Minimax Theory and its Applications
Volume 7 (2022), No. 2, 277–302

Rayleigh Quotients of the Level Set Manifolds Related to the Nonlinear PDE

Yavdat Il'yasov
Institute of Mathematics, Ufa Federal Research Centre, Russian Academy of Sciences, Ufa, Russia; and: Instituto de Matemática, Univ. Federal de Goiás, Goiânia, Brazil
ilyasov02@gmail.com

Received: June 14, 2021
Accepted: September 17, 2021

The main topic of this note is a discussion of applicability conditions of the Nehari manifold method depending on the value of parameters of equations. As the main tool, we apply the nonlinear generalized Rayleigh quotient method.

Keywords: Nehari manifold, NG-Rayleigh quotient, energy level set, zero frequency problem.

2010 Mathematics Subject Classification: 35J50, 35J55, 35J60.

1. Introduction

Let W be a real Banach space, $\Phi_\lambda : W \to \mathbb{R}$ be a twice Fréchet-differentiable functional, and λ be a real parameter. The Nehari manifold method introduced by Z. Nehari [37, 38] in 1960 is a powerful tool for the investigation of equations of the variational form

$$D\Phi_\lambda(u) = 0, \quad u \in W, \tag{1}$$

which consists of finding an extremal point $\hat{u}$ of the functional $\Phi_\lambda(u)$ subject to the *Nehari manifold*

$$\mathcal{N}_\lambda = \{u \in W \setminus 0 : \quad \Phi'_\lambda(u) := D\Phi_\lambda(u)(u) = 0\}.$$

This may lead to a solution to the problem (1). Indeed, under some general assumption the Lagrange multiplier rule due to Lusternik [36] yields

$$\mu_0 D\Phi_\lambda(\hat{u}) + \mu_1(D\Phi_\lambda(\hat{u}) + D^2\Phi_\lambda(\hat{u})(\hat{u}, \cdot)) = 0,$$

for some μ_0, μ_1 such that $|\mu_0| + |\mu_1| \neq 0$. Testing this equality by $\hat{u}$ and taking into account that $D\Phi_\lambda(\hat{u})(\hat{u}) = 0$ we obtain $\mu_1 D^2\Phi_\lambda(\hat{u})(\hat{u}, \hat{u}) = 0$. Hence, if $D^2\Phi_\lambda(\hat{u})(\hat{u}, \hat{u}) \neq 0$, then we have successively $\mu_1 = 0$, $\mu_0 \neq 0$ and $D\Phi_\lambda(u) = 0$. Thus, one has the following *applicability condition of the Nehari manifold method*

$$\Phi''_\lambda(u) := D^2\Phi_\lambda(u)(u, u) \neq 0, \quad \forall u \in \mathcal{N}_\lambda. \tag{2}$$

The feasibility of this condition may depend on parameter λ. This leads us to the problem of finding the so-called *extreme values of the Nehari manifold method*, namely, the limit points of the set of λ where the applicability condition of the Nehari manifold method (2) is satisfied.

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

In [27], the so-called nonlinear generalized Rayleigh (NG-Rayleigh) quotient method has been introduced which allows one to find the extreme values of the Nehari manifold method. The method is based on the analysis of the so-called nonlinear generalized Rayleigh quotients whose critical values correspond to the extreme values of the Nehari manifold method. To specify the principal idea of the method, let us consider equation (1) in the following particular form

$$Dh(u) - \lambda Dg(u) = 0,$$

with $\Phi_\lambda(u) = h(u) - \lambda g(u)$. Observe that $g'(u) := Dg(u)(u) = 0$ if $u = 0$. Let us assume that $g'(u) \neq 0$, $\forall u \in W \setminus 0$. Then testing the equation by $u \in W$ and solving it with respect to λ we obtain the following parameter independent functional:

$$\mathcal{R}(u) = \frac{Dh(u)(u)}{Dg(u)(u)} = \frac{h'(u)}{g'(u)}, \quad u \in W \setminus 0, \tag{3}$$

which we call the *Rayleigh quotient*. Note that u belongs to $\mathcal{N}_\lambda$ if and only if it lies on the level set $\mathcal{R}^{-1}(\lambda) = \{u \in W \setminus 0 : \ \mathcal{R}(u) = \lambda\}$. Consider the one variable function $t \mapsto \mathcal{R}(tu)$, $u \in W$, which we call the fibering function following Pohozaev [40, 41]. Then

$$\mathcal{R}'(u) := \frac{d}{dt}\mathcal{R}(tu)|_{t=1} = \frac{1}{g'(u)}\Phi''_\lambda(u), \quad \forall u \in \mathcal{N}_\lambda.$$

Consequently, $\mathcal{R}'(u) \neq 0 \Leftrightarrow \Phi''_\lambda(u) \neq 0$, for any $u \in \mathcal{N}_\lambda$. Thus the finding of the applicable values λ of the Nehari manifold method is reduced to the determining of the regular value of of the fibering Rayleigh quotient $\mathcal{R}(tu)$, i.e., $\mathcal{R}'(tu) \neq 0$, $t \in \mathbb{R}^+ \setminus 0$, $u \in W \setminus 0$. This reasoning leads us to an idea that the extreme values of the Nehari manifold method can be obtained by means of the critical values of the fibering Rayleigh quotient.

Observe that the condition $\mathcal{R}'(u) \neq 0$ for $u \in W \setminus 0$ implies that the map $D\mathcal{R}(u) : W \to \mathbb{R}$ is surjective, that is u is a regular point of $\mathcal{R}$. The study of regularity (or lack thereof) of level sets of functionals is based on the now classical theory (see, e.g., [32, 47]). It is important to emphasize that the question of the regularity (or lack thereof) of level sets of functional associated with equations arises, not only in connection with Nehari manifold but in many other cases. Below we will deal with finding the so-called solutions with prescribed energy E, i.e., with solutions from the level set $\{u \in W : \Phi_\lambda(u) = E\}$ (see [17, 28]). Another type of level sets arises in the investigation of equations using the Pohozaev manifold (see [16, 29, 26, 39]), and it seems this does not exhaust all the examples.This problem takes on a different shade if the functional (equation) depends on a parameter. Note that the Nehari manifold $\mathcal{N}_\lambda$ is nothing more than a zero-level set of the functional Φ'_λ in $W \setminus 0$, whereas the applicability condition (2) implies the regularity of this set.

Remark 1.1. In some cases, the extreme values of the Nehari manifold method can also be found using the so-called *spectral analysis with respect to the fibering procedure* introduced in [23, 24]. In this approach, the extreme values of the Nehari manifold method are found by solving the system

$$\begin{cases} h'(tu) - \lambda g'(tu) = 0, \\ h''(tu) - \lambda g''(tu) = 0, \end{cases}$$

with respect to unknowns t, λ (see, e.g., [13, 15, 20, 21, 23, 24, 25, 29]). However, in our opinion, the using of the NG-Rayleigh quotient method has a clear geometric meaning, which simplifies calculations and, ultimately, allows solving more complex problems.

Remark 1.2. Below we will see that the application of the NG-Rayleigh quotient method also makes it possible to constructively find useful variational formulations associated with equations. Research in this direction was motivated by the works Pohozaev in [40, 41], where a fibering method was introduced to constructively find constrained minimization problems.

The paper is organized as follows. In Section 2, we present the concept of the nonlinear Rayleigh quotient in an abstract setting. Section 3 is devoted to an example of the application of the NG-Rayleigh quotient method to a problem with convex-concave nonlinearity. In particular, in this section, we discuses finding of the limit point for the branch of ground states obtained in the framework of the Nehari manifold method. In Section 4, we present an example of the recursive application of the NG-Rayleigh quotient method to a problem where the fibering function $\Phi_\lambda(tu)$ may have more than two critical points. Section 5 deals with the energy level Rayleigh quotient method which allow us to show the existence and nonexistence of solution with prescribed energy for a zero mass (zero frequency) problem.

2. Nonlinear generalized Rayleigh quotient

We recall first some definitions from the theory of manifolds. Let W be a Banach space. Consider a Fréchet differentiable map $f : \mathcal{O}(f) \subseteq W \to \mathbb{R}$. Here $\mathcal{O}(f)$ is a domain of definition of f. We denote the *Fréchet derivative* by Df. For $c \in \mathbb{R}$, the set $f^{-1}(c) = \{u \in \mathcal{O}(f) : \ f(u) = c\}$ is said to be a *c-level set of f*. We call $u \in \mathcal{O}(f)$ a *regular point* of f if $Df(u) : W \to \mathbb{R}$ is surjective; it is a *critical point* of f otherwise. A point $c \in \mathbb{R}$ is said to be a *regular value* of f if every point of the level set $f^{-1}(c)$ is a regular point, and a *critical value* otherwise. We denote by $T_u(f^{-1}(c))$ the *tangent space* of $f^{-1}(c)$ at $u \in f^{-1}(c)$ ([32, 47]). In what follows, we use the notation $f'(tu) := \frac{d}{dt} f(tu) = Df(tu)(u)$, $t \in \mathbb{R}^+$, $u \in W$.

Let us consider the family of maps of the following form $f_\lambda = h - \lambda g : W \to \mathbb{R}$, where $h, g \in C^1(W; \mathbb{R})$, $\lambda \in \mathbb{R}$. For a given $c \in \mathbb{R}$, we are interested in how the regularity of $f_\lambda^{-1}(c)$ depends on the value of parameter λ. This problem can be investigate using the following parameter independent functional

$$\mathcal{R}(c; u) := \frac{h(u) - c}{g(u)}, \quad u \in \mathcal{O}(\mathcal{R}),$$

which we call the *Rayleigh quotient of the c-level manifold* (*Rayleigh quotient* for short) of f_λ. Here $\mathcal{O}(\mathcal{R}) = \{u \in W : g(u) \neq 0\}$

Let $u \in \mathcal{O}(\mathcal{R})$. Assume that $\mathcal{R}(c; u) = \lambda$ for some $\lambda \in \mathbb{R}$. Then

$$D\mathcal{R}(c; u) = \frac{1}{g(u)}(Dh(u) - \mathcal{R}(c; u)Dg(u)) = \frac{1}{g(u)} Df_\lambda(u).$$

Thus, for $u \in \mathcal{O}(\mathcal{R})$, the map $Df_\lambda(u)$ is surjective if and only if $D\mathcal{R}(c; u)$ is surjective. From this we have

Corollary 2.1. *Suppose that* λ *is a regular value of* $\mathcal{R}(c;\cdot)$, *and* $f_\lambda^{-1}(c) \subset \mathcal{O}(\mathcal{R})$. *Then the level set* $f_\lambda^{-1}(c)$ *is a* C^1*-manifold in* W. *Moreover* $T_u(f_\lambda^{-1}(c)) = \mathrm{Ker} D\mathcal{R}(c;u)$ *for every* $u \in f_\lambda^{-1}(c)$.

Proof. By the Regular Value Theorem [32, 47] the set $(\mathcal{R}(c;\cdot))^{-1}(\lambda)$ is a C^1-manifold and there holds $T_u((\mathcal{R}(c;\cdot))^{-1}(\lambda)) = \mathrm{Ker} D\mathcal{R}(c;u)$, $\forall u \in (\mathcal{R}(c;\cdot))^{-1}(\lambda)$. Since $f_\lambda^{-1}(c) \subset \mathcal{O}(\mathcal{R})$, we have $f_\lambda^{-1}(c) = (\mathcal{R}(c;\cdot))^{-1}(\lambda)$, which implies the proof. □

Define $f'_\lambda(u) := \dfrac{d}{dt}(h(tu) - \lambda g(tu))|_{t=1} \equiv Dh(u)(u) - \lambda Dg(u)(u)$ for $u \in W$.

The following zero-level set

$$\mathcal{N}_\lambda := (f'_\lambda)^{-1}(0) = \{u \in W \setminus 0 : \ f'_\lambda(u) = 0\}$$

in $W \setminus 0$ is called a Nehari manifold associated with $f_\lambda(u)$. A local minimum or maximum point of the function f_λ subject to $\mathcal{N}_\lambda$ is called the extremal point on the Nehari manifold.

Lemma 2.2. *Let* $\hat{u} \in \mathcal{N}_\lambda$ *be an extremal point of* f_λ *on the Nehari manifold. Assume that* $f'_\lambda(u)$ *is Fréchet differentiable in an open neighborhood of* $\hat{u}$ *and* $Df'_\lambda(\hat{u})$ *is continuous at* $\hat{u}$. *Suppose that* $f''_\lambda(\hat{u}) := Df'_\lambda(\hat{u})(\hat{u}) \neq 0$. *Then* $Df_\lambda(\hat{u}) = 0$.

Proof. Due to the assumption we may apply the Lagrange multiplier rule [36] (see also Proposition 43.19. in [47]). Then, $Df_\lambda(\hat{u}) + \mu Df'_\lambda(\hat{u}) = 0$, for some $\mu \in \mathbb{R}$. Testing this equality by $\hat{u}$ we obtain $\mu Df'_\lambda(\hat{u})(\hat{u}) = \mu f''_\lambda(\hat{u}) = 0$. Since $f''_\lambda(\hat{u}) \neq 0$, $\mu = 0$, and therefore, $Df_\lambda(\hat{u}) = 0$. □

Definition 2.3. We call λ the *applicable value* of the Nehari manifold method (applicable value of the Nehari manifold for short) if $f''_\lambda(u) \neq 0$ for any $u \in \mathcal{N}_\lambda$.

Consider the so-called *Rayleigh quotient of the Nehari manifold*

$$\mathcal{R}^n(u) = \frac{h'(u)}{g'(u)}, \ u \in \mathcal{O}(\mathcal{R}^n),$$

where $\mathcal{O}(\mathcal{R}^n) = \{u \in W : g'(u) \neq 0\}$. Note $g'(0) = 0$ for any $g \in C^1(W;\mathbb{R})$. Hence, $(\mathcal{R}^n)^{-1}(\lambda) := \{u \in \mathcal{O}(\mathcal{R}^n) : \ \mathcal{R}^n(u) = \lambda\} \subseteq \mathcal{N}_\lambda$, and consequently, $\mathcal{N}_\lambda \neq \emptyset$, for any $\lambda \in \mathrm{Im}\, \mathcal{R}^n$. Moreover, if $\mathcal{N}_\lambda \subset \mathcal{O}(\mathcal{R}^n)$, $\forall \lambda \in \mathbb{R}$, then $\mathcal{N}_\lambda = (\mathcal{R}^n)^{-1}(\lambda)$, and consequently $\mathcal{N}_\lambda \neq \emptyset$ if and only if $\lambda \in \mathrm{Im}\, \mathcal{R}^n$. Consider the fibering function $\mathcal{R}^n(tu)$ defined for each $tu \in \mathcal{O}(\mathcal{R}^n)$. We call $u \in \mathcal{O}(\mathcal{R}^n)$ the fibering regular point of $\mathcal{R}^n(u)$ if $(\mathcal{R}^n)'(u) := d\mathcal{R}^n(tu)/dt|_{t=1} \neq 0$. A point $\lambda \in \mathbb{R}$ is said to be a fibering regular value of $\mathcal{R}^n$ if every point of the level set $(\mathcal{R}^n)^{-1}(\lambda)$ is fibering regular. We write $Z(\mathcal{R}^n)$ for the set of all fibering regular values of $\mathcal{R}^n$.

Remark 2.4. Evidently, the fibering regularity implies ordinary regularity in the above sense. On the other hand, in general, it is possible that $\lambda \in \mathbb{R}$ is a regular value of $\mathcal{R}^n$, while it is not fibering regular.

Corollary 2.5. *Assume that* $f_\lambda, f'_\lambda \in C^1(W)$*. Suppose that* λ *is a fibering regular value of* $\mathcal{R}^n$ *and* $\mathcal{N}_\lambda \subset \mathcal{O}(\mathcal{R}^n)$*. Then* λ *is the applicable value of the Nehari manifold method. Furthermore, if* $\mathcal{N}_\lambda \subset \mathcal{O}(\mathcal{R}^n)$*,* $\forall \lambda \in Z(\mathcal{R}^n)$*, then the set of applicable values of the Nehari manifold method coincides with* $Z(\mathcal{R}^n)$*.*

The proof is straightforward.

Let us show how this can be used by a simple example. Suppose that

(S): $\forall u \in \mathcal{O}(\mathcal{R}^n)$*,* $tu \in \mathcal{O}(\mathcal{R}^n)$*,* $\forall t > 0$ *and* $\mathcal{R}^n(tu)$ *has no fibering critical points in* $\mathbb{R}^+ \setminus 0$ *except of global minimum or maximum points of* $\mathcal{R}^n(tu)$*.*

Define
$$\lambda^{n,*}_{min} = \sup_{u\in\mathcal{O}(\mathcal{R}^n)} \lambda^n_{min}(u) \equiv \sup_{u\in\mathcal{O}(\mathcal{R}^n)} \inf_{t\in\mathbb{R}^+} \mathcal{R}^n(tu), \tag{4}$$

and
$$\lambda^{n,*}_{max} = \inf_{u\in\mathcal{O}(\mathcal{R}^n)} \lambda^n_{max}(u) \equiv \inf_{u\in\mathcal{O}(\mathcal{R}^n)} \sup_{t\in\mathbb{R}^+} \mathcal{R}^n(tu). \tag{5}$$

The functionals given by

$$\lambda_{min}(u) := \inf_{t\in\mathbb{R}^+} \mathcal{R}^n(tu), \quad \lambda_{max}(u) := \sup_{t\in\mathbb{R}^+} \mathcal{R}^n(tu), \quad u \in \mathcal{O}(\mathcal{R}^n), \tag{6}$$

are called the *Nehari manifold nonlinear generalized Rayleigh quotients* (*Nehari manifold NG-Rayleigh quotients*) [27].

Lemma 2.6. *Assume* $f_\lambda, f'_\lambda \in C^1(W)$*. Suppose* (S) *and* $-\infty \le \lambda^{n,*}_{min} < \lambda^{n,*}_{max} \le +\infty$*. Assume furthermore that* $\mathcal{N}_\lambda \subset \mathcal{O}(\mathcal{R}^n)$*,* $\forall \lambda \in (\lambda^{n,*}_{min}, \lambda^{n,*}_{max})$*. Then* $(\lambda^{n,*}_{min}, \lambda^{n,*}_{max})$ *is an interval of applicability of the Nehari manifold method.*

Proof. The proof immediately follows from Corollary 2.5. □

Suppose that for any $u \in \mathcal{O}(\mathcal{R}^n)$ the one-dimensional fibering function $\mathcal{R}^n(tu)$ is well-defined for all $t > 0$ and has a countable (or finite) set of extreme points $t_1(u), t_2(u)... \in \mathbb{R}^+\setminus 0$ such that the following maps $t_i(\cdot) : \mathcal{O}(\mathcal{R}^n) \to \mathbb{R}^+\setminus 0$, $i = 1, \dots$, are well-defined and we are able to introduce

$$\lambda_i(u) := \mathcal{R}(t_i(u)u), \quad u \in \mathcal{O}(\mathcal{R}^n), \quad i = 1, \dots .$$

Then we call $\lambda_i(u), i = 1, \dots$ the *the Nehari manifold NG-Rayleigh quotients* [27]. There is the following conjecture:

The set of extreme values of the Nehari manifold method $\sigma_{\mathcal{N}}$ *can be found by means of the set of critical values of the Nehari manifold NG-Rayleigh quotients* $(\lambda_i(u))_{i=1}^{\infty}$.

Notice that $\lambda_i(u), i = 1, \dots$ are 0-homogeneous functionals, that is $\lambda_i(tu) = \lambda_i(u)$, $\forall t > 0$, $u \in \mathcal{O}(\mathcal{R}^n)$. Observe that 0-homogeneity is a basic property of the classical Rayleigh quotient used in the linear theory [31].

In general, the critical points of the Nehari manifold NG-Rayleigh quotients do not necessarily correspond to the critical points of $f_\lambda(u)$. Indeed, consider, for instance, the above introduced Nehari manifold NG-Rayleigh quotients $\lambda_{min}(u)$. Suppose there exists $\hat{u} \in \mathcal{O}(\mathcal{R}^n)$ such that $D\lambda_{min}(\hat{u}) = 0$ in W. It is not hard to show that this implies $D\mathcal{R}^n(\hat{u}) = 0$, and consequently, $D^2 f_{\hat{\lambda}}(\hat{u}) = 0$ with $\hat{\lambda} = \lambda_{min}(\hat{u})$.

Thus if we suppose that $\hat{u}$ is a critical point of $f_{\hat{\lambda}}(u)$ as well, then $\hat{u}$ satisfies to the two equations $Df_{\hat{\lambda}}(\hat{u}) = 0$, $D^2 f_{\hat{\lambda}}(\hat{u}) = 0$. This is possible for linear equations, but impossible, in general, for nonlinear problems. Thus, we come to the conclusion that with respect to this property, Nehari manifold NG-Rayleigh quotients manifold differ from the classical Rayleigh quotient from the linear theory.

However, there are nonlinear generalized Rayleigh quotients which are useful in finding solutions to nonlinear problems (see below and [12, 28]). Indeed, consider the so-called *energy level Rayleigh quotient* corresponding to the equation

$$Df_\lambda(u) \equiv Dh(u) - \lambda Dg(u) = 0,$$

that is
$$\mathcal{R}^e(E;u) := \frac{h(u) - E}{g(u)}, \quad u \in \mathcal{O}(\mathcal{R}^e),$$

where $\mathcal{O}(\mathcal{R}^e) := \{u \in W : \ g(u) \neq 0\}$. Here the value E is called energy (or action) of the solution. Observe, if $D\mathcal{R}^e(E;u) = 0$, $\lambda = \mathcal{R}^e(E;u)$, then $Df_\lambda(u) = 0$ and $f_\lambda(u) = E$, that is any critical point of $\mathcal{R}^e(E;u)$ with $\lambda = \mathcal{R}^e(E;u)$ corresponds to the solution of the equation $Df_\lambda(u) = 0$ with prescribed energy E, [28].

For $u \in \mathcal{O}(\mathcal{R}^e)$, consider the fibering function $\mathcal{R}^e(E;tu)$, $t > 0$. Suppose that $\forall u \in \mathcal{O}(\mathcal{R}^e)$, (S) is satisfied with $\mathcal{R}^e(E;tu)$ instead of $\mathcal{R}^n(tu)$. Then as above we are able to introduce the following *energy level nonlinear generalized Rayleigh quotients*

$$\lambda^e_{min}(E;u) := \inf_{t \in \mathbb{R}^+} \mathcal{R}^e(E;tu), \quad \lambda^e_{max}(E;u) := \sup_{t \in \mathbb{R}^+} \mathcal{R}^e(E;tu).$$

Hence, we have

Corollary 2.7. *The energy level nonlinear generalized Rayleigh quotients* $\lambda^e_{min}(E;u)$, $\lambda^e_{max}(E;u)$ *are* 0*-homogeneous functionals. Any critical point* $\hat{u}$ *of* $\lambda^e_{min}(E;u)$ *or* $\lambda^e_{max}(E;u)$ *is satisfied to equation* $Df_\lambda(\hat{u}) = 0$ *with prescribed energy level* $f_\lambda(\hat{u}) = E$ *and* $\lambda = \lambda^e_{min}(E;\hat{u})$ *or* $\lambda = \lambda^e_{max}(E;\hat{u})$*, respectively.*

Remark 2.8. Corollary 2.7 means that the energy level nonlinear generalized Rayleigh quotients $\lambda^e_{min}(E;u)$, $\lambda^e_{max}(E;u)$ have all the basic properties that it has the classical Rayleigh quotient from the linear theory.

Remark 2.9. When dealing with linear eigenfunction problems such as $f_\lambda(u) := Au - \lambda u = 0$, where A is, say, a compact Hermitian operator in the Hilbert space H, we always arrive to zero energy solutions, i.e. $E = \langle Au, u\rangle - \lambda \langle u, u\rangle = 0$, $u \in H \setminus 0$. Importantly, the corresponding energy-level NG-Rayleigh quotient (with $E = 0$) and the Nehari manifold NG-Rayleigh quotient coincide and given by $\langle Au, u\rangle / \langle u, u\rangle$.

Remark 2.10. The application of some global methods, such as the critical point theory or Ekeland's variational principle requires that the Nehari manifold be C^1-manifold (see, e.g., [4, 5, 14, 19, 24, 43, 46]). In this regard, it is important to emphasize that Corollary 2.1 implies that $\mathcal{N}_\lambda$ is C^1-manifold for any applicable values λ of the Nehari manifold, and which in turn can be determined using the NG-Rayleigh quotient method.

Remark 2.11. Importantly, the Nehari manifold method provides obtaining ground states of problems. In addition, in the case of parametric equations, this allows one to construct branches of solutions modulo values of the functional f_λ [9, 7, 21, 23], which is a useful tool to study various problems, for instance, in the study of the stability of ground states [28], construction of solution at limit points [7, 30].

3. Extreme values of the convex-concave problem

The complexity of Φ_λ may be ranked depending on the number of critical points of the fibering functions $t \mapsto \Phi_\lambda(tu)$ $u \in W$. The simplest case is when $\Phi_\lambda(tu)$ has at most one critical point for any $u \in W$ and $\lambda \in \mathbb{R}$. In this case, under general assumptions Nehari manifold method is applicable for any $\lambda \in \mathbb{R}$. However, when $\Phi_\lambda(tu)$ may have two or more critical points for some $u \in W$ and $\lambda \in \mathbb{R}$, the problem becomes more complicated, because of the Nehari manifold $\mathcal{N}_\lambda$ may contains a point u where $\Phi''_\lambda(u) = 0$. This difficulty can be overcome by finding the corresponding extreme values of the Nehari manifold method. In this section, we show an application of the nonlinear generalized Rayleigh quotient method to the case when $\Phi_\lambda(tu)$ has at most two critical points.

Consider

$$\begin{cases} -\Delta_p u = \lambda |u|^{q-2}u + |u|^{\gamma-2}u & \text{in } \ \Omega, \\ \ u = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{7}$$

Here $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 < q < p < \gamma < p^*$, $p^* = \frac{pN}{N-p}$ if $p < N$, $p^* = +\infty$ if $p \geq N$. In what follows, the norm on the Sobolev space $W^{1,p}_0 := W^{1,p}_0(\Omega)$ we denote by $\|\cdot\|_1$.

By a *weak solution* of (7) we mean a critical point $u \in W^{1,p}_0$ of the energy functional

$$\Phi_\lambda(u) := \frac{1}{p}\int |\nabla u|^p \, dx - \lambda \frac{1}{q}\int |u|^q \, dx - \frac{1}{\gamma}\int |u|^\gamma \, dx.$$

We construct solutions via the following two Nehari manifolds minimization problems

$$\hat{\Phi}^+_\lambda := \min\{\Phi_\lambda(u) : \ \ u \in \mathcal{N}^+_\lambda\}, \tag{M^+_λ}$$

$$\hat{\Phi}^-_\lambda := \min\{\Phi_\lambda(u) : \ \ u \in \mathcal{N}^-_\lambda\}. \tag{M^-_λ}$$

Here

$$\mathcal{N}^+_\lambda := \{u \in W^{1,p}_0 \setminus 0 : \ \Phi'_\lambda(u) = 0, \ \Phi''_\lambda(u) \geq 0\},$$

$$\mathcal{N}^-_\lambda := \{u \in W^{1,p}_0 \setminus 0 : \ \Phi'_\lambda(u) = 0, \ \Phi''_\lambda(u) \leq 0\},$$

where $\mathcal{N}_\lambda = \mathcal{N}^+_\lambda \cup \mathcal{N}^-_\lambda$ are the Nehari manifolds. A sequence $(u^\pm_m) \subset \mathcal{N}^\pm_\lambda$ is said to be *minimizing* of $(M^\pm_\lambda)$ if $\Phi_\lambda(u^\pm_m) \to \hat{\Phi}^\pm_\lambda$ as $m \to +\infty$.

Observe that for $u \in W^{1,p}_0 \setminus 0$, the fibering function $\Phi_\lambda(tu)$ may have at most two critical points: $t^+_\lambda(u)$, $t^-_\lambda(u)$ such that $0 < t^+_\lambda(u) \leq t^-_\lambda(u) < +\infty$, $\Phi''_\lambda(t^+_\lambda(u)u) \geq 0$, $\Phi''_\lambda(t^-_\lambda(u)u) \leq 0$, and thus $t^\pm_\lambda(u)u \in \mathcal{N}^\pm_\lambda$. Furthermore, $t^+_\lambda(u) = t^-_\lambda(u)$ if and only if $\Phi''_\lambda(t^\pm_\lambda(u)u) = 0$.

Evidently, $$\hat{\Phi}^+_\lambda := \min\{\Phi_\lambda(u) : \quad u \in \mathcal{N}_\lambda\}, \qquad (M_\lambda)$$
and thus any minimizer of (M^+_λ) is a ground state of (7). By Lemma 2.2, we have

Corollary 3.1. *Let $\lambda > 0$. If a minimizer $\bar{u}^\pm_\lambda$ of $(M^\pm_\lambda)$ satisfies $\Phi''_\lambda(\bar{u}^\pm_\lambda) \neq 0$, then $\bar{u}^\pm_\lambda$ is a weak solution of* (7).

Furthermore, we have

Lemma 3.2. *Let $\lambda > 0$. Any minimizing sequence of $(M^\pm)$ has a nonzero limit point $u^\pm_\lambda \in W \setminus 0$ in the weak topology of $W^{1,p}_0$ and in the strong topology of L^r, $1 < r < p^*$. Moreover, $\Phi_\lambda(u^\pm_\lambda) \leq \hat{\Phi}^\pm_\lambda$.*

Proof. The weak lower-semicontinuity of the norm of $W^{1,p}_0$ and the Sobolev theorem imply that $\Phi_\lambda(u)$ is weakly lower-semicontinuous on $W^{1,p}_0$. Furthermore, the functional $\Phi_\lambda(u)$ is coercive on $\mathcal{N}_\lambda$ since by the Sobolev embedding we have

$$\Phi_\lambda(u) \geq \frac{\gamma - p}{p\gamma}\|u\|^p_1 - \lambda C\frac{\gamma - q}{q\gamma}\|u\|^q_1, \quad \forall u \in \mathcal{N}_\lambda, \ \forall \lambda \in \mathbb{R},$$

for some constant $C < +\infty$. Since $p > q$, for each $\lambda > 0$, $\hat{\Phi}^\pm_\lambda > -\infty$ and any minimizing sequence of $(M^\pm_\lambda)$ contains a subsequence $(u^\pm_m)$ which weakly in $W^{1,p}_0$ and strongly in L^r, $1 < r < p^*$ converges to some limit point $u^\pm_\lambda$. Since

$$(p - q)\|u^-_m\|^p_1 \leq (\gamma - q)\|u^-_m\|^\gamma_{L^\gamma}, \quad m = 1, \dots,$$

this implies by the Sobolev embedding

$$(p - q)\|u^-_m\|^p_1 \leq (\gamma - q)\|u^-_m\|^\gamma_{L^\gamma} \leq C\|u^-_m\|^\gamma_1, \ \ m = 1, \dots,$$

for some constant $C < +\infty$ which does not depend on $m = 1, \dots$. Hence, we have $\|u^-_m\|^{\gamma-p}_1 > (p - q)/C$, and consequently, $\|u^-_m\|^\gamma_{L^\gamma} \geq (p - q)((p - q)/C)^{p/(\gamma-p)} > 0$, $m = 1, \dots$. Thus, $\|u^-_\lambda\|^\gamma_{L^\gamma} \geq (p - q)((p - q)/C)^{p/(\gamma-p)} > 0$ and therefore, $u^-_\lambda \neq 0$. Note that $\lim_{m\to+\infty} \Phi_\lambda(u^+_m) \geq 0$ if $u^+_\lambda = 0$. Hence, $u^+_\lambda \neq 0$ since $\Phi_\lambda(u^+_m) \to \hat{\Phi}^+_\lambda < 0$. Evidently, $\Phi_\lambda(u^\pm_\lambda) \leq \hat{\Phi}^\pm_\lambda$. □

To continue we need the following Rayleigh quotient

$$\mathcal{R}^n(u) = \frac{\int |\nabla u|^p dx - \int |u|^\gamma dx}{\int |u|^q dx}, \ \ u \in W^{1,p}_0 \setminus 0.$$

Observe $\mathcal{N}_\lambda = \{u \in W^{1,p}_0 \setminus 0 : \mathcal{R}^n(u) = \lambda\}$. The only critical point of $\mathcal{R}^n(su)$ is a global maximum point of the function $\mathcal{R}^n(su)$ which can be found precisely:

$$s_{max}(u) = \left(\frac{(p - q)\int |\nabla u|^p dx}{(\gamma - q)\int |u|^\gamma dx}\right)^{\frac{1}{\gamma-p}}.$$

Substituting $s_{max}(u)$ into $\mathcal{R}^n(su)$ we obtain the following nonlinear generalized Rayleigh quotient

$$\lambda(u) := c_{p,q}\frac{(\int |\nabla u|^p dx)^{\frac{\gamma-q}{\gamma-p}}}{\int |u|^q dx \, (\int |u|^\gamma dx)^{\frac{p-q}{\gamma-p}}},$$

where $c_{p,q} = \frac{\gamma-p}{p-q}\left(\frac{p-q}{\gamma-q}\right)^{\frac{\gamma-q}{\gamma-p}}$.

Notice that $$\lambda'(u)=0,\ \lambda(u)=\lambda\ \Leftrightarrow\ \Phi'_\lambda(u)=0,\ \Phi''_\lambda(u)=0. \tag{8}$$

Consider the nonlinear generalized Rayleigh extremal value:

$$\lambda^*=\inf_{u\in W_0^{1,p}\setminus 0}\lambda(u)=c_{p,q}\inf_{u\in W_0^{1,p}\setminus 0}\frac{(\int|\nabla u|^p dx)^{\frac{\gamma-q}{\gamma-p}}}{\int|u|^q dx\,(\int|u|^\gamma dx)^{\frac{p-q}{\gamma-p}}}. \tag{9}$$

The Sobolev inequalities imply that $0<\lambda^*<+\infty$. Now we are able to prove the following

Lemma 3.3. *Let* $\lambda>0$.

(1) *If* $\lambda\le\lambda(u^\pm_\lambda)$, *then* $u^\pm_\lambda$ *is a nonzero minimizer of* $(M^\pm_\lambda)$.

(2) *If* $\lambda<\lambda(u^\pm_\lambda)$, *then* $u^\pm_\lambda$ *is a nonzero weak solution of* (7).

Proof. Let us show (1). Let $(u^\pm_m)$ be a minimizing sequence of $(M^\pm)$ such that $u^\pm_\lambda\in W\setminus 0$ is its limit point in the weak topology of $W_0^{1,p}$ and in the strong topology of L^r, $1<r<p^*$.

The assumption $\lambda\le\lambda(u^\pm_\lambda)$ implies the existence of $t^\pm_\lambda(u^\pm_\lambda)>0$. Hence,

$$\lambda=\mathcal{R}^n(t^\pm_\lambda(u^\pm_\lambda)u^\pm_\lambda)\le\liminf_{m\to+\infty}\mathcal{R}^n(t^\pm(u^\pm_\lambda)u^\pm_m). \tag{10}$$

Since $\mathcal{R}^n(u^+_m)=\lambda$, $(\mathcal{R}^n)'(u^+_m)\ge 0$, $m=1,\dots$, (10) yields that $1\le t^+(u^+_\lambda)$, and thus

$$\Phi_\lambda(t^+(u^+_\lambda)u^+_\lambda)\le\Phi_\lambda(u^+_\lambda)\le\liminf_{m\to+\infty}\Phi_\lambda(u^+_m)=\hat\Phi^+_\lambda.$$

In view of that $t^+(u^+_\lambda)u^+_\lambda\in\mathcal{N}^+_\lambda$, this implies that u^+_λ is a minimizer of (M^+_λ). Similarly, (10) implies that $\Phi_\lambda(t^-(u^-_\lambda)u^-_m)\le\Phi_\lambda(u^-_m)$, $m=1,\dots$, and thus

$$\Phi_\lambda(t^-(u^-_\lambda)u^-_\lambda)\le\liminf_{m\to+\infty}\Phi_\lambda(t^-(u^-_\lambda)u^-_m)\le\liminf_{m\to+\infty}\Phi_\lambda(u^-_m)=\hat\Phi^-_\lambda,$$

which yields that u^-_λ is a minimizer of (M^-_λ). From here and Corollary 3.1 it follows (2). □

Theorem 3.4. *Assume that* $\lambda\in(0,\lambda^*)$. *Then* (7) *has two distinct weak positive solutions* u^+_λ, u^-_λ *such that* $\Phi''_\lambda(u^+_\lambda)>0$, $\Phi''_\lambda(u^-_\lambda)<0$. *Moreover,* u^+_λ *is a ground state of* (7), $u^\pm_\lambda\in C^{1,\alpha}(\overline{\Omega})$, $\alpha\in(0,1)$.

Proof. Observe for $\lambda\in(0,\lambda^*)$, we have $\lambda<\lambda^*\le\lambda(u^\pm_\lambda)$. Hence Lemma 3.3 implies that $u^\pm_\lambda$ is a weak solution of (7). Since $\Phi_\lambda(|u|)=\Phi_\lambda(u)$, $\Phi'_\lambda(|u|)=\Phi'_\lambda(u)$ for $u\in W_0^{1,p}$, we may assume that $u^\pm_\lambda\ge 0$ in Ω. By the bootstrap argument and the Sobolev embedding theorem it follows that $u^\pm_\lambda\in L^\infty$. Then $C^{1,\alpha}$ -regularity results of DiBenedetto [18] and Tolksdorf [44] (interior regularity) combined with Lieberman [33] (regularity up to the boundary) yield $u^\pm_\lambda\in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha\in(0,1)$. Finally, the Harnack inequality due to Trudinger [45] implies that $u^\pm_\lambda>0$ in Ω. □

We need the following lemma on a continuation to a limit point $\lambda^o>0$.

Lemma 3.5. *Assume that* $\lambda^o > 0$. *Let* $\lambda_m \to \lambda^o$ *as* $m \to +\infty$, *and* $(u^\pm_{\lambda_m})$ *is a sequence of minimizers of* $(M^\pm_{\lambda_m})$ *such that* $D\Phi_{\lambda_m}(u^\pm_{\lambda_m}) = 0$, $m = 1, \ldots$. *Then there exists a weak solution* $u^\pm_{\lambda^o}$ *of* (7), *which is a minimizer of* $(M^\pm_{\lambda^o})$. *Moreover,* $\hat{\Phi}^\pm_{\lambda_m} \to \hat{\Phi}^\pm_{\lambda^o}$ *as* $m \to +\infty$.

Proof. Let $\lambda_m \to \lambda^o$ as $m \to +\infty$, and $(u^\pm_{\lambda_m})$ be a sequence of minimizers of $(M^\pm_{\lambda_m})$ such that $D\Phi_{\lambda_m}(u^\pm_{\lambda_m}) = 0$, $m = 1, \ldots$. Analysis similar to that in the proof of Lemma 3.2 shows that there exist nonzero weak limit points $u^\pm_{\lambda^o} \in W^{1,p}_0 \setminus 0$ of the sequences $(u^\pm_{\lambda_m})$. Hence passing to the limit in $D\Phi_{\lambda_m}(u^\pm_{\lambda_m}) = 0$ we obtain $D\Phi_{\lambda^o}(u^\pm_{\lambda^o}) = 0$. Thus, $u^\pm_{\lambda^o}$ is a weak solution of (7).

Moreover, since $0 = \Phi'_{\lambda^o}(u^\pm_{\lambda^o}) \leq \liminf_{m\to+\infty} \Phi'_{\lambda_m}(u^\pm_{\lambda_m}) = 0$, it can be concluded that $u^\pm_{\lambda_m} \to u^\pm_{\lambda^o}$ strongly in $W^{1,p}_0$, and, in particular, $\hat{\Phi}^\pm_{\lambda_m} \to \Phi_{\lambda^o}(u^\pm_{\lambda^o})$ as $m \to +\infty$. To prove that $u^\pm_{\lambda^o}$ is a minimize of $(M^\pm_{\lambda^o})$, it is sufficient to show that $\hat{\Phi}^\pm_{\lambda^o} = \Phi_{\lambda^o}(u^\pm_{\lambda^o})$. Conversely, suppose that $\hat{\Phi}^\pm_{\lambda^o} < \Phi_{\lambda^o}(u^\pm_{\lambda^o})$. Then there exists $w^\pm \in \mathcal{N}^\pm_{\lambda^o}$ such that $\Phi_{\lambda^o}(w^\pm) = \Phi_{\lambda^o}(u^\pm_{\lambda^o}) - \kappa$ for some $\kappa > 0$. Evidently, for any $\epsilon > 0$, one can find m_ϵ such that $|\Phi_{\lambda^o}(w^\pm) - \Phi_{\lambda_m}(t^\pm_{\lambda_m}(w^\pm)w^\pm)| < \epsilon$ and $|\hat{\Phi}^\pm_{\lambda_m} - \Phi_{\lambda^o}(u^\pm_{\lambda^o})| < \epsilon$ for every $m > m_\epsilon$. Hence we have

$$\Phi_{\lambda^o}(u^\pm_{\lambda^o}) - \epsilon < \hat{\Phi}^\pm_{\lambda_m} \leq \Phi_{\lambda_m}(t^\pm_{\lambda_m}(w^\pm)w^\pm) < \Phi_{\lambda^o}(w^\pm) + \epsilon = \Phi_{\lambda^o}(u^\pm_{\lambda^o}) - \kappa + \epsilon$$

which implies a contradiction. Thus we have $\hat{\Phi}^\pm_{\lambda^o} = \Phi_{\lambda^o}(u^\pm_{\lambda^o})$, and $\hat{\Phi}^\pm_{\lambda_m} \to \hat{\Phi}^\pm_{\lambda^o}$ as $m \to +\infty$. □

We need also the zero-level energy Rayleigh quotient

$$\mathcal{R}^e(u) := \mathcal{R}^e(0; u) = \frac{\frac{1}{p}\int |\nabla u|^p dx - \frac{1}{\gamma}\int |u|^\gamma dx}{\frac{1}{q}\int |u|^q dx}, \quad u \in W^{1,p}_0 \setminus 0.$$

Notice that $\mathcal{R}^e(u) = \lambda \;\Leftrightarrow\; \Phi_\lambda(u) = 0$. A computation similar to that has been used above for $\mathcal{R}^n(u)$ shows that the unique critical point of $\mathcal{R}^e(su)$ in $s > 0$ is a global maximum point $s^e_{max}(u)$ and one can introduce the corresponding nonlinear generalized Rayleigh quotient $\lambda^e(u) = \mathcal{R}^e(s^e_{max}(u)u)$. Moreover, for $u \in W^{1,p}_0 \setminus 0$,

$$D\lambda^e(u) = 0,\ \lambda^e(u) = \lambda^e \;\Leftrightarrow\; D\Phi_{\lambda^e}(s^e_{max}(u)u) = 0,\ \Phi_{\lambda^e}(s^e_{max}(u)u) = 0 \tag{11}$$

$$\lambda^e(u) = c_{pq}\lambda(u), \text{ where } c_{pq} = qp^{\frac{(p-q)}{(\gamma-p)}}/p^{\frac{(\gamma-q)}{(\gamma-p)}}. \tag{12}$$

It is not hard to show that $\lambda^e(u) < \lambda(u)$, $\forall u \in W^{1,p}_0 \setminus 0$.

Theorem 3.6. *For* $\lambda = \lambda^*$, (7) *has two distinct weak positive solutions* $u^+_{\lambda^*}$, $u^-_{\lambda^*}$ *such that* $\Phi''_{\lambda^*}(u^+_{\lambda^*}) > 0$, $\Phi''_{\lambda^*}(u^-_{\lambda^*}) < 0$, $\hat{\Phi}^+_{\lambda^*} < \hat{\Phi}^-_{\lambda^*}$. *Moreover,* $u^+_{\lambda^*}$, $u^-_{\lambda^*}$ *are minimizers of* (M^+_λ), (M^-_λ), *respectively;* $u^+_{\lambda^*}$ *is a ground state of* (7); $u^\pm_\lambda \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$.

Proof. The existence of the weak solutions $u^+_{\lambda^*}$, $u^-_{\lambda^*}$ which are minimizers of (M^+_λ), (M^-_λ), respectively, follows from Theorem 3.4 and Lemma 3.5.

Suppose, contrary to our claim, that $\Phi''_{\lambda^*}(u^\pm_{\lambda^*}) = 0$. Then by (8), $\lambda^* = \lambda(u^\pm_{\lambda^*})$. Because of (9), this implies $D\lambda(u^\pm_{\lambda^*}) = 0$, and consequently $D\lambda^e(u^\pm_{\lambda^*}) = 0$. Hence

$D\Phi_{\lambda^e}(s^e_{max}(u^{\pm}_{\lambda^*})u^{\pm}_{\lambda^*}) = 0$, where $\lambda^e := \lambda^e(u^{\pm}_{\lambda^*})$. Since we have also $D\Phi_{\lambda^*}(u^{\pm}_{\lambda^*}) = 0$ and $\lambda^e < \lambda^*$, we get a contradiction. The rest of the proof works as the proof of Theorem 3.4. □

In the case $\lambda > \lambda^*$, we have

Theorem 3.7. *There exists* $\bar\lambda \in (\lambda^*, +\infty]$ *such that for any* $\lambda \in (\lambda^*, \bar\lambda)$ *problem* (7) *has a ground state* u^+_λ *such that* $\Phi''_\lambda(u^+_\lambda) > 0$. *Moreover* u^+_λ *is a minimizer of* M_λ, $u^+_\lambda \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, $u^+_\lambda > 0$ *in* Ω.

Proof. By Lemma 3.3, it is sufficient to show that there exists $\bar\lambda > \lambda^*$ such that $\lambda < \lambda(u^+_\lambda)$, for any $\lambda \in (\lambda^*, \bar\lambda)$.

Suppose this is false. Then one can find a sequence $\lambda_m > \lambda^*$, $m = 1, \dots$ such that $\lambda_m \to \lambda^*$ as $m \to +\infty$ and $\lambda(u^+_{\lambda_m}) \le \lambda_m$, $\forall m = 1, \dots$. An analysis similar to that in the proof of Lemma 3.2 shows that there exists a nonzero weak limit point $\tilde u^+ \in W^{1,p}_0 \backslash 0$ and a subsequence, which we still denote by $(u^+_{\lambda_m})$, such that $u^+_{\lambda_m} \to \tilde u^+$ weakly in $W^{1,p}_0$ and strongly in L^β, $1 < \beta < 2^*$. Moreover, we may assume that $\lim_{m\to+\infty} \hat\Phi^+_{\lambda_m} = \tilde\phi^+$ for some $\tilde\phi^+ \le 0$.

Suppose that the convergence $u^+_{\lambda_m} \to \tilde u^+$ is not strong in $W^{1,p}_0$. Since $\lambda^* \le \lambda(\tilde u^+)$, there exists $t^+_{\lambda^*}(\tilde u^+) > 0$ such that

$$\lambda^* = \mathcal{R}^n(t^+_{\lambda^*}(\tilde u^+)\tilde u^+) < \liminf_{m\to+\infty} \mathcal{R}^n(t^+_{\lambda^*}(\tilde u^+)u^+_{\lambda_m}) \le \liminf_{m\to+\infty} \lambda(u^+_{\lambda_m}) \le \liminf_{m\to+\infty} \lambda_m = \lambda^*.$$

We get a contradiction, and thus $u^+_{\lambda_m} \to \tilde u^+$ strongly in $W^{1,p}_0$. Consequently, $\Phi_{\lambda_m}(u^+_{\lambda_m}) \to \Phi_{\lambda^*}(\tilde u^+) = \tilde\phi^+$ as $m \to +\infty$.

Let us show that $\Phi_{\lambda^*}(\tilde u^+) \le \hat\Phi^+_{\lambda^*}$. Indeed, by Theorem 3.6, there exists a minimizer $u^+_{\lambda^*}$ of $(M^+_{\lambda^*})$ and $\lambda^* < \lambda(u^*_\lambda)$. Then for sufficiently large m, $\lambda_m < \lambda(u^*_\lambda)$, and thus there exists $t^+_{\lambda_m}(u^+_{\lambda^*}) > 0$. Evidently, $t^+_{\lambda_m}(u^+_{\lambda^*}) \to 1$ as $m \to +\infty$. Hence,

$$\begin{aligned}\Phi_{\lambda^*}(\tilde u^+) &= \lim_{m\to+\infty} \Phi_{\lambda_m}(u^+_{\lambda_m}) \le \lim_{m\to+\infty} \hat\Phi^+_{\lambda_m} \\ &\le \lim_{m\to+\infty} \Phi_{\lambda_m}(t^+_{\lambda_m}(u^+_{\lambda^*})u^+_{\lambda^*}) \le \lim_{m\to+\infty} \Phi_{\lambda^*}(t^+_{\lambda_m}(u^+_{\lambda^*})u^+_{\lambda^*}) = \hat\Phi^+_{\lambda^*},\end{aligned}$$

and thus, $\Phi_{\lambda^*}(\tilde u^+) \le \hat\Phi^+_{\lambda^*}$. Observe, $\lambda(\tilde u^+) = \lim_{m\to+\infty} \lambda(u^+_{\lambda_m}) \le \lambda^*$ since by the assumption $\lambda(u^+_{\lambda_m}) \le \lambda_m$, $\forall m = 1, \dots$. In view of (9), this implies $\lambda(\tilde u^+) = \lambda^*$, and consequently,$\Phi'_{\lambda^*}(\tilde u^+) = 0$, $\Phi''_{\lambda^*}(\tilde u^+) = 0$. Hence, we have $\tilde u^+ \in \mathcal{N}^+_{\lambda^*} \cap \mathcal{N}^-_{\lambda^*}$. Using Theorem 3.6, we conclude that $\Phi_{\lambda^*}(\tilde u^+) = \hat\Phi^+_{\lambda^*} < \hat\Phi^-_{\lambda^*} \le \Phi_{\lambda^*}(\tilde u^+)$, which is a contradiction. □

One can make the following conclusion about a limit point of the branch of ground states obtained by Nehari manifold method. For $\lambda > 0$, denote by G_λ the set of ground states of (7) and by

$$G^n_\lambda := \{u \in W^{1,p}_0 \setminus 0 : \hat\Phi^+_\lambda = \Phi_\lambda(u),\ \lambda < \lambda(u)\}$$

the subset of minimizes of (M_λ), and define

$$\tilde G^n_\lambda := \{u \in W^{1,p}_0 \setminus 0 : \hat\Phi^+_\lambda = \Phi_\lambda(u),\ \lambda \le \lambda(u)\}.$$

The above results yield that for any $\lambda > 0$, $G^n_\lambda = G_\lambda$ if $G^n_\lambda \neq \emptyset$. On the other hand, by in large, it is possible that $G_\lambda \neq \emptyset$, while $G^n_\lambda = \emptyset$. Thus, it makes sense to consider the ground states obtained by means of Nehari manifold method as a particular branch. The family of set G^n_λ which satisfies $G^n_\lambda \neq \emptyset$, we call the Nehari manifold ground states branch.

It is known that the convex-concave problem (7) possesses upper bound of the set positive solutions, namely, there exists $\lambda^b \in (0, +\infty)$ such that for any $\lambda > \lambda^b$, (7) has no positive solutions. For the proof of this assertion, in the case $p = 2$, we refer the reader to Ambrosetti, Brezis, Cerami [3]. The case $p \neq 2$, can be handled in much the same way using Picone's identity [2, 22].

Theorem 3.8. *There exists a limit value $\lambda^f \in (\lambda^*, +\infty)$ of the branch of the Nehari manifold ground states such that:*

(i) $G^n_\lambda \neq \emptyset$ *for any* $\lambda \in (0, \lambda^f)$*;*

(ii) *there exists a weak solution* $u^+_{\lambda^f}$ *of* (7) *and* $u^+_{\lambda^f} \in \tilde{G}^n_\lambda$*;*

(iii) *there exists a sequence* $(\lambda_m)_{m=1}^\infty \subset (\lambda^f, +\infty)$ *such that* $\lambda_m \downarrow \lambda^f$*, and* $G^n_{\lambda_m} = \emptyset$*, for all* $m = 1, \dots$.

Proof. Define $\lambda^f = \inf\{\lambda > 0 : \ G^n_\lambda = \emptyset\}$. By the above results, $\lambda^f > \lambda^*$, moreover, G^n_λ contains a weak positive solution of (7), and thus, $\lambda^f \leq \lambda^b < +\infty$. Lemma 3.5 yields that there exists a weak solution $u^+_{\lambda^f}$ of (7) which is a minimizer of $(M^+_{\lambda^f})$, and thus, $u^+_{\lambda^f} \in \tilde{G}^n_{\lambda^f}$. Hence, we get (i), (ii). The proof of (iii) is straightforward. □

Remark 3.9. We anticipate that λ^f coincides with λ^b.

Remark 3.10. The existence of two distinct positive solutions of (7) can be obtained by the Nehari manifold method without finding the Nehari manifold extreme value λ^*, if λ is sufficiently close to 0, where it can be shown by appropriate estimates that $\mathcal{N}^0_\lambda := \{u \in W^{1,p}_0 \setminus 0 : \ \Phi'_\lambda(u) = 0, \ \Phi''_\lambda(u) = 0\} = \emptyset$ (see, e.g, [11]). In this regard, note that the existence results in Theorems 3.6, 3.7, 3.8 for $\lambda \in [\lambda^*, \lambda^f]$ correspond to the case $\mathcal{N}^0_\lambda \neq \emptyset$.

Remark 3.11. For some other type of equations, results similar in Theorems 3.6, 3.7, that is in the case $\mathcal{N}^0_\lambda \neq \emptyset$, can also be obtained using spectral analysis with respect to the fibering procedure (see, for instance, [30, 42]).

4. Recursive nonlinear generalized Rayleigh quotient method

This section contains some extensions of the results in [12]. We consider a problem when the fibering function $\Phi_\lambda(tu)$ may have more than two critical points for some $u \in W$ and $\lambda \in \mathbb{R}$. In this case, a direct application of the nonlinear generalized Rayleigh quotient method may not be sufficient. Moreover, for such problems, it may require taking into account more than one of parameters of the problem. Below we show how this difficulty can be overcome by using, introduced in [12], recursive application of the nonlinear generalized Rayleigh quotient method.

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, $N \geq 1$. Consider the following boundary value problem

$$\begin{cases} -\Delta_p u = |u|^{\gamma-2}u + \mu|u|^{\alpha-2}u - \lambda|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

Here $1 < q < \alpha < p < \gamma < p^*$, $p^* = \frac{pN}{N-p}$ if $p < N$, $p^* = +\infty$ if $p \geq N$, and $\lambda, \mu \in \mathbb{R}$. By a weak solution of (13) we mean a critical point $u \in W_0^{1,p}$ of the energy functional

$$\Phi_{\lambda,\mu}(u) = \frac{1}{p}\int |\nabla u|^p\,dx + \frac{\lambda}{q}\int |u|^q\,dx - \frac{\mu}{\alpha}\int |u|^\alpha - \frac{1}{\gamma}\int |u|^\gamma\,dx.$$

It easily seen that for $u \in W_0^{1,p} \setminus 0$, the fibering function $\Phi_{\lambda,\mu}(su)$ may have at most three nonzero critical points

$$0 < s^0_{\lambda,\mu}(u) \leq s^1_{\lambda,\mu}(u) \leq s^2_{\lambda,\mu}(u) < \infty$$

such that (see Figure 4.1)

$$\Phi''_{\lambda,\mu}(s^0_{\lambda,\mu}(u)u) \leq 0, \quad \Phi''_{\lambda,\mu}(s^1_{\lambda,\mu}(u)u) \geq 0, \quad \Phi''_{\lambda,\mu}(s^2_{\lambda,\mu}(u)u) \leq 0.$$

Observe that the case $s^j_{\lambda,\mu}(u) = s^k_{\lambda,\mu}(u)$ may occur for some $j,k \in \{0,1,2\}$, $j \neq k$, and so $\Phi''_{\lambda,\mu}(s^j(u)u) = 0$ would be true. An additional difficulty, in application of the Nehari manifold method, is that the critical points $s^0_{\lambda,\mu}(u)$, $s^2_{\lambda,\mu}(u)$ satisfy the same condition $\Phi''_{\lambda,\mu}(s^0_{\lambda,\mu}(u)u) \leq 0$, $\Phi''_{\lambda,\mu}(s^2_{\lambda,\mu}(u)u) \leq 0$.

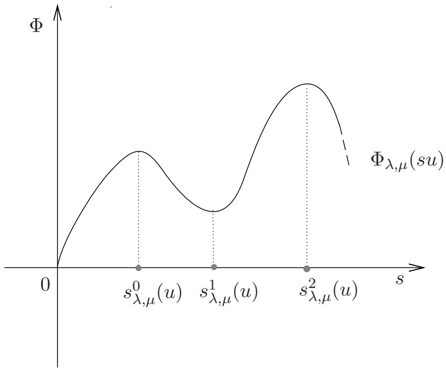

Figure 4.1: *Fibering function* $\Phi_{\lambda,\mu}(su)$, $s \geq 0$, $u \in W_0^{1,p}$.

To apply the Nehari manifold method, we need to find the values of λ, μ for which the strong inequalities $0 < s^0_{\lambda,\mu}(u) < s^1_{\lambda,\mu}(u) < s^2_{\lambda,\mu}(u)$ hold. We solve this problem by recursively application of the nonlinear generalized Rayleigh quotient method so that the simplest problem (the so-called problem of zero codimension of degeneracies) will be used at the final step of the recursion.

In the first step of the recursive procedure, we consider

$$\mathcal{R}^n_\lambda(u) = \frac{\int |\nabla u|^p\,dx + \lambda \int |u|^q - \int |u|^\gamma\,dx}{\int |u|^\alpha\,dx}, \quad u \in W_0^{1,p} \setminus 0, \ \lambda \in \mathbb{R}. \tag{14}$$

Notice that for $u \in W_0^{1,p} \setminus 0$, $\mathcal{R}^n_\lambda(u) = \mu \Leftrightarrow \Phi'_{\lambda,\mu}(u) = 0$. Furthermore, it readily cheek that $\mathcal{R}^n_\lambda(u) = \mu$, $(\mathcal{R}^n_\lambda)'(u) < 0\,(> 0)\,(= 0) \Leftrightarrow \Phi'_{\lambda,\mu}(u) = 0$, $\Phi''_{\lambda,\mu}(u) < 0\,(> 0)\,(= 0)$.

A simple analysis shows that for any $u \in W_0^{1,p} \setminus 0$, the fibering function $\mathcal{R}^n_\lambda(tu)$ may have at most two non-zero fibering critical points such that

$$0 < t^{n,+}_\lambda(u) \leq t^{n,-}_\lambda(u) < +\infty,$$

where $t^{n,+}_\lambda(u)$ is a local minimum, $t^{n,-}_\lambda(u)$ is a local maximum point of $\mathcal{R}^n_\lambda(tu)$, and

$$0 < s^0_{\lambda,\mu}(u) \le t^{n,+}_\lambda(u) \le s^1_{\lambda,\mu}(u) \le t^{n,-}_\lambda(u) \le s^2_{\lambda,\mu}(u) < \infty. \tag{15}$$

Thus, $(\mathcal{R}^n_\lambda)'(t^{n,\pm}_\lambda(u)u) = 0$ and $\mathcal{R}^n_\lambda(s^k_{\lambda,\mu}(u)u) = \mu$, $k = 0, 1, 2$ (see Figure 4.2).

In the second step of the recursive procedure, we apply the nonlinear generalized Rayleigh quotient method to the functional $\mathcal{R}^n_\lambda$ with respect to the parameter λ, i.e., we consider

$$\Lambda^n(u) := \frac{(p-\alpha)\int|\nabla u|^p\,dx - (\gamma-\alpha)\int|u|^\gamma\,dx}{(\alpha-q)\int|u|^q\,dx}, \quad u \in W^{1,p}_0 \setminus 0. \tag{16}$$

Notice that for any $u \in W^{1,p}_0 \setminus 0$, $(\mathcal{R}^n_\lambda)'(tu) = 0 \Leftrightarrow \Lambda^n(tu) = \lambda$. Observe that the only solution of $\frac{d}{dt}\Lambda^n(tu) = 0$ is a global maximum point of the function $\Lambda^n(tu)$ (see Figure 4.3) which is defined by

$$t^n(u) := \left(C_n\frac{\int|\nabla u|^p\,dx}{\int|u|^\gamma\,dx}\right)^{1/(\gamma-p)}, \quad \forall u \in W^{1,p}_0 \setminus 0,$$

where
$$C_n = \frac{(p-\alpha)(p-q)}{(\gamma-\alpha)(\gamma-q)}.$$

Consider the corresponding *nonlinear generalized Rayleigh λ-quotient*

$$\lambda^n(u) := \Lambda^n(t^n(u)u) = c^n_{q,\gamma}\frac{(\int|\nabla u|^p\,dx)^{\frac{\gamma-q}{\gamma-p}}}{(\int|u|^q\,dx)(\int|u|^\gamma\,dx)^{\frac{p-q}{\gamma-p}}}, \tag{17}$$

which has the following *nonlinear generalized Rayleigh λ-extremal value*

$$\lambda^n = \inf_{u\in W^{1,p}_0\setminus 0}\sup_{t>0}\Lambda^n(tu) = c^n_{q,\gamma}\inf_{u\in W^{1,p}_0\setminus 0}\frac{(\int|\nabla u|^p\,dx)^{\frac{\gamma-q}{\gamma-p}}}{(\int|u|^q\,dx)(\int|u|^\gamma\,dx)^{\frac{p-q}{\gamma-p}}}, \tag{18}$$

where
$$c^n_{q,\gamma} = \frac{(p-\alpha)^{\frac{\gamma-q}{\gamma-p}}(p-q)^{\frac{p-q}{\gamma-q}}(\gamma-p)}{(\alpha-q)(\gamma-\alpha)^{\frac{p-q}{\gamma-p}}(\gamma-q)^{\frac{\gamma-q}{\gamma-p}}}.$$

It is not hard to show the following (see, e.g., [12])

Proposition 4.1. *For any $\lambda \in (0, \lambda^n)$ and $u \in W^{1,p}_0 \setminus 0$, the function $\mathcal{R}^n_\lambda(tu)$ has precisely two distinct critical points such that $0 < t^{n,+}_\lambda(u) < t^{n,-}_\lambda(u)$. Moreover, $(\mathcal{R}^n_\lambda)''(t^{n,+}_\lambda(u)u) > 0$ and $(\mathcal{R}^n_\lambda)''(t^{n,-}_\lambda(u)u) < 0$.*

Observe that this and (15) imply that $0 < s^0_{\lambda,\mu}(u) < s^2_{\lambda,\mu}(u) < \infty$ for any $\lambda \in (0, \lambda^n)$.

Notice that for $\lambda \in (0, \lambda^n)$ we are able to introduce the following *nonlinear generalized Rayleigh μ-quotients*

$$\mu^{n,+}_\lambda(u) := \mathcal{R}^n_\lambda(t^{n,+}_\lambda(u)u), \quad \mu^{n,-}_\lambda(u) := \mathcal{R}^n_\lambda(t^{n,-}_\lambda(u)u), \quad u \in W^{1,p}_0 \setminus 0,$$

resulting in the following *nonlinear generalized Rayleigh μ-extremal values*

$$\mu^{n,+}_\lambda = \inf_{u\in W^{1,p}_0\setminus 0}\mu^{n,+}_\lambda(u), \tag{19}$$

$$\mu^{n,-}_\lambda = \inf_{u\in W^{1,p}_0\setminus 0}\mu^{n,-}_\lambda(u). \tag{20}$$

Proposition 4.2. *Assume that* $\lambda \in (0, \lambda^n)$ *and* $\mu < \mu_\lambda^{n,-}$. *Then there exists* $s^2_{\lambda,\mu}(u) > t_\lambda^{n,-}(u)$ *such that* $(\mathcal{R}^n_\lambda)'(s^2_{\lambda,\mu}(u)u) < 0$, *i.e.,* $\Phi''_{\lambda,\mu}(s^2_{\lambda,\mu}(u)u) < 0$ *for any* $u \in W_0^{1,p} \setminus 0$.

Proof. Let $u \in W_0^{1,p} \setminus 0$. If $\mu < \mu_\lambda^{n,-}$, then $\mu < \mu_\lambda^{n,-} \le \mu_\lambda^{n,-}(u) = \mathcal{R}^n_\lambda(t_\lambda^{n,-}(u)u)$. Hence and since the function $\mathcal{R}^n_\lambda(tu)$ monotone decreases on $(t_\lambda^{n,-}(u), +\infty)$, there exists a solution $s^2_{\lambda,\mu}(u) \in (t_\lambda^{n,-}(u), +\infty)$ of the equation $\mathcal{R}^n_\lambda(su) = \mu$ such that $(\mathcal{R}^n_\lambda)'(s^2_{\lambda,\mu}(u)u) < 0$, and consequently, $\Phi''_{\lambda,\mu}(s^2_{\lambda,\mu}(u)u) < 0$. □

However, this assertion is not sufficient to construct a solution corresponding to the critical point $s^2_{\lambda,\mu}(u)$ by the Nehari manifold method since $\Phi_{\lambda,\mu}(su)$ may have another critical point $s^0_{\lambda,\mu}(u)$ which satisfies $\Phi''_{\lambda,\mu}(s^0_{\lambda,\mu}(u)u) \le 0$.

We overcome this difficulty by using the following zero-level energy Rayleigh quotient

$$\mathcal{R}^e_\lambda(u) = \frac{\frac{1}{p}\int |\nabla u|^p\,dx + \frac{\lambda}{q}\int |u|^q\,dx - \frac{1}{\gamma}\int |u|^\gamma\,dx}{\frac{1}{\alpha}\int |u|^\alpha\,dx}, \quad u \in W_0^{1,p} \setminus 0,$$

which is characterized by the fact that

$$\mathcal{R}^e_\lambda(u) = \mu \quad \Leftrightarrow \quad \Phi_{\lambda,\mu}(u) = 0.$$

The Rayleigh quotient $\mathcal{R}^e_\lambda(u)$ possesses similar properties to that $\mathcal{R}^n_\lambda(u)$. In particular, the fibering function $\mathcal{R}^e_\lambda(tu)$ may have at most two non-zero fibering critical points

$$0 < t_\lambda^{e,+}(u) \le t_\lambda^{e,-}(u) < +\infty$$

so that $t_\lambda^{e,+}(u)$ is a local minimum, while $t_\lambda^{e,-}(u)$ is a local maximum point of $\mathcal{R}^e_\lambda(tu)$.

Moreover, the same conclusion as for $\Lambda^n(u)$ can be drawn for the Rayleigh quotient

$$\Lambda^e(u) := q\frac{\frac{(p-\alpha)}{p}\int |\nabla u|^p\,dx - \frac{(\gamma-\alpha)}{\gamma}\int |u|^\gamma\,dx}{(\alpha - q)\int |u|^q\,dx}. \tag{21}$$

which is characterized by the fact that $(\mathcal{R}^e_\lambda)'(tu) = 0 \Leftrightarrow \Lambda^e(tu) = 0$, for $u \in W_0^{1,p} \setminus 0$.

The unique solution of $\frac{d}{dt}\Lambda^e(tu) = 0$ is a global maximum point of the function $\Lambda^e(tu)$ defined by

$$t^e(u) := \left(C_e \frac{\|u\|_1^p}{\|u\|_{L^\gamma}^\gamma}\right)^{1/(\gamma-p)}, \quad \forall u \in W_0^{1,p} \setminus 0, \tag{22}$$

where
$$C_e = \frac{\gamma(p-\alpha)(p-q)}{p(\gamma-\alpha)(\gamma-q)}.$$

Thus we have the following additional *nonlinear generalized Rayleigh λ-extremal value*

$$\lambda^e = \inf_{u \in W_0^{1,p}\setminus 0} \sup_{t>0} \Lambda^e(tu) = c^e_{q,\gamma} \inf_{u \in W_0^{1,p}\setminus 0} \frac{\|u\|_1^{p\frac{\gamma-q}{\gamma-p}}}{\|u\|_{L^q}^q \|u\|_{L^\gamma}^{\gamma\frac{p-q}{\gamma-p}}},. \tag{23}$$

which makes it possible to split the extremal points of the functionals $\mathcal{R}^e_\lambda$, indeed, we have

Proposition 4.3. *For any $\lambda \in (0,\lambda^e)$ and $u \in W^{1,p}_0 \setminus 0$, the function $\mathcal{R}^e_\lambda(tu)$ has precisely two distinct critical points such that $0 < t^{e,+}_\lambda(u) < t^{e,-}_\lambda(u)$. Moreover, $(\mathcal{R}^e_\lambda)''(t^{e,+}_\lambda(u)u) > 0$, $(\mathcal{R}^e_\lambda)''(t^{e,-}_\lambda(u)u) < 0$.*

Thus, for $\lambda \in (0,\lambda^e)$, we have the following *nonlinear generalized Rayleigh μ-quotients*

$$\mu^{e,+}_\lambda(u) := \mathcal{R}^e_\lambda(t^{e,+}_\lambda(u)u), \quad \mu^{e,-}_\lambda(u) := \mathcal{R}^n_\lambda(t^{e,-}_\lambda(u)u), \quad u \in W^{1,p}_0 \setminus 0$$

with corresponding *nonlinear generalized Rayleigh μ-extremal values*

$$\mu^{e,+}_\lambda = \inf_{u \in W^{1,p}_0 \setminus 0} \mu^{e,+}_\lambda(u), \quad \mu^{e,-}_\lambda = \inf_{u \in W^{1,p}_0 \setminus 0} \mu^{e,-}_\lambda(u). \tag{24}$$

The relationships among the NG-Rayleigh quotients is given by the following lemma

Lemma 4.4. *Assume that $1 < q < \alpha < p < \gamma$, $u \in W^{1,p}_0 \setminus 0$, $t > 0$.*
(i) $\Lambda^e(tu) = \Lambda^n(tu) \quad \Leftrightarrow \quad t = t^e(u)$,
(ii) $\mathcal{R}^e_\lambda(tu) = \mathcal{R}^n_\lambda(tu) \Leftrightarrow t = t^{e,+}_\lambda(u)$ *or* $t = t^{e,-}_\lambda(u)$, $\forall \lambda \in (0,\lambda^e)$,
(iii) $t^{n,+}_\lambda(u) < t^{e,+}_\lambda(u) < t^e(u) < t^{n,-}_\lambda(u) < t^{e,-}_\lambda(u)$, $\forall \lambda \in (0,\lambda^e)$.

Lemma 4.4 can be illustrated by the Figures 4.2 and 4.3.

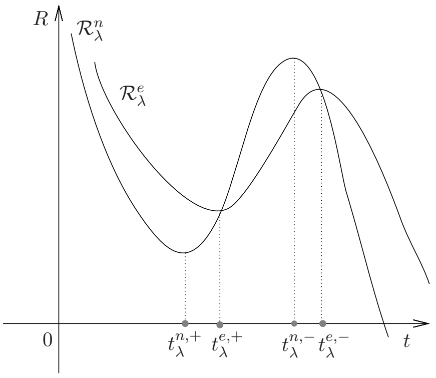

Figure 4.2: The functions $\mathcal{R}^e_\lambda(tu)$, $\mathcal{R}^n_\lambda(tu)$

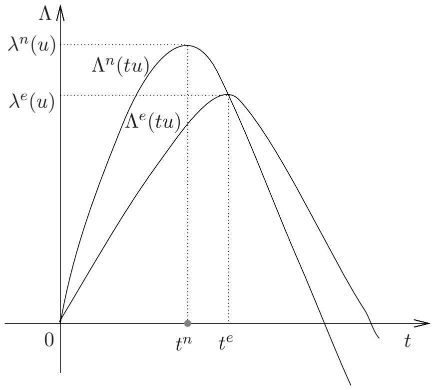

Figure 4.3: The functions $\Lambda^n(tu)$, $\Lambda^e(tu)$

Proof. The equality $\Lambda^e(tu) = \Lambda^n(tu)$ is equivalent to

$$t^{p-q}\|u\|^p_1 - \frac{(\gamma-\alpha)}{(p-\alpha)} t^{\gamma-q}\|u\|^\gamma_{L^\gamma} = q\left(t^{p-q}\frac{1}{p}\|u\|^p_1 - t^{\gamma-q}\frac{(\gamma-\alpha)}{\gamma(p-\alpha)}\|u\|^\gamma_{L^\gamma}\right).$$

Hence,

$$0 = \frac{(p-q)(p-\alpha)}{p} t^{1-q}\|u\|^p_1 - \frac{(\gamma-q)(\gamma-\alpha)}{\gamma} t^{\gamma-q-1}\|u\|^\gamma_{L^\gamma} = (\Lambda^e(tu))',$$

which implies (i).

Observe, $\mathcal{R}^e_\lambda(tu) = \mathcal{R}^n_\lambda(tu)$ for $t > 0$ if and only if

$$t^{p-\alpha}\|u\|^p_1 + \lambda t^{q-\alpha}\|u\|^q_{L^q} - t^{\gamma-\alpha}\|u\|^\gamma_{L^\gamma} = \frac{\alpha t^{p-\alpha}}{p}\|u\|^p_1 + \frac{\lambda\alpha t^{q-\alpha}}{q} - \frac{\alpha t^{\gamma-\alpha}}{\gamma}\|u\|^\gamma_{L^\gamma},$$

which is equivalent to

$$0 = \frac{(p-\alpha)}{p} t^{p-\alpha}\|u\|_1^p - \frac{\gamma-\alpha}{\gamma} t^{\gamma-\alpha}\|u\|_{L^\gamma}^\gamma - \frac{\lambda(\alpha-q)}{q} t^{q-\alpha}\|u\|_{L^q}^q$$
$$= \frac{(\alpha-q)\|u\|_{L^q}^q t^{q-\alpha}}{q}\left(\Lambda_e(tu)-\lambda\right).$$

Since $(\mathcal{R}_\lambda^e)'(tu) = 0 \Leftrightarrow \Lambda^e(tu) = 0$, we get (ii). The proof of (iii) follows from (i) and (ii). □

Lemma 4.5. *Assume that* $1 < q < \alpha < p < \gamma < p^*$. *Then*

(1) $0 < \lambda^e < \lambda^n < +\infty$,

(2) $0 < \mu_\lambda^{n,+} < \mu_\lambda^{e,+} < \mu_\lambda^{e,-} < \mu_\lambda^{n,-} < +\infty$ *for* $\lambda \in (0, \lambda^e)$,

(3) $0 < \mu_\lambda^{n,+} < \mu_\lambda^{n,-} < +\infty$ *for* $\lambda \in (0, \lambda^n)$.

The proof is similar to Theorem 2.1 in [12].

Theorem 4.6. *Assume that* $1 < q < \alpha < p < \gamma < p^*$. *Then*

(1) *For any* $\lambda \in (0, \lambda^e)$, $\mu \in (\mu_\lambda^{e,+}, \mu_\lambda^{n,-})$, *problem* (13) *possesses a weak positive solution* $u^1_{\lambda,\mu} \in C^{1,\kappa}(\overline{\Omega})$, $\kappa \in (0,1)$ *such that*

$$\Phi_{\lambda,\mu}(u^1_{\lambda,\mu}) < 0, \quad \Phi''_{\lambda,\mu}(u^1_{\lambda,\mu}) > 0. \tag{25}$$

Moreover, $u^1_{\lambda,\mu}$ *is a ground state of* (13).

(2) *For any* $\lambda \in (0, \lambda^n)$, $\mu \in (\mu_\lambda^{e,-}, \mu_\lambda^{n,-})$, *problem* (13) *has a weak positive solution* $u^2_{\lambda,\mu} \in C^{1,\kappa}(\overline{\Omega})$, $\kappa \in (0,1)$ *such that*

$$\Phi_{\lambda,\mu}(u^2_{\lambda,\mu}) < 0, \quad \Phi''_{\lambda,\mu}(u^2_{\lambda,\mu}) < 0. \tag{26}$$

Proof. Consider the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} := \{u \in W_0^{1,p} \setminus 0 : \ \mathcal{R}^n_\lambda(u) = \mu\} \equiv \{u \in W_0^{1,p} \setminus 0 : \ \Phi'_{\lambda,\mu}(u) = 0\}.$$

Observe that $\Phi_{\lambda,\mu}$ is coercive on $\mathcal{N}_{\lambda,\mu}$, $\forall \lambda > 0$, $\forall \mu \in \mathbb{R}$. Indeed, by the Sobolev inequality,

$$\Phi_{\lambda,\mu}(u) \geq \frac{\gamma - p}{p\gamma}\|u\|_1^p - \mu C\|u\|_1^\alpha, \quad \forall u \in \mathcal{N}_{\lambda,\mu},$$

where $C < +\infty$ does not depend on $u \in \mathcal{N}_{\lambda,\mu}$. Since $\alpha < p$, this implies that $\Phi_{\lambda,\mu}(u) \to +\infty$ if $\|u\|_1 \to +\infty$ and $u \in \mathcal{N}_{\lambda,\mu}$.

(1) Define $\mathcal{RN}^1_{\lambda,\mu} := \{u \in \mathcal{N}_{\lambda,\mu} : \ (\mathcal{R}^n_\lambda)'(u) > 0\}$ and consider

$$\hat{\Phi}^1_{\lambda,\mu} = \min\{\Phi_{\lambda,\mu}(u) : \ u \in \mathcal{RN}^1_{\lambda,\mu}\}. \tag{27}$$

Observe $\hat{\Phi}^1_{\lambda,\mu} < 0$. Indeed, we have $\mu < \mu_\lambda^{n,-} < \mu_\lambda^{n,-}(u)$, $\forall u \in W_0^{1,p} \setminus 0$, and therefore, for each $\mu \in (\mu_\lambda^{e,+}, \mu_\lambda^{n,-})$, there exists $\tilde{u} \in W_0^{1,p} \setminus 0$ such that

$$\mu_\lambda^{e,+}(\tilde{u}) < \mu < \mu_\lambda^{n,-}(\tilde{u}).$$

Lemma 4.4 implies that $s^1_{\lambda,\mu}(\tilde{u}) \in (t^{e,+}_\lambda(\tilde{u}), t^{e,-}_\lambda(\tilde{u}))$ and $\mathcal{R}^n_\lambda(t\tilde{u}) > \mathcal{R}^e_\lambda(t\tilde{u})$ for all $t \in (t^{e,+}_\lambda(\tilde{u}), t^{e,-}_\lambda(\tilde{u}))$. Hence, $\mu = \mathcal{R}^n_\lambda(s^1_{\lambda,\mu}(\tilde{u})\tilde{u}) > \mathcal{R}^e_\lambda(s^1_{\lambda,\mu}(\tilde{u})\tilde{u})$, and thus,

$$\hat{\Phi}^1_{\lambda,\mu} \le \Phi_{\lambda,\mu}(s^1_{\lambda,\mu}(\tilde{u})\tilde{u}) < 0.$$

In addition, we derive that $\forall \mu \in (\mu^{e,+}_\lambda, \mu^{n,-}_\lambda)$, we have $\mathcal{RN}^1_{\lambda,\mu} \neq \emptyset$, $\forall \lambda \in (0, \lambda^e)$, and $\mathcal{R}^e_\lambda(u) < \mu$ for any $u \in \mathcal{RN}^1_{\lambda,\mu}$.

Let u_m be a minimizing sequence of (27), i.e.,

$$\Phi_{\lambda,\mu}(u_m) \to \hat{\Phi}^1_{\lambda,\mu}, \quad \Phi'_{\lambda,\mu}(u_m) = 0, \ (\mathcal{R}^n_\lambda)'(u_m) > 0, \quad m = 1, \ldots.$$

By the coercivity of $\Phi_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, the sequence (u_m) is bounded in $W^{1,p}_0$ and thus, up to a subsequence,

$$u_m \to \bar{u}^1 \ \text{ strongly in } \ L^r, \ \text{ and weakly in } W^{1,p}_0,$$

for some $\bar{u}^1 \in W^{1,p}_0$, where $r \in (1, p^*)$. Observe that we have

$$\Phi_{\lambda,\mu}(\bar{u}^1) \le \liminf_{m\to\infty} \Phi_{\lambda,\mu}(u_m) = \hat{\Phi}^1_{\lambda,\mu} < 0,$$

and thus $\bar{u}^1 \neq 0$.

Notice that $\lambda < \lambda^e < \lambda^n < \lambda^e(\bar{u}^1) < \lambda^n(\bar{u}^1)$. Consequently, there exist nonlinear generalized Rayleigh μ-quotients $\mu^{e,+}_\lambda(\bar{u}^1), \mu^{n,-}_\lambda(\bar{u}^1)$ such that $\mu^{e,+}_\lambda(\bar{u}^1) < \mu^{n,-}_\lambda(\bar{u}^1)$.

If $\mu < \mu^{e,+}_\lambda(\bar{u}^1)$, then $\mathcal{R}^n_\lambda(\bar{u}^1) \le \mu$ and $\mathcal{R}^e_\lambda(\bar{u}^1) \le \mu$ imply $0 < s^2_{\lambda,\mu}(\bar{u}^1) < 1$. Moreover, by Lemma 4.4, $t^{e,+}_\lambda(\bar{u}^1) < t^{e,-}_\lambda(\bar{u}^1) < s^2_{\lambda,\mu}(\bar{u}^1) < 1$. On the other hand,

$$0 = (\mathcal{R}^e_\lambda)'(t^{e,+}_\lambda(\bar{u}^1)\bar{u}^1) \le \liminf_{m\to\infty} (\mathcal{R}^e_\lambda)'(t^{e,+}_\lambda(\bar{u}^1)u_m),$$

and thus, $t^{e,+}_\lambda(u_m) < t^{e,+}_\lambda(\bar{u}^1) < t^{e,-}_\lambda(u_m)$, $m = 1, 2, \ldots$. From this and since

$$\mu < \mu^{e,+}_\lambda(\bar{u}^1) = \mathcal{R}^e_\lambda(t^{e,+}_\lambda(\bar{u}^1)\bar{u}^1) \le \liminf_{m\to\infty} \mathcal{R}^e_\lambda(t^{e,+}_\lambda(\bar{u}^1)u_m)$$

and $\mathcal{R}^e_\lambda(u_m) < \mu$ for $m = 1, 2, \ldots$, it follows that $1 < t^{e,+}_\lambda(\bar{u}^1)$. We thus get a contradiction: $1 < t^{e,+}_\lambda(\bar{u}^1) < s^2_{\lambda,\mu}(\bar{u}^1) < 1$.

Thus $\mu^{e,+}_\lambda(\bar{u}^1) < \mu < \mu^{n,-}_\lambda(\bar{u}^1)$, and there exists $s^1_{\lambda,\mu}(\bar{u}^1) \in (s^0_{\lambda,\mu}(\bar{u}^1), s^2_{\lambda,\mu}(\bar{u}^1))$ with

$$\mu = \mathcal{R}^n_\lambda(s^1_{\lambda,\mu}(\bar{u}^1)\bar{u}^1) \le \liminf_{m\to\infty} \mathcal{R}^n_\lambda(s^1_{\lambda,\mu}(\bar{u}^1)u_m),$$

$$0 < (\mathcal{R}^n_\lambda)'(s^1_{\lambda,\mu}(\bar{u}^1)\bar{u}^1) \le \liminf_{m\to\infty} (\mathcal{R}^n_\lambda)'(s^1_{\lambda,\mu}(\bar{u}^1)u_m).$$

This means that $1 = s^1_{\lambda,\mu}(u_m) \le s^1_{\lambda,\mu}(\bar{u}^1) < s^3_{\lambda,\mu}(u_m)$, $m = 1, 2, \ldots$. Hence, in view of that $\mathcal{R}^n_\lambda(\bar{u}^1) \le \mu$ and $\mu = \mathcal{R}^n_\lambda(s^1_{\lambda,\mu}(\bar{u}^1)\bar{u}^1)$, we obtain $\mathcal{R}^n_\lambda(t\bar{u}^1) < \mu$, $\forall t \in (1, s^1_{\lambda,\mu}(\bar{u}^1))$, that is $\Phi'_{\lambda,\mu}(t\bar{u}^1) < 0$ for any $t \in (1, s^1_{\lambda,\mu}(\bar{u}^1))$. Hence

$$\Phi_{\lambda,\mu}(s^1_{\lambda,\mu}(\bar{u}^1)\bar{u}^1) \le \Phi_{\lambda,\mu}(\bar{u}^1) \le \liminf_{m\to\infty} \Phi_{\lambda,\mu}(u_m) = \hat{\Phi}^1_{\lambda,\mu},$$

which yields that $\bar{u}^1$ is a minimizer of (27). In view of $\mu < \mu^{n,-}_\lambda < \mu^{n,-}_\lambda(\bar{u}^1)$ we have $(\mathcal{R}^n_\lambda)'(u) > 0$, and consequently, we infer that $u^1_{\lambda,\mu} := \bar{u}^1$ weakly satisfies (13).

A trivial verification shows that $u^1_{\lambda,\mu}$ is a ground state. The rest of the proof is similar to that in Theorem 3.4.

(2) Consider
$$\hat{\Phi}^2_{\lambda,\mu} = \min\{\Phi_{\lambda,\mu}(u) \ : \ u \in \mathcal{RN}^2_{\lambda,\mu}\}, \tag{28}$$
where
$$\mathcal{RN}^2_{\lambda,\mu} = \{u \in \mathcal{N}_{\lambda,\mu} : \ (\mathcal{R}^n_\lambda)'(u) < 0, \ \mathcal{R}^e_\lambda(u) < \mu\}.$$

The assumption $\mu \in (\mu^{e,-}_\lambda, \mu^{n,-}_\lambda)$ implies that there exists $\tilde{u} \in W^{1,p}_0 \setminus 0$ such that $\mu^{e,-}_\lambda(\tilde{u}) < \mu < \mu^{n,-}_\lambda(\tilde{u})$. Hence, by Proposition 4.2, there exists $s^2_{\lambda,\mu}(u) > t^{n,-}_\lambda(u)$ such that $(\mathcal{R}^n_\lambda)'(s^2_{\lambda,\mu}(\tilde{u})\tilde{u}) < 0$. Moreover, $\mu > \mu^{e,-}_\lambda(\tilde{u}) > \mathcal{R}^e_\lambda(s^2_{\lambda,\mu}(\tilde{u})\tilde{u})$, and thus, $s^2_{\lambda,\mu}(\tilde{u})\tilde{u} \in \mathcal{RN}^2_{\lambda,\mu}$. Hence, $\mathcal{RN}^2_{\lambda,\mu} \neq \emptyset$, $\forall \lambda \in (0, \lambda^n)$, $\forall \mu \in (\mu^{e,-}_\lambda, \mu^{n,-}_\lambda)$. Note that the inequality $\mathcal{R}^e_\lambda(u) < \mu$, for $u \in \mathcal{RN}^2_{\lambda,\mu}$ implies $\Phi_{\lambda,\mu}(u) < 0$, and consequently, $\hat{\Phi}^2_{\lambda,\mu} < 0$.

Let (u_m) be a minimizing sequence of (28). Similar to the proof of (1) one deduces that there exists a subsequence, which we again denote by (u_m), and a nonzero limit point $\bar{u}^2$ such that
$$u_m \to \bar{u}^2 \ \text{ strongly in } L^r, \ r \in (1, p^*), \ \text{ and weakly in } W^{1,p}_0.$$
As above in the proof of (1), it follows that $\lambda < \lambda^n(\bar{u}^2)$ and $\mu < \mu^{n,-}_\lambda(\bar{u}^2)$. Consequently, there exist $s^2_{\lambda,\mu}(\bar{u}) > 0$ and $t^{n,\pm}_\lambda(\bar{u}^2) > 0$ with $t^{n,+}_\lambda(\bar{u}^2) < t^{n,-}_\lambda(\bar{u}^2)$, and
$$\mu = \mathcal{R}^n_\lambda(s^2_{\lambda,\mu}(\bar{u}^2)\bar{u}^2) \le \liminf_{m\to\infty} \mathcal{R}^n_\lambda(s^2_{\lambda,\mu}(\bar{u}^2)u_m),$$
$$0 = (\mathcal{R}^n_\lambda)'(t^{n,\pm}_\lambda(\bar{u}^2)\bar{u}^2) \le \liminf_{m\to\infty} (\mathcal{R}^n_\lambda)'(t^{n,\pm}_\lambda(\bar{u}^2)u_m).$$
The first inequality implies that $s^2_{\lambda,\mu}(\bar{u}^2) \in (0, s^0_{\lambda,\mu}(u_m)) \cup (s^1_{\lambda,\mu}(u_m), s^2_{\lambda,\mu}(u_m) = 1)$, whereas from the second we get $t^{n,-}_\lambda(\bar{u}^2) \in (t^{n,+}_\lambda(u_m), t^{n,-}_\lambda(u_m))$, $m = 1, 2, \ldots$.

Hence we get $s^0_{\lambda,\mu}(u_m) < t^{n,+}_\lambda(u_m) < t^{n,-}_\lambda(\bar{u}^2) < s^2_{\lambda,\mu}(\bar{u}^2)$, and thus we have
$$s^2_{\lambda,\mu}(\bar{u}^2) \in (s^1_{\lambda,\mu}(u_m), s^2_{\lambda,\mu}(u_m)), \ \ m = 1, 2, \ldots.$$
Since $\Phi'_{\lambda,\mu}(tu_m) > 0$ for any $t \in (s^1_{\lambda,\mu}(u_m), s^2_{\lambda,\mu}(u_m))$, $m = 1, 2, \ldots$, we have
$$\begin{aligned}\Phi_{\lambda,\mu}(s^2_{\lambda,\mu}(\bar{u}^2)\bar{u}^2) &\le \liminf_{m\to\infty} \Phi_{\lambda,\mu}(s^2_{\lambda,\mu}(\bar{u}^2)u_m) \\ &\le \liminf_{m\to\infty} \Phi_{\lambda,\mu}(s^2_{\lambda,\mu}(u_m)u_m) = \hat{\Phi}^2_{\lambda,\mu} < 0.\end{aligned} \tag{29}$$
The inequality $\Phi_{\lambda,\mu}(s^2_{\lambda,\mu}(\bar{u}^2)\bar{u}^2) < 0$ implies $\mathcal{R}^e_\lambda(s_2(\bar{u}^2)\bar{u}^2) < \mu$. Thus we obtain $s_2(\bar{u}^2)\bar{u}^2 \in \mathcal{RN}^2_{\lambda,\mu}$, and $s_2(\bar{u}^2)\bar{u}^2$ is a minimizer of (28). By (29), this implies that $u_m \to \bar{u}^2$ strongly in $W^{1,p}_0$ and $s^2_{\lambda,\mu}(\bar{u}^2) = 1$. Consequently, $u^2_{\lambda,\mu} := \bar{u}^2$ is a minimizer of (28). The rest of the proof works as before. □

Corollary 4.7. *Assume that* $1 < q < \alpha < p < \gamma < p^*$. *Then for any* $\lambda \in (0, \lambda^e)$ *and* $\mu \in (\mu^{e,-}_\lambda, \mu^{n,-}_{\underline{\lambda}})$, *problem* (13) *possesses two distinct weak positive solutions* $u^1_{\lambda,\mu}, u^2_{\lambda,\mu} \in C^{1,\kappa}(\overline{\Omega})$, $\kappa \in (0,1)$ *such that*
$$\Phi_{\lambda,\mu}(u^1_{\lambda,\mu}) < 0, \ \ \Phi''_{\lambda,\mu}(u^1_{\lambda,\mu}) > 0, \ \ \Phi_{\lambda,\mu}(u^2_{\lambda,\mu}) < 0, \ \ \Phi''_{\lambda,\mu}(u^2_{\lambda,\mu}) < 0.$$
Moreover, $u^1_{\lambda,\mu}$ *is a ground state of* (13).

Remark 4.8. We anticipate that (13) has a third positive solution. This leads us to the conjecture that the branch of solutions to (13) has of S-shape type bifurcation curve, see Figure 4.4.

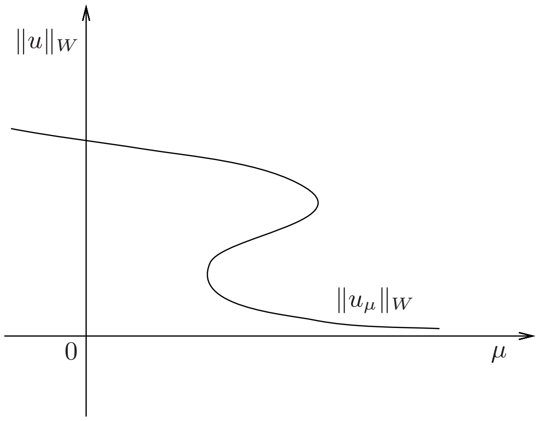

Figure 4.4: *The branches of solutions to* (13) *with fix* $\lambda \in (0, \lambda^e)$.

Remark 4.9. Lubushev in [35], Bobkov, Drábek, Hernández in [8] using global and local minimizations, fibering method and mountain pass theorem as well, obtained the existence of three positive solution for equation (13) in the case $1 < q < p < \alpha < \gamma < p^*$, for some $\lambda, \mu \in (-\infty, 0)$. However, the geometry of the fibering function $\Phi_{\lambda,\mu}(tu)$ in this case is different than that have been considered above for (13). This fact prevents a direct application of the result of [8, 35] to our problem.

Remark 4.10. We wonder if one can find solution $u^2_{\lambda,\mu}$ as in Theorem 4.6 without using the Rayleigh quotients $\mathcal{R}^n_\lambda(u)$, $\mathcal{R}^e_\lambda(u)$ as in (27) and (28). Note that the function $\Phi'''_{\lambda,\mu}(su)$ can change sign twice on $(0, +\infty)$ if $\gamma < \alpha + 1$ (see e.g. [12]), that can cause difficulties in the splitting the critical points of $\Phi_{\lambda,\mu}(u)$ if the NG-Rayleigh quotients are not used.

5. Prescribed energy solution of the zero frequency problem

In this section, we use the energy level Rayleigh quotient method to prove an existence of solution of the following zero-frequency problem [28] (which is also called the "zero mass" case problem (cf. [6]):

$$-\Delta u - \mu|u|^{p-2}u + |u|^{q-2}u = 0, \quad x \in \mathbb{R}^N, \tag{30}$$

where $p, q \in (2, 2^*)$, $N \geq 3$ and $\mu \in \mathbb{R}$. The problem has a variational structure and the associated energy functional

$$E_\mu(u) = \int \left(\frac{1}{2}|\nabla u|^2 - \frac{\mu}{p}|u|^p + \frac{1}{q}|u|^q\right) dx$$

is defined in $\mathcal{D} := \mathcal{D}^{1,2}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, where

$$\mathcal{D}^{1,2} := \mathcal{D}^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

the space with norm $\|u\|_{\mathcal{D}^{1,2}} := \int |\nabla u|^2 \, dx$ (cf. [6, 46]).

For $u \in \mathcal{D}$, we denote $u_\sigma := u(x/\sigma)$, $x \in \mathbb{R}^N$, $\sigma > 0$, and

$$T(u) := \int |\nabla u|^2 \, dx, \quad A(u) := \int |u|^p \, dx, \quad B(u) := \int |u|^q \, dx.$$

For $E \geq 0$, consider the energy level Rayleigh quotient

$$R^E(u) := \frac{\frac{1}{2}T(u) + \frac{1}{q}B(u) - E}{\frac{1}{p}A(u)}. \tag{31}$$

Let $u \in \mathcal{D} \setminus 0$, $E > 0$, $\sigma > 0$, consider

$$r_u^E(\sigma) := R^E(u_\sigma) = \frac{\sigma^{-2}\frac{1}{2}T(u) + \frac{1}{q}B(u) - \sigma^{-N}E}{\frac{1}{p}A(u)}.$$

Then $$\frac{d}{d\sigma} r_u^E(\sigma) = 0 \;\Leftrightarrow\; \sigma = \sigma^E(u) := \left(\frac{NE}{T(u)}\right)^{\frac{1}{N-2}}.$$

Hence, we are able to introduce the following nonlinear generalized Rayleigh quotient

$$M^E(u) := R^E(u_{\sigma^E(u)}) = \frac{p}{A(u)}\left(\frac{c_N^E}{2} T^{\frac{N}{(N-2)}}(u) + \frac{1}{q}B(u)\right),$$

where $$c_N^E = \frac{(N-2)}{N^{\frac{N}{(N-2)}} E^{\frac{2}{(N-2)}}}.$$

Observe that $M^E(u)$ is a 0-homogeneous functional with respect to the scale change $\sigma \mapsto u_\sigma$. Furthermore, it easy to see that

$$DM^E(u) = 0, M^E(u) = \mu, \sigma^E(u) = 1 \quad \Leftrightarrow \quad DE_\mu(u) = 0, E_\mu(u) = E. \tag{32}$$

For every $u \in \mathcal{D} \setminus 0$, consider the fibering function

$$M^E(tu) := \frac{\frac{c_N^E}{2} t^{2^*-p} T^{\frac{N}{(N-2)}}(u) + \frac{1}{q} t^{q-p} B(u)}{\frac{1}{p}A(u)}, \quad t > 0.$$

Notice that if $2 < p < q < 2^*$, then $(M^E)'(tu) \neq 0$, $\forall t > 0$, $\forall u \in \mathcal{D}$. Hence, by (32) we have

Corollary 5.1. *If* $2 < p < q < 2^*$, *then* (30) *has no weak solution in* $\mathcal{D} \setminus 0$ *with any energy level* $E > 0$.

Assume that $2 < q < p < 2^*$. It is easily seen that the function $t \mapsto M^E(tu)$ attains its global minimum at the unique point

$$t^E(u) = \left(\frac{B(u)}{c_{p,q,N,E} T^{\frac{N}{(N-2)}}(u)}\right)^{1/(2^*-q)},$$

where $c_{p,q,N,E} = c_N^E q(2^* - p)/(2(p-q))$. Define

$$\mu^E(u) := M^E(t^E(u)u) = \min_{t\geq 0} M^E(tu) = C_{p,q,N,E}\frac{B^{\frac{2^*-p}{2^*-q}}(u)T^{\frac{2^*(p-q)}{2(2^*-q)}}(u)}{A(u)}, \tag{33}$$

where
$$C_{p,q,N,E} = \frac{c(p,q,N)}{E^{\frac{2(p-q)}{(2^*-q)(N-2)}}}, \tag{34}$$

and
$$c(p,q,N) = \left(\frac{(N-2)}{N^{\frac{N}{(N-2)}}}\frac{q(2^*-p)}{2(p-q)}\right)^{\frac{(p-q)}{(2^*-q)}}\frac{p(2^*-q)}{q(2^*-p)}. \tag{35}$$

Corollary 5.2.

(1) $D\mu^E(u) = 0$, $\mu^E(u) = \mu$, $\sigma^E(u) = t^E(u) = 1 \Leftrightarrow DE_\mu(u) = 0$, $E_\mu(u) = E$;

(2) $\mu^E(u)$ *is a multi-homogeneous functional on* $\mathcal{D}$, *namely:*
$\mu^E(u) = \mu^E(u_\sigma) = \mu^E(tu)$, $\forall\sigma > 0$, $\forall t > 0$, $\forall u \in \mathcal{D}\setminus 0$.

Define $\beta := \frac{2q(2^*-p)}{2^*(p-q)}$, $\rho := \frac{2p(2^*-q)}{2^*(p-q)}$ and

$$\mu(u) := \frac{\|u\|^\beta_{L^q}\|\nabla u\|^2_{L^2}}{\|u\|^\rho_{L^p}(u)} \equiv \frac{E^{\frac{2(p-q)}{(2^*-q)(N-2)}}}{c(p,q,N))^{\rho/p}}\mu^E(u), \quad u \in \mathcal{D}\setminus 0.$$

Consider
$$\bar\mu = \inf_{u\in\mathcal{D}\setminus 0}\mu(u). \tag{36}$$

Define
$$\hat\mu^E := \frac{c(p,q,N)}{E^{\frac{2(p-q)}{(2^*-q)(N-2)}}}\bar\mu^{p/\rho}, \quad \forall E > 0. \tag{37}$$

By the Gagliardo–Nirenberg interpolation inequality

$$\int |u|^p\,dx \leq C_{gn}(\int |\nabla u|^2\,dx)^{\frac{2^*(p-q)}{2(2^*-q)}}(\int |u|^q\,dx)^{\frac{2^*-p}{2^*-q}}$$
$$\Leftrightarrow A(u) \leq C_{gn}(T(u))^{\frac{2^*(p-q)}{2(2^*-q)}}(B(u))^{\frac{2^*-p}{2^*-q}}, \tag{38}$$

where the constant C_{gn} does not depend on $u \in \mathcal{D}$. Hence $0 < \bar\mu < +\infty$.

Theorem 5.3. *Assume that* $2 < q < p < 2^*$, $N \geq 3$. *There exists a minimizer* $\hat u^E_{\hat\mu^E}$ *of* (36) *which is a weak solution of* (30) *with* $\mu = \hat\mu^E$ *and prescribed energy* $E_{\hat\mu^E}(u) = E$. *Moreover,* $\hat u^E_{\hat\mu^E} > 0$ *in* $\mathbb{R}^N$, $\hat u^E_{\hat\mu^E} \in C^2(\mathbb{R}^N)$.

Proof. Let (v_i) be a minimizing sequence of (36), i.e., $\mu(v_i) \to \bar\mu$ as $i \to \infty$. Set $u_i = t_i \cdot (v_i)_{\sigma_i}$, $i = 1, 2, \ldots$, where

$$t_i = (\|v_i\|^q_{L^q}/\|v_i\|^p_{L^p})^{1/(p-q)}, \quad \sigma_i = (\|v_i\|^{pq}_{L^q}/\|v_i\|^{qp}_{L^p})^{1/N(p-q)}.$$

Then $\|u_i\|_{L^p} = 1$ and $\|u_i\|_{L^q} = 1$, $i = 1, 2, \ldots$, and by the homogeneity of $\mu(u)$, (u_i) is a minimizing sequence of (36). Since $\bar\mu < +\infty$, $(\|\nabla u_i\|_{L^2})$ is bounded, and thus (u_i) is bounded in $\mathcal{D}$ and in $W^{1,2}_{loc}(\mathbb{R}^N)$. Thus, by the Banach–Alaoglu and Sobolev embedding theorems, there exists a subsequence, still denoted by (u_i), such that

$$u_i \rightharpoonup \hat u^* \text{ in } \mathcal{D}, \quad u_i \to \hat u^* \text{ in } L^\gamma_{loc},\ 1 \leq \gamma < 2^*, \text{ and } u_i \to \hat u^* \text{ a.e. on } \mathbb{R}^N,$$

for some $\hat u^* \in \mathcal{D}$.

Let $r > 0$. Observe that

$$\delta := \liminf_{i\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y;r)} |u_i|^\gamma \, dx > 0,$$

for each $\gamma \in (1,p)$. Indeed, if this is not true, then by the Lions lemma (see Lemma I.1 p.231, in [34]), $u_i \to 0$ in $L^p(\mathbb{R}^N)$, which is impossible since $\|u_i\|_{L^p} = 1$, $i = 1,\dots$. Thus, passing to a subsequence if necessary, we may assume that there exists $(y_i) \subset \mathbb{R}^N$ such that there holds $\int_{B(y_i;r)} |u_i|^q \, dx > \delta/2$, $\forall i = 1,\dots$. Hence, redefining $u_i := u_i(\cdot + y_n)$ if necessary, we have

$$\int_{B(0;r)} |u_i|^q \, dx > \delta/2, \; i = 1,\dots,$$

which implies that $\hat{u}^* \neq 0$.

Recall the Brezis-Lieb Lemma (see [34]):

Lemma 5.4. *Assume (u_n) is bounded in $L^\gamma(\mathbb{R}^N)$, $1 \le \gamma < +\infty$ and $u_n \to u$ a. e. on $\mathbb{R}^N$, then*

$$\lim_{n\to+\infty} \|u_n\|^\gamma_{L^\gamma} = \|u\|^\gamma_{L^\gamma} + \lim_{n\to+\infty} \|u_n - u\|^\gamma_{L^\gamma}.$$

Hence, we have

$$\|\nabla\hat{u}^*\|^2_{L^2} = \lim_{i\to\infty} \|\nabla u_i\|^2_{L^2} - \lim_{i\to\infty} \|\nabla(u_i - \hat{u}^*)\|^2_{L^2}, \tag{39}$$

$$\|\hat{u}^*\|^p_{L^p} = \lim_{i\to\infty} \|u_i\|^p_{L^p} - \lim_{i\to\infty} \|u_i - \hat{u}^*\|^p_{L^p}, \tag{40}$$

$$\|\hat{u}^*\|^q_{L^q} = \lim_{i\to\infty} \|u_i\|^q_{L^q} - \lim_{i\to\infty} \|u_i - \hat{u}^*\|^q_{L^q}. \tag{41}$$

Suppose that $\lim_{i\to\infty} \|\nabla(u_i - \hat{u}^*)\|^2_{L^2} > 0$. Hence,

$$\begin{aligned}\bar{\mu} &= \lim_{i\to\infty} \|\nabla u_i\|^2_{L^2} = \|\nabla\hat{u}^*\|^2_{L^2} + \lim_{i\to\infty} \|\nabla(u_i - \hat{u}^*)\|^2_{L^2} \\ &\ge \bar{\mu}\left(\frac{\|\hat{u}^*\|^\rho_{L^p}}{\|\hat{u}^*\|^\beta_{L^q}} + \lim_{i\to\infty} \frac{\|u_i - \hat{u}^*\|^\rho_{L^p}}{\|u_i - \hat{u}^*\|^\beta_{L^q}}\right) = \bar{\mu}\left(\frac{\|\hat{u}^*\|^\rho_{L^p}}{\|\hat{u}^*\|^\beta_{L^q}} + \frac{(1 - \|\hat{u}^*\|^p_{L^p})^{\rho/p}}{(1 - \|\hat{u}^*\|^q_{L^q})^{\beta/q}}\right) > \bar{\mu},\end{aligned} \tag{42}$$

which implies a contradiction. Hence

$$\|\nabla\hat{u}^*\|^2_{L^2} = \lim_{i\to\infty} \|\nabla u_i\|^2_{L^2} = \bar{\mu}, \quad \|\hat{u}^*\|^p_{L^p} = \lim_{i\to\infty} \|u_i\|^p_{L^p}, \quad \|\hat{u}^*\|^q_{L^q} = \lim_{i\to\infty} \|u_i\|^q_{L^q},$$

and consequently, $\hat{u}^*$ is a minimizer of (36).

Due to the homogeneity of $\mu^E(u)$, we can find a minimizer $\hat{u}^E_{\hat{\mu}^E} \in \mathcal{D}$ of (36) which satisfies $\sigma^E(\hat{u}^E_{\hat{\mu}^E}) = 1$ and $t^E(\hat{u}^E_{\hat{\mu}^E}) = 1$. From this it follows that $\hat{u}^E_{\hat{\mu}^E}$ is weak solution of (30) with $\mu = \hat{\mu}^E$ and prescribed energy $E_{\hat{\mu}^E}(u) = E$. Since $\mu^E(|u|) = \mu^E(u)$ for $u \in \mathcal{D} \setminus 0$, we may assume that $\hat{u}^S_\mu \ge 0$ in $\mathbb{R}^N$. The Brezis & Kato Theorem [10] and the L^γ estimates for the elliptic problems [1] yield that $\hat{u}^E_\mu \in W^{2,\gamma}_{loc}(\mathbb{R}^N)$, for any $\gamma \in (1,+\infty)$, and whence by the regularity theory of the solutions of the elliptic problems, $\hat{u}_\mu \in C^2(\mathbb{R}^N)$. Consequently, the Harnack inequality [45] implies that $\hat{u}^E_\mu > 0$ in $\mathbb{R}^N$. □

Remark 5.5. The existence of spherically symmetric ground states of the "zero-mass" case problem (30) including more general form

$$-\Delta u = g(u), \quad u \in \mathcal{D}^{1,2}$$

was proved by Berestycki & Lions in [6]. This result is obtained in [6] under certain hypothesis including the following sufficient condition

$$\limsup_{s \to 0+} \frac{g(s)}{s^{2^*-1}} \leq 0. \tag{43}$$

Notice that in the case $2 < p < q < 2^*$, $\lim_{s \to 0+} g(s)/s^{2^*-1} = +\infty$. Thus, Corollary 5.1 yields that condition (43) is also necessary.

References

[1] S. Agmon, A. L. Douglis, L. Nirenberg: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, Comm. Pure Appl. Math. 12/4 (1959) 623–727.

[2] W. Allegretto, Y. X. Huang: *A Picone's identity for the p-Laplacian and applications*, Nonlinear Analysis 32/7 (1998) 819–830.

[3] A. Ambrosetti, H. Brezis, G. Cerami: *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Analysis 122/2 (1994) 519–543.

[4] T. Bartsch: *Topological Methods for Variational Problems with Symmetries*, Springer, Berlin (2006).

[5] T. Bartsch, T. Weth, M. Willem: *Partial symmetry of least energy nodal solutions to some variational problems*, J. d'Analyse Math. 96/1 (2005) 1–18.

[6] H. Berestycki, P. L. Lions: *Nonlinear scalar field equations. I: Existence of a ground state*, Arch. Ration. Mech. Analysis 82/4 (1983), 313–345.

[7] V. Bobkov: *Least energy nodal solutions for elliptic equations with indefinite nonlinearity*, Electron. J. Qual. Theory Diff. Equations 56 (2014) 1–15.

[8] V. Bobkov, P. Drábek, J. Hernández: *Existence and multiplicity results for a class of semilinear elliptic equations*, Nonlinear Analysis, Ser. A Theory Methods 200 (2020), art. no. 112017, 24 p.

[9] V. Bobkov, Y. Il'yasov: *Asymptotic behaviour of branches for ground states of elliptic systems*, Electron J. Diff. Equations 212 (2013) 1–21.

[10] H. Brezis, T. Kato: *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures Appl. 58/2 (1979) 137–151.

[11] K. J. Brown, T.-F. Wu: *A fibering map approach to a potential operator equation and its applications*, Diff. Int. Equations 22/11-12 (2009) 1097–1114.

[12] M. L. Carvalho, Y. Il'yasov, C. A. Santos: *Separating solutions of nonlinear problems using nonlinear generalized Rayleigh quotients*, Topol. Meth. Nonlinear Analysis (2021), 1–28.

[13] L. Cherfils, Y. Il'yasov: *On the stationary solutions of generalized reaction diffusion equations with p&q-Laplacian*, Comm. Pure Appl. Analysis 4/1 (2005) 9–22.

[14] D. C. Clark: *A variant of the Ljusternik–Schnirelmann theory*, Indiana Univ. Math. J. 22 (1972) 65–74.

[15] J. I. Díaz, J. Hernández, Y. S. Ilyasov: *On the existence of positive solutions and solutions with compact support for a spectral nonlinear elliptic problem with strong absorption*, Nonlinear Analysis 119 (2015) 484–500.

[16] J. I. Díaz, J. Hernández, Y. S. Ilyasov: *Flat solutions of some non-Lipschitz autonomous semilinear equations may be stable for $N \geq 3$*, Chin. Ann. Math. Ser. B 38/1 (2017) 345–378.

[17] J. I. Díaz, J. Hernández, Y. S. Ilyasov: *On the exact multiplicity of stable ground states of non-Lipschitz semilinear elliptic equations for some classes of starshaped sets*, Adv. Nonlinear Analysis 9/1 (2019) 1046–1065.

[18] E. DiBenedetto: *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Analysis 7/8 (1983) 827–850.

[19] I. Ekeland: *On the variational principle*, J. Math. Anal. Appl. 47/2 (1974) 324–353.

[20] F. Faraci, K. Silva: *On the Brezis–Nirenberg problem for a Kirchhoff type equation in high dimension*, Calc. Var. Partial Diff. Equations 60/1 (2021) 1–33.

[21] Y. Il'yasov: *On a procedure of projective fibering of functionals on Banach spaces*, Proc. Steklov Inst. Math. 232 (2001) 150–156.

[22] Y. Il'yasov: *On positive solutions of indefinite elliptic equations*, C. R. Acad. Sci. Paris Sér. I Math. 333/6 (2001) 533–538.

[23] Y. Il'yasov: *Non-local investigation of bifurcations of solutions of non-linear elliptic equations*, Izv. RAN. Ser. Mat. 66/6 (2002) 1103–1130.

[24] Y. Il'yasov: *On nonlocal existence results for elliptic equations with convex-concave nonlinearities*, Nonlinear Analysis 61/1-2 (2005) 211–236.

[25] Y. Il'yasov: *On critical exponent for an elliptic equation with non-Lipschitz nonlinearity*, in: Conference Publications 2011, American Institute of Mathematical Sciences, Springfield (2011) 698–706.

[26] Y. S. Il'yasov, P. Takác: *Optimal $W^{2,2}_{loc}$-regularity, Pohozhaev's identity, and nonexistence of weak solutions to some quasilinear elliptic equations*, J. Diff. Equations 252/3 (2012) 2792–2822.

[27] Y. S. Ilyasov: *On extreme values of Nehari manifold method via nonlinear Rayleigh's quotient*, Topol. Meth. Nonlinear Analysis 49/2 (2017) 683–714.

[28] Y. Il'yasov: *On orbital stability of the physical ground states of the NLS equations*, arXiv: 2103.16353 (2021).

[29] Y. Il'yasov, Y. Egorov: *Hopf boundary maximum principle violation for semilinear elliptic equations*, Nonlinear Analysis 72/7-8 (2010) 3346–3355.

[30] Y. Ilyasov, K. Silva: *On branches of positive solutions for p-Laplacian problems at the extreme value of the Nehari manifold method*, Proc. Amer. Math. Soc. 146/7 (2018) 2925–2935.

[31] P. Lancaster, M. Tismenetsky: *The Theory of Matrices: with Applications*, Elsevier, Amsterdam (1985).

[32] S. Lang: *Fundamentals of Differential Geometry*, Graduate Texts in Mathematics 191, Springer, New York (1999).

[33] G. M. Lieberman: *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Analysis 12/11 (1988) 1203–1219.

[34] P. L. Lions: *The concentration-compactness principle in the calculus of variations. The locally compact case*, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 109–145, 223–283.

[35] V. Lubyshev: *Three positive solutions of a nonlinear Dirichlet problem with competing power nonlinearities*, arXiv: 1501.03870 (2015).

[36] L. Lusternik: *Sur les extrémés relatifs des fonctionnelles*, Mat. Sbornik 41/3 (1934) 390–401.

[37] Z. Nehari: *On a class of nonlinear second-order differential equations*, Trans. Amer. Math. Soc. 95/1 (1960) 101–123.

[38] Z. Nehari: *Characteristic values associated with a class of nonlinear second-order differential equations*, Acta Math. 105 (1961) 141–175.

[39] S. I. Pohozaev: *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$ *(Russian)*, Dokl. Akad. Nauk SSSR 165 (1965) 36–39.

[40] S. I. Pohozaev: *On an approach to nonlinear equations*, Dokl. Acad. Sci. USSR 247 (1979) 1327–1331.

[41] S. I. Pohozaev: *On the method of fibering a solution in nonlinear boundary value problems*, Proc. Steklov Inst. Math. 192 (1992) 157–173.

[42] G. Siciliano, K. Silva: *The fibering method approach for a non-linear Schrödinger equation coupled with the electromagnetic field*, Publ. Mat. 64/2 (2020) 373–390.

[43] A. Szulkin, T. Weth: *The method of Nehari manifold*, in: *Handbook of Nonconvex Analysis and Applications*, D. Y. Gao and D. Motreanu (eds.), International Press, Boston (2010) 597–632.

[44] P. Tolksdorf: *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Equations 51 (1984) 126–150.

[45] N. S. Trudinger: *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. 20/4 (1967) 721–747.

[46] M. Willem: *Minimax Theorems*, Birkhäuser, Boston (1996).

[47] E. Zeidler: *Nonlinear Functional Analysis and its Applications. III: Variational Methods and Optimization*, Springer, Cham (2013).