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Nonrelativistic Limit of Ground State Solutions for Nonlinear Dirac-Klein-Gordon Systems

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We study the nonrelativistic limit and some properties of the solutions

$$(\psi, \phi) := (u, v, \phi) \in \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{R}$$

for the following nonlinear Dirac-Klein-Gordon systems:

$$\begin{cases} ic \displaystyle\sum_{k=1}^{3} \alpha_k \partial_k \psi - mc^2 \beta \psi - \omega \psi - \lambda \phi \beta \psi = |\psi|^{p-2} \psi, \\ -\Delta \phi + c^2 M^2 \phi = 4\pi \lambda (\beta \psi) \cdot \psi, \end{cases}$$

where $p \in [\frac{12}{5}, \frac{8}{3}]$, c denotes the speed of light, $m > 0$ is the mass of the electron. We show that the first component u and the last one ϕ of ground state solutions for nonlinear Dirac-Klein-Gordon systems converge to zero and the second one v converges to corresponding solutions of a coupled system of nonlinear Schrödinger equations as the speed of light tends to infinity for electrons with small mass. Moreover, we also prove the uniform boundedness and the exponential decay properties of the solutions for the nonlinear Dirac-Klein-Gordon systems with respect to the speed of light c.

Keywords: Nonlinear Dirac-Klein-Gordon systems, nonrelativistic limit, ground state solution.

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1. Introduction and main results

Relativistic quantum mechanics is quantum mechanics applied with special relativity. It is applicable to Dirac fermions with velocities comparable to the speed of light. Nonrelativistic quantum mechanics refers to the mathematical formulation of quantum mechanics applied in Galilean relativity. The aim of this paper is to extend the study of the wave function and study in depth the nonrelativistic limits of the nonlinear Dirac-Klein-Gordon systems in relativistic quantum mechanics:

$$\begin{cases} ic \displaystyle\sum_{k=1}^{3} \alpha_k \partial_k \psi - mc^2 \beta \psi - \omega \psi - \lambda \phi \beta \psi = g(|\psi|) \psi, \\ -\Delta \phi + c^2 M^2 \phi = 4\pi \lambda (\beta \psi) \cdot \psi, \end{cases} \tag{1}$$

where $\psi : \mathbb{R}^3 \to \mathbb{C}^4, \phi : \mathbb{R}^3 \to \mathbb{R}, \partial_k = \frac{\partial}{\partial x_k}$, c denotes the speed of light, $\lambda > 0$ is a coupling constant, $m > 0$ is the mass of the electron and M is the mass of the meson (we use the notation $\psi \cdot \varphi$ to express the inner product of $\psi, \varphi \in \mathbb{C}$).

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We want to study the model with the involved velocities are much smaller than the light velocity c, so that relativistic effects can be neglected. Here $\alpha_1, \alpha_2, \alpha_3$ and β are 4×4 Pauli-Dirac matrices (in 2×2 blocks):

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,$$

with
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In addition, it is easily verified that β and α_k satisfy the following anticommutation relations

$$\begin{cases} \alpha_k \alpha_l + \alpha_l \alpha_k = 2\delta_{kl} I_4, \\ \alpha_k \beta + \beta \alpha_k = 0, \\ \beta^2 = I_4. \end{cases}$$

Physically, system (1) describes the Dirac and Klein–Gordon equations coupled through the Yukawa interaction between a Dirac field $\psi \in \mathbb{C}^4$ and a scalar field $\phi \in \mathbb{R}$ (see [6]). This system is inspired by approximate descriptions of the external force involving only functions of fields. The nonlinear self-coupling $g(|\psi|)\psi$, which describes a self-interaction in Quantum electrodynamics, gives a closer description of many particles found in the real world. Other nonlinearities are considered to be possible basis models for unified field theories (see [18, 19, 23], etc. and references therein). Its most general form is

$$\begin{cases} i\hbar\partial_t \hat{\psi} + ic\hbar \sum_{k=1}^{3} \alpha_k \partial_k \hat{\psi} - mc^2 \beta \hat{\psi} - \lambda \phi \beta \hat{\psi} = G_{\hat{\psi}}(x, \hat{\psi}), \\ \frac{\hbar^2}{c^2} \partial_t^2 \phi - \hbar^2 \Delta \phi + c^2 M^2 \phi = 4\pi\lambda(\beta\hat{\psi}) \cdot \hat{\psi}, \end{cases} \tag{2}$$

for $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, where $\hbar$ is Planck's constant. Assuming that $G(x, e^{i\theta}\hat{\psi}) = G(x, \hat{\psi})$ for all $\theta \in [0, 2\pi]$, a standing wave solution of (2) is a solution of the form

$$\begin{cases} \hat{\psi}(t, x) = \psi(x) e^{-i\omega t/\hbar}, \quad \omega \in \mathbb{R}, \\ \phi = \phi(x) \end{cases}$$

It is clear that $(\hat{\psi}(t, x), \phi(t, x))$ solves (2) if and only if $(\psi(x), \phi(x))$ solves the nonlinear Dirac-Klein-Gordon system

$$\begin{cases} ic\hbar \sum_{k=1}^{3} \alpha_k \partial_k \psi - mc^2 \beta \psi - \omega \psi - \lambda \phi \beta \psi = g(x, \psi), \\ -\hbar^2 \Delta \phi + c^2 M^2 \phi = 4\pi\lambda(\beta\psi) \cdot \psi. \end{cases} \tag{3}$$

For small $\hbar$, the solitary waves are referred to semi-classical states. In [14], Ding and Xu considered the semi-classical problem for nonlinear Dirac-Klein-Gordon systems. They were devoted to the existence of stationary semi-classical solutions to the Dirac-Klein-Gordon system with some general subcritical self-coupling nonlinearity. Moreover, these soluntions converge to a ground state solution of auto-nonlinear Dirac-Klein-Gordon systems (the limit equation). As for the semi-classical problem, we also refer the reader to [11, 12, 13] and references therein.

For large c, that is the nonrelativistic limit problem. There are many papers about the nonrelativistic limit for different systems. In [26], by Strichartz estimates the authors study the nonrelativistic limit of the Cauchy problem for the nonlinear Klein-Gordon equation and prove that any finite energy solution converges to the corresponding solution of the nonlinear Schrödinger equation in the energy space, after the infinite oscillation in time is removed. Then, in [4] a solution of the Dirac-Maxwell system without nonlinearity in the nonrelativistic limit $c \to \infty$, yields a solution of the Schrödinger-Poisson system, where the spin and magnetic field no longer appear. The proof relies on modifications of the bilinear null form estimates of Klainerman and Machedon [25]. Moreover, in [17] the Hartree-Fock equations are proved to be the nonrelativistic limit of the Dirac-Fock equations as far as convergence of stationary states is concerned. As for the Dirac equation, on noncompact metric graphs with localized Kerr nonlinearities, the existence and multiplicity of the bound states (arising as critical points of the nonlinear Dirac equation action functional) has been discussed in [7] and they proved that, in the L^2-subcritical case, these bound states converge to the bound states of the nonlinear Schrödinger equation in the nonrelativistic limit. In [27, 28], the authors considered the following Cauchy problem for the nonlinear Dirac equation

$$i\partial_t\psi + ic\alpha\nabla\psi - c^2\beta\psi + 2\lambda(\beta\psi,\psi)\beta\psi = 0, \qquad \psi(0) = \psi_0.$$

They proved that the solutions of the nonlinear Dirac equation after multiplication by a phase factor (dependent on c) converge to the solutions of a coupled system of nonlinear Schrödinger type equations by the Strichartz estimates. Recently, Ding, Dong and Guo in [10] obtained that the solutions of nonlinear Dirac equation converge to the corresponding solutions of a coupled system of nonlinear Schrödinger equations as the speed of light tends to infinity. Moreover, they prove the uniform boundedness and the exponential decay properties of the solutions for the nonlinear Dirac equation with respect to the speed of light c. In [1], Bao and the collaborators compare numerically spatial/temporal resolution of various numerical methods for the discretization of the Dirac equation in the nonrelativistic limit regime. Then, the authors in [35] proposed and compared numerically spatial/temporal resolution of various efficient numerical methods for solving the Klein-Gordon-Dirac system in the nonrelativistic limit regime. Unfortunately, none of these numerical approaches can handle standing state systems.

As a consequence of the above observation, it aroused our interest to study the nonrelativistic limit of standing wave for the nonlinear Dirac-Klein-Gordon systems (DKGS). We are devoted to obtain that the limit equation is a coupled system of nonlinear Schrödinger equations (NLSE), rather than the semi-classical limit equation (auto-nonlinear Dirac-Klein-Gordon systems).

More precisely, we are concerned with the nonrelativistic limit that, as $c, \omega \to \infty$, the stationary solution (ψ, ϕ) (where $\psi := (u, v) \in \mathbb{C}^4$) of (1) converge to the corresponding solutions $(0, v, 0)$ of a coupled system of nonlinear Schrödinger type equations

$$\begin{cases} -\Delta v_1 + 2\nu v_1 = 2m|v|^{p-2}v_1, \\ -\Delta v_2 + 2\nu v_2 = 2m|v|^{p-2}v_2, \end{cases} \tag{4}$$

where $v = (v_1, v_2) : \mathbb{R}^3 \to \mathbb{C}^2$ is the wave functions correspond to positive energy values, and $\nu > 0$ is a constant. Additionally, it is natural to ask if the nonlinear Dirac-Klein-Gordon systems (1) would have more regularities with respect to the speed of light c, i.e. the uniform boundedness and exponential decay of corresponding solutions.

Without loss of generality, let $\hbar = 1$, we consider the nonlinearity $g(|\psi|) = |\psi|^{p-2}$. For notational convenience, writing $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha \cdot \nabla = \sum_{k=1}^{3} \alpha_k \partial_k$, we rewrite system (1) as

$$\begin{cases} ic\alpha \cdot \nabla\psi - mc^2\beta\psi - \omega\psi - \lambda\phi\beta\psi = |\psi|^{p-2}\psi, \\ -\Delta\phi + c^2M^2\phi = 4\pi\lambda(\beta\psi)\psi. \end{cases} \tag{5}$$

Before presenting our results, we recall that solutions of the nonlinear Dirac-Klein-Gordon systems (5) can be obtained by variational methods if $\omega \in (-mc^2, mc^2)$, see [2]. Then, we get the following connection between the nonlinear Dirac-Klein-Gordon systems (5) and the nonlinear Schrödinger equation (4) without the initial value conditions.

Theorem 1.1. *Let $\nu > 0$, $p \in [\frac{12}{5}, \frac{8}{3}]$. Take two real sequences $\{c_n\}, \{\omega_n\}$ such that*

$$0 < c_n, \omega_n \to +\infty, \tag{6}$$

$$0 < \omega_n < mc_n^2, \tag{7}$$

$$mc_n^2 - \omega_n \to \frac{\nu}{m}, \tag{8}$$

as $n \to \infty$. If $\{(\psi_n, \phi_n)\}$ (where $\psi_n := (u_n, v_n) \in \mathbb{C}^4$) is a sequence of solutions for system (5) with frequency ω_n at speed of light c_n, there exists a mass m_0, λ_0, such that for $m \leq m_0$, $\lambda \leq \lambda_0$, up to a subsequence,

$$u_n \to 0, \quad v_n \to v, \quad \text{in } H^1(\mathbb{R}^3, \mathbb{C}^2),$$
$$\phi_n \to 0, \quad \text{in } H^1(\mathbb{R}^3, \mathbb{R}),$$

as $n \to \infty$, where $v : \mathbb{R}^3 \to \mathbb{C}^2$ is a solution for the NLSE (4) with frequency ν.

Remark 1.2. Just as [32], in nonrelativistic quantum mechanics, the expression "speed of light c_n, with $c_n \to \infty$ " has to be meant as if it should be "infinitely large" compared to the other velocities. Moreover, the frequency ω has the same scale as c^2. □

Then, we obtain the uniform boundedness and exponential decay by maximum principle and sub-solution estimate.

Theorem 1.3. *Under the assumptions of Theorem 1.1, the sequence $\{\psi_n\}$ is bounded in $L^\infty(\mathbb{R}^3, \mathbb{C}^4)$, and the sequence $\{\phi_n\}$ is bounded in $L^\infty(\mathbb{R}^3, \mathbb{R})$ uniformly with respect to n. Moreover, there exists $C, \tilde{C} > 0$ such that*

$$|\psi_n(x)| \leq Ce^{-\tilde{C}|x|}, \quad \text{for all } x \in \mathbb{R}^3, \quad \text{uniformly in } n \in \mathbb{N}.$$

The idea of Theorem 1.1 is derived from Esteban and Séré in [17] for the Dirac-Fock equation. In the sequel, Borrelli, Carlone and Tentarelli in [7] used this idea for the nonlinear Dirac equation on metric graphs. Ding, Dong and Guo [10] studied nonlinear Dirac equations using similar methods. In [17], the sequence of the Lagrange multipliers of bound states with L^2-norm fixed was estimated due to the constraint conditions. In [7], the Gagliardo-Nirenberg inequality (about the L^∞-norm) was used to estimate the uniform H^1-norm boundedness only for DKGS on one-dimensional metric graphs. Moreover, the authors dealt with a nonlinearity localized on a compact part of the graph, which can overcome the lack of compactness. Recently, they considered the nonlinear Dirac equation with Kerr-type nonlinearity on noncompact metric graphs with a finite number of edges, see [8]. In [10], the general Gagliardo-Nirenberg was used to estimate the uniform H^1-norm boundedness for the nonlinear Dirac equations. However, in this paper, we no longer have L^∞-norm estimate in $\mathbb{R}^3$. Therefore we need more accurate estimate for H^1-norm and make the study of the geometry of the energy functional a bit more delicate. In this process, we have to consider the effect of the Klein-Gordon equations and get more properties about the second equation. For example, the method of L^2-bounded estimate is invalid in [10]. Hence, we introduce and estimate a new norm to overcome the diffculty. For Theorem 1.3, similar to [10, 14, 16], we study the uniform L^∞-boundedness and exponential decay by maximum principle and sub-solution estimate. Unlike [15, 16], we cannot use the translation technique to obtain the uniform boundedness with respect to n. Thus, we need the subtle estimation for the norm of v_n in order to get L^∞-norm boundedness uniformly.

Remark 1.4. In general, due to system (3) is linear with respect to $\hbar$, the method for the semi-classical limit is that letting $\varphi(x) = \psi(\hbar x)$ and $\tilde{\phi}(x) = \phi(\hbar x)$, system (3) is equivalent to

$$\begin{cases} ic\sum_{k=1}^{3}\alpha_k\partial_k\varphi - mc^2\beta\varphi - \omega\varphi - \lambda\tilde{\phi}\beta\varphi = g(\hbar x, \varphi), \\ -\Delta\tilde{\phi} + c^2M^2\tilde{\phi} = 4\pi\lambda(\beta\varphi)\cdot\varphi. \end{cases}$$

However, the fact that system (3) is quadratic with respect to c makes the above method invalid when we deal with nonrelativistic limit. □

This paper is organized as follows. In the next section, we present some preliminary notions on the Dirac-Klein-Gordon systems and some basic results which will be used later. In Section 3, we get the uniform boundedness. The proofs of Theorem 1.1 and Theorem 1.3 are obtained in Section 4.

Notation. Throughout this paper, we make use of the following notations.

- For any $R > 0$ and for any $x \in \mathbb{R}^3$, $B_R(x)$ denotes the ball of radius R centered at x;
- $\|\cdot\|_{L^q}$ denotes the usual norm of the space $L^q(\mathbb{R}^3, \mathbb{C}^4)$, $L^q(\mathbb{R}^3, \mathbb{C}^2)$ or $L^q(\mathbb{R}^3, \mathbb{R})$, $1 \le q \le \infty$;
- $\|\cdot\|_{H^1}$ denotes the usual norm of the space $H^1(\mathbb{R}^3, \mathbb{C}^4)$, $H^1(\mathbb{R}^3, \mathbb{C}^2)$ or $H^1(\mathbb{R}^3, \mathbb{R})$;
- $\psi \cdot \varphi$ denotes the scalar product in $\mathbb{C}^4$ of ψ and φ, i.e., $\psi\cdot\varphi = \sum_{i=1}^{4}\psi_i\bar{\varphi}_i$;
- C or $C_i (i = 1, 2, \cdots)$ are some positive constants may change from line to line.

2. The functional-analytic setting and preliminary results

2.1. The functional-analytic setting

For convenience, let $H_n := ic_n\alpha \cdot \nabla - mc_n^2\beta$ denote the Dirac operator.

It is well known that H_n is a selfadjoint operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with the domain $D(H_n) = H^1(\mathbb{R}^3, \mathbb{C}^4)$, see [9]. Then, the following lemma about the spectrum and essential spectrum of the Dirac operator H_n can be obtained directly by Fourier analysis, see [16].

Lemma 2.1. *$\sigma(H_n) = \sigma_e(H_n) = \mathbb{R} \setminus (-mc_n^2, mc_n^2)$, where $\sigma(\cdot)$ and $\sigma_e(\cdot)$ denote the spectrum and the essential spectrum.*

Remark 2.2. Let $H_{\omega_n} = H_n - \omega_n$, we have

$$\sigma(H_{\omega_n}) = (-\infty, -mc_n^2 - \omega_n] \cup [mc_n^2 - \omega_n, +\infty).$$

Under the assumptions of c_n and ω_n, we have

$$mc_n^2 + \omega_n \to +\infty,\ mc_n^2 - \omega_n \to \frac{\nu}{m} \ \text{ as } n \to +\infty.$$

Thus, from the view of spectrum, for n tends to infinity, the behavior of H_{ω_n} turns to be a positive definite operator, i.e. the Schrödinger operator. The phenomenon is also called spectral concentration of Dirac operators and it is studied by many authors from various points of view, see [21, 22, 33, 34]. □

The space $L^2(\mathbb{R}^3, \mathbb{C}^4)$ possesses the orthogonal decomposition:

$$L^2 = L^- \oplus L^+, \quad \psi = \psi^- + \psi^+,$$

so that H_n is negative definite on L^- and positive definite on L^+. Let $|H_n|$ denote the absolute value of H_n and $|H_n|^{\frac{1}{2}}$ its square root. Define $E := D(|H_n|^{\frac{1}{2}}) = H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{C}^4)$ as the Hilbert space with the inner product

$$(\psi, \tilde{\psi})_n = \Re(|H_n|^{\frac{1}{2}}\psi, |H_n|^{\frac{1}{2}}\tilde{\psi})_{L^2}$$

and the induced norm $\|\psi\|_n = (\psi, \psi)_n^{\frac{1}{2}}$, where $\Re$ stands for the real part of a complex number. Since $\sigma(H_n) = \mathbb{R} \setminus (-mc_n^2, mc_n^2)$, one has

$$mc_n^2\|\psi\|_{L^2}^2 \le \|\psi\|_n^2, \quad \text{for all } \psi \in E. \tag{9}$$

Moreover, it is clear that E possesses the following decomposition

$$E = E^- \oplus E^+, \quad \text{where } E^\pm = E \cap L^\pm,$$

which is the orthogonal decomposition with respect to inner products $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$. Furthermore, from [15, Proposition 2.1], this decomposition of E also induces a natural decomposition of $L^q(\mathbb{R}^3, \mathbb{C}^4), q \in [2, 3]$, hence there is $c_q > 0$ such that

$$c_q\|\psi^\pm\|_{L^q}^q \le \|\psi\|_{L^q}^q, \quad \text{for all } \psi \in E.$$

Let $H^1(\mathbb{R}^3, \mathbb{R})$ be equipped with the equivalent norm

$$\|v\|_{H^1} = \left(\int_{\mathbb{R}^3} (|\nabla v|^2 + M^2 v^2)dx\right)^{1/2}, \quad \forall v \in H^1\left(\mathbb{R}^3, \mathbb{R}\right).$$

Then (5) can be reduced to a single equation with a nonlocal term. Actually, for any $v \in H^1(\mathbb{R}^3, \mathbb{R})$,

$$\begin{aligned}\left|4\pi \int_{\mathbb{R}^3} \lambda(\beta\psi)\psi \cdot v dx\right| &\leq 4\pi\lambda \int_{\mathbb{R}^3} |\psi|^2 |v| dx \leq 4\pi\lambda \|\psi\|^2_{L^{\frac{12}{5}}} \|v\|_{L^6} \\ &\leq 4\pi\lambda S^{-1/2} \|\psi\|^2_{L^{\frac{12}{5}}} \|v\|_{H^1}, \end{aligned} \tag{10}$$

where S is the Sobolev embedding constant: $S\|v\|^2_{L^6} \leq \|v\|^2_{H^1}$ for all $v \in H^1(\mathbb{R}^3, \mathbb{R})$. Hence there exists a unique $\phi_{n,\psi} \in H^1$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_{n,\psi} \cdot \nabla z + c_n^2 M^2 \phi_{n,\psi} z dx = 4\pi \int_{\mathbb{R}^3} \lambda(\beta\psi)\psi \cdot z dx \tag{11}$$

for all $z \in H^1$. It follows that $\phi_{n,\psi}$ satisfies the Schrödinger type equation

$$-\Delta\phi_{n,\psi} + c_n^2 M^2 \cdot \phi_{n,\psi} = 4\pi\lambda(\beta\psi)\psi$$

and that

$$\phi_{n,\psi}(x) = \int_{\mathbb{R}^3} \lambda \frac{[(\beta\psi)\psi](y)}{|x-y|} e^{-c_n M|x-y|} dy.$$

Substituting $\phi_{n,\psi}$ in (5) , we are led to the equation

$$H_{\omega_n}\psi - \lambda\phi_{n,\psi}\beta\psi = |\psi|^{p-2}\psi.$$

On E, we define the energy functional Φ_n corresponding to equation (5) with $c := c_n$, $\omega := \omega_n$ by

$$\Phi_n(\psi) = \frac{1}{2}\int_{\mathbb{R}^3} \langle \psi, H_n\psi \rangle dx - \frac{\omega_n}{2}\int_{\mathbb{R}^3} |\psi|^2 dx - \Gamma_n(\psi) - \int_{\mathbb{R}^3} |\psi|^p dx, \tag{12}$$

for $\psi = \psi^+ + \psi^- \in E$, where

$$\Gamma_n(\psi) = \frac{1}{4}\int_{\mathbb{R}^3} \lambda\phi_{n,\psi} \cdot (\beta\psi)\psi dx = \frac{1}{4}\iint \frac{\lambda[(\beta\psi)\psi](x)\lambda[(\beta\psi)\psi](y)}{|x-y|} e^{-c_n M|x-y|} dy dx.$$

Note that $\Gamma_n'(\psi)\varphi = \frac{d}{dt}\Gamma_n(\psi + t\varphi)\big|_{t=0}$, so

$$\begin{aligned}\Gamma_n'(\psi)\varphi &= \frac{\lambda^2}{2}\Re \iint \frac{e^{-c_n M|x-y|}}{|x-y|}([(\beta\psi)\psi](x)[(\beta\psi)\varphi](y) + [(\beta\psi)\psi](y)[(\beta\psi)\varphi](x)) dy dx \\ &= \lambda \int_{\mathbb{R}^3} \phi_{n,\psi} \cdot \Re(\beta\psi)\varphi dx.\end{aligned}$$

It follows by standard arguments that $\Phi_n \in \mathcal{C}^1(E, \mathbb{R})$. Moreover, in [13, Lemma 2.1] it is proved that critical points of Φ_n are weak solutions of nonlinear Dirac-Klein-Gordon system (5).

Notice that $\phi_{n,\psi}$ satisfies the equation $-\Delta\phi_{n,\psi} + c_n^2 M^2 \phi_{n,\psi} = 4\pi\lambda(\beta\psi)\psi$.

Hence, using ϕ_n as a test function, we have

$$\|\phi_{n,\psi}\|^2_{H^1} \leq 4C_1\pi\lambda \int_{\mathbb{R}^3} |\phi_{n,\psi}| \cdot |\psi|^2 dx. \tag{13}$$

By Hölder's inequality we have

$$\|\phi_{n,\psi}\|_{L^6}^2 \le C_2 \|\phi_{n,\psi}\|_{H^1}^2 \le C_3 \|\phi_{n,\psi}\|_{L^6} \|\psi\|_{L^p} \|\psi\|_{L^q},$$

that is,

$$\|\phi_{n,\psi}\|_{L^6} \le C \|\psi\|_{L^p} \|\psi\|_{L^q}, \tag{14}$$

where $2 < q = \dfrac{6p}{5p-6} < 3$. Then, we have the following

Proposition 2.3. [14] Γ_λ *is non-negative and weakly sequentially lower semi-continuous.*

2.2. Existence of solutions for DKGS

Let $A = \int_{\mathbb{R}} \lambda dE_\lambda$ be the spectral representation of the selfadjoint operator A in H where $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the unique spectral family of the selfadjoint operator A. Recall that a family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of orthogonal projections in a separable complex Hilbert space H is called a spectral family (or a resolution of the identity) if it satisfies:

(i) $E_\lambda E_\mu = E_{\min\{\lambda,\mu\}}$, for $\lambda, \mu \in \mathbb{R}$.

(ii) $E_{-\infty} = 0$, $E_{+\infty} = I$, where $E_{\pm\infty} u = \lim\limits_{\lambda\to\pm\infty} E_\lambda u$, $\forall u \in H$.

(iii) $E_{\lambda+0} = E_\lambda$, where $E_{\lambda+0} u = \lim\limits_{\epsilon\to 0+} E_{\lambda+\epsilon} u$, $\forall u \in H$,

where the limits are taken in the norm of H. A further description of spectral family can be found in [24]. Then by the spectral theorem, there exists a bijection between the set of spectral families and the set of selfadjoint operators.

According to [14], we can obtain the existence of the least energy solutions for the DKGS (5). Here, we just outline the main points of the proof. It is point out that we will reconstruct the linking structure of the energy functional a bit more delicate.

Let V be the space of the spinors

$$\eta = \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix}, \quad \text{where } \eta_2 \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{C}^2),$$

which is clearly a subset of E. Moreover, a simple computation shows that

$$\int_{\mathbb{R}^3} \langle \eta, H_n \eta \rangle dx = mc_n^2 \int_{\mathbb{R}^3} |\eta_2|^2 dx. \tag{15}$$

The assumptions (6), (7) and (8) imply that

$$0 < C_1 \le mc_n^2 - \omega_n \le C_2. \tag{16}$$

Obviously, by (16), there exists a $\bar{\gamma} > \gamma_0 := mc_n^2 - \omega_n$, which is independent on n, such that

$$V_0 := V \cap (E_{\bar{\gamma}} - E_{\gamma_0}) L^2 \ne \emptyset,$$

where $(E_\gamma)_{\gamma\in\mathbb{R}}$ denote the spectral families of H_n. We can choose a element $e \in V$ such that $e^+ \in V_0 \subset E^+$ (independent on n) with $\|e^+\| = 1$. Define $\hat{E} := E^- \oplus \mathbb{R}^+ e^+$, where $\mathbb{R}^+ := [0, \infty)$.

Then, by [10, 14], we can easily get the following linking lemma.

Lemma 2.4. *For every n, we have the following linking structure:*

(i) *there exist constants $\rho, r^* > 0$ such that $\kappa_n := \inf \Phi_n(\partial B_\rho \cap E^+) \geq r^* > 0$, where $B_\rho = \{\psi \in E : \|\psi\| \leq \rho\}$.*

(ii) $\sup \Phi_n(\hat{E}) < \infty$, *and there is a constant $R > 0$ such that* $\sup \Phi_n(\hat{E} \setminus B_R) \leq 0$, *where $B_R = \{\psi \in \hat{E} : \|\psi\| \leq R\}$.*

For $z \in E^+$, $E_z := E^- \oplus \mathbb{R}^+ z$. Now let us define ([3, 31]),

$$\iota_n := \inf_{z \in E^+ \setminus \{0\}} \max_{\psi \in E_z} \Phi_n(\psi), \tag{17}$$

As a consequence of Lemma 2.4, we have

$$\kappa := \inf \kappa_n \leq \iota_n \leq \sup \Phi_n(\hat{E}). \tag{18}$$

Denote the critical set and the least energy of Φ_n as

$$\mathscr{K}_n := \{\psi \in E : \Phi_n'(\psi) = 0\} \quad \text{and} \quad \gamma_n := \inf \{\Phi_n(\psi) : \psi \in \mathscr{K}_n \setminus \{0\}\}.$$

Additionally, it is standard to see that there exists a unique $h_n : E^+ \to E^-$ with

$$\Phi_n(\psi + h_n(\psi)) = \max_{\tilde{\psi} \in E^-} \Phi_n(\psi + \tilde{\psi}).$$

In the sequel, we fix λ in the interval $(0, \lambda_0]$. Next, setting $I_n : E^+ \to \mathbb{R}$ by

$$I_n(\psi) = \Phi_n(\psi + h_n(\psi))$$

and the Nehari manifold for the functional I_n in E^+ is

$$\mathscr{N}_n = \left\{\psi \in E^+ \setminus \{0\} : I_n'(\psi)\psi = 0\right\}.$$

Using the concentration compactness principle and the invariance of the norm and of the functional under the $\mathbb{Z}^3$-action (see [14]) one obtains the following

Proposition 2.5. *There hold*

(1) $\mathscr{K}_n \neq \varnothing$ *and* $\gamma_n > 0$ *is attained.*

(2) $\iota_n = \gamma_n = \inf_{\psi \in \mathscr{N}_n} I_n(\psi)$.

That is, for every n, there exists a critical point $\psi_n \neq 0$ such that $\Phi_n(\psi_n) = \iota_n$. Moreover, by the standard regular estimation, for every n, we have $\psi_n \in W^{1,q}(\mathbb{R}^3, \mathbb{C}^4)$, for any $q \in [2, +\infty)$. Then, let

$$\phi_n(x) = \int_{\mathbb{R}^3} \lambda \frac{[(\beta\psi_n)\psi_n](y)}{|x-y|} e^{-c_n M|x-y|} dy, \tag{19}$$

for every n, (ψ_n, ϕ_n) is a ground state solution of (5).

Finally, we recall the following Gagliardo-Nirenberg inequalities [29], which will be used later.

Lemma 2.6. *For every $q \in [2, 6)$, there exists $\mu_q > 0$ such that*

$$\|\psi\|_{L^q} \leq \mu_q \|\psi\|_{L^2}^{q_1} \|\nabla\psi\|_{L^2}^{q_2}, \quad \forall\, \psi \in H^1(\mathbb{R}^3, \mathbb{C}^4). \tag{20}$$

where $q_1 = \frac{6-q}{2q}$ *and* $q_2 = \frac{3q-6}{2q}$.

3. Uniform boundedness of the solutions for DKGS

In the section, we prove H^1-boundedness of the solutions of the DKGS (5). Firstly, we prove that the sequence $\{\psi_n\}$ defined above is bounded in $L^p(\mathbb{R}^3, \mathbb{C}^4)$ uniformly with respect to n.

Lemma 3.1. *The sequence $\{\psi_n\}$ is bounded in $L^p(\mathbb{R}^3, \mathbb{C}^4)$ uniformly with respect to n, as well as the associated minimax levels $\{\iota_n\}$.*

Proof. For every spinor $\psi \in \hat{E} := E^- \oplus \mathbb{R}^+ e^+$, one has $\psi = \varphi^\perp + \lambda_e e$, where $\lambda_e \in \mathbb{R}, e = e^- + e^+$ and $\varphi^\perp \in E^-$ is orthogonal to $\lambda_e e$. Hence, by (15) and (16),

$$\begin{aligned}\Phi_n(\psi) &\le \frac{1}{2}\int_{\mathbb{R}^3} \langle \lambda_e e, (H_n - \omega_n)\lambda_e e\rangle dx - \frac{1}{p}\int_{\mathbb{R}^3} |\lambda_e e^+|^p dx \\ &= \frac{mc_n^2 - \omega_n}{2}\int_{\mathbb{R}^3} |\lambda_e e|^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |\lambda_e e^+|^p dx \\ &\le \frac{C_1|\lambda_e|^2}{2}\int_{\mathbb{R}^3} |e|^2 dx - \frac{|\lambda_e|^p}{p}\int_{\mathbb{R}^3} |e^+|^p dx \\ &\le |\lambda_e|^2(C_2 - C_3|\lambda_e|^{p-2}) \le C_4,\end{aligned}$$

where C_i are constant independent on n.

Thus, by the fact that $0 \le 4\Gamma_n(\psi_n) = \Gamma_n'(\psi_n)\psi_n$ and (18), we have

$$\begin{aligned}C \ge \max_{\hat{E}} \Phi_n \ge \iota_n &= \Phi_n(\psi_n) - \frac{1}{2}\langle \Phi_n'(\psi_n), \psi_n\rangle = \Gamma_n(\psi_n) + (\frac{1}{2} - \frac{1}{p})\int_{\mathbb{R}^3} |\psi_n|^p dx \\ &\ge (\frac{1}{2} - \frac{1}{p})\int_{\mathbb{R}^3} |\psi_n|^p dx. \end{aligned} \tag{21}$$

This completes the proof. □

Lemma 3.2. *$\{\psi_n\}$ is uniformly bounded in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with respect to n.*

Proof. Let $q = \frac{6p}{5p-6}$, then $2 < q \le p < 3$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{6} = 1$. Set

$$\zeta = \begin{cases} 0, & \text{if } q = p, \\ \frac{2(p-q)}{q(p-2)}, & \text{if } q < p, \end{cases}$$

we deduce that $\zeta < 1$ and

$$\|u\|_{L^q} \le \|u\|_{L^2}^{\zeta} \cdot \|u\|_{L^p}^{1-\zeta}, \quad \text{if } 2 < q \leqslant p. \tag{22}$$

Thus, in virtue of (14), (22), Lemma 3.1 and Hölder's inequality, one gets

$$\begin{aligned}|\Gamma_n'(\psi_n)(\psi_n^+ - \psi_n^-)| &\le \lambda\|\phi_n\|_{L^6}\|\psi_n\|_{L^p}\|\psi_n^+ - \psi_n^-\|_{L^q} \le C_1\|\psi_n\|_{L^p}^{4-2\zeta}\|\psi_n\|_{L^2}^{2\zeta} \\ &\le C_2\|\psi_n\|_n^{2\zeta}. \end{aligned} \tag{23}$$

Now, by (9), (23), Hölder's inequality and the form of Φ_n', we have

$$\begin{aligned}
&o(1)\|\psi_n\|_n \geq \langle \Phi_n'(\psi_n), \psi_n^+ - \psi_n^- \rangle \\
&= \int_{\mathbb{R}^3} (\psi_n^+ - \psi_n^-, (H_n - \omega_n)\psi_n)dx - \Gamma_n'(\psi_n)(\psi_n^+ - \psi_n^-) - \int_{\mathbb{R}^3} |\psi_n|^{p-2}\psi_n(\psi_n^+ - \psi_n^-)dx \\
&\geq \|\psi_n\|_n^2 - \omega_n\|\psi_n\|_{L^2}^2 - \Gamma_n'(\psi_n)(\psi_n^+ - \psi_n^-) - \int_{\mathbb{R}^3} |\psi_n|^{p-2}\psi_n(\psi_n^+ - \psi_n^-)dx \\
&\geq (1 - \frac{\omega_n}{mc_n^2})\|\psi_n\|_n^2 - C_2\|\psi_n\|_n^{2\zeta} - C_3\|\psi_n\|_p^p,
\end{aligned}$$

which, together with (7), Lemma 3.1 and $\zeta < 1$, implies

$$\|\psi_n\|_n \leq C, \tag{24}$$

where the constant C is independent on n. Therefore, the combinition of (9) shows that $\{\psi_n\}$ is uniformly bounded in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ with respect to n. □

As a consequence, we have the boundedness of the sequence $\{\psi_n\}$ in $H^1(\mathbb{R}^3, \mathbb{C}^4)$.

Lemma 3.3. *$\{\psi_n\}$ is uniformly bounded in $H^1(\mathbb{R}^3, \mathbb{C}^4)$ with respect to n.*

Proof. Observe that ψ_n satisfy

$$H_n\psi_n - \lambda\phi_n\beta\psi_n = \omega_n\psi_n + |\psi_n|^{p-2}\psi_n. \tag{25}$$

Moreover,
$$\|H_n\psi_n - \lambda\phi_n\beta\psi_n\|_{L^2}^2 = \|\omega_n\psi_n + |\psi_n|^{p-2}\psi_n\|_{L^2}^2. \tag{26}$$

An easy compution shows that

$$\|\omega_n\psi_n + |\psi_n|^{p-2}\psi_n\|_{L^2}^2 = \omega_n^2\|\psi_n\|_{L^2}^2 + \int_{\mathbb{R}^3} |\psi_n|^{2p-2}dx + 2\omega_n \int_{\mathbb{R}^3} |\psi_n|^p dx. \tag{27}$$

Now, we estimate the right-hand side of (27). By Lemma 2.6 with $q = 2p - 2$ and Lemma 3.2, we have

$$\int_{\mathbb{R}^3} |\psi_n|^{2p-2}dx \leq \mu_{2p-2}\|\psi_n\|_{L^2}^{4-p}\|\nabla\psi_n\|_{L^2}^{3p-6} \leq C\|\nabla\psi_n\|_{L^2}^{3p-6}. \tag{28}$$

Similarly to (23), we deduce from (24) that

$$\begin{aligned}
\int_{\mathbb{R}^3} |\phi_n||\psi_n|^2dx &\leq \|\phi_n\|_{L^6}\|\psi_n\|_{L^p}\|\psi_n\|_{L^q} \leq C_1\|\psi_n\|_{L^p}^{4-2\zeta}\|\psi_n\|_{L^2}^{2\zeta} \\
&\leq C_2\|\psi_n\|_n^{2\zeta} \leq C,
\end{aligned} \tag{29}$$

where $q = \dfrac{6p}{5p-6}$. On the other hand, the left-hand side of (27) can be rewritten as

$$\begin{aligned}
&\|H_n\psi_n - \lambda\phi_n\beta\psi_n\|_{L^2}^2 \\
&= c_n^2\|\nabla\psi_n\|_{L^2}^2 + m^2c_n^4\|\psi_n\|_{L^2}^2 + \lambda^2 \int_{\mathbb{R}^3} \phi_n^2\psi_n^2dx - 2mc_n^2\lambda \int_{\mathbb{R}^3} \phi_n\psi_n^2dx.
\end{aligned} \tag{30}$$

Combining (26)–(30), we obtain that

$$\begin{aligned}\|\nabla\psi_n\|_{L^2}^2 &\le \|\nabla\psi_n\|_{L^2}^2 + \frac{m^2c_n^4-\omega_n^2}{c_n^2}\|\psi_n\|_{L^2}^2 + \frac{\lambda^2}{c_n^2}\int_{\mathbb{R}^3}\phi_n^2\psi_n^2dx \\ &= \frac{2mc_n^2\lambda}{c_n^2}\int_{\mathbb{R}^3}\phi_n\psi_n^2dx + \frac{1}{c_n^2}\int_{\mathbb{R}^3}|\psi_n|^{2p-2}dx + \frac{2\omega_n}{c_n^2}\int_{\mathbb{R}^3}|\psi_n|^p dx \\ &\le C + \frac{C}{c_n^2}\|\nabla\psi_n\|_{L^2}^{3p-6} + \frac{C\omega_n}{c_n^2},\end{aligned}$$

which, together with $p\in[\frac{12}{5},\frac{8}{3}]$ and $c_n\to\infty$, implies

$$\|\nabla\psi_n\|_{L^2}\le C. \tag{31}$$

Finally, the Lemma follows from Lemma 3.2 and (31). □

By (13), (14) and Hölder's inequality, Lemma 3.3 implies the following lemma.

Lemma 3.4. *$\{\phi_n\}$ is uniformly bounded in $H^1(\mathbb{R}^3,\mathbb{R})$ with respect to n.*

4. Proof of the main theorem

4.1. Proof of Theorem 1.1

This section is devoted to studying the limit of the solutions $\{(\psi_n,\phi_n)\}$ (where $\psi_n := (u_n,v_n)\in\mathbb{C}^4$) of (5) as $n\to\infty$. We will prove that the first components $\{u_n\}$ and the last sequence $\{\phi_n\}$ converge to zero and the second one $\{v_n\}$ converge to a solution of the NLSE (4) in corresponding work space.

In addition, we define

$$a_n := (mc_n^2-\omega_n)b_n \quad \text{and} \quad b_n := \frac{mc_n^2+\omega_n}{c_n^2}, \quad \text{for all } \ n\in\mathbb{N}.$$

It follows from (6)–(8) that

$$b_n\to 2m \quad \text{and} \quad a_n\to 2\nu \quad \text{as } \ n\to\infty. \tag{32}$$

First we prove the first components of the sequence $\{\psi_n\}$ converge to zero, that is $u_n\to 0$ in $H^1(\mathbb{R}^3,\mathbb{C}^2)$. It is worth to point out that we can also estimate the speed of convergence of $\{u_n\}$.

Lemma 4.1. *The sequence $\{u_n\}$ converges to 0 in $H^1(\mathbb{R}^3,\mathbb{C}^2)$ as $n\to\infty$. Moreover, there holds*

$$\|u_n\|_{H^1} = \mathcal{O}(\frac{1}{c_n}) \quad \text{as } n\to\infty. \tag{33}$$

Proof. Since $\psi_n=(u_n,v_n)$ is a solution of the DKGS (5) with $c=c_n,\omega=\omega_n$, for every $n\in\mathbb{N}$, then we can rewrite the equation as follows:

$$ic_n\sigma\cdot\nabla v_n - mc_n^2u_n - \omega_nu_n - \lambda\phi_nu_n = (|u_n|^2+|v_n|^2)^{\frac{p-2}{2}}u_n, \tag{34}$$

$$ic_n\sigma\cdot\nabla u_n + mc_n^2v_n - \omega_nv_n + \lambda\phi_nv_n = (|u_n|^2+|v_n|^2)^{\frac{p-2}{2}}v_n, \tag{35}$$

where $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ and $\sigma\cdot\nabla=\sum_{k=1}^3\sigma_k\partial_k$.

Dividing (35) by c_n, then its L^2-norm squared reads

$$\|\nabla u_n\|_{L^2}^2 = \left\| \frac{mc_n^2 - \omega_n}{c_n} v_n + \frac{1}{c_n}\lambda\phi_n v_n - \frac{1}{c_n}(|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} v_n \right\|_{L^2}^2 .$$

Therefore, by (16), Lemma 3.3 and Lemma 3.4 , we have

$$\begin{aligned}
\|\nabla u_n\|_{L^2}^2 \le& \frac{(mc_n^2-\omega_n)^2}{c_n^2}\|v_n\|_{L^2}^2 + \frac{\lambda^2}{c_n^2}\int_{\mathbb{R}^3}\phi_n^2 v_n^2 dx + \frac{1}{c_n^2}\int_{\mathbb{R}^3}(|u_n|^2+|v_n|^2)^{p-2}|v_n|^2 dx \\
&+ \frac{2(mc_n^2-\omega_n)\lambda}{c_n^2}\int_{\mathbb{R}^3}\phi_n v_n^2 dx - \frac{2\lambda}{c_n^2}\int_{\mathbb{R}^3}\phi_n(|u_n|^2+|v_n|^2)^{\frac{p-2}{2}}|v_n|^2 dx \\
\le& \frac{C}{c_n^2}\|v_n\|_{L^2}^2 + \frac{C}{c_n^2}\int_{\mathbb{R}^3}\phi_n^2 v_n^2 dx + \frac{C}{c_n^2}\int_{\mathbb{R}^3}(|u_n|^2+|v_n|^2)^{p-2}|v_n|^2 dx + \frac{C}{c_n^2}\int_{\mathbb{R}^3}\phi_n v_n^2 dx \\
\le& \frac{C}{c_n^2}\|v_n\|_{L^2}^2 + \frac{C}{c_n^2}\|\phi_n\|_{L^6}^2\|v_n\|_{L^3}^2 + \frac{C}{c_n^2}\int_{\mathbb{R}^3}(|v_n|^{2(p-1)} + |u_n|^{2(p-2)}|v_n|^2) dx \\
&+ \frac{C}{c_n^2}\|\phi_n\|_{L^6}\|v_n\|_{L^{\frac{12}{5}}}^2 \le \frac{C}{c_n^2},
\end{aligned}$$

that is

$$\|\nabla u_n\|_{L^2} = \mathcal{O}(\frac{1}{c_n}). \tag{36}$$

On the other hand, dividing (34) by c_n^2 and using Lemma 3.3, we can infer that

$$\left\| i\frac{1}{c_n}\sigma\cdot\nabla v_n - \frac{mc_n^2+\omega_n}{c_n^2}u_n \right\|_{L^2}^2 = \left\| \frac{1}{c_n^2}\lambda\phi_n u_n + \frac{1}{c_n^2}(|u_n|^2+|v_n|^2)^{\frac{p-2}{2}}u_n \right\|_{L^2}^2 \le \frac{C}{c_n^4}.$$

Therefore, by (16) and Lemma 3.3, we have

$$\begin{aligned}
\frac{mc_n^2+\omega_n}{c_n^2}\|u_n\|_{L^2} \le& \|i\frac{1}{c_n}\sigma\cdot\nabla v_n - \frac{mc_n^2+\omega_n}{c_n^2}u_n\|_{L^2} + \frac{1}{c_n}\|\nabla v_n\|_{L^2} \\
\le& \frac{C}{c_n^2} + \frac{C}{c_n} = \mathcal{O}(\frac{1}{c_n}),
\end{aligned}$$

which, together with (36), implies that $\|u_n\|_{H^1} = \mathcal{O}(\frac{1}{c_n})$ as $n \to \infty$. □

Next the proof of the convergence of the second components v_n will be divided into several parts. As a first step, we prove the sequence $\{v_n\}$ is bounded away from zero in $H^1(\mathbb{R}^3, \mathbb{C}^2)$.

Lemma 4.2. *There exists $\varrho > 0$ such that*

$$\inf_{n\in\mathbb{N}} \|v_n\|_{H^1} \ge \varrho > 0. \tag{37}$$

Proof. To the contrary, let us assume that, up to a subsequence,

$$\lim_{n\to\infty} \|v_n\|_{H^1} = 0. \tag{38}$$

Dividing (35) by c_n, one has

$$i\sigma\cdot\nabla u_n = \frac{1}{c_n}\left[(|u_n|^2+|v_n|^2)^{\frac{p-2}{2}}v_n - (mc_n^2-\omega_n)v_n - \lambda\phi_n v_n\right]. \tag{39}$$

Thus, by Hölder's inequality, Lemma 2.6 and Lemma 3.3, we can deduce from (39):

$$\begin{aligned}
\|\nabla u_n\|_{L^2}^2 &\le \frac{C}{c_n^2}\|v_n\|_{L^2}^2 + \frac{C}{c_n^2}\int_{\mathbb{R}^3}(|u_n|^2+|v_n|^2)^{p-2}|v_n|^2dx + \frac{C}{c_n^2}\int_{\mathbb{R}^3}\phi_n^2 v_n^2 dx \\
&\quad + \frac{C}{c_n^2}\int_{\mathbb{R}^3}|\phi_n|v_n^2dx + \frac{C}{c_n^2}\int_{\mathbb{R}^3}|\phi_n|(|u_n|^2+|v_n|^2)^{\frac{p-2}{2}}v_n^2dx \\
&\le \frac{C}{c_n^2}\|v_n\|_{L^2}^2 + \frac{C}{c_n^2}\int_{\mathbb{R}^3}(|v_n|^{2(p-1)}+|u_n|^{2(p-2)}|v_n|^2)dx + \frac{C}{c_n^2}\|\phi_n\|_{L^6}^2\|v_n\|_{L^3}^2 \\
&\quad + \frac{C}{c_n^2}\|V_n\|_{L^6}\|v_n\|_{L^{\frac{12}{5}}}^2 + \frac{C}{c_n^2}\|\phi_n\|_{L^{\frac{2(p-1)}{p-2}}}\|\psi_n\|_{L^{2(p-1)}}^{p-1}\|v_n\|_{L^{2(p-1)}}^2 \\
&\le \frac{C}{c_n^2}\|v_n\|_{L^2}^2 + \frac{C}{c_n^2}\|v_n\|_{L^2}^{4-p}\|\nabla v_n\|_{L^2}^{3p-6} + \frac{C\lambda}{c_n^2}\|v_n\|_{L^3}^2 + \frac{C}{c_n^2}\|v_n\|_{L^{\frac{12}{5}}}^2 \\
&\quad + \frac{C}{c_n^2}\left(\int_{\mathbb{R}^3}|u_n|^{2(p-1)}dx\right)^{\frac{p-2}{p-1}}\left(\int_{\mathbb{R}^3}|v_n|^{2(p-1)}dx\right)^{\frac{1}{p-1}} + \frac{C}{c_n^2}\|v_n\|_{L^{2(p-1)}}^2 \\
&\le \frac{C}{c_n^2}\|v_n\|_{H^1}^2 + \frac{C}{c_n^2}\|v_n\|_{H^1}^{2p-2} + \frac{C}{c_n^2}\left(\|v_n\|_{L^2}^{4-p}\|\nabla v_n\|_{L^2}^{3p-6}\right)^{\frac{1}{p-1}} \\
&\le \frac{C}{c_n^2}\|v_n\|_{H^1}^2 + \frac{C}{c_n^2}\|v_n\|_{H^1}^{2p-2} + \frac{C}{c_n^2}\|v_n\|_{H^1}^2,
\end{aligned} \tag{40}$$

which, together with $p\in[\frac{12}{5},\frac{8}{3}]$ and the boundedness of $\|v_n\|_{H^1}$, implies that

$$\|\nabla u_n\|_{L^2}^2 \le \frac{C}{c_n^2}\|v_n\|_{H^1}^2. \tag{41}$$

In addition, (34) is equivalent to the following equation

$$u_n\left(1+\frac{\lambda\phi_n+(|u_n|^2+|v_n|^2)^{\frac{p-2}{2}}}{mc_n^2+\omega_n}\right) = i\frac{c_n}{mc_n^2+\omega_n}\sigma\cdot\nabla v_n, \tag{42}$$

so that, combining with Lemma 3.3, Lemma 3.4

$$\begin{aligned}
&\|u_n\|_{L^2}^2 - \frac{C}{mc_n^2+\omega_n}\|u_n\|_{L^{\frac{12}{5}}}^2 - \frac{C}{(mc_n^2+\omega_n)^2}\|u_n\|_{L^6} \\
&\le \|u_n\|_{L^2}^2 - \frac{2\lambda}{mc_n^2+\omega_n}\int_{\mathbb{R}^3}|\phi_n|u_n^2dx - \frac{2\lambda}{(mc_n^2+\omega_n)^2}\int_{\mathbb{R}^3}|\phi_n||\psi_n|^{p-2}|u_n|^2dx \\
&\le \left\|u_n\left(1+\frac{\lambda\phi_n+(|u_n|^2+|v_n|^2)^{\frac{p-2}{2}}}{mc_n^2+\omega_n}\right)\right\|_{L^2}^2 \\
&= \frac{c_n^2}{(mc_n^2+\omega_n)^2}\|\nabla v_n\|_{L^2}^2 \le \frac{C}{c_n^2}\|v_n\|_{H^1}^2,
\end{aligned}$$

which, together with Lemma 4.1, (41), implies that

$$\begin{aligned}
&\|u_n\|_{H^1}^2 - \frac{C}{mc_n^2+\omega_n}\|u_n\|_{H^1}^2 - \frac{Cc_n}{(mc_n^2+\omega_n)^2}\|u_n\|_{H^1}^2 \\
&\le \|\nabla u_n\|_{L^2}^2 + \|u_n\|_{L^2}^2 - \frac{C}{mc_n^2+\omega_n}\|u_n\|_{L^{\frac{12}{5}}}^2 - \frac{C}{(mc_n^2+\omega_n)^2}\|u_n\|_{L^6} \\
&\le \frac{C}{c_n^2}\|v_n\|_{H^1}^2, \quad \text{for } \ n \text{ large.}
\end{aligned}$$

Therefore, for large n, there exists a constant $C > 0$ such that

$$\|u_n\|_{H^1} \leq \frac{C}{c_n}\|v_n\|_{H^1}. \tag{43}$$

In the next equation, plugging (42) into (39), we get

$$\begin{aligned} &-\Delta v_n + a_n v_n \qquad (44)\\ &= \frac{i}{c_n}\sigma \cdot \nabla \left[(|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} u_n + \lambda\phi_n u_n\right] + b_n \left[(|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} v_n - \lambda\phi_n v_n\right]. \end{aligned}$$

Multiplying (44) by $\bar{v}_n$ and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + a_n \int_{\mathbb{R}^3} |v_n|^2 dx \\ &= \frac{i}{c_n}\int_{\mathbb{R}^3} \sigma \cdot \nabla \left[(|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} u_n + \lambda\phi_n u_n\right] \bar{v}_n dx \qquad (45)\\ &\quad + b_n \int_{\mathbb{R}^3} ((|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} |v_n|^2 - \lambda\phi_n v_n^2) dx. \end{aligned}$$

We claim that

$$\begin{aligned} o(\|v_n\|_{H^1}^2) = &\frac{i}{c_n}\int_{\mathbb{R}^3} \sigma \cdot \nabla \left[(|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} u_n + \lambda\phi_n u_n\right] \bar{v}_n dx \\ &+ b_n \int_{\mathbb{R}^3} ((|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} |v_n|^2 - \lambda\phi_n v_n^2) dx. \qquad (46) \end{aligned}$$

Accepting this fact for the moment, combining with (32), we get

$$o(\|v_n\|_{H^1}^2) = \int_{\mathbb{R}^3} |\nabla v_n|^2 dx + a_n \int_{\mathbb{R}^3} |v_n|^2 dx \geq C\|v_n\|_{H^1}^2,$$

which is a contradiction. Therefore, the lemma is obtained.

Now, it remains to show that (46) is valid. Indeed, by (38), (43), Hölder's inequality and Lemma 3.4, one has, for large n,

$$\begin{aligned} &\int_{\mathbb{R}^3} ((|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} |v_n|^2 + \lambda|\phi_n||v_n|^2) dx \\ &\leq \left(\int_{\mathbb{R}^3} (|u_n|^2 + |v_n|^2)^{\frac{p}{2}} dx\right)^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^3} |v_n|^p dx\right)^{\frac{2}{p}} + C_1\|\phi_n\|_{H^1}\|v_n\|_{H^1}^2 \\ &\leq \left(\int_{\mathbb{R}^3} (|u_n|^p + |v_n|^p) dx\right)^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^3} |v_n|^p dx\right)^{\frac{2}{p}} + C_2\|v_n\|_{H^1}^2 \\ &\leq (\|u_n\|_{H^1}^{p-2} + \|v_n\|_{H^1}^{p-2})\|v_n\|_{H^1}^2 + C_2\|v_n\|_{H^1}^2 = o(\|v_n\|_{H^1}^2). \qquad (47) \end{aligned}$$

In addition, by Lemma 3.4 and (43), one gets, for large n,

$$\begin{aligned} &\left|\frac{i}{c_n}\int_{\mathbb{R}^3} \sigma \cdot \nabla \left[\lambda\phi_n u_n\right] \bar{v}_n dx\right| = \left|\frac{\lambda}{c_n}\int_{\mathbb{R}^3} \sigma \cdot \nabla \bar{v}_n \phi_n u_n dx\right| \\ &\leq \frac{C}{c_n}\|\bar{v}_n\|_{H^1}\|\phi_n\|_{H^1}\|u_n\|_{H^1} \leq \frac{C}{c_n^2}\|v_n\|_{H^1}^2\|\phi_n\|_{H^1} = o(\|v_n\|_{H^1}^2). \qquad (48) \end{aligned}$$

Moreover, we have

$$
\begin{aligned}
&\frac{i}{c_n}\int_{\mathbb{R}^3}\sigma\cdot\nabla\left[(|u_n|^2+|v_n|^2)^{\frac{p-2}{2}}u_n\right]\bar{v}_n dx\\
&=\frac{i}{c_n}\int_{\mathbb{R}^3}\sigma\cdot\nabla u_n(|u_n|^2+|v_n|^2)^{\frac{p-2}{2}}\bar{v}_n dx+\frac{i}{c_n}\int_{\mathbb{R}^3}\sigma\cdot\nabla\left[(|u_n|^2+|v_n|^2)^{\frac{p-2}{2}}\right]u_n\bar{v}_n dx\\
&=: I_1+I_2.
\end{aligned}
$$

We can deduce from Lemma 3.3 and (43) that, for large n,

$$
\begin{aligned}
|I_1| &\le \frac{1}{c_n}\|u_n\|_{H^1}(\|u_n\|_{H^1}^{p-2}+\|v_n\|_{H^1}^{p-2})\|v_n\|_{H^1}\\
&\le \frac{C}{c_n^2}\|v_n\|_{H^1}^2=o(\|v_n\|_{H^1}^2).
\end{aligned}
\tag{49}
$$

Observe that for $p\in[\frac{12}{5},\frac{8}{3}]$ we have $(|u_n|^2+|v_n|^2)^{\frac{p-4}{2}}\le 2^{\frac{p-4}{2}}(|u_n||v_n|)^{\frac{p-4}{2}}$.

Therefore, combining with Hölder's inequality and (28), we obtain

$$
\begin{aligned}
|I_2| &= \left|\frac{-i}{c_n}(p-2)\int_{\mathbb{R}^3}(|u_n|^2+|v_n|^2)^{\frac{p-4}{2}}(\sigma\cdot(\nabla u_n\cdot u_n)+\sigma\cdot(\nabla v_n\cdot v_n))u_n\bar{v}_n dx\right|\\
&\le \frac{C}{c_n}\int_{\mathbb{R}^3}|v_n|^{\frac{p}{2}}|u_n|^{\frac{p-2}{2}}|\nabla v_n|dx+\frac{C}{c_n}\int_{\mathbb{R}^3}|v_n|^{\frac{p-2}{2}}|u_n|^{\frac{p}{2}}|\nabla u_n|dx\\
&\le \frac{C}{c_n}\left(\int_{\mathbb{R}^3}|v_n|^{2(p-1)}dx\right)^{\frac{p}{4(p-1)}}\left(\int_{\mathbb{R}^3}|u_n|^{2(p-1)}dx\right)^{\frac{p-2}{4(p-1)}}\left(\int_{\mathbb{R}^3}|\nabla v_n|^2dx\right)^{\frac{1}{2}}\\
&\quad+\frac{C}{c_n}\left(\int_{\mathbb{R}^3}|u_n|^{2(p-1)}dx\right)^{\frac{p}{4(p-1)}}\left(\int_{\mathbb{R}^3}|v_n|^{2(p-1)}dx\right)^{\frac{p-2}{4(p-1)}}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)^{\frac{1}{2}}\\
&\le \frac{C}{c_n^2}\|v_n\|_{H^1}^p=o(\|v_n\|_{H^1}^2).
\end{aligned}
$$

Here $\sigma\cdot(\nabla w\cdot w):=\sum_{k=1}^3\sigma_k(\partial_k w\cdot w)$. This, together with (47)–(49), implies that the claim is true. □

We recall that a function $w:\mathbb{R}^3\to\mathbb{C}^2$ is a solution of the NLSE (4) if and only if it is a critical point of the $\mathcal{C}^2$ energy functional $\Phi: H^1(\mathbb{R}^3,\mathbb{C}^2)\to\mathbb{R}$ defined by

$$
\Phi(w):=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla w|^2dx+\nu\int_{\mathbb{R}^3}|w|^2dx-\frac{2m}{p}\int_{\mathbb{R}^3}|w|^pdx.
$$

For the study of solutions to the NLSE (4), we refer to [5]. Therefore, we have the following Lemma:

Lemma 4.3. *Let $\{w_n\}$ be a bounded sequence in $H^1(\mathbb{R}^3,\mathbb{C}^2)$. For every n, we define the linear functional $\mathcal{A}_n(w_n): H^1(\mathbb{R}^3,\mathbb{C}^2)\to\mathbb{R}$*

$$
\langle\mathcal{A}_n(w_n),\phi\rangle:=\int_{\mathbb{R}^3}\nabla w_n\nabla\bar{\phi}dx+a_n\int_{\mathbb{R}^3}w_n\bar{\phi}dx-b_n\int_{\mathbb{R}^3}|w_n|^{p-2}w_n\bar{\phi}dx.
$$

Then, $\{w_n\}$ is a (PS)-sequence for Φ if and only if

$$
\sup_{\|\phi\|_{H^1}\le 1}\langle\mathcal{A}_n(w_n),\phi\rangle\to 0\quad \text{as}\quad n\to\infty.
\tag{50}
$$

Proof. See [10]. □

Lemma 4.4. *The sequence $\{v_n\}$ is a (PS)-sequence for Φ.*

Proof. From Lemma 4.3, it is sufficient to prove (50). Let us take an arbitrary $\phi \in H^1$ with $\|\phi\|_{H^1} \leq 1$. Multiplying (44) by ϕ and integrating over $\mathbb{R}^3$, one has

$$-\int_{\mathbb{R}^3} \Delta v_n \bar{\phi} dx + a_n \int_{\mathbb{R}^3} v_n \bar{\phi} dx = \frac{i}{c_n} \int_{\mathbb{R}^3} \sigma \cdot \nabla \left[(|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} u_n + \lambda \phi_n u_n \right] \bar{\phi} dx$$
$$+ b_n \int_{\mathbb{R}^3} (|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} v_n \bar{\phi} dx - b_n \int_{\mathbb{R}^3} \lambda \phi_n v_n \bar{\phi} dx =: J_1 + J_2 - J_3. \tag{51}$$

By Hölder's inequality, Lemma 3.3 and Lemma 4.1, we get

$$\begin{aligned}
|J_1| &= \left| \frac{i}{c_n} \int_{\mathbb{R}^3} \left[(|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} u_n + \lambda \phi_n u_n \right] \sigma \cdot \nabla \bar{\phi} dx \right| \\
&\leq \frac{1}{c_n} \int_{\mathbb{R}^3} (|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} |u_n| |\sigma \cdot \nabla \bar{\phi}| dx + \frac{C}{c_n} \|\phi_n\|_{H^1} \|u_n\|_{H_1} \left(\int_{\mathbb{R}^3} |\nabla \bar{\phi}|^2 dx \right)^{\frac{1}{2}} \\
&\leq \frac{1}{c_n} \left(\int_{\mathbb{R}^3} (|u_n|^2 + |v_n|^2)^{p-2} |u_n|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla \bar{\phi}|^2 dx \right)^{\frac{1}{2}} + \frac{C}{c_n} \|u_n\|_{H_1} \\
&\leq \frac{1}{c_n} \left(\left(\int_{\mathbb{R}^3} |v_n|^{2p-2} dx \right)^{\frac{p-2}{p-1}} \left(\int_{\mathbb{R}^3} |u_n|^{2p-2} dx \right)^{\frac{1}{p-1}} + \int_{\mathbb{R}^3} |u_n|^{2p-2} dx \right)^{\frac{1}{2}} + \frac{C}{c_n} \|u_n\|_{H_1} \\
&\leq \frac{C}{c_n} \left(\|u_n\|_{H^1}^2 + \|u_n\|_{H^1}^{2p-2} \right)^{\frac{1}{2}} + \frac{C}{c_n} \|u_n\|_{H_1} \to 0.
\end{aligned} \tag{52}$$

It follows from Lemma 4.1 that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[(|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} - |v_n|^{p-2} \right] |v_n| |\bar{\phi}| dx \leq \int_{\mathbb{R}^3} |u_n|^{p-2} |v_n| |\bar{\phi}| dx \\
&\leq \left(\int_{\mathbb{R}^3} |u_n|^p dx \right)^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^3} |v_n|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^3} |\bar{\phi}|^p dx \right)^{\frac{1}{p}} \leq C \|u_n\|_{H^1}^{p-2} \to 0
\end{aligned}$$

which implies

$$\begin{aligned}
J_2 &= b_n \int_{\mathbb{R}^3} |v_n|^{p-2} v_n \bar{\phi} dx + b_n \int_{\mathbb{R}^3} \left[(|u_n|^2 + |v_n|^2)^{\frac{p-2}{2}} - |v_n|^{p-2} \right] v_n \bar{\phi} dx \\
&= b_n \int_{\mathbb{R}^3} |v_n|^{p-2} v_n \bar{\phi} dx + o(1).
\end{aligned} \tag{53}$$

Finally, by virtue of Hölder's inequality, the form of V_n in (19) and Lemma 3.3, we obtain

$$
\begin{aligned}
|\phi_n(x)| &= \left| \int_{\mathbb{R}^3} \lambda \frac{[(\beta\psi_n)\psi_n](y)}{|x-y|} e^{-c_n M|x-y|} dy \right| \\
&\leq \lambda \left(\int_{\mathbb{R}^3} |\psi_n|^4 dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} \frac{e^{-2c_n M|x-y|}}{|x-y|^2} dy \right)^{\frac{1}{2}} \\
&\leq C\|\psi_n\|_{H^1}^2 \left(\int_{\mathbb{R}^3} \frac{e^{-2c_n M|x-y|}}{|x-y|^2} dy \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{\mathbb{R}^3} \frac{e^{-2c_n M|x-y|}}{|x-y|^2} dy \right)^{\frac{1}{2}}.
\end{aligned}
$$

This, together with (32) and Lemma 3.3, gives us

$$
\begin{aligned}
|J_3| &\leq b_n \int_{\mathbb{R}^3} \lambda |\phi_n||v_n||\bar{\phi}| dx \\
&\leq C \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{-2c_n M|x-y|}}{|x-y|^2} dy dx \right)^{\frac{1}{2}} \|v_n\|_{H^1} \|\phi\|_{H^1}.
\end{aligned} \tag{54}
$$

By Lebesgue dominated convergence theorem, one gets

$$
|J_3| \leq C \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{e^{-2c_n M|x-y|}}{|x-y|^2} dy dx \right)^{\frac{1}{2}} \to 0, \quad \text{as } n \to \infty. \tag{55}
$$

Hence, combining with (32), (51)-(55), we can deduce that

$$
\int_{\mathbb{R}^3} \nabla v_n \nabla \bar{\phi} dx + a_n \int_{\mathbb{R}^3} v_n \bar{\phi} dx = b_n \int_{\mathbb{R}^3} |v_n|^{p-2} v_n \bar{\phi} dx + o(1),
$$

that is: $\sup_{\|\phi\|_{H^1} \leq 1} \langle \mathcal{A}_n(v_n), \phi \rangle \to 0$ as $n \to \infty$. □

Lemma 4.5. *The sequence* $\{\phi_n\}$ *converges to* 0 *in* $H^1(\mathbb{R}^3, \mathbb{R})$.

Proof. Replace z with V_n in (11), we have

$$
C\|\phi_n\|_{H^1}^2 \leq \int_{\mathbb{R}^3} |\nabla \phi_n|^2 + c_n^2 M^2 |\phi_n|^2 dx = 4\pi \int_{\mathbb{R}^3} \lambda (\beta\psi_n)\psi_n \cdot \phi_n dx. \tag{56}
$$

Similarly to (54) and (55), one gets

$$
\int_{\mathbb{R}^3} \lambda (\beta\psi_n)\psi_n \phi_n dx \to 0, \quad \text{as } n \to \infty,
$$

which, together with (56), implies that $\|\phi_n\|_{H^1} \to 0$, as $n \to \infty$. □

Complete proof of Theorem 1.1

Define the linear functional $\mathcal{B}(v) : H^1(\mathbb{R}^3, \mathbb{C}^2) \to \mathbb{R}$

$$
\mathcal{B}(v)w := \int_{\mathbb{R}^3} \nabla v \nabla \bar{w} dx + 2\nu \int_{\mathbb{R}^3} v \bar{w} dx.
$$

By Lemma 3.3 the sequence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3,\mathbb{C}^2)$, which, together with Lemma 4.4 implies that it is a (PS)-sequence for Φ. Thus, up to subsequences, there is a $v \neq 0 \in H^1(\mathbb{R}^3,\mathbb{C}^2)$ such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3,\mathbb{C}^2)$ and $v_n \to v$ in $L^p_{loc}(\mathbb{R}^3,\mathbb{C}^2)$. Therefore, using (32) and the H^1-boundedness of $\{v_n\}$, we get

$$\begin{aligned}
o(1) &= \langle \mathcal{A}_n(v_n) - \mathcal{B}(v_n), v_n - v\rangle \\
&= \int_{\mathbb{R}^3} |\nabla v_n - \nabla v|^2 dx + a_n \int_{\mathbb{R}^3} v_n(\bar{v}_n - \bar{v})dx - b_n \int_{\mathbb{R}^3} |v_n|^{p-2} v_n(\bar{v}_n - \bar{v})dx \\
&\quad - 2\nu \int_{\mathbb{R}^3} v(\bar{v}_n - \bar{v})dx \\
&= \int_{\mathbb{R}^3} |\nabla v_n - \nabla v|^2 dx + 2\nu \int_{\mathbb{R}^3} |v_n - v|^2 dx - 2m \int_{\mathbb{R}^3} |v_n|^{p-2} v_n(\bar{v}_n - \bar{v})dx \\
&\geq C_1\|v_n - v\|^2_{H^1} - 2mC_p\|v_n - v\|^p_{H^1} \geq \|v_n - v\|^2_{H^1}(C_1 - 2mC_2).
\end{aligned}$$

Here we use the fact $\nu > 0, p \in [\frac{12}{5}, \frac{8}{3}]$. Hence, there exists a constant $m_0 > 0$ such that $v_n \to v$ in $H^1(\mathbb{R}^3,\mathbb{C}^2)$ for all $m \leq m_0$. This concludes the whole proof.

4.2. Proof of Theorem 1.3

In this section, we estimate the uniform boundedness and the exponential decay properties of solutions. Let $n \to \infty$, ψ_n be a solution of frequency ω_n of the DKGS (5) at speed of light c_n. To begin with, we estimate the uniform boundedness of the sequence $\{\psi_n\}$ in $L^\infty(\mathbb{R}^3,\mathbb{C}^4)$.

Lemma 4.6. *The sequences $\{\psi_n\}$ is bounded in $L^\infty(\mathbb{R}^3,\mathbb{C}^4)$ uniformly with respect to n. Moreover, $\{\phi_n\}$ is bounded in $L^\infty(\mathbb{R}^3,\mathbb{R})$ uniformly with respect to n.*

Proof. By the Sobolev embedding theorem, we only need to prove that there exist $C_r > 0$ independent of n such that $\|\psi_n\|_{W^{1,r}} \leq C_r$ for any $r \geq 2$. Let $H_a := i\alpha\cdot\nabla - a\beta$ with $a > 0$. Then, (5) is equivalent to

$$H_a\psi_n = (mc_n - a)\beta\psi_n + \frac{\omega_n}{c_n}\psi_n + \frac{1}{c_n}|\psi_n|^{p-2}\psi_n + \frac{\lambda}{c_n}\phi_n\beta\psi_n.$$

It is clear that $0 \notin \sigma(H_a)$. Writing $\psi_n = \psi_n^1 + \psi_n^2 + \psi_n^3$, then

$$\psi_n^1 = H_a^{-1}\left(\frac{1}{c_n}|\psi_n|^{p-2}\psi_n\right) = H_a^{-1}\left(\frac{1}{c_n}|(u_n,v_n)|^{p-2}(u_n,v_n)^T\right),$$

$$\psi_n^2 = H_a^{-1}\left[\left(\frac{\omega_n}{c_n} + (mc_n - a)\beta\right)\psi_n\right] = H_a^{-1}\left[\left(\frac{\omega_n}{c_n} + (mc_n - a)\beta\right)(u_n,v_n)^T\right].$$

$$\psi_n^3 = \lambda H_a^{-1}\left[\frac{1}{c_n}\phi_n\beta\psi_n\right] = \lambda H_a^{-1}\left[\frac{1}{c_n}\phi_n\beta(u_n,v_n)^T\right].$$

We have shown that $\|\psi_n\|_{L^2} \leq C$ and $\|\psi_n\|_{H^1} \leq C$ in Lemma 3.2 and Lemma 3.3. Thus, using the Sobolev embedding theorem, we obtain

$$\|\psi_n\|_{L^q} \leq C_q, \quad \forall\, q \in [2,6]. \tag{57}$$

For any $r \in [2,3]$, by Hölder's inequality, one has

$$\begin{aligned}\|\frac{1}{c_n}|(u_n,v_n)|^{p-2}(u_n,v_n)\|_{L^r}^r &= \|\frac{1}{c_n}|(u_n,v_n)|^{p-2}u_n\|_{L^r}^r + \|\frac{1}{c_n}|(u_n,v_n)|^{p-2}v_n\|_{L^r}^r \\ &\leq \frac{C}{c_n^r}\int_{\mathbb{R}^3}|u_n|^{r(p-1)}dx + \frac{C}{c_n^r}\int_{\mathbb{R}^3}|v_n|^{r(p-1)}dx.\end{aligned}$$

Since $r(p-1) \in [\frac{14}{5},5]$ for $r \in [2,3]$, it follows from (57) that

$$\|\frac{1}{c_n}|(u_n,v_n)|^{p-2}(u_n,v_n)\|_{L^r}^r \leq C, \quad \forall\, r \in [2,3].$$

In addition, from Lemma 4.1 and (57), we can infer that for any $r \in [2,3]$

$$\begin{aligned}&\left\|\left(\frac{\omega_n}{c_n} + (mc_n - a)\beta\right)(u_n,v_n)^T\right\|_{L^r}^r \\ &\quad = \left\|\left(\frac{\omega_n}{c_n^2} + (m - \frac{a}{c_n})\right)c_n u_n\right\|_{L^r}^r + \left\|\left(\frac{\omega_n - mc_n^2}{c_n} + a\right)v_n\right\|_{L^r}^r \\ &\quad \leq C_2(r)c_n\|u_n\|_{L^r}^r + C_1(r)\|v_n\|_{L^r}^r \leq C,\end{aligned}$$

with $C>0$ independent of n. Therefore, there is a $C>0$ such that

$$\|\psi_n^1\|_{W^{1,r}} \leq C, \quad \|\psi_n^2\|_{W^{1,r}} \leq C, \quad \text{for any } r \in [2,3].$$

By Hölder's inequality, for $q \geqslant 2$

$$\|\phi_n\beta\psi_n\|_{L^s} \leq \|\phi_n\|_{L^6}\cdot\|\psi_n\|_{L^q},$$

with $\frac{1}{s} = \frac{1}{6} + \frac{1}{q}$. Hence, we obtain

$$\|\psi_n^3\|_{W^{1,s^*}} \leq C, \quad \text{with } s^* = \frac{3s}{3-s}.$$

A standard bootstrap argument shows that $\{\psi_n\}$ is bounded in $\bigcap_{q\geqslant 2} L^q$, $\{\psi_n^1\}$ is bounded in $\bigcap_{q\geqslant 2} W^{1,q}$, $\{\psi_n^2\}$ is bounded in $\bigcap_{q\geqslant 2} W^{1,q}$ and $\{\psi_n^3\}$ is bounded in $\bigcap_{6>q\geqslant 2} W^{1,q}$.

By Sobolev's embedding theorems, $\{\psi_n\}$ is bounded in $C^{0,\gamma}$ for some $\gamma \in (0,1)$. This, together with the elliptic regularity (see [22]), shows that $\{\phi_n\}$ is bounded in $W^{2,2}_{\text{loc}}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and

$$\|\phi_n\|_{W^{2,2}(B_1(x))} \leq C_2\left(\lambda\|\psi_n\|^2_{L^4(B_2(x))} + \|\phi_n\|_{H^1(B_2(x))}\right),$$

for all $x \in \mathbb{R}^3$, with C_2 independent of x and n, where $B_r(x) = \{y \in \mathbb{R}^3 : |y-x| < r\}$ for $r>0$. Since $W^{2,2}(B_1(x)) \hookrightarrow C^{0,\delta}(B_1(x)), \delta \in \left(0,\frac{1}{2}\right)$, we have

$$\|\phi_n\|_{C^{0,\delta}(B_1(x))} \leq C_3\left(\lambda\|\psi_n\|^2_{L^4(B_2(x))} + \|\phi_n\|_{H^1(B_2(x))}\right), \tag{58}$$

for all $x \in \mathbb{R}^3$ with C_3 independent of x and n.

Consequently $\{\phi_n\}$ is bounded in L^∞, and this yields

$$\|\phi_n\beta\psi_n\|_{L^s} \leq \|\phi_n\|_{L^\infty}\|\psi_n\|_{L^s} \leq C\|\psi_n\|_{L^s}$$

Thus, $\{\psi_n^3\}$ is bounded in $\bigcap_{q\geqslant 2} W^{1,q}$. Combining with the fact that $\{\psi_n^1\}$ and $\{\psi_n^2\}$ are bounded in $\bigcap_{q\geqslant 2} W^{1,q}$, we obtain there is a $C_q > 0$ such that

$$\|\psi_n\|_{W^{1,q}} \leq C_q, \quad \text{for any } q \geq 2. \qquad \square$$

Setting $\mathcal{D} := i\alpha\cdot\nabla$, ψ_n solves (5), that is

$$\mathcal{D}\psi_n = mc_n\beta\psi_n + \frac{\omega_n}{c_n}\psi_n + \frac{\lambda}{c_n}\phi_n\beta\psi_n + \frac{1}{c_n}|\psi_n|^{p-2}\psi_n.$$

Acting the operator $\mathcal{D}$ on the two sides and noting that $\mathcal{D}^2 = -\Delta$, we get

$$\begin{aligned}
-\Delta\psi_n &= -mc_n\beta\mathcal{D}\psi_n + \frac{\omega_n}{c_n}\mathcal{D}\psi_n - \frac{\lambda}{c_n}\phi_n\beta\mathcal{D}\psi_n + \frac{\lambda}{c_n}\mathcal{D}\phi_n\beta\psi_n \\
&\quad + \frac{1}{c_n}(\mathcal{D}|\psi_n|^{p-2}\psi_n + |\psi_n|^{p-2}\mathcal{D}\psi_n) \\
&= (-mc_n\beta + \frac{\omega_n}{c_n} - \frac{\lambda}{c_n}\phi_n\beta + \frac{1}{c_n}|\psi_n|^{p-2}) \\
&\quad \cdot (mc_n\beta\psi_n + \frac{\omega_n}{c_n}\psi_n + \frac{1}{c_n}|\psi_n|^{p-2}\psi_n + \frac{\lambda}{c_n}\phi_n\beta\psi_n) \\
&\quad + \frac{\lambda}{c_n}\mathcal{D}\phi_n\beta\psi_n + \frac{1}{c_n}\mathcal{D}|\psi_n|^{p-2}\psi_n.
\end{aligned}$$

Thus,

$$\Delta\psi_n = (mc_n + \frac{\lambda}{c_n}\phi_n)^2\psi_n - (\frac{\omega_n}{c_n} + \frac{1}{c_n}|\psi_n|^{p-2})^2\psi_n - \frac{\lambda}{c_n}\mathcal{D}\phi_n\beta\psi_n - \frac{1}{c_n}\mathcal{D}|\psi_n|^{p-2}\psi_n.$$

Note that $\Re\left((\mathcal{D}\phi_n\beta\psi_n + \mathcal{D}|\psi_n|^{p-2}\psi_n)\frac{\bar{\psi}_n}{|\psi_n|}\right) = 0.$

By Kato's inequality [9], there holds $\triangle|\psi_n| \geq \Re[\triangle\psi_n(\text{sgn }\psi_n)]$, where $\text{sgn }\psi_n = \frac{\psi_n}{|\psi_n|}$, if $\psi_n \neq 0$; sgn $\psi_n = 0$, if $\psi_n = 0$. By Lemma 4.6, we obtain that there is a $\tau > 0$, independent of n, such that

$$\begin{aligned}
\Delta|\psi_n| &\geq \frac{m^2c_n^4 - \omega_n^2}{c_n^2}|\psi_n| + m\lambda\phi_n|\psi_n| - \frac{2\omega_n}{c_n^2}|\psi_n|^{p-2}|\psi_n| - \frac{1}{c_n^2}|\psi_n|^{2(p-2)}|\psi_n|^2 \\
&\geq -\tau|\psi_n|. \qquad (59)
\end{aligned}$$

It then follows from the sub-solution estimate [20, 30] that

$$|\psi_n(x)| \leq C_0\int_{B_1(x)}|\psi_n(y)|dy,$$

where $C_0 > 0$ independent of x, n and ψ_n.

Lemma 4.7. *$|\psi_n(x)|$ and $|\phi_n(x)|$ vanish at infinity uniformly in n as $|x| \to \infty$.*

Proof. Assume by contradiction that the conclusion of the lemma does not hold. Then, there exist $\bar{r} > 0$ and $x_j \in \mathbb{R}^3$ with $|x_n| \to \infty$ such that

$$\bar{r} \leq |\psi_n(x_n)| \leq C_0 \int_{B_1(x_n)} |\psi_n(x)| dx,$$

which, together with $\psi_n \to \psi$ in $H^1(\mathbb{R}^3, \mathbb{C}^4)$, one obtains

$$\begin{aligned} \bar{r} &\leq C_0 \left(\int_{B_1(x_j)} |\psi_n(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq C_0 \left(\int_{\mathbb{R}^3} |\psi_n - \psi|^2 dx \right)^{\frac{1}{2}} + C_0 \left(\int_{B_1(x_n)} |\psi|^2 dx \right)^{\frac{1}{2}} \quad \longrightarrow \quad 0, \end{aligned}$$

as $n \to \infty$, a contradiction. Now, jointly with (58), one sees also $|\phi_n(x)| \to 0$ as $|x| \to \infty$ uniformly in n. □

Similar to [10], we have the following lemma.

Lemma 4.8. *There exists $C, \tilde{C} > 0$ such that*

$$|\psi_n(x)| \leq C e^{-\tilde{C}|x|}, \quad \forall x \in \mathbb{R}^3, \text{ uniformly in } n \in \mathbb{N}.$$

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