Minimax Theory and its Applications
Volume 7 (2022), No. 2, 303–320

Normalized Solutions for a System of Fractional Schrödinger Equations with Linear Coupling

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Received: February 23, 2021
Accepted: October 5, 2021

We study the normalized solutions of the following fractional Schrödinger system:

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2}u + \beta v & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2 |v|^{q-2}v + \beta u & \text{in } \mathbb{R}^N, \end{cases}$$

with prescribed mass $\int_{\mathbb{R}^N} u^2 = a$ and $\int_{\mathbb{R}^N} v^2 = b$, where $s \in (0,1)$, $2 < p, q \leq 2_s^*$, $\beta \in \mathbb{R}$ and μ_1, μ_2, a, b are all positive constants. Under different assumptions on p, q and $\beta \in \mathbb{R}$, we succeed to prove several existence and nonexistence results about the normalized solutions. Specifically, in the case of mass-subcritical nonlinear terms, we overcome the lack of compactness by establishing the least energy inequality and obtain the existence of the normalized solutions for any given $a, b > 0$ and $\beta \in \mathbb{R}$. While for the mass-supercritical case, we use the generalized Pohozaev equality to get the boundedness of the Palais-Smale sequence and obtain the positive normalized solution for any $\beta > 0$. Finally, in the fractional Sobolev critical case i.e., $p = q = 2_s^*$, we give a result about the nonexistence of the positive solution.

Keywords: Fractional Laplacian, Schrödinger system, normalized solutions.

2010 Mathematics Subject Classification: 35R11, 35B09, 35B33.

1. Introduction

In the present paper, we consider the following system of nonlinear fractional Schrödinger equations:

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2}u + \beta v & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2 |v|^{q-2}v + \beta u & \text{in } \mathbb{R}^N, \\ u, v \in H^s(\mathbb{R}^N), \end{cases} \tag{1}$$

satisfying the prescribed mass

$$\int_{\mathbb{R}^N} u^2 = a \quad \text{and} \quad \int_{\mathbb{R}^N} v^2 = b, \tag{2}$$

where $0 < s < 1$, $N \geq 2$, μ_1, μ_2, a, b are positive parameters, $\beta \in \mathbb{R}$, $2 < p, q \leq 2_s^*$, and $2_s^* = \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent.

ISSN 2199-1413 (printed), ISSN 2199-1421 (electronic) / $ 2.50

The fractional Laplacian operator $(-\Delta)^s$ is defined as following

$$(-\Delta)^s u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}dy, \quad \text{for all } x \in \mathbb{R}^N.$$

It can be constructed as the infinitesimal generators of Lévy stable diffusion processes which have been widely applied to diverse physical phenomena, such as anomalous diffusion and molecular dynamics(see [3, 10]), and arises in chemistry, biology, finance areas (see[3, 4] and references therein).

Linearly coupled systems with $s = 1$ arise in nonlinear optics and have been widely investigated, mainly in the fixed frequency case, i.e., $-\lambda_1, -\lambda_2 > 0$ are prescribed, see e.g. [2, 14, 15]. In recent years, considerable attentions have been paid to the problem (1)-(2), that's to search for solutions of (1) having prescribed mass where $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ are unknown and appear as Lagrange multipliers. In the literature such solutions are called normalized solutions. From the physical point of view, prescribed mass represents the law of conservation of mass. Hence, it has particular significance for finding normalized solutions.

In order to obtain the normalized solutions, we usually consider the corresponding energy functional constrained on $S_a \times S_b$, where

$$S_a := \Big\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 = a\Big\}.$$

However, this process will meet some difficulties: in the mass-supercritical case, the authors are hard to obtain the existence and boundedness of the Palais-Smale sequence, the weak limit of the Palais-Smale sequence may not be on $S_a \times S_b$ (even in the radial space), since the embeddings $H^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ and $H^s_r(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ are not compact. Therefore, it becomes much more complex to study the normalized solutions of (1) comparing with the study of (1) with prescribed $(\lambda_1, \lambda_2) \in \mathbb{R}^2$.

As far as we know, many works about normalized solutions of Schrödinger equations or systems are based on the method due to [5, 6, 7, 8, 9, 17, 18, 19, 20, 26, 27] and references therein. Luo and Zhang in [24] studied the scalar fractional Schrödinger equation with combined power nonlinearities, and obtained some existence and nonexistence results for normalized solutions. The existence of normalized solutions for a fractional Schrödinger system with nonlinear coupling terms has been proved in [21] by variational methods.

However, there seems to exist no literature concerned about the system (1) with the mass (2) are prescribed, in spite of the physical relevance of normalized solutions. Based on this fact, we aim to study this type of problems in the current paper.

The existence of the normalized solutions to (1) can be formulated as follows: given $a, b > 0$, our motivation is to look for $(u, v, \lambda_1, \lambda_2) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) \times \mathbb{R} \times \mathbb{R}$ solving that

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2}u + \beta v & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2 |v|^{q-2}v + \beta u & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 = a \quad \text{and} \quad \int_{\mathbb{R}^N} v^2 = b. \end{cases}$$

A natural approach to searching normalized solutions of (1) satisfying the L^2-norm constraints

$$\int_{\mathbb{R}^N} u^2 := |u|_2^2 = a \quad \text{and} \quad \int_{\mathbb{R}^N} v^2 := |v|_2^2 = b, \tag{3}$$

consists in finding critical points $(u,v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ of the C^1 energy functional

$$I(u,v) := \int_{\mathbb{R}^N} \frac{1}{2}(|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2) - \frac{\mu_1}{p}|u|^p - \frac{\mu_2}{q}|v|^q - \beta uv, \tag{4}$$

under the constraint

$$S_a \times S_b := \Big\{(u,v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 = a, \int_{\mathbb{R}^N} |v|^2 = b\Big\}, \tag{5}$$

where $s \in (0,1), N \geq 2$,

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 = \int\int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy, \tag{6}$$

and $$H^s(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 < +\infty\}$$

is a Hilbert space endowed with the norm $||u||^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 + u^2$. Particularly, the parameters λ_1, λ_2 appear as the Lagrange multipliers.

Definition 1.1. We say $(\hat{u}, \hat{v})$ is a *normalized ground state* of (1)–(2), if it is a solution to (1)–(2) having minimal energy among all the normalized solutions:

$$I(\hat{u},\hat{v}) = \inf\{I(u,v) : (u,v) \text{ solves } (1)-(2) \text{ for some } (\lambda_1,\lambda_2) \in \mathbb{R}^2\}. \qquad \square$$

In the present paper, we will be mainly concerned about the ground state solutions. Denote the L^2-critical exponent of the fractional Schrödinger equation by

$$\overline{p} := 2 + \frac{4s}{N}.$$

By a simple calculation, in the mass-subcritical case (i.e., $p, q < \overline{p}$), $I(u,v)$ is coercive and bounded from below on $S_a \times S_b$, that is,

$$M(a,b) := \inf_{S_a \times S_b} I(u,v) > -\infty.$$

In the mass-supercritical case $I(u,v)$ is not bounded from below on $S_a \times S_b$.

Hence $$\inf_{S_a \times S_b} I(u,v) = -\infty.$$

Thus it is impossible to find a minimum of $I(u,v)$ on $S_a \times S_b$ like the mass-subcritical case. We have to search for a critical point with a minimax characterization. For simplicity, let $\gamma_p := \frac{N(p-2)}{2p}$. It is convenient to observe that

$$\frac{p\gamma_p}{2s} = \frac{N(p-2)}{4s} \begin{cases} < 1 & \text{if } 2 < p < \overline{p}, \\ = 1 & \text{if } p = \overline{p}, \\ > 1 & \text{if } \overline{p} < p < 2_s^*. \end{cases}$$

We first analyze the mass-subcritical case: $2 < p, q < \overline{p}$, $\mu_1, \mu_2 > 0$, $\beta \in \mathbb{R}$.

Theorem 1.2. *Let* $0 < s < 1$, $N \geq 2$, $2 < p, q < \overline{p}$ *and* $\mu_1, \mu_2 > 0$, *then for any* $a > 0, b > 0$, $M(a,b)$ *is achieved by* $(\tilde{u}, \tilde{v}) \in S_a \times S_b$ *and* $(\tilde{u}, \tilde{v})$ *is a ground state of* (1)–(2), *with the following properties:*

(i) *if* $\beta > 0$, $\tilde{u}, \tilde{v}$ *are both positive and radially symmetric;*

(ii) *if* $\beta < 0$, $\tilde{u}, \tilde{v}$ *are both radially symmetric, and either* $\tilde{u} > 0$, $\tilde{v} < 0$ *or* $\tilde{u} < 0$, $\tilde{v} > 0$.

If in addition $s \geq \frac{1}{2}$ *and* $N \leq 4s$, *then* $\lambda_1, \lambda_2 < 0$.

Next we consider the mass-supercritical case i.e., $\overline{p} < p, q < 2_s^*$.

Theorem 1.3. *Suppose* $\frac{1}{2} \leq s < 1$, $2 \leq N \leq 4s$, $\overline{p} < p, q < 2_s^*$ *and* $\mu_1, \mu_2, \beta > 0$. *Then for any* $a, b > 0$, *the problem* (1)-(2) *has a solution* $(\hat{u}, \hat{v})$ *with* $\lambda_1, \lambda_2 < 0$. *Furthermore,* $(\hat{u}, \hat{v})$ *is a positive radially symmetric ground state in the sense that*

$$I(\hat{u}, \hat{v}) = \inf\{I(u,v) : dI|_{\mathcal{P}_{a,b}} = 0, (u,v) \in S_a \times S_b\},$$

where $\mathcal{P}_{a,b}$ *is defined in* (22).

Remark 1.4. For every $\beta < 0$, the problem (1)-(2) has no positive ground state solutions at all. Indeed, let us assume that (w_1, w_2) is a positive ground state solution to (1)-(2) with $I(w_1, w_2) = m(a,b)$, where $m(a,b)$ is defined in (24). By a direct calculation, we have $I(-w_1, w_2) = I(w_1, -w_2) < I(w_1, w_2) = m(a,b)$. Since (w_1, w_2) satisfy $P(w_1, w_2) = P(-w_1, w_2) = P(w_1, -w_2) = 0$ and $(-w_1, w_2), (w_1, -w_2) \in S_a \times S_b$, we deduce that $I(-w_1, w_2) = I(w_1, -w_2) \geq m(a,b)$, this is a contradiction.

Finally, we consider the case of $p = q = 2_s^*$.

Theorem 1.5. (Nonexistence) *Let* $\frac{1}{2} \leq s < 1$, $2 \leq N \leq 4s$, $\mu_1, \mu_2, \beta > 0$ *and suppose* $p = q = 2_s^*$, *the problem* (1)–(2) *has no positive solution.*

This paper is organized as follows: In Section 2 we display some preliminary results which will be used from time to time in the paper. The proof of Theorem 1.2 will be completed in Section 3. Section 4 is devoted to the proof of Theorem 1.3. In Section 5, we deal with the fractional Sobolev exponent case and prove Theorem 1.5. Throughout the paper we denote the norm of $L^p(\mathbb{R}^N)$ by $|u|_p$,

$$H_r^s(\mathbb{R}^N) := \{u \in H^s(\mathbb{R}^N) : u(x) = u(|x|),\ x \in \mathbb{R}^N\}.$$

For convenience we write $H^s = H^s(\mathbb{R}^N)$, $H = H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$, $S_{a,r} = S_a \cap H_r^s(\mathbb{R}^N)$, $H_r = H_r^s \times H_r^s, \mathcal{P}_{a,b,r} := \mathcal{P}_{a,b} \cap H_r$. The symbol $||\cdot||$ denotes the norm of H or H^s. Denoting by u^* the symmetric decreasing rearrangement of $u \in H^s$. We recall that (see [22]) for $1 \leq p < +\infty$,

$$|u^*|_p = |u|_p \quad \text{and} \quad \int_{\mathbb{R}^N} uv \leq \int_{\mathbb{R}^N} u^* v^*.$$

2. Preliminary results

In this section, we will give several results and notations for convenience.
Let $u \in H^s(\mathbb{R}^N)$ and $2 < p < 2_s^*$, the fractional Gagliardo-Nirenberg-Sobolev (GNS) inequality ([16])

$$\int_{\mathbb{R}^N} |u|^p \leq C_{N,p,s}\Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\Big)^{\frac{N(p-2)}{4s}}\Big(\int_{\mathbb{R}^N} |u|^2\Big)^{\frac{p}{2}-\frac{N(p-2)}{4s}}. \tag{7}$$

Throughout the paper, we denote by

$$\begin{cases} C_1 := & \frac{1}{sp}\mu_1 C_{N,p,s} a^{\frac{p}{2}-\frac{N(p-2)}{4s}}, \\ C_2 := & \frac{1}{sq}\mu_2 C_{N,q,s} b^{\frac{q}{2}-\frac{N(q-2)}{4s}}, \\ C_3 := & \min\{\frac{p\gamma_p - 2s}{2p\gamma_p}, \frac{q\gamma_q - 2s}{2q\gamma_q}\}. \end{cases} \tag{8}$$

It then follows from (7) that

$$\frac{\mu_1}{p}\int_{\mathbb{R}^N} |u|^p \leq sC_1\Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2\Big)^{\frac{N(p-2)}{4s}}, \tag{9}$$

$$\frac{\mu_2}{q}\int_{\mathbb{R}^N} |v|^q \leq sC_2\Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2\Big)^{\frac{N(q-2)}{4s}}, \tag{10}$$

for any $(u, v) \in S_a \times S_b$. Let $0 < s < 1$. Recall that (see [1, Section 9]) for any $u \in H^s(\mathbb{R}^N)$

$$\int\int_{\mathbb{R}^{2N}} \frac{(u^*(x) - u^*(y))^2}{|x-y|^{N+2s}} dxdy \leq \int\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} dxdy. \tag{11}$$

To proceed, it is necessary to recall some results of scalar fractional Schrödinger equation with mass-subcritical or mass-supercritical nonlinearities. For fixed $a > 0$, $\mu > 0, 2 < p < 2_s^*$, solving the problem

$$\begin{cases} (-\Delta)^s u = \lambda u + \mu |u|^{p-2} u \quad \text{in } \mathbb{R}^N, \\ u > 0, \\ \int_{\mathbb{R}^N} |u|^2 = a, \end{cases} \tag{12}$$

is equivalent to finding the critical points of the energy functional

$$I_{p,\mu} : H^s(\mathbb{R}^N) \to \mathbb{R}, \quad I_{p,\mu}(u) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 - \frac{\mu}{p}\int_{\mathbb{R}^N} |u|^p, \tag{13}$$

which is constrained on S_a. We set

$$M_{p,\mu}(a) := \inf_{S_a} I_{p,\mu}(u). \tag{14}$$

A straightforward calculation shows that $I_{p,\mu}$ is coercive on S_a when $2 < p < \overline{p}$, while $I_{p,\mu}$ is not bounded from below on S_a for $\overline{p} < p < 2_s^*$. Thanks to the homogeneity of the nonlinear term, (12) is equivalent to (15) after suitable scalings,

$$(-\Delta)^s u + u = |u|^{p-2} u \quad \text{in } \mathbb{R}^N. \tag{15}$$

It is well known [16, Theorem 3.4] that, for $2 < p < 2_s^*$, problem (15) has a unique positive radial ground state solution, denoted by $Q_{N,p}$. By [24, Theorem 1.2] we know that if $p \in (2, \overline{p}) \cup (\overline{p}, 2_s^*)$, then (12) admits a unique positive radial solution for any $a > 0$ and $\mu = 1$.

However, if $p = \overline{p}$, (12) has a unique (up to a translation) positive radial solution when $a = |Q_{N,\overline{p}}|_2^2$ and $\mu = 1$. For simplicity, denote $\overline{a}_N := |Q_{N,\overline{p}}|_2^2$. As $p = \overline{p}$, formula (7) allows to characterize the critical mass as

$$\overline{a}_N = \Big(\frac{\overline{p}}{2C_{N,s,\overline{p}}}\Big)^{\frac{N}{2s}}.$$

Using the homogeneous property, the existence and uniqueness of the ground state normalized solution to (12) can be obtained easily, then we have the following Lemma.

Lemma 2.1. *Assume $N \geq 2$, $2 < p < \bar{p}$, $\mu > 0$.*
Then for any $a > 0$, problem (12) *has a unique positive radial minimizer $u_{p,\mu,a} \in S_a$. Moreover, $M_{p,\mu}(a) = I_{p,\mu}(u_{p,\mu,a}) < 0$.*

Proof. Since the Lemma can be proved following closely the method of [24, Theorem 1.2], we only provide the summary of the proof. By [16], $Q_{N,p}$ is the unique positive radial ground state solution of (15), since $2 < p < \bar{p}$. It follows that the unique minimizer of (12) is given by:

$$u_{p,\mu,a} = \mu^{-\frac{1}{p-2}}\beta^{-1}Q_{N,p}(\frac{1}{r}x),$$

with β, r satisfying

$$-\lambda r^{2s} = 1, \quad \beta^{2-p}r^{2s} = 1, \quad \mu^{-\frac{2}{p-2}}\beta^{-2}r^N|Q_{N,p}|_2^2 = a. \tag{16}$$

It is standard to show that $M_{p,\mu}(a) < 0$ for $2 < p < \bar{p}$, here we omit the detail. □

Lemma 2.2. *Assume that $s \in [\frac{1}{2}, 1)$, $2 \leq N \leq 4s$ and $(u, v) \in H_r$ is a solution of* (1) *with $2 < p, q \leq 2_s^*$. Suppose one of the following conditions holds:*

(I) *$\beta > 0$, $u > 0, v > 0$;*

(II) *$\beta < 0$ either $u > 0, v < 0$ or $u < 0, v > 0$.*

Then $\lambda_1, \lambda_2 < 0$.

Proof. (I) By $s \in [\frac{1}{2}, 1)$ and $2 \leq N \leq 4s$, we infer that $2 \leq \frac{N}{N-2s}$. Since $u > 0, v > 0$ satisfy

$$(-\Delta)^s u = \lambda_1 u + \mu_1 u^{p-1} + \beta v,$$

it follows that the right hand side is nonnegative if $\lambda_1 \geq 0$. We have from [21, Lemma 2.7] that $u = 0$. This contradicts the assumption that $u > 0$, which implies that $\lambda_1 < 0$. $\lambda_2 < 0$ can be proved in the similar way.

(II) Adopt a similar argument as the case (I). Since $u > 0, v < 0$, it yields that $(-\Delta)^s u \geq 0$, if $\lambda_1 \geq 0$; and $(-\Delta)^s(-v) \geq 0$, if $\lambda_2 \geq 0$. We thus have that $u = 0, v = 0$. This contradicts the assumption that $u > 0, v < 0$, which implies that $\lambda_1 < 0$, $\lambda_2 < 0$. The proof of the other part is analogous. □

We will need the following version of Brézis-Lieb Lemmas (see [11]) in the working space H and $H^s(\mathbb{R}^N)$ respectively. Since their proofs are standard, we would like to drop them.

Lemma 2.3. *Assume $(u_n,v_n)\subset H$ is a bounded sequence and $(u_n,v_n)\to(u,v)$ a.e. in $\mathbb{R}^N$, then for $2\le p\le 2_s^*$, we have*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|u_n|^p-|u|^p-|u_n-u|^p=0,$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|v_n|^q-|v|^q-|v_n-v|^q=0,$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}u_nv_n-uv-(u_n-u)(v_n-v)=0.$$

Lemma 2.4. *Let $H^s(\mathbb{R}^N)$ be the Hilbert space with $||u||_{H^s}^2=\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2+u^2$ as norm. If $\{u_n\}\subset H^s(\mathbb{R}^N)$ is a bounded sequence and $u_n\to u$ a.e. in $\mathbb{R}^N$, then*

$$\lim_{n\to+\infty}\Big(||u_n||_{H^s}^2-||u||_{H^s}^2-||u_n-u||_{H^s}^2\Big)=0,$$

$$\lim_{n\to+\infty}\Big(||u_n||_{L^2}^2-||u||_{L^2}^2-||u_n-u||_{L^2}^2\Big)=0.$$

Further, we infer that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_n|^2-|(-\Delta)^{\frac{s}{2}}u|^2-|(-\Delta)^{\frac{s}{2}}(u_n-u)|^2=0.$$

3. Mass-subcritical case

In this section, for $2<p,q<\overline{p}$, $\mu_1,\mu_2>0$, $\beta\in\mathbb{R}$, we prove Theorem 1.2.

Lemma 3.1. (1) *For any $a,b>0$, it follows that $M(a,b)\in(-\infty,0)$ with*

$$M(a,b):=\inf_{S_a\times S_b}I(u,v).$$

(2) *For any $a_1,a_2,b_1,b_2\ge 0$, there holds $M(a_1+a_2,b_1+b_2)\le M(a_1,b_1)+M(a_2,b_2)$.*

Proof. (1) For any $(u,v)\in S_a\times S_b$, we have from (9), (10) that

$$I(u,v)\ge\frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2+|(-\Delta)^{\frac{s}{2}}v|^2-C_1s\Big(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\Big)^{\frac{N(p-2)}{4s}}$$
$$-C_2s\Big(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}v|^2\Big)^{\frac{N(q-2)}{4s}}-|\beta|\sqrt{ab}.$$

Since $2<p,q<\overline{p}$, then $\frac{N(p-2)}{4s},\frac{N(q-2)}{4s}<1$, this implies that for every choice of $a>0,b>0$, $I(u,v)$ is coercive and bounded below on $S_a\times S_b$. Particularly, $M(a,b)>-\infty$. On the other hand,

$$I(u,v)=I_{p,\mu_1}(u)+I_{q,\mu_2}(v)-\beta\int_{\mathbb{R}^N}uv.$$

Since $I_{p,\mu_1}(u_{p,\mu_1,a})=I_{p,\mu_1}(-u_{p,\mu_1,a})$ and $I_{q,\mu_2}(v_{q,\mu_2,b})=I_{1,\mu_2}(-v_{q,\mu_2,b})$, hence for $\beta\in\mathbb{R}$ we immediately get that $M(a,b)\le M_{p,\mu_1}(a)+M_{q,\mu_2}(b)$, where I_{p,μ_1},I_{q,μ_2} are defined in (13); $M_{p,\mu_1}(a),M_{q,\mu_2}(b)$ are defined in (14). It follows from Lemma 2.1 that $M_{p,\mu_1}(a)<0$, $M_{q,\mu_2}(b)<0$. We thus derive that $M(a,b)<0$.

(2) Inspired by [17], for any $\varepsilon > 0$, based on the fact that $C_0^\infty(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$, there always exist $(\phi_1, \phi_2), (\varphi_1, \varphi_2) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ satisfying $(\phi_1, \phi_2) \in S_{a_1} \times S_{b_1}$ and $(\varphi_1, \varphi_2) \in S_{a_2} \times S_{b_2}$. In particular, we may require that ϕ_1, φ_1 have the same sign and ϕ_2, φ_2 have the same sign. They satisfy

$$I(\phi_1, \phi_2) \leq M(a_1, b_1) + \varepsilon \quad \text{and} \quad I(\varphi_1, \varphi_2) \leq M(a_2, b_2) + \varepsilon.$$

Let us suppose that $supp\, \phi_i \cap supp\, \varphi_j = \emptyset$ (for $i, j = 1, 2$). We thus obtain that $(\phi_1 + \varphi_1, \phi_2 + \varphi_2) \in S_{a_1+a_2} \times S_{b_1+b_2}$ and that

$$I(\phi_1 + \varphi_1, \phi_2 + \varphi_2) \leq I(\phi_1, \phi_2) + I(\varphi_1, \varphi_2) \leq M(a_1, b_1) + M(a_2, b_2) + 2\varepsilon.$$

Since ε is arbitrary, it then follows that

$$M(a_1 + a_2, b_1 + b_2) \leq M(a_1, b_1) + M(a_2, b_2). \qquad \square$$

Lemma 3.2. *Let* $M_r(a, b) := \inf\limits_{S_{a,r} \times S_{b,r}} I(u, v)$, *then holds that* $M_r(a, b) = M(a, b)$.

Proof. The inequality $M_r(a, b) \geq M(a, b)$ is obvious since $S_{a,r} \times S_{b,r} \subset S_a \times S_b$. For any $(u, v) \in S_a \times S_b$, we have from (11) that $(u^*, v^*) \in S_{a,r} \times S_{b,r}$ and

$$I(u, v) \geq I(u^*, v^*).$$

It implies that $m(a, b) \geq m_r(a, b)$ and then $m_r(a, b) = m(a, b)$. $\square$

Lemma 3.3. *Let* $\{(u_n, v_n)\} \subset S_a \times S_b$ *be a minimizing sequence for* $I|_{S_a \times S_b}$ *at the level* $M(a, b)$.

(i) *If* $\beta > 0$, *then* $\{(|u_n|, |v_n|)\}$ *is also a minimizing sequence.*

(ii) *If* $\beta < 0$, *then* $\{(-|u_n|, |v_n|)\}$ *or* $\{(|u_n|, -|v_n|)\}$ *is also a minimizing sequence.*

Proof. (i) Obviously, $(|u_n|, |v_n|) \in S_a \times S_b$. We derive from (6) that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} |u_n||^2 \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2,$$

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} |v_n||^2 \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \quad \text{and} \quad \int_{\mathbb{R}^N} u_n v_n \leq \int_{\mathbb{R}^N} |u_n||v_n|.$$

Since $\beta > 0$, it follows that $I(|u_n|, |v_n|) \leq I(u_n, v_n)$. Clearly, $\{(|u_n|, |v_n|)\}$ is also a minimizing sequence.

(ii) Adopt a similar argument as that of case (i), for $\beta < 0$, we have that

$$(-|u_n|, |v_n|), (|u_n|, -|v_n|) \in S_a \times S_b; \quad I(-|u_n|, |v_n|) = I(|u_n|, -|v_n|) \leq I(u_n, v_n),$$

and then (ii) holds true. $\square$

Considering the minimizing sequence $(u_n, v_n) \in S_{a,r} \times S_{b,r}$ of $I|_{S_a \times S_b}$ at level $M(a, b)$, it follows from Lemma 3.1 that $\{(u_n, v_n)\}$ is bounded in H. Since $N \geq 2$, the embedding $H_r^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $2 < p < 2_s^*$ (see [23]), up to a subsequence, we deduce that there exists $(\tilde{u}, \tilde{v}) \in H_r$ such that $(u_n, v_n) \rightharpoonup (\tilde{u}, \tilde{v})$ in H_r, $(u_n, v_n) \to (\tilde{u}, \tilde{v})$ in $L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$ and that $(u_n, v_n) \to (\tilde{u}, \tilde{v})$ a.e. in $\mathbb{R}^N$.

Lemma 3.4. *Let* $\{(u_n, v_n)\}$ *be a minimizing sequence for* $M(a,b)$ *such that*

$$(u_n, v_n) \rightharpoonup (\tilde{u}, \tilde{v}) \text{ in } H, \ (u_n, v_n) \to (\tilde{u}, \tilde{v}) \text{ in } L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \text{ for } 2 \le p, q < 2_s^*.$$

Then $(u_n, v_n) \to (u,v)$ *in* H.

Proof. Due to $(u_n, v_n) \to (\tilde{u}, \tilde{v})$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$, we have that

$$\int_{\mathbb{R}^N} (u_n - \tilde{u})(v_n - \tilde{v}) dx \le |u_n - \tilde{u}|_2 |v_n - \tilde{v}|_2 \to 0. \tag{17}$$

Then, combining (9), (10), (17) with $(u_n, v_n) \to (\tilde{u}, \tilde{v})$ in $L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$, it follows that

$$M(a,b) \le I(\tilde{u}, \tilde{v}) \le \varliminf_{n \to +\infty} I(u_n, v_n) = M(a,b).$$

Clearly $||u_n|| \to ||\tilde{u}||$, $||v_n|| \to ||\tilde{v}||$. Based on the fact that H^s is a Hilbert space, it is not difficult to check that $(u_n, v_n) \to (u,v)$ in H. ☐

We need further compactness of $\{(u_n, v_n)\}$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ to proceed with the compactness of $\{(u_n, v_n)\}$ in H.

Lemma 3.5. $u_n \to \tilde{u}$ *in* $L^2(\mathbb{R}^N)$, $v_n \to \tilde{v}$ *in* $L^2(\mathbb{R}^N)$.

Proof. Due to the weakly lower semi-continuity, we have that $|\tilde{u}|_2^2 := a_1 \le a$, $|\tilde{v}|_2^2 := b_1 \le b$. Let $a_2 := a - a_1$, $b_2 := b - b_1$. Next we divide the proof into two main steps.

Step 1: If $a_1 < a$, $b_1 < b$. We derive from Lemmas 2.3 and 2.4 that

$$\begin{aligned} M(a,b) + o(1) = I(u_n, v_n) &= I(\tilde{u}, \tilde{v}) + I(u_n - \tilde{u}, v_n - \tilde{v}) + o(1) \\ &\ge M(a_1, b_1) + I(u_n - \tilde{u}, v_n - \tilde{v}) + o(1), \end{aligned} \tag{18}$$

where $(u_n - \tilde{u}, v_n - \tilde{v}) \rightharpoonup (0,0)$ in H_r; $u_n - \tilde{u} \to 0$ in $L^p(\mathbb{R}^N)$, $v_n - \tilde{v} \to 0$ in $L^q(\mathbb{R}^N)$; $|u_n - \tilde{u}|_2^2 \to a_2 > 0$, $|v_n - \tilde{v}|_2^2 \to b_2 > 0$. It follows that

$$\begin{aligned} &I(u_n - \tilde{u}, v_n - \tilde{v}) = \\ &= \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} (u_n - \tilde{u})|^2 + |(-\Delta)^{\frac{s}{2}} (v_n - \tilde{v})|^2 - \beta \int_{\mathbb{R}^N} (u_n - \tilde{u})(v_n - \tilde{v}) + o(1) \\ &\ge -|\beta| \sqrt{a_2 b_2}. \end{aligned} \tag{19}$$

Next we estimate $-|\beta|\sqrt{a_2 b_2}$. If $a_2 \le b_2$, then for every $v \in S_{b_2}$, we have obviously $(\sqrt{\frac{a_2}{b_2}} v, \pm v) \in S_{a_2} \times S_{b_2}$. It yields from Lemma 2.1 that $I_{q, \frac{\mu_2}{2}}(v_{q, \frac{\mu_2}{2}, b_2}) < 0$. Now, take $v = v_{q, \frac{\mu_2}{2}, b_2}$, then for $\beta > 0$, we see that

$$\begin{aligned} &I(\sqrt{\frac{a_2}{b_2}} v, v) = \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \frac{a_2}{b_2} |(-\Delta)^{\frac{s}{2}} v|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 - \frac{\mu_2}{q} |v|_q^q - \frac{\mu_1}{p} (\frac{a_2}{b_2})^{\frac{p}{2}} |v|_p^p - |\beta| \sqrt{a_2 b_2} \\ &\le 2 \Big(\frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 - \frac{\mu_2}{2q} |v|_q^q \Big) - |\beta| \sqrt{a_2 b_2} \\ &= 2 I_{q, \frac{\mu_2}{2}}(v_{q, \frac{\mu_2}{2}, b_2}) - |\beta| \sqrt{a_2 b_2} < -|\beta| \sqrt{a_2 b_2}, \end{aligned}$$

while for $\beta < 0$, we get that

$$
\begin{aligned}
&I(\sqrt{\frac{a_2}{b_2}}v, -v) = \\
&= \frac{1}{2}\int_{\mathbb{R}^N} \frac{a_2}{b_2}|(-\Delta)^{\frac{s}{2}}v|^2 + |(-\Delta)^{\frac{s}{2}}v|^2 - \frac{\mu_2}{q}|v|_q^q - \frac{\mu_1}{p}(\frac{a_2}{b_2})^{\frac{p}{2}}|v|_p^p - |\beta|\sqrt{a_2b_2} \\
&\leq 2\Big(\frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}v|^2 - \frac{\mu_2}{2q}|v|_q^q\Big) - |\beta|\sqrt{a_2b_2} \\
&= 2I_{q,\frac{\mu_2}{2}}(v_{q,\frac{\mu_2}{2},b_2}) - |\beta|\sqrt{a_2b_2} < -|\beta|\sqrt{a_2b_2}.
\end{aligned}
$$

Both cases imply that
$$M(a_2, b_2) \leq I(\sqrt{\frac{a_2}{b_2}}v, \pm v) < -|\beta|\sqrt{a_2b_2}. \tag{20}$$

Substituting (20) into (19), (18) and by Lemma 3.1(2), we infer that

$$M(a, b) \geq M(a_1, b_1) - |\beta|\sqrt{a_2b_2} > M(a_1, b_1) + M(a_2, b_2) \geq M(a_1 + a_2, b_1 + b_2),$$

this is a contradiction. If $a_2 > b_2$, similarly we can also get a contradiction.

Step 2: If $a_1 = a$, $b_1 < b$, then we have $a_2 = 0$ and $b_2 > 0$. For convenience, let $M(a_2, 0) := M_{p,\mu_1}(a_2)$, $M(0, b_2) := M_{q,\mu_2}(b_2)$. Analogously, it follows that

$$
\begin{aligned}
M(a, b) \geq& M(a_1, b_1) + I(u_n - \tilde{u}, v_n - \tilde{v}) + o(1) \\
\geq& M(a_1, b_1) + M(0, b_2) - I_{q,\mu_2}(v_{q,\mu_2,b_2}) + o(1) > M(a, b).
\end{aligned}
$$

Similarly if $a_1 < a$, $b_1 = b$, we obtain that

$$M(a, b) \geq M(a_1, b_1) + I(u_n - \tilde{u}, v_n - \tilde{v}) + o(1) > M(a, b).$$

Summing up, we get a contradiction which implies the strong L^2-convergence as desired. ☐

Now we begin to proceed with the proof of Theorem 1.2.

Proof of Theorem 1.2. (i) Since having established Lemmas 3.1, 3.2, 3.3 and Lemmas 3.4, 3.5, we choose $\{(u_n, v_n)\} \subset S_{a,r} \times S_{b,r}$ as the nonnegative minimizing sequence for $I|_{S_a \times S_b}$ at the level $M(a, b)$. Up to a subsequence, we deduce that $(u_n, v_n) \to (\tilde{u}, \tilde{v})$ in H_r. Particularly $(\tilde{u}, \tilde{v})$ is a minimizer for $M(a, b)$, that means, $I(\tilde{u}, \tilde{v}) = M(a, b)$. Hence, by the Lagrange multiplier rule, there exists $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that

$$
\begin{cases}
(-\Delta)^s\tilde{u} = \lambda_1 u + \mu_1|\tilde{u}|^{p-2}\tilde{u} + \beta\tilde{v} & \text{in } \mathbb{R}^N, \\
(-\Delta)^s\tilde{v} = \lambda_2 u + \mu_2|\tilde{v}|^{q-2}\tilde{v} + \beta\tilde{u} & \text{in } \mathbb{R}^N.
\end{cases} \tag{21}
$$

Since $\tilde{u} \geq 0, \not\equiv 0, \tilde{v} \geq 0, \not\equiv 0$, it follows from the strong maximum principle (see [25, Proposition 2.17]) that $\tilde{u}, \tilde{v} > 0$. We thus derive from Lemma 2.2 that $\lambda_1, \lambda_2 < 0$.

(ii) Let $\{(-|u_n|, |v_n|)\} \subset S_{a,r} \times S_{b,r}$ or $\{(|u_n|, -|v_n|)\} \subset S_{a,r} \times S_{b,r}$ be a minimizing sequence for $M(a, b)$. Using the same notation $(\tilde{u}, \tilde{v})$, analogously we have that either $\tilde{u} > 0, \tilde{v} < 0$ or $\tilde{u} < 0, \tilde{v} > 0$. Then we obtain from Lemma 2.2 that $\lambda_1, \lambda_2 < 0$. ☐

4. Mass-supercritical case

In this section, we consider the case $\overline{p} < p, q < 2_s^*,\ \mu_1, \mu_2, \beta > 0$ and prove Theorem 1.3.

4.1. Existence of a radial Palais-Smale sequence

We define
$$\mathcal{P}_{a,b} = \big\{(u,v) \in S_a \times S_b : P(u,v) = 0\big\}, \tag{22}$$
where
$$P(u,v) = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 + (-\Delta)^{\frac{s}{2}}v|^2 - \frac{N(p-2)\mu_1}{2sp}|u|^p - \frac{N(q-2)\mu_2}{2sq}|v|^q. \tag{23}$$

As a consequence of the Pohozaev identity (see [12, Appendix]), all solutions of (1)–(2) stay in $\mathcal{P}_{a,b}$. Hence if $(u,v) \in \mathcal{P}_{a,b}$ is a minimizer of the constraint minimization
$$m(a,b) = \inf_{(u,v)\in P_{a,b}} I(u,v), \tag{24}$$
and (u,v) solves (1) for some λ_1, λ_2, then (u,v) is a normalized ground state of (1)–(2). Clearly $m(a,b) > -\infty$ (see Lemma 4.1).

To investigate the minimization problem (24), we introduce a transformation preserving the L^2-norm: for $u \in S_a$ and $\tau \in \mathbb{R}^+$,
$$\tau \star u(x) := \tau^{\frac{N}{2}}u(\tau x), \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Then $\tau \star u \in S_a$. Define $\tau \star (u,v) := (\tau \star u, \tau \star v)$ and the fiber maps
$$\Phi(\tau,u,v) := \Phi_{(u,v)}(\tau) := I(\tau \star (u,v))$$
$$= \frac{\tau^{2s}}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2 - \frac{\tau^{p\gamma_p}\mu_1}{p}\int_{\mathbb{R}^N}|u|^p - \frac{\tau^{q\gamma_q}\mu_2}{q}\int_{\mathbb{R}^N}|v|^q - \beta\int_{\mathbb{R}^N} uv.$$

We easily get that $\Phi(\tau,u,v)$ is a C^1-functional. A straightforward calculation yields
$$P(\tau \star (u,v)) = \tau^{2s}\Big[\int\limits_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2 - \frac{\gamma_p}{s}\tau^{p\gamma_p-2s}\mu_1|u|_p^p - \frac{\gamma_q}{s}\tau^{q\gamma_q-2s}\mu_2|v|_q^q\Big],$$
and $\Phi'_{(u,v)}(\tau) = \frac{s}{\tau}P(\tau \star (u,v))$. Since being $p\gamma_p, q\gamma_q > 2s$, for any $(u,v) \in S_a \times S_b$, we observe that there exactly exists an unique $\tau_{(u,v)} \in (0,+\infty)$, which satisfies $P(\tau \star (u,v)) > 0$ iff $0 < \tau < \tau_{u,v}$; $P(\tau_{(u,v)} \star (u,v)) = 0$ and $P(\tau \star (u,v)) < 0$ iff $\tau > \tau_{u,v}$. The map $(u,v) \in S_a \times S_b \mapsto \tau_{(u,v)} \in \mathbb{R}^+$ is of class C^1(see Proposition 4.2).

Define the functional $\varphi : S_{a,r} \times S_{b,r} \to \mathbb{R} \cup \{+\infty\}$ by
$$\varphi(u,v) = \max_{\tau>0} I(\tau \star (u,v)) = \max_{\tau>0}\Phi(\tau,u,v) = \max_{\tau>0}\Phi_{u,v}(\tau).$$

Then φ is continuous and bounded from below.

Lemma 4.1. *For any $(u,v) \in \mathcal{P}_{a,b}$, there exists a positive constant $C_* > 0$ such that $\inf\limits_{(u,v)\in\mathcal{P}_{a,b}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2 \geq C_* > 0$. Furthermore, $m(a,b) > -\infty$.*

Proof. Let $\rho_0 := \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2$. It follows from (9) and (10) that

$$\begin{aligned}\rho_0 &= \frac{1}{s}\gamma_p\mu_1\int_{\mathbb{R}^N}|u|^p + \frac{1}{s}\gamma_q\mu_2\int_{\mathbb{R}^N}|v|^q\\ &\le C_1p\gamma_p\Big(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\Big)^{\frac{N(p-2)}{4s}} + C_2q\gamma_q\Big(\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}v|^2\Big)^{\frac{N(q-2)}{4s}}.\end{aligned} \tag{25}$$

for any $(u,v)\in\mathcal{P}_{a,b}$. Since $\frac{N(q-2)}{4s}, \frac{N(p-2)}{4s} > 1$, this implies that there exists a positive constant $C_* > 0$ such that

$$\inf_{(u,v)\in\mathcal{P}_{a,b}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2 \ge C_* > 0. \tag{26}$$

In addition, we derive from (8) and (25) that

$$\begin{aligned}I(u,v) &= \frac{\rho_0}{2} - \frac{\mu_1}{p}|u|_p^p - \frac{\mu_2}{q}|v|_q^q - \beta\int_{\mathbb{R}^N}uv\\ &\ge \min\{\frac{p\gamma_p-2s}{2p\gamma_p}, \frac{q\gamma_q-2s}{2q\gamma_q}\}\Big[\frac{\gamma_p\mu_1}{s}|u|_p^p + \frac{\gamma_q\mu_2}{s}|v|_q^q\Big] - \beta\int_{\mathbb{R}^N}uv\\ &\ge C_3\inf_{(u,v)\in\mathcal{P}_{a,b}}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2 - \beta\sqrt{ab}\\ &\ge C_3C_* - \beta\sqrt{ab}.\end{aligned}$$

Therefore $m(a,b) > -\infty$. □

Proposition 4.2. *The map* $(u,v)\in S_a\times S_b \mapsto \tau_{(u,v)}\in\mathbb{R}^+$ *is of class* C^1.

Proof. Set

$$\begin{aligned}\mathcal{P}^0_{a,b} &= \{(u,v)\in\mathcal{P}_{a,b} : \Phi''_{(u,v)}(1) = 0\}\\ &= \{(u,v)\in\mathcal{P}_{a,b} : 2s^2\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2 = \mu_1p\gamma_p^2|u|_p^p + \mu_2q\gamma_q^2|v|_q^q\}.\end{aligned}$$

Let us assume that there exists $(u,v)\in\mathcal{P}^0_{a,b}$. Then combining $P(u,v)=0$ defined in (22) with $\Phi''_{(u,v)}(1)=0$, we deduce that

$$\mu_1\gamma_p(p\gamma_p-2s)\int_{\mathbb{R}^N}|u|^p = \mu_2\gamma_q(2s-q\gamma_q)\int_{\mathbb{R}^N}|v|^q,$$

which implies $u\equiv 0, v\equiv 0$, since $p\gamma_p, q\gamma_q > 2s$, $\mu_1,\mu_2>0$. This is obviously a contradiction. Thus $\mathcal{P}^0_{a,b}=\emptyset$. Using the fact that $\Phi'_{(u,v)}(\tau)|_{\tau_{(u,v)}} = 0$, $\Phi''_{(u,v)}\tau|_{\tau_{(u,v)}} < 0$ and $\mathcal{P}^0_{a,b}=\emptyset$, by the implicit function theorem on $\Phi'_{(u,v)}(\tau)$, it follows that the map $(u,v)\mapsto\tau_{(u,v)}$ is of class C^1. □

Lemma 4.3. *Let* $m_r(a,b) := \inf_{\mathcal{P}_{a,b,r}} I(u,v)$, *then* $m_r(a,b) = m(a,b)$.

Proof. Obviously $m_r(a,b)\ge m(a,b)$. Using the properties of symmetric rearrangement, for any $(u,v)\in\mathcal{P}_{a,b}$, we know that $(u^*,v^*)\in S_{a,r}\times S_{b,r}$.

It follows from (11) that $I(\tau\star(u,v))\geq I(\tau\star(u^*,v^*))$, for any $\tau>0$. Hence,

$$I(u,v)=\max_{\tau>0}I(\tau\star(u,v))\geq\max_{\tau>0}I(\tau\star(u^*,v^*))\geq m_r(a,b),$$

this implies $m(a,b)\geq m_r(a,b)$ and then $m_r(a,b)=m(a,b)$. □

As an immediate corollary, we have

Corollary 4.4. $m_r(a,b)=m(a,b)=\inf\limits_{(u,v)\in S_{a,r}\times S_{b,r}}\varphi(u,v)$.

Lemma 4.5. *There exists a nonnegative radially symmetric Palais-Smale sequence* $\{(u_n,v_n)\}\subset S_{a,r}\times S_{b,r}$ *such that* $P(u_n,v_n)=0$ *and*

$$I(u_n,v_n)\to m(a,b),\quad I'|_{S_{a,r}\times S_{b,r}}(u_n,v_n)\to 0 \text{ as } n\to+\infty.$$

Proof. Based on [13, Lemma 3.1], we can find a nonnegative sequence $\{(\hat{u}_n,\hat{v}_n)\}\subset S_{a,r}\times S_{b,r}$ such that $\varphi(\hat{u}_n,\hat{v}_n)\to m(a,b)$ and $\varphi'(\hat{u}_n,\hat{v}_n)\to 0$ as $n\to\infty$. From the definition and properties of $\varphi(u,v)$, we know that, for any $(u,v)\in S_{a,r}\times S_{b,r}$, there exists a unique $\tau_{(u,v)}>0$ such that

$$\varphi(u,v)=\max_{\tau>0}I(\tau\star(u,v))=\max_{\tau>0}\Phi_{u,v}(\tau)=\Phi_{u,v}(\tau_{(u,v)})=\Phi(\tau_{(u,v)},u,v).$$

Let $\tau_n:=\tau_{(\hat{u}_n,\hat{v}_n)}$, it yields that

$$\varphi(\hat{u}_n,\hat{v}_n)=\max_{\tau>0}I(\tau\star(\hat{u}_n,\hat{v}_n))=I(\tau_n\star(\hat{u}_n,\hat{v}_n))=\Phi_{\hat{u}_n,\hat{v}_n}(\tau_n)=\Phi(\tau_n,\hat{u}_n,\hat{v}_n).$$

We immediately infer that $P(\tau_n\star(\hat{u}_n,\hat{v}_n))=0$. We have from Proposition 4.2 that $\partial_{(u,v)}t_{(u,v)}|_{(\hat{u}_n,\hat{v}_n)}$ exists. Now, take $(u_n,v_n)=(\tau_n\star(\hat{u}_n,\hat{v}_n))$, it results that $(u_n,v_n)\in S_{a,r}\times S_{b,r}$ and $I(u_n,v_n)\to m(a,b)$. Recall that

$$\Phi'_{(\hat{u}_n,\hat{v}_n)}(\tau)|_{\tau=\tau_n}=\frac{s}{\tau_n}P(\tau_n\star(\hat{u}_n,\hat{v}_n))=\frac{s}{\tau_n}P(u_n,v_n)=0$$

and $\varphi'(\hat{u}_n,\hat{v}_n)\to 0$, we have

$$\begin{aligned}\varphi'(\hat{u}_n,\hat{v}_n)=&\partial_\tau\Phi(\tau,\hat{u}_n,\hat{v}_n)|_{\tau_n}\cdot\partial_{(u,v)}\tau_{(u,v)}|_{(\hat{u}_n,\hat{v}_n)}+\partial_{(u,v)}\Phi(\tau_n,u,v)|_{(\hat{u}_n,\hat{v}_n)}\\&=\partial_{(u,v)}I(\tau_n\star(u,v)|_{(\hat{u}_n,\hat{v}_n)}\\&=I'_{S_{a,r}\times S_{b,r}}(\tau_n\star(u,v)|_{(\hat{u}_n,\hat{v}_n)}=I'_{S_{a,r}\times S_{b,r}}(u_n,v_n).\end{aligned}$$

Therefore, $I'|_{S_{a,r}\times S_{b,r}}(u_n,v_n)\to 0$. This completes the proof. □

4.2. Compactness of the Palais-Smale sequence

Lemma 4.6. *Let* $\{(u_n,v_n)\}\subset S_{a,r}\times S_{b,r}$ *be a nonnegative Palais-Smale sequence for* $I|_{S_{a,r}\times S_{b,r}}$ *at level* $m(a,b)$ *and suppose* $P(u_n,v_n)=0$, *then* $\{(u_n,v_n)\}\subset H$ *is a bounded sequence.*

Proof. Let $\rho := \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2$. We have from $P(u_n, v_n) = 0$ that

$$\rho = \frac{1}{s}\gamma_p \mu_1 \int_{\mathbb{R}^N} |u_n|^p + \frac{1}{s}\gamma_q \mu_2 \int_{\mathbb{R}^N} |v_n|^q. \tag{27}$$

Combining (27) with $I(u_n, v_n) \to m(a, b)$, there exists a constant $C > 0$ such that

$$\begin{aligned} C \geq & m(a,b) + \beta \int_{\mathbb{R}^N} u_n v_n + o(1) \\ = & \Big(\frac{N(p-2)}{4s} - 1\Big)\frac{\mu_1}{p} \int_{\mathbb{R}^N} |u_n|^p + \Big(\frac{N(q-2)}{4s} - 1\Big)\frac{\mu_2}{q} \int_{\mathbb{R}^N} |v_n|^q. \end{aligned}$$

Both the coefficients inside the bracket are positive and $\mu_1, \mu_2 > 0$, we obtain the boundedness of $|u_n|_p$ and $|v_n|_q$. We observe from (27) that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2$$

is bounded. Therefore we proved that $\{(u_n, v_n)\} \subset H$ is bounded. □

Lemma 4.7. *Suppose* $\frac{1}{2} \leq s < 1$ *and* $2 \leq N \leq 4s$. *Let* $\{(u_n, v_n)\} \subset S_{a,r} \times S_{b,r}$ *be a nonnegative Palais-Smale sequence for* $I|_{S_{a,r}\times S_{b,r}}$ *at level* $m(a, b)$ *satisfying* $P(u_n, v_n) = 0$. *Then, going if necessary to a subsequence,* $(u_n, v_n) \to (\hat{u}, \hat{v})$ *in* H, *and* $(\hat{u}, \hat{v})$ *is a positive radial solution of* (1)–(2) *with* $\lambda_1 < 0, \lambda_2 < 0$.

Proof. We know from Lemma 4.6 that $(u_n, v_n) \in H_r$ is bounded. Since $N \geq 2$, the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $p \in (2, 2_s^*)$, we deduce that there exists $(\hat{u}, \hat{v}) \in H_r$ such that, up to a subsequence,

$$\begin{aligned} (u_n, v_n) &\rightharpoonup (\hat{u}, \hat{v}) \quad \text{in } H_r, \\ (u_n, v_n) &\to (\hat{u}, \hat{v}) \quad \text{in } L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N), \\ (u_n, v_n) &\to (\hat{u}, \hat{v}) \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{28}$$

In addition $I|'_{S_{a,r}\times S_{b,r}}(u_n, v_n) \to 0$, by the Lagrange multipliers rule, there exists $(\lambda_{1,n}, \lambda_{2,n}) \in \mathbb{R}^2$ such that, as n large enough,

$$\begin{aligned} o(1)||(\phi, \varphi)||_H = & \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \phi + \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \varphi \\ & - \beta \int_{\mathbb{R}^N} (u_n \varphi + v_n \phi) dx - \lambda_{1,n} \int_{\mathbb{R}^N} u_n \phi dx - \lambda_{2,n} \int_{\mathbb{R}^N} v_n \varphi dx \\ & - \mu_1 \int_{\mathbb{R}^N} |u_n|^{p-2} u_n \phi dx - \mu_2 \int_{\mathbb{R}^N} |v_n|^{p-2} v_n \varphi dx, \end{aligned} \tag{29}$$

for every $(\phi, \varphi) \in H$. Now, take $(\phi, \varphi) = (u_n, 0)$ as a test function in (29), we have that

$$\lambda_{1,n} a = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 - \mu_1 \int_{\mathbb{R}^N} |u_n|^p dx - \beta \int_{\mathbb{R}^N} u_n v_n dx + o(1).$$

Combining Hölder's inequality and the boundedness of the sequences $(u_n, v_n) \in H_r$, $(u_n, v_n) \in L^p \times L^q$, it implies that $\lambda_{1,n} \in \mathbb{R}$ is bounded. Similarly, take $(\phi, \varphi) = (0, v_n)$ as a test function in (29), we obtain that $\lambda_{2,n} \in \mathbb{R}$ is bounded as well. Therefore, up to a subsequence, we have that $(\lambda_{1,n}, \lambda_{2,n}) \to (\lambda_1, \lambda_2) \in \mathbb{R}^2$.

We will claim that $\hat{u}, \hat{v} \neq 0$. We argue by contradiction and assume that $(\hat{u}, \hat{v}) = (0, 0)$. Since the strong L^p convergence of u_n and the strong L^q convergence of v_n i.e., $\int_{\mathbb{R}^N} |u_n|^p dx \to 0$, $\int_{\mathbb{R}^N} |v_n|^q dx \to 0$, we infer that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 = \frac{1}{s}\gamma_p \mu_1 |u_n|_p^p + \frac{1}{s}\gamma_q \mu_1 |v_n|_q^q \to 0,$$

this contradicts (26). Therefore $(\hat{u}, \hat{v})$ is a radial nonnegative nontrivial solution to (1). Furthermore, $\hat{u} = 0$ implies that $\hat{v} = 0$; $\hat{v} = 0$ implies that $\hat{u} = 0$. This gives immediately a contradiction for $\hat{u}, \hat{v} \neq 0$. Since $\hat{u} \geq 0, \not\equiv 0$ and $\hat{v} \geq 0, \not\equiv 0$, we derive from the strong maximum principle that $\hat{u} > 0$ and $\hat{v} > 0$. It follows from Lemma 2.2 (I) that $\lambda_1 < 0, \lambda_2 < 0$.

Recall that $P(u_n, v_n) = 0$, we thus have from (28) that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 = $$
$$= \frac{1}{s}\gamma_p \mu_1 |u_n|_p^p + \frac{1}{s}\gamma_q \mu_2 |v_n|_q^q = \frac{1}{s}\gamma_p \mu_1 |u|_p^p + \frac{1}{s}\gamma_q \mu_2 |v|_q^q + o(1).$$

Notice that $P(\hat{u}, \hat{v}) = 0$, then we deduce that, for n large enough

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 \to \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \hat{u}|^2 + |(-\Delta)^{\frac{s}{2}} \hat{v}|^2. \tag{30}$$

By Lemma 2.4 we obtain

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 \to \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \hat{u}|^2, \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \to \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \hat{v}|^2. \tag{31}$$

Next we prove strong L^2 convergence. Let $|\hat{u}|_2^2 := a_1$, $|\hat{v}|_2^2 =: b_1$. Then we have that $0 < a_1 \leq a$, $0 < b_1 \leq b$. Testing (29) with $(u_n, 0)$, $(0, v_n)$, and using (28)–(31), we conclude

$$-\lambda_1(a - a_1) = -\lambda_2(b - b_1) = \beta \lim_{n \to +\infty} \int_{\mathbb{R}^N} (u_n - \hat{u})(v_n - \hat{v}) dx. \tag{32}$$

We argue by contradiction and assume that $a_1 < a$. Then, it follows from (32) and $\lambda_1, \lambda_2 < 0$ that $b_1 < b$. By Lemma 2.3, Schwarz' inequality and Hölder's inequality,

$$2\sqrt{(-\lambda_1)(-\lambda_2)}\sqrt{(a - a_1)(b - b_1)} \leq -\lambda_1(a - a_1) + (-\lambda_2)(b - b_1)$$
$$= 2\beta \lim_{n \to +\infty} \int_{\mathbb{R}^N} (u_n - \hat{u})(v_n - \hat{v}) dx \leq 2\beta\sqrt{(a - a_1)(b - b_1)},$$

therefore

$$\beta \geq \sqrt{(-\lambda_1)(-\lambda_2)}. \tag{33}$$

Define $(\overline{u}, \overline{v}) := (\sqrt{-\lambda_2}\hat{u}, \sqrt{-\lambda_1}\hat{v})$. It is easy to see that $(\overline{u}, \overline{v}) \in H$ is a positive radial function and satisfies

$$(-\Delta)^s \overline{u} \geq \lambda_1 \overline{u} - \lambda_2 \overline{v} \quad \text{and} \quad (-\Delta)^s \overline{v} \geq \lambda_2 \overline{v} - \lambda_1 \overline{u}. \tag{34}$$

It follows from (34) that

$$(-\Delta)^s(\overline{u} + \overline{v}) \geq 0.$$

From the proof process of Lemma 2.2, we see that $\overline{u} = 0$, $\overline{v} = 0$, which is a contradiction. This proves that $a_1 = a$, $b_1 = b$ i.e., $(u_n, v_n) \to (\hat{u}, \hat{v})$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. Similarly as that in Lemma 3.4, we deduce that $(u_n, v_n) \to (\hat{u}, \hat{v})$ in H_r as desired. ☐

Proof of Theorem 1.3. We finish the proof of Theorem 1.3 by Lemmas 4.5, 4.6 and 4.7. ☐

5. Fractional Sobolev critical case

In this section we deal with the case $p = q = 2_s^*$, $\mu_1, \mu_2, \beta > 0$, and prove Theorem 1.5.

Lemma 5.1. *Suppose* $\frac{1}{2} \leq s < 1$, $2 \leq N \leq 4s$ *and* $\mu_1, \mu_2, \beta > 0$, *then*

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{2_s^*-2} u + \beta v & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2 |v|^{2_s^*-2} v + \beta u & \text{in } \mathbb{R}^N, \\ u, v \in H^s(\mathbb{R}^N), \end{cases} \tag{35}$$

has no positive solution.

Proof. Let us assume that (U, V) is a positive solution of the problem (35) satisfying $\int_{\mathbb{R}^N} |U|_2^2 = a > 0$, $\int_{\mathbb{R}^N} |V|_2^2 = b > 0$. It follows from Lemma 2.2 that $\lambda_1 < 0, \lambda_2 < 0$. By the Pohozaev identity ([12, Appendix]) and the definition of weak solutions to problem (35), we have that

$$\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} U|^2 + |(-\Delta)^{\frac{s}{2}} V|^2 \right) =$$

$$= \lambda_1 |U|_2^2 + \lambda_2 |V|_2^2 + \mu_1 |U|_{2_s^*}^{2_s^*} + \mu_2 |V|_{2_s^*}^{2_s^*} + 2\beta \int_{\mathbb{R}^N} UV,$$

and

$$\int_{\mathbb{R}^N} \left(|(-\Delta)^{\frac{s}{2}} U|^2 + |(-\Delta)^{\frac{s}{2}} V|^2 \right) =$$

$$= 2_s^* \left[\frac{\lambda_1}{2} |U|_2^2 + \frac{\lambda_2}{2} |V|_2^2 + \beta \int_{\mathbb{R}^N} UV \right] + \mu_1 |U|_{2_s^*}^{2_s^*} + \mu_2 |V|_{2_s^*}^{2_s^*}.$$

Hence

$$(-\lambda_1) \int_{\mathbb{R}^N} |U|^2 + (-\lambda_2) \int_{\mathbb{R}^N} |V|^2 = 2\beta \int_{\mathbb{R}^N} UV \leq 2\beta\sqrt{ab},$$

which implies that $\beta \geq \sqrt{\lambda_1 \lambda_2}$.

On the other hand, $(u_1, v_1) := (\sqrt{-\lambda_2}U, \sqrt{-\lambda_1}V)$, then (u_1, v_1) satisfies $u_1 > 0$, $v_1 > 0$ and

$$\begin{cases} (-\Delta)^s u_1 = \lambda_1 u_1 + \mu_1(-\lambda_2)^{\frac{-2s}{N-2s}}|u_1|^{2_s^*-2}u_1 + \beta\sqrt{\frac{\lambda_2}{\lambda_1}}v_1 & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v_1 = \lambda_2 v_1 + \mu_2(-\lambda_1)^{\frac{-2s}{N-2s}}|v_1|^{2_s^*-2}v_1 + \beta\sqrt{\frac{\lambda_1}{\lambda_2}}u_1 & \text{in } \mathbb{R}^N, \\ |u_1|_2^2 = -\lambda_2 a, \quad |v_1|_2^2 = -\lambda_1 b. \end{cases} \tag{36}$$

It follows from (5) and (36) that

$$(-\Delta)^s(u_1 + v_1) \geq 0.$$

We deduce from the proof process of Lemma 2.2 that $(u_1, v_1) = (0, 0)$, that means, $(U, V) = (0, 0)$. This is clearly a contradiction. □

Proof of Theorem 1.5. Theorem 1.5 follows from Lemma 5.1, then we finish the proof. □

Acknowledgements. The authors thank the anonymous reviewers and editors very much for their suggestions and comments which improved the manuscript consideralby.

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