_\varepsilon)} (V(x) - V(a_i)) v^i_{\varepsilon, y^i_\varepsilon} \varphi_\varepsilon + \int_{B^c_d(y^i_\varepsilon)} (V(x) - V(a_i)) v^i_{\varepsilon, y^i_\varepsilon} \varphi_\varepsilon$$
$$=: \mathcal{P}_{41} + \mathcal{P}_{42}.$$

Combining the assumption (V_3) and (2.8), we derive by Hölder's inequality that

$$\begin{aligned}
|\mathcal{P}_{41}| &= \left| \int_{B_d(y^i_\varepsilon)} (V(x) - V(a_i)) v^i_{\varepsilon, y^i_\varepsilon} \varphi_\varepsilon \right| \le C \int_{B_d(y^i_\varepsilon)} |x - a_i|^m v^i_{\varepsilon, y^i_\varepsilon} |\varphi_\varepsilon| \\
&\le C \left(\int_{B_d(y^i_\varepsilon)} |x - a_i|^{2m} \left(v^i_{\varepsilon, y^i_\varepsilon} \right)^2 \right)^{\frac{1}{2}} \left(\int_{B_d(y^i_\varepsilon)} \varphi^2_\varepsilon \right)^{\frac{1}{2}} \\
&\le C \varepsilon^{\frac{3q}{2}} \left(\int_{B_{\frac{d}{\varepsilon^q}}(0)} |\varepsilon^q z + y^i_\varepsilon - a_i|^{2m} \left(v^i(z) \right)^2 \right)^{\frac{1}{2}} \|\varphi_\varepsilon\|_\varepsilon \\
&\le C \varepsilon^{\frac{3q}{2}} \left(\varepsilon^{qm} + |y^i_\varepsilon - a_i|^m \right) \|\varphi_\varepsilon\|_\varepsilon.
\end{aligned}$$

Meanwhile, by the exponential decay of $v^i_{\varepsilon, y^i_\varepsilon}$ and the assumption (V_1), we deduce from Lemma A.1 that, for any $\gamma > 0$,

$$|\mathcal{P}_{42}| = \left| \int_{B^c_d(y^i_\varepsilon)} (V(x) - V(a_i)) v^i_{\varepsilon, y^i_\varepsilon} \varphi_\varepsilon \right| \le C \varepsilon^\gamma \|\varphi_\varepsilon\|_\varepsilon.$$

Hence
$$|\mathcal{P}_4| \le C \varepsilon^{2 + \frac{3q}{2}} \left(\varepsilon^{qm} + \sum_{i=1}^k |y^i_\varepsilon - a_i|^m \right) \|\varphi_\varepsilon\|_\varepsilon.$$

Combining the above estimates of $\mathcal{P}_1$–$\mathcal{P}_4$ and (5.8), we obtain

$$|\langle \mathcal{I}_\varepsilon \varphi_\varepsilon, \varphi_\varepsilon \rangle| = \varepsilon^{2 + \frac{3q}{2}} O \left(\varepsilon^{qm} + \sum_{i=1}^k |y^i_\varepsilon - a_i|^m \right) \|\varphi_\varepsilon\|_\varepsilon.$$

From Lemma 4.1, we also have

$$\rho \|\varphi_\varepsilon\|^2_\varepsilon \le |\langle \mathcal{I}_\varepsilon \varphi_\varepsilon, \varphi_\varepsilon \rangle| = \varepsilon^{2 + \frac{3q}{2}} O \left(\varepsilon^{qm} + \sum_{i=1}^k |y^i_\varepsilon - a_i|^m \right) \|\varphi_\varepsilon\|_\varepsilon,$$

which implies that
$$\|\varphi_\varepsilon\|_\varepsilon = \varepsilon^{2 + \frac{3q}{2}} O \left(\varepsilon^{qm} + \sum_{i=1}^k |y^i_\varepsilon - a_i|^m \right).$$

The proof of (i) is complete.

(ii) It is sufficient to verify (5.3) for fixed $i \in \{1, 2, \cdots, k\}$ with i satisfying

$$|y^i_\varepsilon - a_i| = \max\{|y^j_\varepsilon - a_j| : 1 \le j \le k\}.$$

(In other words, it is also enough to verify (5.3) for each fixed i.)

Recall the local Pohozaev identity derived as in Proposition 2.4, that is,

$$\varepsilon^2\int_\Omega \frac{\partial V}{\partial x_\alpha}w^2 = \left(a\varepsilon^2 + b\varepsilon^q\int_{\mathbb{R}^3}|\nabla w|^2\right)\int_{\partial\Omega}\left(|\nabla w|^2\nu_\alpha - 2\frac{\partial w}{\partial \nu}\frac{\partial w}{\partial x_\alpha}\right) + \int_{\partial\Omega}\left(1+\varepsilon^2V(x)\right)w^2\nu_\alpha - \frac{2}{p+1}\int_{\partial\Omega}w^{p+1}\nu_\alpha$$

for each $\alpha = 1,2,3$, where $\nu = (\nu_1,\nu_2,\nu_3)$ is the unit outward normal of $\partial\Omega$. We will use the Pohozaev identity to w_ε with $\Omega = B_d(y_\varepsilon^i)$ to deduce the conclusion (5.3). We choose the radius d as follows.

Notice that a_i are distinct points, let $d_1 = \min_{i\neq j}\{1, |a_i - a_j|/4\}$ and $d_2 = d_1/4$. For $\varepsilon > 0$ sufficiently small, $|y_\varepsilon^i - a_i| < d_2$, i.e., $B_{d_2}(y_\varepsilon^i) \subset B_{d_1}(a_i)$ and $|y_\varepsilon^i - y_\varepsilon^j| > 2d_1$. Here, we also point out that d_2 can be chosen as small as we wish. Set $f = \varepsilon^{2q}|\nabla\varphi_\varepsilon|^2 + |\varphi_\varepsilon|^2 + |\varphi_\varepsilon|^{p+1}$. Using the formula of polar coordinates, we have

$$\int_0^{d_2}\int_{\partial B_r(y_\varepsilon^i)} f = \int_{B_{d_2}(y_\varepsilon^i)} f,$$

we can choose $d \in (0, d_3)$, where $d_3 = \min\{d_2, d_0\}$ and d_0 is defined as in Lemma A.1, such that

$$\int_{\partial B_d(y_\varepsilon^i)}\left(\varepsilon^{2q}|\nabla\varphi_\varepsilon|^2 + |\varphi_\varepsilon|^2 + |\varphi_\varepsilon|^{p+1}\right) = O(\|\varphi_\varepsilon\|_\varepsilon^2). \tag{5.9}$$

It is obvious that d can be chosen as small as we wish. Applying the local Pohozaev identity (2.9) derived in Proposition 2.4 to w_ε with $\Omega = B_d(y_\varepsilon^i)$, we obtain

$$\varepsilon^2\int_{B_d(y_\varepsilon^i)}\frac{\partial V}{\partial x_\alpha}w_\varepsilon^2 = \left(\varepsilon^2 a + \varepsilon^q b\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2\right)\int_{\partial B_d(y_\varepsilon^i)}\left(|\nabla w_\varepsilon|^2\nu_\alpha - 2\frac{\partial w_\varepsilon}{\partial\nu}\frac{\partial w_\varepsilon}{\partial x_\alpha}\right) + \int_{\partial B_d(y_\varepsilon^i)}\left(1+\varepsilon^2V(x)\right)w_\varepsilon^2\nu_\alpha - \frac{2}{p+1}\int_{\partial B_d(y_\varepsilon^i)}w_\varepsilon^{p+1}\nu_\alpha =: \sum_{i=1}^3 I_i, \tag{5.10}$$

where
$$I_1 = \left(\varepsilon^2 a + \varepsilon^q b\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2\right)\int_{\partial B_d(y_\varepsilon^i)}\left(|\nabla w_\varepsilon|^2\nu_\alpha - 2\frac{\partial w_\varepsilon}{\partial\nu}\frac{\partial w_\varepsilon}{\partial x_\alpha}\right),$$
$$I_2 = \int_{\partial B_d(y_\varepsilon^i)}\left(1+\varepsilon^2V(x)\right)w_\varepsilon^2\nu_\alpha \text{ and } I_3 = -\frac{2}{p+1}\int_{\partial B_d(y_\varepsilon^i)}w_\varepsilon^{p+1}\nu_\alpha.$$

We estimate the left and right hand side of Eq. (5.10), respectively.

On one hand, for the right hand side of Eq. (5.10), we estimate I_i $(i = 1,2,3)$ term by term. To estimate I_1, notice that

$$\varepsilon^2 a + \varepsilon^q b\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2 = O(\varepsilon^{2q}),$$

and then by Lemma A.2, we obtain for any $\gamma > 0$

$$|I_1| = \left(\varepsilon^2 a + \varepsilon^q b\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2\right)\left|\int_{\partial B_d(y_\varepsilon^i)}\left(|\nabla w_\varepsilon|^2\nu_\alpha - 2\frac{\partial w_\varepsilon}{\partial\nu}\frac{\partial w_\varepsilon}{\partial x_\alpha}\right)\right|$$

$$\begin{aligned}
&\le C\varepsilon^{2q}\left|\int_{\partial B_d(y_\varepsilon^i)}\left(|\nabla w_\varepsilon|^2\nu_\alpha - 2\frac{\partial w_\varepsilon}{\partial\nu}\frac{\partial w_\varepsilon}{\partial x_\alpha}\right)\right| \\
&\le C\varepsilon^{2q}\int_{\partial B_d(y_\varepsilon^i)}\left(\sum_{i=1}^k |\nabla v^i_{\varepsilon,y_\varepsilon^i}|^2 + |\nabla\varphi_\varepsilon|^2\right) \\
&= O(\varepsilon^\gamma + \|\varphi_\varepsilon\|_\varepsilon^2)
\end{aligned}$$

Similarly, we derive by the assumption (V_1) and Lemma A.2 that

$$I_2 = \int_{\partial B_d(y_\varepsilon^i)}\left(1+\varepsilon^2 V(x)\right) w_\varepsilon^2 \nu_\alpha \le C\int_{\partial B_d(y_\varepsilon^i)} w_\varepsilon^2 = O(\varepsilon^\gamma + \|\varphi_\varepsilon\|_\varepsilon^2).$$

Furthermore, by (5.9), we have

$$\begin{aligned}
|I_3| = \frac{2}{p+1}\left|\int_{\partial B_d(y_\varepsilon^i)} w_\varepsilon^{p+1}\nu_\alpha\right| &\le C\int_{\partial B_d(y_\varepsilon^i)}\left(\sum_{l=1}^k\left(v^l_{\varepsilon,y_\varepsilon^l}\right)^{p+1} + |\varphi_\varepsilon|^{p+1}\right) \\
&= O(\varepsilon^\gamma + \|\varphi_\varepsilon\|_\varepsilon^2).
\end{aligned}$$

Combining the above estimates, we conclude that, for any given $\gamma > 0$,

$$\varepsilon^2\int_{B_d(y_\varepsilon^i)}\frac{\partial V}{\partial x_\alpha} w_\varepsilon^2 = O(\varepsilon^\gamma + \|\varphi_\varepsilon\|_\varepsilon^2). \tag{5.11}$$

On the other hand, we estimate the left hand side of Eq. (5.10). Notice that

$$\int_{B_d(y_\varepsilon^i)}\frac{\partial V}{\partial x_\alpha}w_\varepsilon^2 = \int_{B_d(y_\varepsilon^i)}\frac{\partial V}{\partial x_\alpha}\left(v^i_{\varepsilon,y_\varepsilon^i}\right)^2 + 2\int_{B_d(y_\varepsilon^i)}\frac{\partial V}{\partial x_\alpha}v^i_{\varepsilon,y_\varepsilon^i}\varphi_\varepsilon + O(\varepsilon^\gamma + \|\varphi_\varepsilon\|_\varepsilon^2) \tag{5.12}$$

for any given $\gamma > 0$. By the assumption (V_3) and Hölder's inequality, there holds

$$\begin{aligned}
\left|\int_{B_d(y_\varepsilon^i)}\frac{\partial V}{\partial x_\alpha}v^i_{\varepsilon,y_\varepsilon^i}\varphi_\varepsilon\right| &\le C\int_{B_d(y_\varepsilon^i)}|x-a_i|^{m-1}v^i_{\varepsilon,y_\varepsilon^i}|\varphi_\varepsilon| \\
&\le C\left(\int_{B_d(y_\varepsilon^i)}|x-a_i|^{2(m-1)}\left(v^i_{\varepsilon,y_\varepsilon^i}\right)^2\right)^{\frac12}\left(\int_{B_d(y_\varepsilon^i)}\varphi_\varepsilon^2\right)^{\frac12} \\
&\le C\varepsilon^{\frac{3q}{2}}\left(\int_{B_{\frac{d}{\varepsilon^q}}(0)}|\varepsilon^q z + y_\varepsilon^i - a_i|^{2(m-1)}\left(v^i(z)\right)^2\right)^{\frac12}\|\varphi_\varepsilon\|_\varepsilon \\
&\le C\varepsilon^{\frac{3q}{2}}\left(\varepsilon^{q(m-1)} + |y_\varepsilon^i - a_i|^{m-1}\right)\|\varphi_\varepsilon\|_\varepsilon.
\end{aligned} \tag{5.13}$$

Similarly, by assumption (V_3), we deduce

$$\begin{aligned}
\int_{B_d(y_\varepsilon^i)}\frac{\partial V}{\partial x_\alpha}\left(v^i_{\varepsilon,y_\varepsilon^i}\right)^2 = mc_{i,\alpha}&\int_{B_d(y_\varepsilon^i)}|x_\alpha - a_{i,\alpha}|^{m-2}(x_\alpha - a_{i,\alpha})\left(v^i_{\varepsilon,y_\varepsilon^i}\right)^2 \\
&+ O\left(\int_{B_d(y_\varepsilon^i)}|x_\alpha - a_{i,\alpha}|^m\left(v^i_{\varepsilon,y_\varepsilon^i}\right)^2\right)
\end{aligned}$$

$$= mc_{i,\alpha}\varepsilon^{3q}\int_{B_{\frac{d}{\varepsilon^q}}(0)}|\varepsilon^q x_\alpha + y^i_{\varepsilon,\alpha} - a_{i,\alpha}|^{m-2}(\varepsilon^q x_\alpha + y^i_{\varepsilon,\alpha} - a_{i,\alpha})\left(v^i\right)^2$$
$$+\,\varepsilon^{3q}O\left(\varepsilon^{qm} + |y^i_\varepsilon - a_i|^m\right), \tag{5.14}$$

where $y^i_{\varepsilon,\alpha}$ is the α-th exponent of y^i_ε. From (5.2), we have

$$\|\varphi_\varepsilon\|_\varepsilon = \varepsilon^{2+\frac{3q}{2}}O\left(\varepsilon^{qm} + |y^i_\varepsilon - a_i|^m\right). \tag{5.15}$$

Therefore, combining (5.11)–(5.15) yields

$$\int_{B_{\frac{d}{\varepsilon^q}}(0)}|\varepsilon^q x_\alpha + y^i_{\varepsilon,\alpha} - a_{i,\alpha}|^{m-2}(\varepsilon^q x_\alpha + y^i_{\varepsilon,\alpha} - a_{i,\alpha})\left(v^i\right)^2 = O\left(\varepsilon^{qm} + |y^i_\varepsilon - a_i|^m\right). \tag{5.16}$$

Note that, for any $e, f \in \mathbb{R}^1, m > 1, m^* = \min\{m, 2\}$, we have the inequality

$$\big||e+f|^m - |e|^m - m|e|^{m-2}ef\big| \le C\left(|e|^{m-m^*}|f|^{m^*} + |f|^m\right) \tag{5.17}$$

where the constant C is independent of e, f. Applying (5.17) to $e = \varepsilon^q x_\alpha + y^i_{\varepsilon,\alpha} - a_{i,\alpha}$ and $f = -\varepsilon^q x_\alpha$, we have

$$|y^i_{\varepsilon,\alpha} - a_{i,\alpha}|\int_{B_{\frac{d}{\varepsilon^q}}(0)}|\varepsilon^q x_\alpha + y^i_{\varepsilon,\alpha} - a_{i,\alpha}|^{m-2}(\varepsilon^q x_\alpha + y^i_{\varepsilon,\alpha} - a_{i,\alpha})\left(v^i\right)^2$$
$$\ge \frac{1}{m}|y^i_{\varepsilon,\alpha} - a_{i,\alpha}|^m\int_{B_{\frac{d}{\varepsilon^q}}(0)}\left(v^i\right)^2 \tag{5.18}$$
$$-\frac{C}{m}\int_{B_{\frac{d}{\varepsilon^q}}(0)}\left(|\varepsilon^q x_\alpha|^m + |\varepsilon^q x_\alpha|^{m^*}|y^i_{\varepsilon,\alpha} - a_{i,\alpha}|^{m-m^*}\right)\left(v^i\right)^2.$$

Take some $\alpha = i_0 \in \{1,2,3\}$ such that

$$|y^i_{\varepsilon,i_0} - a_{i,i_0}| \ge \frac{|y^i_\varepsilon - a_i|}{\sqrt{3}}.$$

Note also that $|y^i_{\varepsilon,i_0} - a_{i,i_0}| \le |y^i_\varepsilon - a_i|$. Thus, combined with (5.16) and (5.18), we get

$$|y^i_\varepsilon - a_i|^m = |y^i_\varepsilon - a_i|O\left(\varepsilon^{qm} + |y^i_\varepsilon - a_i|^m\right) + O\left(\varepsilon^{qm} + |y^i_\varepsilon - a_i|^{m-m^*}\varepsilon^{m^*}\right)$$
$$= |y^i_\varepsilon - a_i|O\left(\varepsilon^{qm} + |y^i_\varepsilon - a_i|^m\right) + O\left(\varepsilon^{qm}\right) + \frac{1}{2}|y^i_\varepsilon - a_i|^m,$$

which implies that
$$|y^i_\varepsilon - a_i| = O(\varepsilon^q).$$

Then, from (5.16), we have

$$\int_{B_{\frac{d}{\varepsilon^q}}(0)}\left|x_\alpha + \frac{y^i_{\varepsilon,\alpha} - a_{i,\alpha}}{\varepsilon^q}\right|^{m-2}\left(x_\alpha + \frac{y^i_{\varepsilon,\alpha} - a_{i,\alpha}}{\varepsilon^q}\right)\left(v^i\right)^2 = O(\varepsilon^q).$$

Since $|y^i_\varepsilon - a_i| = O(\varepsilon^q)$, we suppose that $\frac{y^i_\varepsilon - a_i}{\varepsilon^q} \to t$ as $\varepsilon \to 0$. Then we have

$$\int_{\mathbb{R}^3}|x_\alpha + t_\alpha|^{m-2}(x_\alpha + t_\alpha)\left(U^i\right)^2 = 0.$$

where t_α is the α-th component of t for $\alpha = 1, 2, 3$.

Notice that U^i is radially symmetric and decreasing, then we conclude that $t = 0$. From this, we deduce (5.3). Then, combining (5.2) and (5.3) yields (5.4). The proof of (ii) is complete.

(iii) By Theorem 1.2, we know that $\|\varphi_\varepsilon\|_\varepsilon = O(\varepsilon^{2+\frac{3q}{2}})$. Thus, a straightforward computation gives

$$\|w_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{3q}{2}}). \tag{5.19}$$

For any fixed $i \in \{1, 2, \cdots, k\}$, set $\widetilde{w}_\varepsilon(x) = w_\varepsilon(\varepsilon^q x + y^i_\varepsilon)$.

Then $\widetilde{w}_\varepsilon > 0$ solves

$$-\left(a\varepsilon^{2-2q} + b\int_{\mathbb{R}^3} |\nabla \widetilde{w}_\varepsilon|^2\right)\Delta \widetilde{w}_\varepsilon + \left(1 + \varepsilon^2 V(\varepsilon^q x + y^i_\varepsilon)\right)\widetilde{w}_\varepsilon = \widetilde{w}^p_\varepsilon \quad \text{in } \mathbb{R}^3. \tag{5.20}$$

Moreover, there holds by (5.19)

$$\int_{\mathbb{R}^3} \left(a|\nabla \widetilde{w}_\varepsilon|^2 + \widetilde{w}^2_\varepsilon\right) = \varepsilon^{-3q}\|w_\varepsilon\|^2_\varepsilon = O(1). \tag{5.21}$$

By the assumption (V_1), $(1 + \varepsilon^2 V(\varepsilon^q x + y^i_\varepsilon))$ is bounded uniformly with respect to ε, and

$$\inf_{x\in\mathbb{R}^3} \left(1 + \varepsilon^2 V(\varepsilon^q x + y^i_\varepsilon)\right) > 1.$$

Therefore, $\widetilde{w}_\varepsilon$ satisfies

$$-B_\varepsilon \Delta \widetilde{w}_\varepsilon + \widetilde{w}_\varepsilon \le \widetilde{w}^p_\varepsilon \quad \text{in } \mathbb{R}^3, \quad \text{and} \quad \sup_\varepsilon \|\widetilde{w}_\varepsilon\|_{H^1(\mathbb{R}^3)} \le C < +\infty,$$

where $B_\varepsilon = \left(a\varepsilon^{2-2q} + b\int_{\mathbb{R}^3} |\nabla \widetilde{w}_\varepsilon|^2\right) \to bc_* > 0$ as $\varepsilon \to 0$.

Using the comparison principal, we infer that

$$\widetilde{w}_\varepsilon(x) \le Ce^{-\kappa|x|}, \quad x \in \mathbb{R}^3 \tag{5.22}$$

holds for some constants $C, \kappa > 0$ independent of $\varepsilon > 0$.

Notice that (5.22) is equivalent to

$$w_\varepsilon(x) \le Ce^{-\frac{\kappa|x-y^i_\varepsilon|}{\varepsilon^q}}, \quad x \in \mathbb{R}^3,$$

which means that w_ε concentrates at a_i rapidly as $\varepsilon \to 0$. In particular, under the additional assumption (V_3), we have estimate (5.3), that is $|y^i_\varepsilon - a_i| = o(\varepsilon^q)$, which in turn implies that

$$w_\varepsilon(x) \le Ce^{-\frac{\kappa|x-a_i+a_i-y^i_\varepsilon|}{\varepsilon^q}} \le Ce^{-\frac{\kappa|x-a_i|-\kappa|a_i-y^i_\varepsilon|}{\varepsilon^q}} \le Ce^{-\frac{\kappa|x-a_i|}{\varepsilon^q}}, \quad x \in \mathbb{R}^3.$$

This shows that the solutions concentrate around the minima of $V(x)$.

Furthermore, the standard potential theory and iteration arguments shows that $\widetilde{w}_\varepsilon \in L^\infty(\mathbb{R}^3)$ and

$$\|\widetilde{w}_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \le C < +\infty \tag{5.23}$$

holds for some constant $C > 0$ uniformly with respect to ε.

As a consequence of this estimate (5.23) and the assumption (V_1), we infer from Eq. (5.20) that

$$\|\Delta\widetilde{w}_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \le C \tag{5.24}$$

holds for some constant $C > 0$ uniformly with respect to $\varepsilon > 0$. The proof of (iii) is complete. □

Proposition 5.2. *We have*

$$\|\xi_\varepsilon\|_\varepsilon = O(\varepsilon^{\frac{3q}{2}}). \tag{5.25}$$

Proof. Since $w_\varepsilon^{(j)}$ $(j = 1, 2)$ are assumed to satisfy Eq. (1.6), we obtain

$$\begin{aligned}-&\left(\varepsilon^2 a + \varepsilon^q b\int_{\mathbb{R}^3}|\nabla w_\varepsilon^{(1)}|^2\right)\Delta\xi_\varepsilon + (1+\varepsilon^2 V(x))\xi_\varepsilon\\ &- \varepsilon^q b\left(\int_{\mathbb{R}^3}\nabla(w_\varepsilon^{(1)} + w_\varepsilon^{(2)})\cdot\nabla\xi_\varepsilon\right)\Delta w_\varepsilon^{(2)} = C_\varepsilon(x)\xi_\varepsilon\end{aligned} \tag{5.26}$$

and

$$\begin{aligned}-&\left(\varepsilon^2 a + \varepsilon^q b\int_{\mathbb{R}^3}|\nabla w_\varepsilon^{(2)}|^2\right)\Delta\xi_\varepsilon + (1+\varepsilon^2 V(x))\xi_\varepsilon\\ &- \varepsilon^q b\left(\int_{\mathbb{R}^3}\nabla(w_\varepsilon^{(1)} + w_\varepsilon^{(2)})\cdot\nabla\xi_\varepsilon\right)\Delta w_\varepsilon^{(1)} = C_\varepsilon(x)\xi_\varepsilon,\end{aligned} \tag{5.27}$$

where

$$C_\varepsilon(x) = p\int_0^1 (tw_\varepsilon^{(1)}(x) + (1-t)w_\varepsilon^{(2)}(x))^{p-1}\mathrm{d}t.$$

Combining (5.26) with (5.27), we have

$$\begin{aligned}&-\left(2\varepsilon^2 a + \varepsilon^q b\int_{\mathbb{R}^3}\left(|\nabla w_\varepsilon^{(1)}|^2 + |\nabla w_\varepsilon^{(2)}|^2\right)\right)\Delta\xi_\varepsilon + 2(1+\varepsilon^2 V(x))\xi_\varepsilon\\ &- \varepsilon^q b\left(\int_{\mathbb{R}^3}\nabla(w_\varepsilon^{(1)} + w_\varepsilon^{(2)})\cdot\nabla\xi_\varepsilon\right)\Delta(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}) = 2C_\varepsilon(x)\xi_\varepsilon.\end{aligned} \tag{5.28}$$

Multiply by ξ_ε on both sides of (5.28) and integrate by parts over $\mathbb{R}^3$. As the terms containing a and $V(x)$ are positive, we can throw them away, and get

$$\varepsilon^q b\int_{\mathbb{R}^3}\left(|\nabla w_\varepsilon^{(1)}|^2 + |\nabla w_\varepsilon^{(2)}|^2\right)\int_{\mathbb{R}^3}|\nabla\xi_\varepsilon|^2 + 2\int_{\mathbb{R}^3}|\xi_\varepsilon|^2 \le 2\int_{\mathbb{R}^3}C_\varepsilon(x)\xi_\varepsilon^2\mathrm{d}x, \tag{5.29}$$

Notice that for each $j = 1, 2$, there holds

$$\varepsilon^q\int_{\mathbb{R}^3}|\nabla w_\varepsilon^{(j)}|^2 = \varepsilon^q\int_{\mathbb{R}^3}\left(\sum_{i=1}^k|\nabla v^i_{\varepsilon,y_\varepsilon^{i(j)}}|^2 + |\nabla\varphi_\varepsilon^{(j)}|^2\right) + o(\varepsilon^{2q}) = c_\varepsilon\varepsilon^{2q}, \tag{5.30}$$

where $c_\varepsilon \to c_* > 0$ as $\varepsilon \to 0$. From (5.29)–(5.30), we conclude that

$$\|\xi_\varepsilon\|_\varepsilon^2 \le C\int_{\mathbb{R}^3}C_\varepsilon(x)\xi_\varepsilon^2\mathrm{d}x$$

holds for some constant $C > 0$ independent of ε.

Since $|\xi_\varepsilon| \le 1$, then for $j = 1, 2$, a simple scaling argument implies that

$$\int_{\mathbb{R}^3}(v^i_{\varepsilon,y_\varepsilon^{i(j)}})^{p-1}\xi_\varepsilon^2 \le C\varepsilon^{3q},$$

and by applying Hölder's inequality and Lemma 2.3 that

$$\int_{\mathbb{R}^3}(\varphi^{(j)})^{p-1}\xi_\varepsilon^2 \le \|\varphi_\varepsilon^{(j)}\|^{p-1}_{L^{p+1}(\mathbb{R}^3)}\|\xi_\varepsilon\|^2_{L^{p+1}(\mathbb{R}^3)} = o(\varepsilon^{2(p-1)})\|\xi_\varepsilon\|^2_\varepsilon.$$

Hence

$$\|\xi_\varepsilon\|_\varepsilon^2 \le C\int_{\mathbb{R}^3} C_\varepsilon\xi_\varepsilon^2\mathrm{d}x \le \int_{\mathbb{R}^3} C\left(\sum_{i,j=1}^{2}\left(v^i_{\varepsilon,y_\varepsilon^{i(j)}}\right)^{p-1} + \sum_{j=1}^{2}|\varphi_\varepsilon^{(j)}|^{p-1}\right)\xi_\varepsilon^2\mathrm{d}x = O(\varepsilon^{3q}),$$

which gives (5.25). The proof of Proposition 5.2 is complete. □

Next we study the asymptotic behavior of $\widetilde{\xi}_\varepsilon$.

Proposition 5.3. *Let* $\widetilde{\xi}_\varepsilon(x) = \xi_\varepsilon(\varepsilon^q x + y_\varepsilon^{i_0(1)})$. *Then there exist* $d_\omega \in \mathbb{R}^1$, $\omega = 1,2,3$, *such that (up to a subsequence)*

$$\widetilde{\xi}_\varepsilon \to \sum_{\omega=1}^{3} d_\omega \partial_{x_\omega} U^{i_0} \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^3)$$

as $\varepsilon \to 0$. *Here* i_0 *is the index used in the definition of* $\widetilde{\xi}_\varepsilon$ *defined at the beginning of this section.*

Proof. It is straightforward to deduce from (5.26) that $\widetilde{\xi}_\varepsilon$ solves

$$-\left(a\varepsilon^{2-2q} + \varepsilon^{-q}b\int_{\mathbb{R}^3}|\nabla w_\varepsilon^{(1)}|^2\right)\Delta\widetilde{\xi}_\varepsilon + (1+\varepsilon^2 V(\varepsilon^q x + y_\varepsilon^{i_0(1)}))\widetilde{\xi}_\varepsilon \tag{5.31}$$
$$-\varepsilon^{-q}b\left(\int_{\mathbb{R}^3}\nabla(w_\varepsilon^{(1)} + w_\varepsilon^{(2)})\cdot\nabla\xi_\varepsilon\right)\Delta w_\varepsilon^{(2)}(\varepsilon^q x + y_\varepsilon^{i_0(1)}) = C_\varepsilon(\varepsilon^q x + y_\varepsilon^{i_0(1)})\widetilde{\xi}_\varepsilon.$$

For convenience, we introduce

$$\widetilde{w}_\varepsilon^{(i)}(x) = w_\varepsilon^{(i)}(\varepsilon^q x + y_\varepsilon^{i_0(1)}) \ \text{ and } \ \widetilde{\varphi}_\varepsilon^{(i)}(x) = \varphi_\varepsilon^{(i)}(\varepsilon^q x + y_\varepsilon^{i_0(1)})$$

for $i = 1, 2$. Then, we have

$$\begin{aligned}\varepsilon^{-q}\int_{\mathbb{R}^3}|\nabla w_\varepsilon^{(1)}|^2 &= \varepsilon^{-q}\int_{\mathbb{R}^3}|\sum_{i=1}^{k}\nabla v^i_{\varepsilon,y_\varepsilon^{i(1)}} + \nabla\varphi_\varepsilon^{(1)}|^2\\ &= \sum_{i=1}^{k}\int_{\mathbb{R}^3}|\nabla v^i|^2 + O(\varepsilon^{2+qm}) + O(e^{-\frac{\gamma_0}{\varepsilon^q}})\\ &= c_\varepsilon + O(\varepsilon^{2+qm}) + O(e^{-\frac{\gamma_0}{\varepsilon^q}}) \to c_* > 0,\end{aligned} \tag{5.32}$$

and
$$\varepsilon^{-q}b\Big(\int_{\mathbb{R}^3}\nabla(w_\varepsilon^{(1)} + w_\varepsilon^{(2)})\cdot\nabla\xi_\varepsilon\Big) = b\int_{\mathbb{R}^3}\nabla(\widetilde{w}_\varepsilon^{(1)} + \widetilde{w}_\varepsilon^{(2)})\cdot\nabla\widetilde{\xi}_\varepsilon, \tag{5.33}$$

which are uniformly bounded for $\varepsilon > 0$ by (5.21) and by below (5.34),

$$\int_{\mathbb{R}^3}|\nabla\widetilde{\xi}_\varepsilon|^2 = \varepsilon^{-q}\int_{\mathbb{R}^3}|\nabla\xi_\varepsilon|^2 = O(1). \tag{5.34}$$

We also have used the following estimates for $i = 1, 2$:

$$\int_{\mathbb{R}^3}|\nabla\widetilde{\varphi}_\varepsilon^{(i)}|^2 = \varepsilon^{-3q}\left(\|\varphi_\varepsilon^{(i)}\|_\varepsilon^2\right) = O(\varepsilon^{4+2qm}).$$

Thus, in view of $\|\widetilde{\xi}_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1$ and (5.24) and estimates in the above, the elliptic regularity theory implies that $\widetilde{\xi}_\varepsilon$ is locally uniformly bounded with respect to ε in $C^{1,\varrho}_{\text{loc}}(\mathbb{R}^3)$ for some $\varrho \in (0,1)$. In consequence, we assume (up to a subsequence) that

$$\widetilde{\xi}_\varepsilon \to \widetilde{\xi} \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^3).$$

We claim that $\widetilde{\xi} \in \operatorname{Ker} \mathcal{L}^{i_0}$, that is,

$$-b\sum_{i=1}^{k}\int_{\mathbb{R}^3}|\nabla U^i|^2\Delta\widetilde{\xi} - 2b\left(\int_{\mathbb{R}^3}\nabla U^{i_0}\cdot\nabla\widetilde{\xi}\right)\Delta U^{i_0} + \widetilde{\xi} = p(U^{i_0})^{p-1}\widetilde{\xi}. \tag{5.35}$$

Then $\widetilde{\xi} = \sum_{\omega=1}^{3} d_\omega \partial_{x_\omega} U^{i_0}$ for some $d_\omega \in \mathbb{R}^1$ ($\omega = 1,2,3$), and thus Proposition 5.3 is proved.

To deduce (5.35), it is sufficient to show that (5.35) is the limiting equation of Eq. (5.31). It follows from (5.32) that

$$\begin{aligned}
&\left(a\varepsilon^{2-2q} + b\varepsilon^{-q}\int_{\mathbb{R}^3}|\nabla w_\varepsilon^{(1)}|^2\right) - b\sum_{i=1}^{k}\int_{\mathbb{R}^3}|\nabla U^i|^2 \\
&= a\varepsilon^{2-2q} + b\sum_{i=1}^{k}\int_{\mathbb{R}^3}|\nabla v^i|^2 + O(\varepsilon^{2+qm}) + O(e^{-\frac{\gamma_0}{\varepsilon^q}}) - bc_* \\
&= a\varepsilon^{2-2q} + bc_\varepsilon + O(\varepsilon^{2+qm}) + O(e^{-\frac{\gamma_0}{\varepsilon^q}}) - bc_* \to 0
\end{aligned} \tag{5.36}$$

as $\varepsilon \to 0$. Similarly, by (5.33) and (5.34), we have

$$\begin{aligned}
&\varepsilon^{-q}\left(\int_{\mathbb{R}^3}\nabla(w_\varepsilon^{(1)} + w_\varepsilon^{(2)})\cdot\nabla\xi_\varepsilon\right) - 2\left(\int_{\mathbb{R}^3}\nabla U^{i_0}\cdot\nabla\widetilde{\xi}_\varepsilon\right) \\
&= \int_{\mathbb{R}^3}\nabla(\widetilde{w}_\varepsilon^{(1)} + \widetilde{w}_\varepsilon^{(2)})\cdot\nabla\widetilde{\xi}_\varepsilon - 2\left(\int_{\mathbb{R}^3}\nabla U^{i_0}\cdot\nabla\widetilde{\xi}_\varepsilon\right) \\
&= \int_{\mathbb{R}^3}\left(\sum_{j\neq i_0}\nabla v^j\Big(z + \frac{y_\varepsilon^{i_0(1)} - y_\varepsilon^{j(1)}}{\varepsilon^q}\Big) + \sum_{j\neq i_0}\nabla v^j\Big(z + \frac{y_\varepsilon^{i_0(1)} - y_\varepsilon^{j(2)}}{\varepsilon^q}\Big)\right)\cdot\nabla\widetilde{\xi}_\varepsilon \\
&\quad + \int_{\mathbb{R}^3}\left(\nabla v^{i_0} + \nabla v^{i_0}\Big(z + \frac{y_\varepsilon^{i_0(1)} - y_\varepsilon^{i_0(2)}}{\varepsilon^q}\Big) - 2\nabla U^{i_0}\right)\cdot\nabla\widetilde{\xi}_\varepsilon \\
&\quad + \int_{\mathbb{R}^3}\left(\nabla\widetilde{\varphi}_\varepsilon^{(1)} + \nabla\widetilde{\varphi}_\varepsilon^{(2)}\right)\cdot\nabla\widetilde{\xi}_\varepsilon \\
&= O(e^{-\frac{\gamma_0}{\varepsilon^q}}) + o_\varepsilon(1) + O(\varepsilon^{2+qm}) \to 0
\end{aligned} \tag{5.37}$$

as $\varepsilon \to 0$. Here we have used Proposition 2.2 (3) and Lemma B.3.

Similarly, for any $\Phi \in C_0^\infty(\mathbb{R}^3)$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left(\nabla w_\varepsilon^{(2)}(\varepsilon^q x + y_\varepsilon^{i_0(1)}) - \nabla U^{i_0}\right)\cdot\nabla\Phi \\
&= \int_{\mathbb{R}^3}\left(\nabla v^{i_0}\Big(z + \frac{y_\varepsilon^{i_0(1)} - y_\varepsilon^{i_0(2)}}{\varepsilon^q}\Big) - \nabla U^{i_0}\right)\cdot\nabla\Phi
\end{aligned}$$

$$+\sum_{j\neq i_0}\int_{\mathbb{R}^3}\nabla v^j(z+\frac{y_\varepsilon^{i_0(1)}-y_\varepsilon^{j(2)}}{\varepsilon^q})\cdot\nabla\Phi+\int_{\mathbb{R}^3}\nabla\widetilde{\varphi}_\varepsilon^{(2)}\cdot\nabla\Phi$$
$$=o_\varepsilon(1)+O(e^{-\frac{\gamma_0}{\varepsilon^q}})+O(\varepsilon^{2+qm})\to 0 \tag{5.38}$$

as $\varepsilon\to 0$. Combining (5.37) and (5.38) we conclude that

$$\varepsilon^{-q}b\left(\int_{\mathbb{R}^3}\nabla(w_\varepsilon^{(1)}+w_\varepsilon^{(2)})\cdot\nabla\xi_\varepsilon\right)\Delta w_\varepsilon^{(2)}(\varepsilon^q x+y_\varepsilon^{i_0(1)})\to 2b\left(\int_{\mathbb{R}^3}\nabla U^{i_0}\cdot\nabla\widetilde{\xi}\right)\Delta U^{i_0} \tag{5.39}$$

in $H^{-1}(\mathbb{R}^3)$. By the assumption (V_1) we have

$$1+\varepsilon^2 V(\varepsilon^q x+y_\varepsilon^{i_0(1)})\to 1\ \text{ locally uniformly} \tag{5.40}$$

as $\varepsilon\to 0$. Now, we estimate $C_\varepsilon(\varepsilon^q x+y_\varepsilon^{i_0(1)})$. For each $j=1,2,\cdots,k$,

$$v^j\Big(\frac{x-y_\varepsilon^{j(1)}}{\varepsilon^q}\Big)-v^j\Big(\frac{x-y_\varepsilon^{j(2)}}{\varepsilon^q}\Big)=O\Big(\frac{y_\varepsilon^{j(1)}-y_\varepsilon^{j(2)}}{\varepsilon^q}\cdot\nabla v^j\Big(\frac{x-y_\varepsilon^{j(1)}}{\varepsilon^q}\Big)\Big)$$
$$=o_\varepsilon(1)O\Big(\nabla v^j\Big(\frac{x-y_\varepsilon^{j(1)}}{\varepsilon^q}\Big)\Big). \tag{5.41}$$

Then we have

$$w_\varepsilon^{(1)}(x)-w_\varepsilon^{(2)}(x)=o_\varepsilon(1)\sum_{j=1}^k\nabla v^j\left(\frac{x-y_\varepsilon^{j(1)}}{\varepsilon^q}\right)+O(|\varphi_\varepsilon^{(1)}(x)|+|\varphi_\varepsilon^{(2)}(x)|). \tag{5.42}$$

For any $x\in B_d(y_\varepsilon^{i_0(1)})$, by Lemma A.2, we deduce that, for any $\gamma>0$, there holds

$$C_\varepsilon(x)=p\int_0^1(tw_\varepsilon^{(1)}(x)+(1-t)w_\varepsilon^{(2)}(x))^{p-1}\mathrm{d}t$$
$$=p\int_0^1\Big(t\Big(\sum_{i=1}^k v^i_{\varepsilon,y_\varepsilon^{i(1)}}-\sum_{i=1}^k v^i_{\varepsilon,y_\varepsilon^{i(2)}}\Big)+\sum_{i=1}^k v^i_{\varepsilon,y_\varepsilon^{i(2)}}+t\varphi_\varepsilon^{(1)}(x)+(1-t)\varphi_\varepsilon^{(2)}(x)\Big)^{p-1}\mathrm{d}t$$
$$=p\Big(v^{i_0}\Big(\frac{x-y_\varepsilon^{i_0(2)}}{\varepsilon^q}\Big)\Big)^{p-1}+o_\varepsilon(1)\nabla v^{i_0}\Big(\frac{x-y_\varepsilon^{i_0(1)}}{\varepsilon^q}\Big)+O\left(|\varphi_\varepsilon^{(1)}(x)|+|\varphi_\varepsilon^{(2)}(x)|\right)^{p-1}+O(\varepsilon^\gamma).$$

Hence it yields that

$$C_\varepsilon(\varepsilon^q x+y_\varepsilon^{i_0(1)})=p\Big(v^{i_0}\Big(x+\frac{y_\varepsilon^{i_0(1)}-y_\varepsilon^{i_0(2)}}{\varepsilon^q}\Big)\Big)^{p-1}+O\left(|\widetilde{\varphi}_\varepsilon^{(1)}(x)|+|\widetilde{\varphi}_\varepsilon^{(2)}(x)|\right)^{p-1}+o_\varepsilon(1)$$

holds for $x\in B_{\frac{d}{\varepsilon^q}}(0)$. We deduce from Proposition 2.2 (3) and Lemma B.3 that for any $\Phi\in C_0^\infty(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3}C_\varepsilon(\varepsilon^q x+y_\varepsilon^{i_0(1)})\widetilde{\xi}_\varepsilon\Phi\to p\int_{\mathbb{R}^3}(U^{i_0})^{p-1}\widetilde{\xi}\Phi\ \text{ as }\varepsilon\to 0. \tag{5.43}$$

Finally, combining (5.36) and (5.39)–(5.43), we obtain (5.35), and the proof of Proposition 5.3 is complete. □

Lemma 5.4. *Let* d_ω *be defined as in Proposition* 5.3. *Then* $d_\omega = 0$ *for* $\omega = 1, 2, 3$.

Proof. Applying the local Pohozaev identity (2.9) to $w_\varepsilon^{(1)}, w_\varepsilon^{(2)}$ with $\Omega = B_d(y_\varepsilon^{i_0(1)})$, where d is chosen in the same way as in Proposition 5.1, we combine this with (5.4), (5.9) and Proposition 5.2 to obtain

$$\int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(\varepsilon^{2q}|\nabla\varphi_\varepsilon^{(i)}|^2 + |\varphi_\varepsilon^{(i)}|^2 + |\varphi_\varepsilon^{(i)}|^{p+1}\right) = O(\|\varphi_\varepsilon^{(i)}\|_\varepsilon^2) = O(\varepsilon^{4+3q+2qm}) \tag{5.44}$$

for $i = 1, 2$, and
$$\int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(\varepsilon^{2q}|\nabla\xi_\varepsilon|^2 + |\xi_\varepsilon|^2\right) = O(\varepsilon^{3q}). \tag{5.45}$$

Furthermore, we obtain for $i = 1, 2$,

$$\begin{aligned}
&\varepsilon^2 \int_{B_d(y_\varepsilon^{i_0(1)})} \frac{\partial V}{\partial x_\alpha} \left(w_\varepsilon^{(i)}\right)^2 \mathrm{d}x \\
&= \left(\varepsilon^2 a + \varepsilon^q b \int_{\mathbb{R}^3} \left|\nabla w_\varepsilon^{(i)}\right|^2\right) \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(\left|\nabla w_\varepsilon^{(i)}\right|^2 \nu_\alpha - 2\frac{\partial w_\varepsilon^{(i)}}{\partial \nu}\frac{\partial w_\varepsilon^{(i)}}{\partial x_\alpha}\right) \\
&\quad + \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(1 + \varepsilon^2 V(x)\right) \left(w_\varepsilon^{(i)}\right)^2 \nu_\alpha - \frac{2}{p+1} \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(w_\varepsilon^{(i)}\right)^{p+1} \nu_\alpha,
\end{aligned}$$

where $\alpha = 1, 2, 3$, from which we deduce

$$\begin{aligned}
&\varepsilon^2 \int_{B_d(y_\varepsilon^{i_0(1)})} \frac{\partial V}{\partial x_\alpha} \left(\left(w_\varepsilon^{(1)}\right)^2 - \left(w_\varepsilon^{(2)}\right)^2\right) \mathrm{d}x \\
&= \left(\varepsilon^2 a + \varepsilon^q b \int_{\mathbb{R}^3} \left|\nabla w_\varepsilon^{(1)}\right|^2\right) \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(\left|\nabla w_\varepsilon^{(1)}\right|^2 \nu_\alpha - 2\frac{\partial w_\varepsilon^{(1)}}{\partial \nu}\frac{\partial w_\varepsilon^{(1)}}{\partial x_\alpha}\right) \\
&\quad - \left(\varepsilon^2 a + \varepsilon^q b \int_{\mathbb{R}^3} \left|\nabla w_\varepsilon^{(2)}\right|^2\right) \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(\left|\nabla w_\varepsilon^{(2)}\right|^2 \nu_\alpha - 2\frac{\partial w_\varepsilon^{(2)}}{\partial \nu}\frac{\partial w_\varepsilon^{(2)}}{\partial x_\alpha}\right) \\
&\quad + \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(1 + \varepsilon^2 V(x)\right) \left(\left(w_\varepsilon^{(1)}\right)^2 - \left(w_\varepsilon^{(2)}\right)^2\right) \nu_\alpha \\
&\quad - \frac{2}{p+1} \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(\left(w_\varepsilon^{(1)}\right)^{p+1} - \left(w_\varepsilon^{(2)}\right)^{p+1}\right) \nu_\alpha.
\end{aligned}$$

In terms of ξ_ε, we get

$$\begin{aligned}
&\varepsilon^2 \int_{B_d(y_\varepsilon^{i_0(1)})} \frac{\partial V}{\partial x_\alpha} \left(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}\right) \xi_\varepsilon \mathrm{d}x \\
&= \left(\varepsilon^2 a + \varepsilon^q b \int_{\mathbb{R}^3} \left|\nabla w_\varepsilon^{(1)}\right|^2\right) \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \nabla \left(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}\right) \cdot \nabla\xi_\varepsilon \nu_\alpha \\
&\quad + \varepsilon^q b \Big(\int_{\mathbb{R}^3} \nabla \left(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}\right) \cdot \nabla\xi_\varepsilon\Big) \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(\left|\nabla w_\varepsilon^{(2)}\right|^2 \nu_\alpha - 2\frac{\partial w_\varepsilon^{(1)}}{\partial \nu}\frac{\partial w_\varepsilon^{(1)}}{\partial x_\alpha}\right) \\
&\quad + \left(\varepsilon^2 a + \varepsilon^q b \int_{\mathbb{R}^3} \left|\nabla w_\varepsilon^{(2)}\right|^2\right) \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(2\frac{\partial w_\varepsilon^{(2)}}{\partial \nu}\frac{\partial w_\varepsilon^{(2)}}{\partial x_\alpha} - 2\frac{\partial w_\varepsilon^{(1)}}{\partial \nu}\frac{\partial w_\varepsilon^{(1)}}{\partial x_\alpha}\right) \\
&\quad + \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(1 + \varepsilon^2 V(x)\right) \left(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}\right) \xi_\varepsilon \nu_\alpha - 2 \int_{\partial B_d(y_\varepsilon^{i_0(1)})} A_\varepsilon \xi_\varepsilon \nu_\alpha,
\end{aligned} \tag{5.46}$$

where $1 \le \alpha \le 3$ and $\quad A_\varepsilon = \displaystyle\int_0^1 \left(tw_\varepsilon^{(1)} + (1-t)w_\varepsilon^{(2)}\right)^p.$

We estimate (5.46) term by term. Notice that

$$\varepsilon^2 a + \varepsilon^q b \int_{\mathbb{R}^3} \left|\nabla w_\varepsilon^{(i)}\right|^2 = O(\varepsilon^{2q})$$

holds for each $i = 1, 2$. Moreover,

$$\int_{\partial B_d(y_\varepsilon^{i_0(1)})} |\nabla w_\varepsilon^{(i)}|^2 = O(\varepsilon^\gamma + \|\nabla \varphi_\varepsilon^{(i)}\|^2_{L^2(\mathbb{R}^3)})$$

for any $\gamma > 0$. Thus, by Hölder's inequality, (5.4) and (5.25), we have

$$\begin{aligned}
&\left(\varepsilon^2 a + \varepsilon^q b \int_{\mathbb{R}^3} \left|\nabla w_\varepsilon^{(1)}\right|^2\right) \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left|\nabla \left(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}\right) \cdot \nabla \xi_\varepsilon \nu_\alpha\right| \\
&= O(\varepsilon^\gamma + \sum_{i=1}^2 \|\varphi_\varepsilon^{(i)}\|_\varepsilon) O(\varepsilon^{\frac{3q}{2}}) = O(\varepsilon^{2+3q+qm})
\end{aligned}$$

by choosing γ sufficiently large. Similarly, we obtain

$$\begin{aligned}
&\varepsilon^q b \left(\int_{\mathbb{R}^3} \nabla \left(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}\right) \cdot \nabla \xi_\varepsilon\right) \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left| \left|\nabla w_\varepsilon^{(2)}\right|^2 \nu_\alpha - 2\frac{\partial w_\varepsilon^{(1)}}{\partial \nu}\frac{\partial w_\varepsilon^{(1)}}{\partial x_\alpha}\right| \\
&\le C\varepsilon^q \sum_{i=1}^2 \left(\int_{\mathbb{R}^3} |\nabla w_\varepsilon^{(i)}|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla \xi_\varepsilon|^2\right)^{\frac{1}{2}} \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(|\nabla w_\varepsilon^{(1)}|^2 + |\nabla w_\varepsilon^{(2)}|^2\right) \\
&\le C\varepsilon^{2q} \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(|\nabla w_\varepsilon^{(1)}|^2 + |\nabla w_\varepsilon^{(2)}|^2\right) = O(\varepsilon^\gamma + \sum_{i=1}^2 \|\varphi_\varepsilon^{(i)}\|^2_\varepsilon) = O(\varepsilon^{2+3q+qm})
\end{aligned}$$

and

$$\begin{aligned}
&\left(\varepsilon^2 a + \varepsilon^q b \int_{\mathbb{R}^3} \left|\nabla w_\varepsilon^{(2)}\right|^2\right) \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left| 2\frac{\partial w_\varepsilon^{(2)}}{\partial \nu}\frac{\partial w_\varepsilon^{(2)}}{\partial x_\alpha} - 2\frac{\partial w_\varepsilon^{(1)}}{\partial \nu}\frac{\partial w_\varepsilon^{(1)}}{\partial x_\alpha}\right| \\
&\le C\varepsilon^{2q} \int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(|\nabla w_\varepsilon^{(1)}|^2 + |\nabla w_\varepsilon^{(2)}|^2\right) \\
&= O(\varepsilon^\gamma + \sum_{i=1}^2 \|\varphi_\varepsilon^{(i)}\|^2_\varepsilon) = O(\varepsilon^{2+3q+qm}).
\end{aligned}$$

Combining (5.44) and (5.45) yields

$$\begin{aligned}
&\int_{\partial B_d(y_\varepsilon^{i_0(1)})} \left(1 + \varepsilon^2 V(x)\right) \left(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}\right) \xi_\varepsilon \nu_\alpha \\
&\le C \sum_{i=1}^2 \left(\int_{\partial B_d(y_\varepsilon^{i_0(1)})} |w_\varepsilon^{(i)}|^2\right)^{\frac{1}{2}} \left(\int_{\partial B_d(y_\varepsilon^{i_0(1)})} |\xi_\varepsilon|^2\right)^{\frac{1}{2}} \\
&= O(\varepsilon^\gamma + \sum_{i=1}^2 \|\varphi_\varepsilon^{(i)}\|_\varepsilon) O(\varepsilon^{\frac{3q}{2}}) = O(\varepsilon^{2+3q+qm}).
\end{aligned}$$

As to A_ε, by (5.41)–(5.42) and Lemma A.2, for any $x\in\partial B_d(y_\varepsilon^{i_0(1)})$, one can verify

$$\begin{aligned}
A_\varepsilon &= \int_0^1 (tw_\varepsilon^{(1)}(x)+(1-t)w_\varepsilon^{(2)}(x))^p \mathrm{d}t \\
&= \int_0^1 \left(t\left(\sum_{i=1}^k v^i_{\varepsilon,y_\varepsilon^{i(1)}} - \sum_{i=1}^k v^i_{\varepsilon,y_\varepsilon^{i(2)}}\right) + \sum_{i=1}^k v^i_{\varepsilon,y_\varepsilon^{i(2)}} + t\varphi_\varepsilon^{(1)}(x) + (1-t)\varphi_\varepsilon^{(2)}(x)\right)^p \mathrm{d}t \\
&= O(\varepsilon^\gamma + \sum_{j=1}^2 |\varphi_\varepsilon^{(j)}(x)|^p)
\end{aligned}$$

for any $\gamma>0$. Hence we can deduce that

$$\begin{aligned}
\int_{\partial B_d(y_\varepsilon^{i_0(1)})} |A_\varepsilon \xi_\varepsilon \nu_\alpha| &\le C\varepsilon^\gamma \left(\int_{\partial B_d(y_\varepsilon^{i_0(1)})} |\xi_\varepsilon|^2\right)^{\frac12} + C\sum_{i=1}^2 \left(\int_{\partial B_d(y_\varepsilon^{i_0(1)})} |\varphi_\varepsilon^{(i)}|^p |\xi_\varepsilon|\right) \\
&\le C\varepsilon^{\gamma+\frac{3q}{2}} + O\left(\sum_{i=1}^2 \left(\varepsilon^{-q\left(\frac{3}{p+1}-\frac32\right)} \|\varphi_\varepsilon^{(i)}\|_\varepsilon\right)^p \left(\varepsilon^{-q\left(\frac{3}{p+1}-\frac32\right)}\|\xi_\varepsilon\|_\varepsilon\right)\right) \\
&= O(\varepsilon^{2+3q+qm}).
\end{aligned}$$

Therefore, by the above estimates, there holds

$$\varepsilon^2 \int_{B_d(y_\varepsilon^{i_0(1)})} \frac{\partial V}{\partial x_\alpha}\left(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}\right)\xi_\varepsilon \mathrm{d}x = O(\varepsilon^{2+3q+qm}). \tag{5.47}$$

Next, we estimate the left hand side of (5.46). By the assumption (V_3), it holds

$$\begin{aligned}
&\int_{B_d(y_\varepsilon^{i_0(1)})} \frac{\partial V}{\partial x_\alpha}\left(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}\right)\xi_\varepsilon \\
&= mc_{i_0,\alpha}\int_{B_d(y_\varepsilon^{i_0(1)})} |x_\alpha - a_{i_0,\alpha}|^{m-2}(x_\alpha - a_{i_0,\alpha})\left(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}\right)\xi_\varepsilon \\
&\quad + O\left(\int_{B_d(y_\varepsilon^{i_0(1)})} |x-a_{i_0}|^m \left(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}\right)\xi_\varepsilon\right).
\end{aligned} \tag{5.48}$$

Observe that

$$\begin{aligned}
&mc_{i_0,\alpha}\int_{B_d(y_\varepsilon^{i_0})} |x_\alpha - a_{i_0,\alpha}|^{m-2}(x_\alpha - a_{i_0,\alpha})\left(w_\varepsilon^{(1)} + w_\varepsilon^{(2)}\right)\xi_\varepsilon \\
&= mc_{i_0,\alpha}\varepsilon^{3q}\int_{B_{\frac{d}{\varepsilon^q}}(0)} |\varepsilon^q z_\alpha + y_{\varepsilon,\alpha}^{i_0(1)} - a_{i_0,\alpha}|^{m-2}(\varepsilon^q z_\alpha + y_{\varepsilon,\alpha}^{i_0(1)} - a_{i_0,\alpha}) \\
&\quad \cdot\left(\sum_{i=1}^k v^i\left(z+\frac{y_\varepsilon^{i_0(1)} - y_\varepsilon^{i(1)}}{\varepsilon^q}\right) + \sum_{i=1}^k v^i\left(z+\frac{y_\varepsilon^{i_0(1)} - y_\varepsilon^{i(2)}}{\varepsilon^q}\right)\right)\widetilde{\xi}_\varepsilon \\
&\quad + mc_{i_0,\alpha}\int_{B_d(y_\varepsilon^{i_0})} |x_\alpha - a_{i_0,\alpha}|^{m-2}(x_\alpha - a_{i_0,\alpha})\left(\varphi_\varepsilon^{(1)} + \varphi_\varepsilon^{(2)}\right)\xi_\varepsilon \\
&=: \mathfrak{I}_1 + \mathfrak{I}_2.
\end{aligned} \tag{5.49}$$

Since U^{i_0} is a positive and radially symmetric function, there holds

$$\begin{aligned}\mathfrak{I}_1 &= mc_{i_0,\alpha}\varepsilon^{3q+q(m-1)}\int_{B_{\frac{d}{\varepsilon^q}}(0)}\Big|z_\alpha+\frac{y^{i_0(1)}_{\varepsilon,\alpha}-a_{i_0,\alpha}}{\varepsilon^q}\Big|^{m-2}\Big(z_\alpha+\frac{y^{i_0(1)}_{\varepsilon,\alpha}-a_{i_0,\alpha}}{\varepsilon^q}\Big)\\ &\quad\cdot\Big[\sum_{i\neq i_0}v^i\Big(z+\frac{y^{i_0(1)}_\varepsilon-y^{i(1)}_\varepsilon}{\varepsilon^q}\Big)+\sum_{i\neq i_0}v^i\Big(z+\frac{y^{i_0(1)}_\varepsilon-y^{i(2)}_\varepsilon}{\varepsilon^q}\Big)\\ &\qquad\qquad+v^{i_0}(z)+v^{i_0}\Big(z+\frac{y^{i_0(1)}_\varepsilon-y^{i_0(2)}_\varepsilon}{\varepsilon^q}\Big)\Big]\widetilde{\xi}_\varepsilon\\ &= 2mc_{i_0,\alpha}\varepsilon^{q(m+2)}\sum_{\omega=1}^{3}d_\omega\int_{\mathbb{R}^3}|z_\alpha|^{m-2}z_\alpha U^{i_0}(z)\partial_{z_\omega}U^{i_0}+o(\varepsilon^{q(m+2)})\\ &= D_\alpha d_\alpha\varepsilon^{q(m+2)}+o(\varepsilon^{q(m+2)})\end{aligned}$$

for $\varepsilon>0$ sufficiently small, where

$$D_\alpha=2mc_{i_0,\alpha}\int_{\mathbb{R}^3}|z_\alpha|^{m-2}z_\alpha U^{i_0}(z)\partial_{z_\alpha}U^{i_0}\neq 0.$$

On the other hand, by Hölder's inequality, we have

$$\begin{aligned}\mathfrak{I}_2 &= mc_{i_0,\alpha}\int_{B_d(y^{i_0}_\varepsilon)}|x_\alpha-a_{i_0,\alpha}|^{m-2}(x_\alpha-a_{i_0,\alpha})\left(\varphi^{(1)}_\varepsilon+\varphi^{(2)}_\varepsilon\right)\xi_\varepsilon\\ &= O\left(\sum_{i=1}^{2}\int_{\mathbb{R}^3}|\varphi^{(i)}_\varepsilon||\xi_\varepsilon|\right)=O\left(\sum_{i=1}^{2}\|\varphi^{(i)}_\varepsilon\|_\varepsilon\|\xi_\varepsilon\|_\varepsilon\right)=O(\varepsilon^{2+3q+qm}).\end{aligned}$$

Combining the above estimates of $\mathfrak{I}_1$, $\mathfrak{I}_2$ and (5.48)–(5.49), we deduce

$$\begin{aligned}mc_{i_0,\alpha}&\int_{B_d(y^{i_0}_\varepsilon)}|x_\alpha-a_{i_0,\alpha}|^{m-2}(x_\alpha-a_{i_0,\alpha})\left(w^{(1)}_\varepsilon+w^{(2)}_\varepsilon\right)\xi_\varepsilon\\ &= D_\alpha d_\alpha\varepsilon^{q(m+2)}+o(\varepsilon^{q(m+2)}),\end{aligned}\tag{5.50}$$

where $D_\alpha\neq 0$. Similar arguments give

$$O\left(\int_{B_d(y^{i_0(1)}_\varepsilon)}|x-a_{i_0}|^m\left(w^{(1)}_\varepsilon+w^{(2)}_\varepsilon\right)\xi_\varepsilon\right)=O(\varepsilon^{q(m+3)}).\tag{5.51}$$

Hence, combining (5.50) and (5.51), we obtain the estimate of the left hand side of (5.46), that is

$$\varepsilon^2\int_{B_d(y^{i_0(1)}_\varepsilon)}\frac{\partial V}{\partial x_\alpha}\left(w^{(1)}_\varepsilon+w^{(2)}_\varepsilon\right)\xi_\varepsilon\mathrm{d}x=D_\alpha d_\alpha\varepsilon^{2+q(m+2)}+o(\varepsilon^{2+q(m+2)}).\tag{5.52}$$

Finally, combining (5.47) and (5.52), we deduce

$$O(\varepsilon^{2+3q+qm})=D_\alpha d_\alpha\varepsilon^{2+q(m+2)}+o(\varepsilon^{2+q(m+2)}),$$

which implies $d_\alpha=0$ for each $\alpha=1,2,3$. The proof of Lemma 5.4 is complete. □

As a consequence of Proposition 5.3 and Lemma 5.4, we have the following asymptotic estimates.

Proposition 5.5. *For any fixed $R > 0$, there holds*

$$\xi_\varepsilon(x) = o_\varepsilon(1), \quad \textit{for } x \in \cup_{i=1}^k B_{R\varepsilon^q}(y_\varepsilon^{i(1)}).$$

Proof. By Proposition 5.3 and Lemma 5.4, we have

$$\widetilde{\xi}_{\varepsilon,i} := \xi_\varepsilon(\varepsilon^q x + y_\varepsilon^{i(1)}) \to 0 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^3),\ i = 1, 2, \cdots, k.$$

as $\varepsilon \to 0$. Then for any fixed $R > 0$,

$$\widetilde{\xi}_{\varepsilon,i} = o_\varepsilon(1) \ \text{ in } B_R(0),\ i = 1, 2, \cdots, k,$$

which implies that $\xi_\varepsilon(x) = o_\varepsilon(1)$ for $x \in B_{R\varepsilon^q}(y_\varepsilon^{i(1)})$, $i = 1, 2, \cdots, k$.
The proof of Proposition 5.5 is complete. □

Proposition 5.6. *For large $R > 0$, we have*

$$\xi_\varepsilon(x) = o_\varepsilon(1), \quad \textit{for } x \in \mathbb{R}^3 \setminus \cup_{i=1}^k B_{R\varepsilon^q}(y_\varepsilon^{i(1)}).$$

Proof. This proposition can be proved by same argument as that of Proposition 3.5 of Cao, Li and Luo [7]. We omit the details. □

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. Suppose to the contrary that there exist two distinct solutions $w_\varepsilon^{(i)}, i = 1, 2$, and let ξ_ε be defined as above. Then

$$\|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1. \tag{5.53}$$

However, from Proposition 5.5 and Proposition 5.6, for small ε, we have

$$\|\xi_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = o_\varepsilon(1),$$

a contradiction to (5.53). The proof of Theorem 1.3 is complete. □

6. Proof of Theorem 1.4

In this section, we give a brief proof of Theorem 1.4.

Proof of Theorem 1.4. Let w_ε be the unique multi-peak solution of Eq. (1.6) derived as in Theorem 1.2. If we set

$$\lambda := \varepsilon^{-2} \ \text{ and } \ u(x) := \lambda^{\frac{1}{p-1}} w_\varepsilon(x),$$

then $u(x)$ satisfies (1.1). Denoting

$$u_\lambda(x) := \frac{u(x)}{\Big(\int_{\mathbb{R}^3} u^2(x)\mathrm{d}x\Big)^{\frac{1}{2}}},$$

then u_λ satisfies (1.2)–(1.3), where $b_\lambda = b\|u\|^2_{L^2(\mathbb{R}^3)}$, $\beta_\lambda = \|u\|^{p-1}_{L^2(\mathbb{R}^3)}$.

Notice that

$$\begin{aligned}\int_{\mathbb{R}^3} u^2(x)\mathrm{d}x &= \lambda^{\frac{2}{p-1}} \int_{\mathbb{R}^3} w_\varepsilon^2(x)\mathrm{d}x = \lambda^{\frac{2}{p-1}} \int_{\mathbb{R}^3} \left(\sum_{i=1}^k v^i_{\varepsilon,y^i_\varepsilon} + \varphi_\varepsilon\right)^2 \mathrm{d}x \\ &= \lambda^{\frac{2}{p-1}} \int_{\mathbb{R}^3} \left(\sum_{i=1}^k \left(v^i_{\varepsilon,y^i_\varepsilon}\right)^2 + 2\sum_{i=1}^k v^i_{\varepsilon,y^i_\varepsilon}\varphi_\varepsilon + \varphi_\varepsilon^2\right) \mathrm{d}x + \lambda^{\frac{2}{p-1}} O(e^{-\frac{\gamma_0}{\varepsilon^q}}) \\ &= \lambda^{\frac{2}{p-1}} \varepsilon^{3q} \sum_{i=1}^k \int_{\mathbb{R}^3} (v^i)^2 + \lambda^{\frac{2}{p-1}} o(\varepsilon^{2+3q}) + \lambda^{\frac{2}{p-1}} o(\varepsilon^{4+3q}) + \lambda^{\frac{2}{p-1}} O(e^{-\frac{\gamma_0}{\varepsilon^q}}) \\ &= \varepsilon^{4q-2} \sum_{i=1}^k \int_{\mathbb{R}^3} (v^i)^2 + o(\varepsilon^{4q}).\end{aligned}$$

Denoting $a_* := \int_{\mathbb{R}^3} (U^i)^2$, $i = 1, 2, \cdots, k$, then by of Proposition 2.2(3) we conclude that $\sum_{i=1}^k \int_{\mathbb{R}^3} (v^i)^2 \to ka_*$, as $\lambda \to +\infty$.

(1) If $4q - 2 \neq 0$, then $p \neq 11/3$ and $\int_{\mathbb{R}^3} u^2(x)\mathrm{d}x = O(\varepsilon^{4q-2}) = O(\lambda^{1-2q})$.

Hence:

if $0 < q < \frac{1}{2}$, i.e., $3 < p < \frac{11}{3}$, $b_\lambda = O(\lambda^{1-2q}) \to +\infty$, and $\beta_\lambda = O(\lambda^{\frac{11-3p}{2}}) \to +\infty$;

if $\frac{1}{2} < q < 1$, i.e., $\frac{11}{3} < p < 5$, $b_\lambda = O(\lambda^{1-2q}) \to 0$, and $\beta_\lambda = O(\lambda^{\frac{11-3p}{2}}) \to 0$.

In this sense, we have

$$\begin{aligned}u_\lambda(x) &= \frac{\lambda^{\frac{1}{p-1}}}{\lambda^{\frac{1-2q}{2}} (ka_*)^{\frac{1}{2}} + o\left(\lambda^{-q}\right)} \left[\sum_{i=1}^k v^i \left(\lambda^{\frac{q}{2}}(x - y^i_\lambda)\right) + \varphi_\lambda(x)\right] \\ &\approx (ka_*)^{-\frac{1}{2}} \lambda^{\frac{3(p-3)}{2(p-1)}} \left[\sum_{i=1}^k v^i \left(\lambda^{\frac{q}{2}}(x - y^i_\lambda)\right) + \varphi_\lambda(x)\right],\end{aligned}$$

where the local maximum points $y^i_\lambda \to a_i$ as $\lambda \to +\infty$ with $|y^i_\lambda - a_i| = o(\lambda^{-\frac{q}{2}})$, and $\varphi_\lambda(x) \in H^1(\mathbb{R}^3)$ with

$$\int_{\mathbb{R}^3} \left[\frac{1}{\lambda^q} |\nabla \varphi_\lambda|^2 + \varphi_\lambda^2\right] = \lambda^{-\left(2+\frac{3q}{2}+qm\right)}.$$

(2) If $4q - 2 = 0$, then $b_\lambda \to b_*$ and $\beta_\lambda \to \beta_*$ for some constants $b_*, \beta_* > 0$. In fact, if $q = 1/2$, i.e., $p = 11/3$, then

$$\int_{\mathbb{R}^3} u^2(x)\mathrm{d}x = \sum_{i=1}^k \int_{\mathbb{R}^3} (v^i)^2 + o(\varepsilon^2) \to \sum_{i=1}^k \int_{\mathbb{R}^3} (U^i)^2 = k(bc_*)^{\frac{3}{2}} \int_{\mathbb{R}^3} Q^2 \mathrm{d}x, \tag{6.1}$$

from which, we deduce

$$b_\lambda = b\|u\|^2_{L^2(\mathbb{R}^3)} \to bk(bc_*)^{\frac{3}{2}} \int_{\mathbb{R}^3} Q^2 \mathrm{d}x = 6^3 b^4 k^4 \left(\int_{\mathbb{R}^3} Q^2\right)^4 =: b_* \tag{6.2}$$

and

$$\beta_\lambda = \|u\|^{p-1}_{L^2(\mathbb{R}^3)} \to 6^4 b^4 k^{\frac{16}{3}} \left(\int_{\mathbb{R}^3} Q^2\right)^{\frac{16}{3}} =: \beta_*. \tag{6.3}$$

One can also rewrite (6.1), (6.2) and (6.3) as

$$\int_{\mathbb{R}^3} u^2(x)\mathrm{d}x = ka_* + o\left(\frac{1}{\lambda}\right),\ b_\lambda = bka_* + o\left(\frac{1}{\lambda}\right),\ \text{and } \beta_\lambda = (ka_*)^{\frac{4}{3}} + o\left(\frac{1}{\lambda^{\frac{4}{3}}}\right).$$

In this sense, we can explicitly express u_λ as below:

$$u_\lambda(x) = \frac{\lambda^{\frac{3}{8}}}{(ka_*)^{\frac{1}{2}} + o\left(\lambda^{-\frac{1}{2}}\right)}\left[\sum_{i=1}^{k} v^i\left(\lambda^{\frac{1}{4}}(x-y_\lambda^i)\right) + \varphi_\lambda(x)\right],$$

where the local maximum points $y_\lambda^i \to a_i$ as $\lambda \to +\infty$ with $|y_\lambda^i - a_i| = o(\lambda^{-\frac{1}{4}})$ and $\varphi_\lambda(x) \in H^1(\mathbb{R}^3)$ with

$$\int_{\mathbb{R}^3}\left[\frac{1}{\lambda^{1/2}}|\nabla\varphi_\lambda|^2 + \varphi_\lambda^2\right] = \lambda^{-\left(2+\frac{3}{4}+\frac{m}{2}\right)}.$$

The proof of Theorem 1.4 is complete. □

A. Some estimates

In this appendix, we give various estimates and results which have been used repeatedly in previous sections. First, from the exponential decay of v^i_{ε,y^i}, if we have $|y^i_\varepsilon - y^j_\varepsilon|/\varepsilon^q \to +\infty$ as $\varepsilon \to 0$, then we have the following asymptotic estimate.

Lemma A.1. *There exists a small constant $d_0 > 0$, such that, for any $\gamma > 0$, we have*

$$v^j_{\varepsilon,y^j_\varepsilon}(x) = O(\varepsilon^\gamma),\ \textit{for } x \in B_d(y^i_\varepsilon),\ j \neq i \textit{ and } 0 < d < d_0, \tag{A.1}$$

and $$v^j_{\varepsilon,y^j_\varepsilon}(x) = O(\varepsilon^\gamma),\ \textit{for } x \in \partial B_d(y^i_\varepsilon),\ i = 1,2,\cdots,k \textit{ and } 0 < d < d_0. \tag{A.2}$$

Proof. It easily deduces (A.1) and (A.2) from (2.7). We omit the details. □

Lemma A.2. ([1], Lemma 3.7) *Suppose that $f,\ g : \mathbb{R}^n \to \mathbb{R}^1$ are two positive radial functions satisfying*

$$f(x) \sim |x|^a e^{-b|x|},\ g(x) \sim |x|^{a'} e^{-b'|x|}\ (|x| \to +\infty),$$

where $a, a' \in \mathbb{R}^1,\ b, b' > 0$. Assume that $\xi \in \mathbb{R}^n$, $|\xi| \to +\infty$, and let $f_\xi(x) = f(x-\xi)$ for $x \in \mathbb{R}^n$, then the following asymptotic estimates hold

(i) *if $b < b'$, then* $\displaystyle\int_{\mathbb{R}^n} f_\xi g \sim e^{-b|\xi|}|\xi|^a.$

(ii) *if $b = b'$, without loss of generality, assume that $a \geq a'$, then*

$$\int_{\mathbb{R}^n} f_\xi g \sim \begin{cases} e^{-b|\xi|}|\xi|^{a+a'+\frac{n+1}{2}}, & \textit{if } a' > -\frac{n+1}{2}; \\ e^{-b|\xi|}|\xi|^a \log|\xi|, & \textit{if } a' = -\frac{n+1}{2}; \\ e^{-b|\xi|}|\xi|^a, & \textit{if } a' < -\frac{n+1}{2}. \end{cases}$$

From Lemma A.2 and estimates (2.7), we conclude that for any $r, s > 0,\ r \neq s$, if $|\frac{y^i - y^j}{\varepsilon^q}| \to +\infty$ as $\varepsilon \to 0$, it holds that

$$\int_{\mathbb{R}^3} \left(v^i_{\varepsilon,y^i}\right)^r \left(v^j_{\varepsilon,y^j}\right)^s \sim \varepsilon^{3q} e^{-\min\{r,s\}\frac{\gamma_{ij}|y^i-y^j|}{\varepsilon^q}} \text{ as } \varepsilon \to 0$$

and
$$\int_{\mathbb{R}^3} |\nabla v^i_{\varepsilon,y^i} \cdot \nabla v^j_{\varepsilon,y^j}| \sim \varepsilon^q e^{-\frac{\gamma_{ij}|y^i-y^j|}{\varepsilon^q}} \text{ as } \varepsilon \to 0,$$

where $\gamma_{ij} > 0$ $(i, j = 1, \cdots, k)$ are constants. In particular, there exists a constant $C > 0$, such that

$$\int_{\mathbb{R}^3} \left(v^i_{\varepsilon,y^i}\right)^r \left(v^j_{\varepsilon,y^j}\right)^s \le C\varepsilon^{3q} e^{-\min\{r,s\}\frac{\gamma_{ij}|y^i-y^j|}{\varepsilon^q}}, \text{ as } \varepsilon \to 0 \tag{A.3}$$

and
$$\int_{\mathbb{R}^3} |\nabla w^i_{\varepsilon,y^i} \cdot \nabla w^j_{\varepsilon,y^j}| \le C\varepsilon e^{-\frac{\gamma_{ij}|y^i-y^j|}{\varepsilon}}, \text{ as } \varepsilon \to 0. \tag{A.4}$$

Notice that, for the case of $r = s \ge 1$, the above estimates hold as well. We deduce from (A.3) and (A.4) that for $\gamma > 0$, there holds

$$\int_{\mathbb{R}^3} \left(v^i_{\varepsilon,y^i}\right)^r \left(v^j_{\varepsilon,y^j}\right)^s = O(\varepsilon^\gamma) \text{ as } \varepsilon \to 0 \tag{A.5}$$

and
$$\int_{\mathbb{R}^3} \varepsilon^{2q} \nabla v^i_{\varepsilon,y^i} \cdot \nabla v^j_{\varepsilon,y^j} \mathrm{d}x = O(\varepsilon^\gamma) \text{ as } \varepsilon \to 0, \tag{A.6}$$

where $i, j = 1, 2, \cdots, k, i \ne j$ and any $r, s \ge 1$.

B. Some analysis results

Lemma B.1. *Let $1 \le p < \infty$ and $f(x) \in L^p(\mathbb{R}^k)$ for some $k \ge 1$. Then we have:*

(i) *For any fixed $t > 0$, the function $f(\frac{x}{t})$ is Lebesgue measurable on $\mathbb{R}^k$ as a function of x.*

(ii) *If $t_0 > 0$ and $\{t_n\}$ is a sequence of positive numbers with $\lim_{n\to\infty} t_n = t_0$, then*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^k} |f(\frac{x}{t_n})|^p \mathrm{d}m_k = \int_{\mathbb{R}^k} |f(\frac{x}{t_0})|^p \mathrm{d}m_k \tag{B.1}$$

and
$$\lim_{n\to+\infty} \int_{\mathbb{R}^k} |f(\frac{x}{t_n}) - f(\frac{x}{t_0})|^p \mathrm{d}m_k = 0, \tag{B.2}$$

where m_k denotes the Lebesgue measure in $\mathbb{R}^k$.

Proof. The proof is standard. We omit the details. □

Next, we consider the translation operator in $L^p(\mathbb{R}^k)$ $(1 \le p < \infty)$. For any $x_0 \in \mathbb{R}^k$ for some $k \ge 1$, define $\tau_{x_0} : L^p(\mathbb{R}^k) \to L^p(\mathbb{R}^k)$ as follows:

$$(\tau_{x_0} f)(x) := f(x - x_0), \quad \text{for any } f \in L^p(\mathbb{R}^k).$$

We say that τ_{x_0} is a translation operator in $L^p(\mathbb{R}^k)$. In the following, we give a well-known result.

Lemma B.2. *Let $1 \le p < \infty$. Suppose that $f(x) \in L^p(\mathbb{R}^k)$ for some $k \ge 1$. Then there holds*

$$\lim_{y\to 0} \|\tau_y f - f\|_{L^p(\mathbb{R}^k)} = 0. \tag{B.3}$$

As a consequence of Lemma B.2, we obtain the following lemma directly.

Lemma B.3. *Let $1 \le p < \infty$. Suppose that $f_n(x), f(x) \in L^p(\mathbb{R}^k)$ for some $k \ge 1$ and $f_n(x) \to f(x)$ in $L^p(\mathbb{R}^k)$. Then, for any sequence $\{y_n\} \subseteq \mathbb{R}^k$ with $|y_n| = o_n(1)$, there holds*

$$f_n(x+y_n) \to f(x) \quad \text{in } L^p(\mathbb{R}^k) \text{ as } n \to \infty. \tag{B.4}$$

Proof. Notice that

$$\|f_n(x+y_n)-f(x)\|_{L^p(\mathbb{R}^k)} \le \|f_n(x+y_n)-f(x+y_n)\|_{L^p(\mathbb{R}^k)}+\|f(x+y_n)-f(x)\|_{L^p(\mathbb{R}^k)}.$$

It is obvious that

$$\|f_n(x+y_n)-f(x+y_n)\|_{L^p(\mathbb{R}^k)} = \|f_n(x)-f(x)\|_{L^p(\mathbb{R}^k)} \to 0 \ \text{ as } n \to \infty, \tag{B.5}$$

since $f_n(x) \to f(x)$ in $L^p(\mathbb{R}^k)$. As a consequence of Lemma B.2, we deduce

$$\|f(x+y_n)-f(x)\|_{L^p(\mathbb{R}^k)} \to 0 \ \text{ as } n \to \infty. \tag{B.6}$$

Combining (B.5) and (B.6) implies (B.4). The proof of Lemma B.3 is complete. □

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